Probabilistic Engineering Design
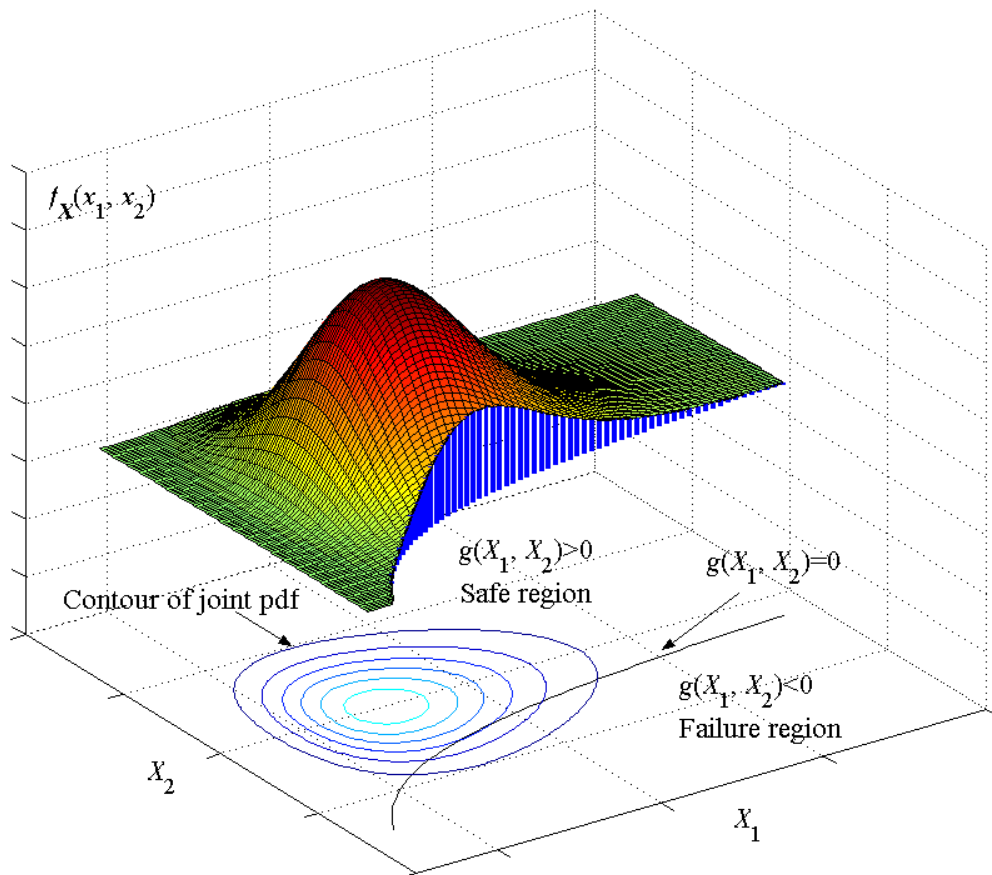
Chapter Seven

First Order and Second Reliability Methods

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$f_{\mathbf{X}}(x_1, x_2)$

Contour of joint pdf

$g(X_1, X_2)>0$
Safe region

$g(X_1, X_2)=0$

$g(X_1, X_2)<0$
Failure region

X_2

X_1

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Chapter 7

First Order and Second Reliability Methods

7.1   Introduction

As discussed in Chapter 6, reliability is defined as the probability of a performance function $g(\mathbf{X})$ greater than zero, i.e. $P\{g(\mathbf{X})>0\}$. In other words, reliability is the probability that the random variables $\mathbf{X}=(X_1,X_2,\cdots,X_n)$ are in the safe region that is defined by $g(\mathbf{X})>0$. The probability of failure is defined as the probability $P\{g(\mathbf{X})<0\}$. Or it is the probability that the random variables $\mathbf{X}=(X_1,X_2,\cdots,X_n)$ are in the failure region that is defined by $g(\mathbf{X})<0$. If the joint *pdf* of $\mathbf{X}$ is $f_\mathbf{x}(\mathbf{x})$, the probability of failure is evaluated with the integral

$$p_f=P\{g(\mathbf{X})<0\}= \int_{g(\mathbf{x})<0} f_\mathbf{x}(\mathbf{x})d\mathbf{x} \qquad (7.1)$$

The reliability is computed by

$$R=1-p_f=P\{g(\mathbf{X})>0\}= \int_{g(\mathbf{x})>0} f_\mathbf{x}(\mathbf{x})d\mathbf{x} \qquad (7.2)$$

In this chapter, two of the most commonly used reliability analysis methods, the First Order Reliability Method (FORM) and the Second Order Reliability Method (SORM) will be presented. The basic idea of the methods is to ease the computational difficulties through simplifying the integrand $f_\mathbf{x}(\mathbf{x})$ and approximating the performance function $g(\mathbf{X})$. With the simplification and approximation, solutions to Eqs. 7.1 and 7.2 will be easily obtained.

All the random variables $\mathbf{X}$ herein are assumed mutually independent. The methods discussed can be extended to problems with correlated random variables after those variables are converted to independent variables.

7.2   First Order Reliability Method

The name of First Order Reliability Method (FORM) comes from the fact that the performance function $g(\mathbf{X})$ is approximated by the first order Taylor expansion (linearization).

The probability integrations in Eqs. 7.1 and 7.2 are visualized with a two-dimensional case in Fig. 7.1. The figure shows the joint *pdf* $f_\mathbf{x}(\mathbf{x})$ and its contours, which are

projections of the surface of $f_x(\mathbf{x})$ on X_1 - X_2 plane. All the points on the contours have the same values of $f_x(\mathbf{x})$ or the same probability density. The integration boundary $g(\mathbf{X})=0$ is also plotted on X_1 - X_2 plane.
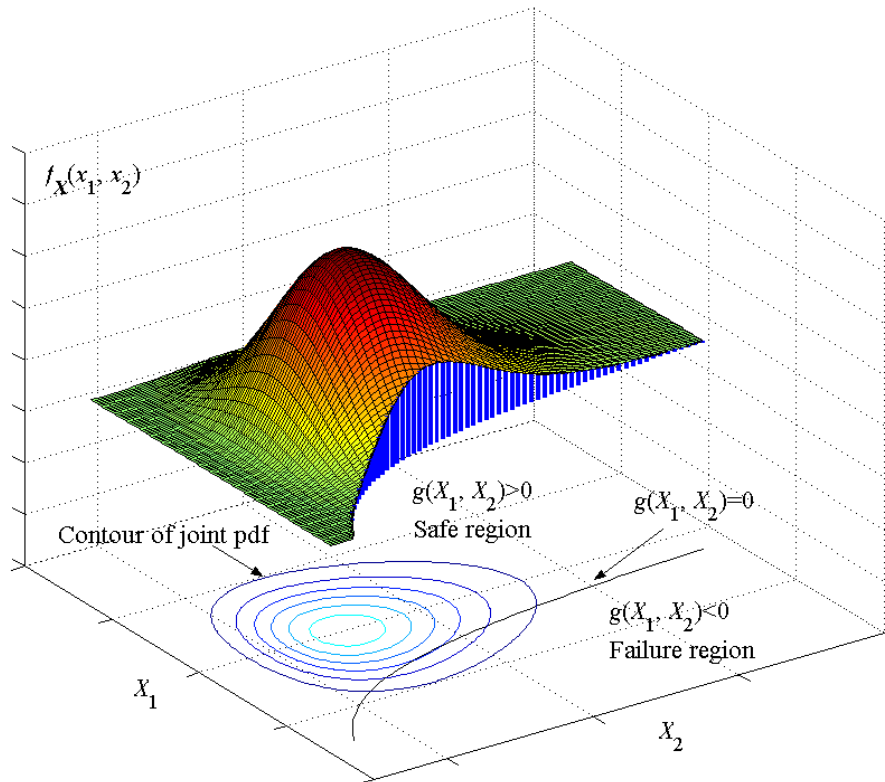

Figure 7.1 Probability Integration

The probability integration in Eqs. 7.1 or 7.2 is the volume underneath the surface (hyper-surface for the higher than 2-D problems) of the joint *pdf* $f_x(\mathbf{x})$ in the failure region $g(\mathbf{X})<0$ or the safe region $g(\mathbf{X})>0$. Imagine that the surface of the integrand $f_x(\mathbf{x})$ forms a "hill". If the hill were cut by a knife that has a blade shaped with the curve $g(\mathbf{X})=0$, the hill would be divided into two parts. If the part on the side of $g(\mathbf{X})<0$ were removed, the part left would be on the side of $g(\mathbf{X})>0$ as shown in Fig. 7.1. The volume left is the probability integration in Eq. 7.2, which represents the reliability. In other words, the reliability is the volume underneath $f_x(\mathbf{x})$ on the side of safe region $g(\mathbf{X})>0$. Of course, the probability of failure will be the volume underneath $f_x(\mathbf{x})$ on the side of failure region $g(\mathbf{X})<0$, the removed part.

To show the integration region more clearly, the contours of integrand $f_x(\mathbf{x})$ and the integration boundary $g(\mathbf{X})=0$ are plotted again in the random variable space (X-space)

in Fig. 7.2, which is X_1 - X_2 plane. The integration for the reliability is performed in the region where $g(\mathbf{X}) > 0$ while the integration for the probability of failure is performed in the region where $g(\mathbf{X}) < 0$.
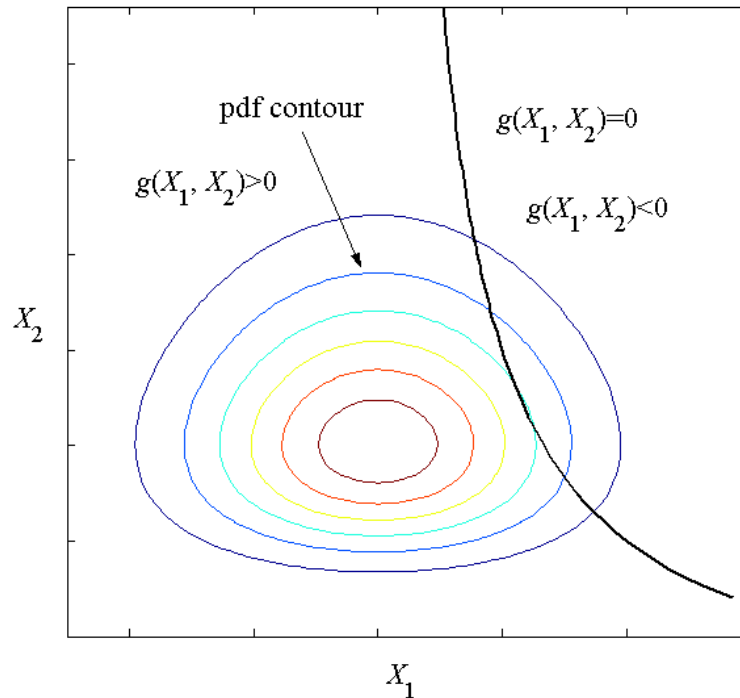

Figure 7.2 Probability Integration in X-Space

The direct evaluation of the probability integration in Eqs 7.1 and 7.2 is extremely difficult. The reasons are multifold. First, since a number of random variables $\mathbf{X}$ are involved, the probability integration is multidimensional. The dimensionality is typically high for engineering applications. Second, the integrand $f_{\mathbf{x}}(\mathbf{x})$ is the joint *pdf* of $\mathbf{X}$ and is generally a nonlinear multidimensional function. Third, the integration boundary $g(\mathbf{X}) = 0$ is also multidimensional and usually a nonlinear function. In many engineering applications, $g(\mathbf{X})$ is a black-box model (or simulation model), and the evaluation of $g(\mathbf{X})$ is computationally expensive. Examples of black-box models include finite element analysis, dynamic simulation, and computational fluid dynamics. Because of the complexities, there is seldom an analytical solution to the probability integration, except for very special cases. It is also unpractical using numerical integration to find the solution due to the high dimensionality in most engineering applications. To this end, approximation methods, such as the First Order Reliability Method (FORM) and Second Order Reliability Method (SORM) have been developed in the area of structural reliability.

Two steps are involved in these approximation methods to make the probability integration easy to be computed. The first step is to simplify the integrand $f_{\mathbf{x}}(\mathbf{x})$ so that its contours become more regular and symmetric, and the second step is to approximate the integration boundary $g(\mathbf{X}) = 0$. After the two steps, an analytical solution to the probability integration will be easily found.

The way to approximate the probability integration divides the methods into two types: the First Order Reliability Method (FORM) and the Second Order Reliability Method (SORM). We will discuss FORM first. The procedure of the First Order Reliability Method (FORM) is described below.

Step One – Simplify the integrand

The simplification is achieved through transforming the random variables from their original random space into a standard normal space. The space that contains the original random variables $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ is called X-space. To make the shape of the integrand $f_{\mathbf{x}}(\mathbf{x})$ *regular*, all the random variables $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ are transformed from X-space to a standard normal space, where the transformed random variables $\mathbf{U} = (U_1, U_2, \cdots, U_n)$ follow the standard normal distribution. The transformed space is termed as U-space. Recall that a standard normal variable has a mean of 0 and a standard deviation of 1.0.

The transformation from $\mathbf{X}$ to $\mathbf{U}$ is based on the condition that the *cdf*s of the random variables remain the same before and after the transformation. This type of transformation is called Rosenblatt transformation [1], which is expressed by

$$F_{X_i}(x_i) = \Phi(u_i) \tag{7.3}$$

in which $\Phi(\cdot)$ is the *cdf* of the standard normal distribution.

The transformed standard normal variable is then given by

$$U_i = \Phi^{-1}\left[F_{X_i}(X_i)\right] \tag{7.4}$$

For example, for a normally distributed random variable $X \sim N(\boldsymbol{m}, \boldsymbol{s})$, Eq. 7.4 yields

$$U = \Phi^{-1}[F_X(X)] = \Phi^{-1}\left[\Phi\left(\frac{X - \boldsymbol{m}}{\boldsymbol{s}}\right)\right] = \frac{X - \boldsymbol{m}}{\boldsymbol{s}} \tag{7.5}$$

Or

$$X = \boldsymbol{m} + \boldsymbol{s}U \tag{7.6}$$

It should be noted that in the above example the transformation from a normal variable to a standard normal variable is linear. The general transformation from a non-normal variable to a standard normal variable, however, is nonlinear,

After the transformation, the performance function becomes

$$Y = g(\mathbf{U}) \tag{7.7}$$

It is worthwhile to mention that after the transformation, the mathematical formulation of the performance function $g(\mathbf{X})$ will change. Without confusion, we will still use $g(\mathbf{U})$ to denote the transformed performance function in U-space in order to avoid introducing an additional symbol for the performance function.

After the transformation, the probability integration becomes

$$p_f = \mathrm{P}\{g(\mathbf{U}) < 0\} = \int_{g(\mathbf{U})<0} f_{\mathbf{U}}(\mathbf{u})d\mathbf{u} \tag{7.8}$$

where $f_{\mathbf{u}}(\mathbf{u})$ is the joint *pdf* of $\mathbf{U}$. Since all the random variables are independent, the joint *pdf* is the product of the individual *pdf*s of standard normal distribution and is then given by

$$f_{\mathbf{U}}(\mathbf{u}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2p}} \exp\left(-\frac{1}{2}u_i^2\right) \tag{7.9}$$

Therefore, the probability integration becomes

$$p_f = \int \cdots \int_{g(u_1,u_2,\cdots,u_n)<0} \prod_{i=1}^{n} \frac{1}{\sqrt{2p}} \exp\left(-\frac{1}{2}u_i^2\right) du_1 du_2 \cdots du_n \tag{7.10}$$

It should be noted that after the transformation, the integration in Eq. 7.10 in U-space is identical to that in Eq. 7.1 in X-space without any loss of accuracy, but the contours of the integrand $f_{\mathbf{U}}$ become concentric circles (or hyperspheres for a higher dimensional problem). The circular contours are shown in Figs. 7.3 and 7.4. It is obvious that the integrand $f_{\mathbf{U}}$ is easier to be integrated.
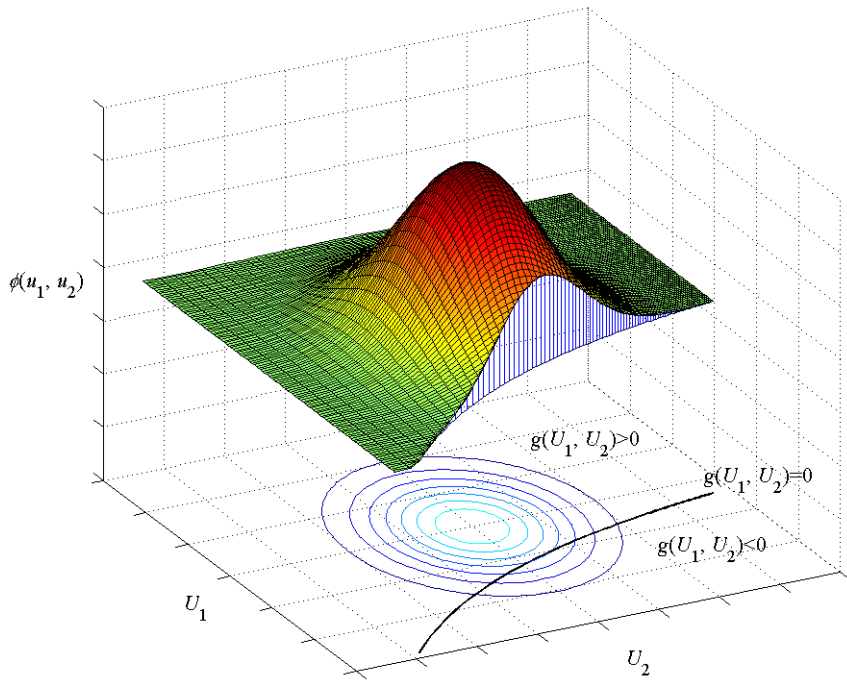
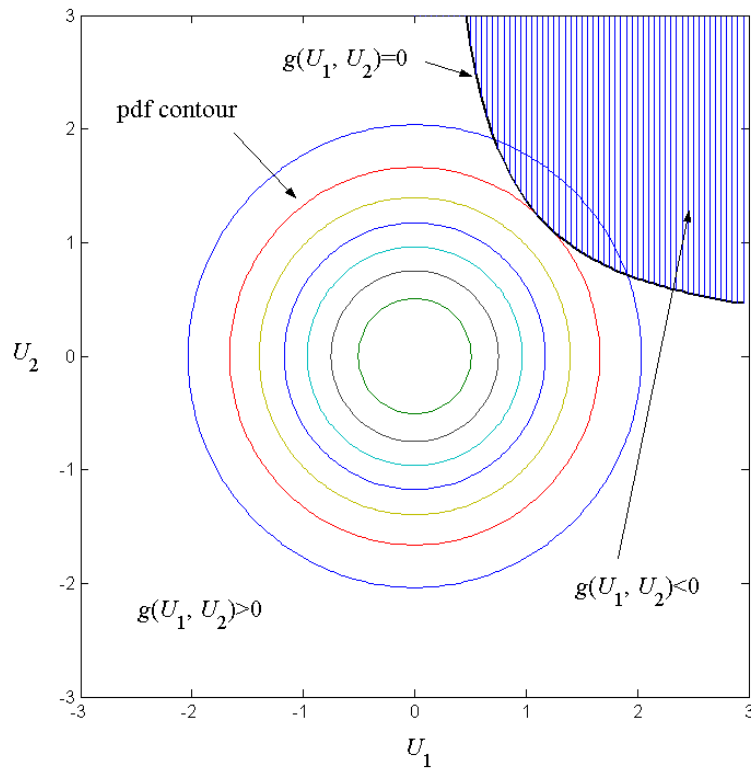**Figure 7.3 Probability Integration after the Transformation**

Figure 7.4 Probability Integration in U - Space

Step Two – Approximate the integration boundary

In order to further make the probability integration easier to be evaluated, in addition to simplifying the shape of the integrand, the integration boundary $g(\mathbf{U}) = 0$ will also be approximated. FORM uses a liner approximation (the first order Taylor expansion) as shown below.

$$g(\mathbf{U}) \approx L(\mathbf{U}) = g(\mathbf{u}^*) + \nabla g(\mathbf{u}^*)(\mathbf{U} - \mathbf{u}^*)^T \tag{7.11}$$

where $L(\mathbf{U})$ is the linearized performance function, $\mathbf{u}^* = \left(u_1^*, u_2^*, \cdots, u_n^*\right)$ is the expansion point, T stands for a transpose, and $\nabla g(\mathbf{u}^*)$ is the gradient of $g(\mathbf{U})$ at $\mathbf{u}^*$. $\nabla g(\mathbf{u}^*)$ is given by

$$\nabla g(\mathbf{u}^*) = \left(\frac{\partial g(\mathbf{U})}{\partial U_1}, \frac{\partial g(\mathbf{U})}{\partial U_2}, \cdots, \frac{\partial g(\mathbf{U})}{\partial U_n} \right)\Bigg|_{\mathbf{u}^*} \tag{7.12}$$

To minimize the accuracy loss, it is natural to expand the performance function $g(\mathbf{U})$ at a point that has the highest contribution to the probability integration. In other word, it is preferable to expand the function at the point that has the highest value of the integrand, namely, the highest probability density. With the integration going away from the expansion point, the integrand function values will quickly diminish. The point that has the highest probability density on the performance $g(\mathbf{U}) = 0$ is termed as the *Most Probable Point* (MPP). Therefore, the performance function will be approximated at the MPP. Maximizing the joint *pdf* $f_{\mathbf{u}}(\mathbf{u})$ at the limit state of $g(\mathbf{U}) = 0$ gives the location of the MPP. The mathematical model for locating the MPP is then given by

$$\begin{cases} \max_{\mathbf{u}} \quad \prod_{i=1}^{n} \frac{1}{\sqrt{2p}} \exp\left(-\frac{1}{2} u_i^2 \right) \\ \text{subject to} \quad g(\mathbf{u}) = 0 \end{cases} \tag{7.13}$$

Since

$$\prod_{i=1}^{n} \frac{1}{\sqrt{2p}} \exp\left(-\frac{1}{2} u_i^2 \right) = \frac{1}{\sqrt{2p}} \exp\left(-\frac{1}{2} \sum_{i=1}^{n} u_i^2 \right) \tag{7.14}$$

maximizing $\prod_{i=1}^{n} \frac{1}{\sqrt{2p}} \exp\left(-\frac{1}{2} u_i^2 \right)$ is equivalent to minimizing $\sum_{i=1}^{n} u_i^2$, the model for the MPP search can be rewritten as

$$\begin{cases} \min_{\mathbf{u}} \|\mathbf{u}\| \\ \text{subject to} \quad g(\mathbf{u}) = 0 \end{cases} \tag{7.15}$$

where $\|\cdot\|$ stands for the norm (length or magnitude) of a vector, namely,

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots u_n^2} = \sum_{i=1}^{n} u_i^2 \tag{7.16}$$

The solution to the model given in Eq. 7.15 is the MPP and is denoted by $\mathbf{u}^* = \left(u_1^*, u_2^*, \cdots u_n^* \right)$. As shown graphically in Figs. 7.5 and 7.6, the MPP is the shortest distance point from the limit state $g(\mathbf{U}) = 0$ to the origin O in U-space. The minimum distance $\boldsymbol{b} = \|\mathbf{u}^*\|$ is called *reliability index*.
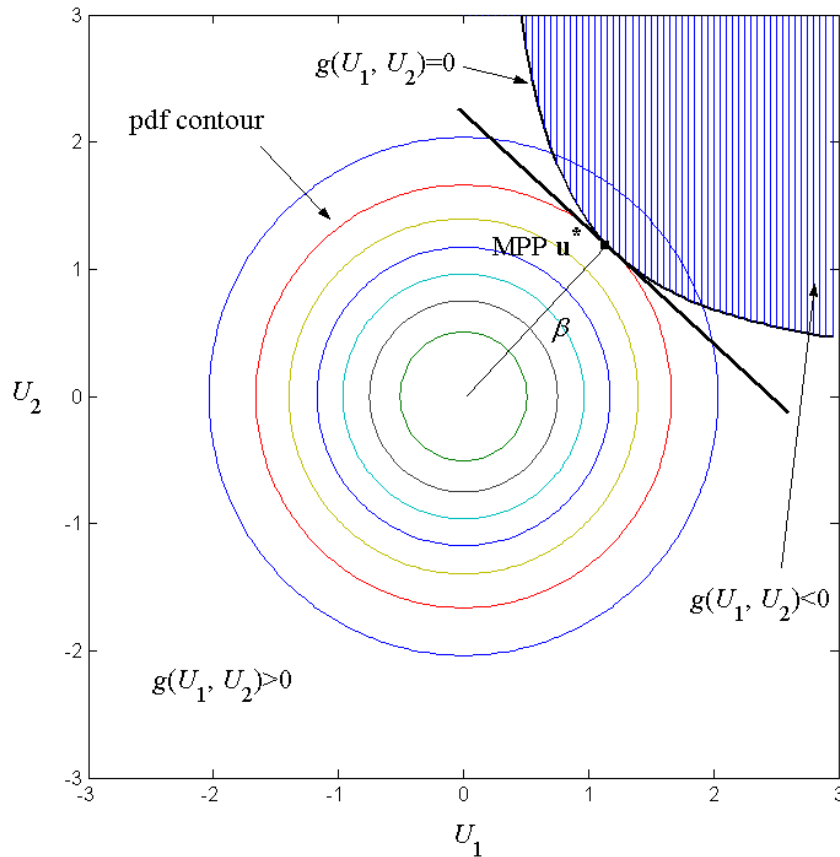
Figure 7.5 Probability Integration in FORM

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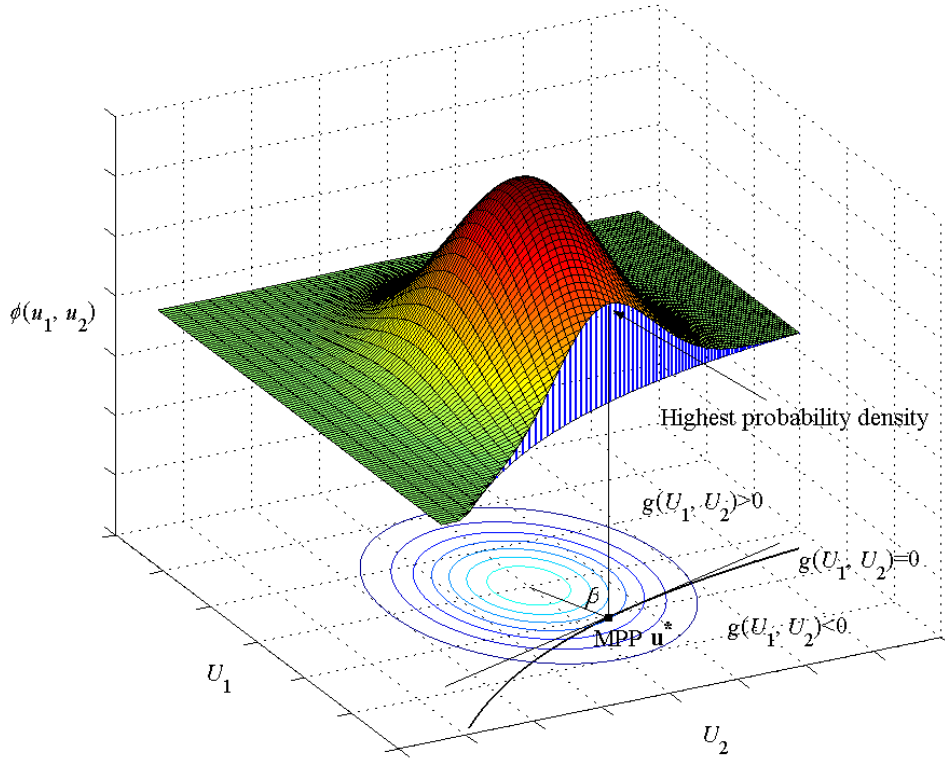
Figure 7.6 Highest Probability Density at the MPP

Since at the MPP $\mathbf{u}^*$, $g(\mathbf{U}) = 0$, Eq. 7.11 becomes

$$L(\mathbf{U}) = \sum_{i=1}^{n} \left. \frac{\partial g(\mathbf{U})}{\partial U_i} \right|_{\mathbf{u}^*} (\mathbf{U}_i - \mathbf{u}_i^*) = a_0 + \sum_{i=1}^{n} a_i U_i \qquad (7.17)$$

where

$$a_0 = -\sum_{i=1}^{n} \left. \frac{\partial g(\mathbf{U})}{\partial U_i} \right|_{\mathbf{u}^*} \mathbf{u}_i^* \qquad (7.18)$$

and

$$a_i = \left. \frac{\partial g(\mathbf{U})}{\partial U_i} \right|_{\mathbf{u}^*} \qquad (7.19)$$

Eq. 7.17 indicates that $L(\mathbf{U})$ is a linear function of standard normal variables. Therefore, $L(\mathbf{U})$ is also normally distributed. Its mean is given by

$$m_L = a_0 = -\sum_{i=1}^{n} \frac{\partial g(\mathbf{U})}{\partial U_i}\bigg|_{\mathbf{u}^*} u_i^* \tag{7.20}$$

and its standard deviation is given by

$$s_L = \sqrt{\sum_{i=1}^{n} a_i^2} = \sqrt{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial U_i}\bigg|_{u_i^*}\right)^2} \tag{7.21}$$

Therefore, using Eq. 4.6, the probability of failure is calculated by

$$p_f \approx P\{L(\mathbf{U}) < 0\} = \Phi(\frac{-m_L}{s_L}) = \Phi\left(\frac{\sum_{i=1}^{n} \frac{\partial g}{\partial U_i}\big|_{\mathbf{u}^*} u_i^*}{\sqrt{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial U_i}\big|_{\mathbf{u}^*}\right)^2}}\right) = \Phi\left(\sum_{i=1}^{n} a_i u_i^*\right) \tag{7.22}$$

where

$$a_i = \frac{\frac{\partial g}{\partial U_i}\big|_{\mathbf{u}^*}}{\sqrt{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial U_i}\big|_{\mathbf{u}^*}\right)^2}} \tag{7.23}$$

Let the vector of a_i be

$$\mathbf{a} = (a_1, a_2, \cdots, a_n) = \frac{\nabla g(\mathbf{u}^*)}{\left\|\nabla g(\mathbf{u}^*)\right\|} \tag{7.24}$$

The probability of failure can also be written as

$$p_f \approx \Phi\left(\sum_{i=1}^{n} a_i u_i^*\right) = \Phi\left(\mathbf{a}\mathbf{u}^{*T}\right) \tag{7.25}$$

in which $\mathbf{a}\mathbf{u}^{*T}$ is the inner (dot) product of the unit vector $\mathbf{a}$ and the vector of the MPP $\mathbf{u}^*$.

As shown in Fig. 7.7, since the MPP $\mathbf{u}^*$ is the shortest distance point from the origin to the performance function curve $g(\mathbf{U})=0$, the MPP is the tangent point of the curve $g(\mathbf{U})=0$ and the circle with the radius of $\boldsymbol{b}$. Therefore, the MPP vector $\mathbf{u}^*$ is perpendicular to the curve $g(\mathbf{U})=0$. The direction of the MPP can be represented by the unit vector $\mathbf{u}^*/\|\mathbf{u}^*\|=\mathbf{u}^*/\boldsymbol{b}$. On the other hand, the direction of the gradient is also perpendicular to the curve at the MPP, and its direction can be represented by the unit vector $\mathbf{a}$ (see Eq. 7.24). Therefore

$$\frac{\mathbf{u}^*}{\boldsymbol{b}}=\mathbf{a} \tag{7.26}$$

or,

$$\mathbf{u}^*=-\boldsymbol{b}\mathbf{a} \tag{7.27}$$
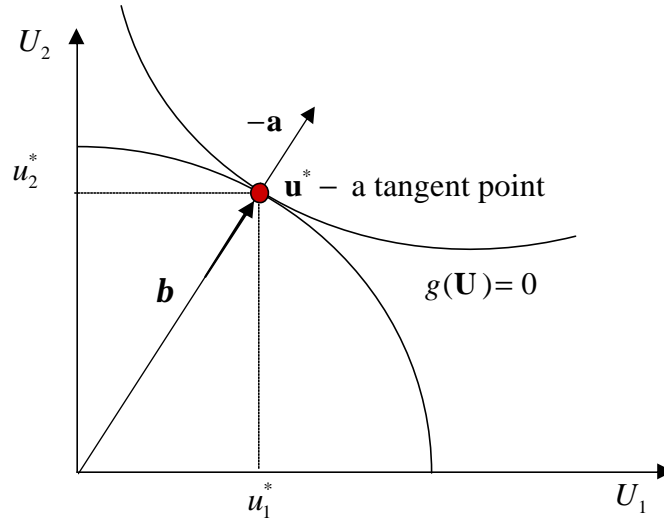
Figure 7.7 The MPP is A Tangent Point

Therefore, the probability of failure is evaluated by

$$p_f \approx P\{L(\mathbf{U})<0\}=\Phi\left(\mathbf{a}\mathbf{u}^{*T}\right)=\Phi\left(-\boldsymbol{b}\mathbf{a}\mathbf{a}^T\right)=\Phi\left(-\boldsymbol{b}\right) \tag{7.28}$$

Note that in the above derivation, $\mathbf{a}\mathbf{a}^T=\sum_{i=1}^{n}a_i^2=1$ is used.

The reliability is then given by

$$R=1-p_f=1-\Phi\left(-\boldsymbol{b}\right)=\Phi\left(\boldsymbol{b}\right) \tag{7.29}$$

The procedure of the FORM is briefly summarized below.

1) Transform the original random variables from X-space to U-space by Rosenblatt transformation.
2) Search the MPP in U-space and calculate the reliability index b.
3) Calculate reliability $R = \Phi(b)$.

7.3   MPP Search

From the above discussion, it is noted that the key to calculating the probability of failure or reliability is to locate the MPP in U-space. Since it is very difficult or even impossible to solve the MPP search model in Eq. 7.15 analytically, many numerical methods have been developed for the MPP search. Next, we will introduce a commonly used MPP search algorithm.

The MPP search algorithm uses a recursive formula and is based on the linearization of the performance function. The procedure is demonstrated in Fig. 7.8.

Figure 7.8 MPP Search

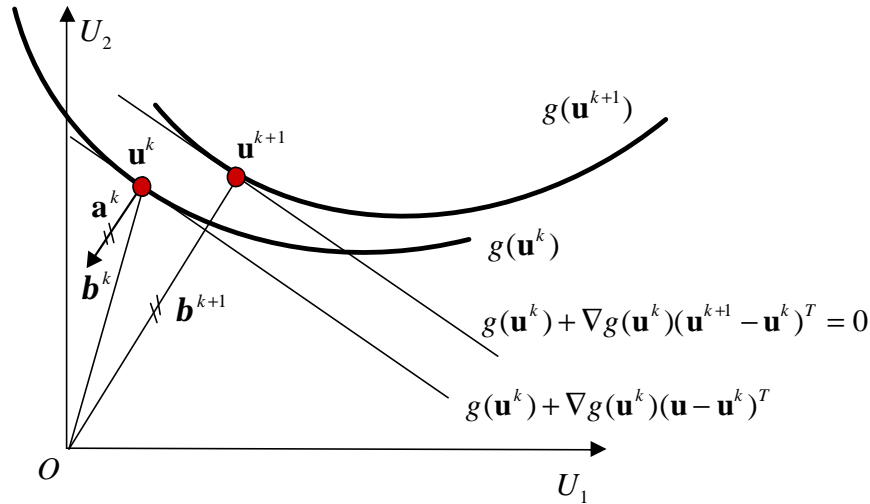Let the MPP in kth iteration be $\mathbf{u}^k$. The performance function is linearized at $\mathbf{u}^k$ as shown by the lower line in Fig. 7.8. The linearized function is given by

$$g(\mathbf{u}) = g(\mathbf{u}^k) + \nabla g(\mathbf{u}^k)(\mathbf{u} - \mathbf{u}^k)^T \tag{7.30}$$

Let the linearized function be zero, then the MPP $\mathbf{u}^{k+1}$ in the next iteration will be on the line, namely,

$$g(\mathbf{u}^{k+1}) = g(\mathbf{u}^k) + \nabla g(\mathbf{u}^k)(\mathbf{u}^{k+1} - \mathbf{u}^k)^T = 0 \tag{7.31}$$

The line is shown as the upper line in Fig. 7.8.

From Eq. 7.27,

$$\mathbf{u}^k = -\boldsymbol{b}^k \mathbf{a}^k \tag{7.32}$$

As shown in Fig. 7.8, since $\mathbf{u}^{k+1}$ is the shortest distance point from the origin to the line, vector $\mathbf{u}^{k+1}$ is perpendicular to the line and is directed from the origin O to $\mathbf{u}^{k+1}$. $\mathbf{a}^k$, which is the unit vector of the gradient, is also perpendicular to the line and in the opposite direction to $\mathbf{u}^{k+1}$. Because the magnitude of $\mathbf{u}^{k+1}$ is the distance from the origin to $\mathbf{u}^{k+1}$ (reliability index), then

$$\mathbf{u}^{k+1} = -\boldsymbol{b}^{k+1} \mathbf{a}^k \tag{7.33}$$

Substituting $\mathbf{u}^k$ in Eq. 7.32 and $\mathbf{u}^{k+1}$ in Eq. 7.33 into Eq. 7.31 yields

$$g(\mathbf{u}^k) + \nabla g(\mathbf{u}^k)(\mathbf{a}^k)^T (\boldsymbol{b}^k - \boldsymbol{b}^{k+1}) = g(\mathbf{u}^k) + \left\| \nabla g(\mathbf{u}^k) \right\| (\boldsymbol{b}^k - \boldsymbol{b}^{k+1}) = 0 \tag{7.34}$$

Rearranging Eq. 7.34 gives

$$\boldsymbol{b}^{k+1} = \boldsymbol{b}^k + \frac{g(\mathbf{u}^k)}{\left\| \nabla g(\mathbf{u}^k) \right\|} \tag{7.35}$$

Therefore, the updated point is given by

$$\mathbf{u}^{k+1} = -\mathbf{a}^k \left\{ \boldsymbol{b}^k + \frac{g(\mathbf{u}^k)}{\left\| \nabla g(\mathbf{u}^k) \right\|} \right\} \tag{7.36}$$

To use the recursive formula in Eqs. 7.35 and 7.36, a starting point $\mathbf{u}^0$ is required. Usually, the origin $\mathbf{u}^0 = \mathbf{0}$ is set as the starting point. Three convergence criteria may be used to terminate the MPP search process.

1) If $\left\| \mathbf{u}^{k+1} - \mathbf{u}^k \right\| \le e_1$, stop;
2) If $\left\| \nabla g(\mathbf{u}^{k+1}) - \nabla g(\mathbf{u}^k) \right\| \le e_2$, stop; or
3) If $\left| \boldsymbol{b}^{k+1} - \boldsymbol{b}^k \right| \le e_3$, stop.

e_1, e_2, and e_3 are very small quantities.

The flowchart of the MPP search is shown in Fig. 7.9.

```
        ┌───────────────────────────┐
       /     Input starting Point    /
      └───────────────────────────┘
                  │ u₀
                  ▼
        ┌─────────────────────┐
        │  u = u₀,  b = ‖u‖    │
        └─────────────────────┘
                  │
                  ▼
        ┌─────────────────────┐
        │       ∇g(u)          │
   ───▶ │  a = ───────         │
        │      ‖∇g(u)‖         │
        └─────────────────────┘
                  │
                  ▼
        ┌─────────────────────┐
        │              g(u)    │
        │ b_new = b + ───────  │
        │             ‖∇g(u)‖  │
        └─────────────────────┘
                  │
                  ▼
        ┌─────────────────────┐
        │   u_new = −a b       │
        └─────────────────────┘
```

$$\mathbf{u} = \mathbf{u}_0, \quad \boldsymbol{b} = \|\mathbf{u}\|$$

$$\mathbf{a} = \frac{\nabla g(\mathbf{u})}{\|\nabla g(\mathbf{u})\|}$$

$$\boldsymbol{b}_{new} = \boldsymbol{b} + \frac{g(\mathbf{u})}{\|\nabla g(\mathbf{u})\|}$$

$$\mathbf{u} = \mathbf{u}_{new}, \boldsymbol{b} = \boldsymbol{b}_{new}$$

$$\mathbf{u}_{new} = -\mathbf{a}\boldsymbol{b}$$

Y.

N.

$$\|\mathbf{u}_{new} - \mathbf{u}\| \leq \boldsymbol{e}_1$$
$$\|\nabla g(\mathbf{u}_{new}) - \nabla g(\mathbf{u})\| \leq \boldsymbol{e}_2$$
$$|\boldsymbol{b}_{new} - \boldsymbol{b}| \leq \boldsymbol{e}_3$$
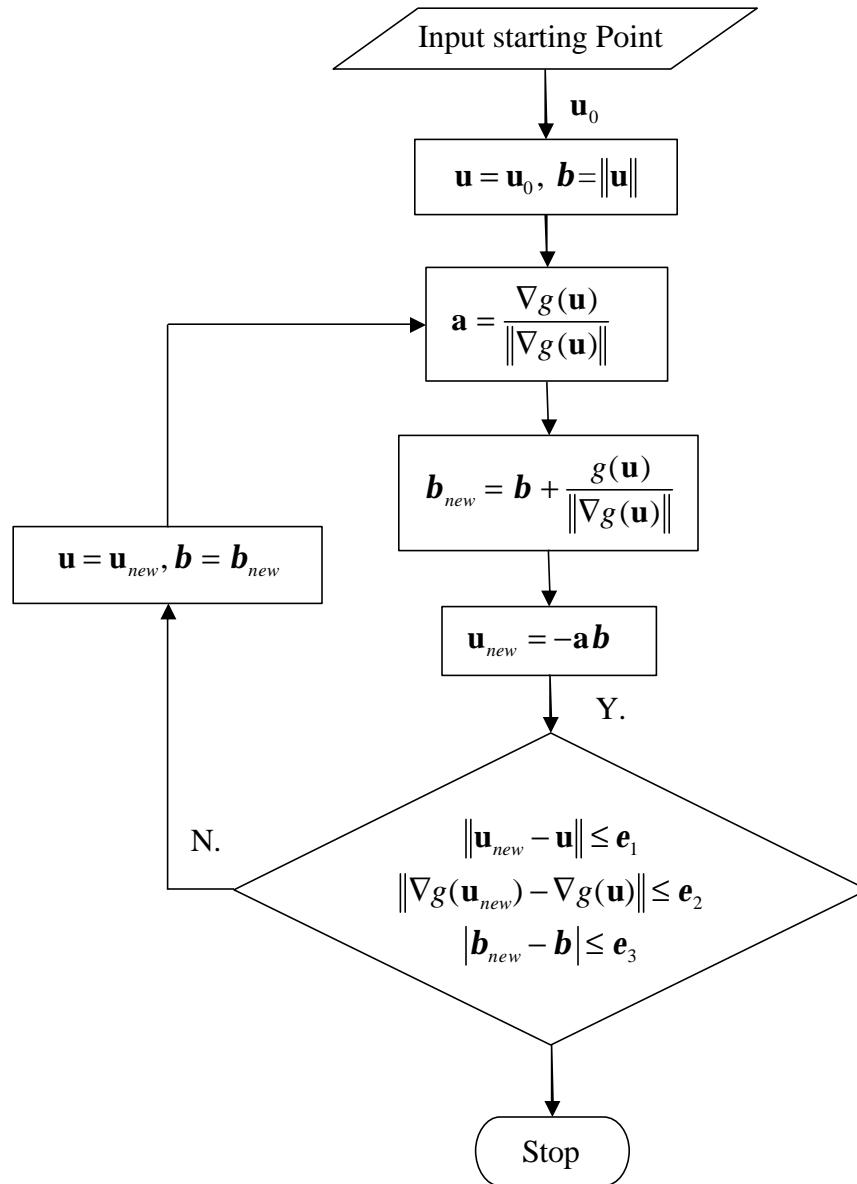
Stop

Figure 7.9 The Flowchart of the MPP Search

The above MPP search algorithm is easy to use and program, and the algorithm converges quickly. Because of the good features, it is widely used in the field of structural reliability and probabilistic engineering design. However, it may fail to converge for some problems, for example, oscillating among two or more points without convergence, or diverging away from the solution. If divergence occurs, one may

consider using other MPP search methods or using optimization techniques. We will discuss optimization later in this book.

Example 7.1

The strength and maximum stress of a mechanical component, X_1 and X_2, are normally distributed. $X_1 \sim N(200,20)\text{MPa}$ and $X_2 \sim N(150,10)\text{MPa}$. Use FORM to compute the probability of failure of the component.

The performance function is given by

$$g(\mathbf{X}) = X_1 - X_2$$

For the normally distributed random variables X_1 and X_2, using the transformation given in Eq. 7.6, the performance function becomes

$$g(\mathbf{U}) = m_1 + s_1 U_1 - (m_2 + s_2 U_2) = 20U_1 - 10U_2 + 50$$

The performance function $g(\mathbf{U})$ is plotted in Fig. 7.10. For this linear performance function, it is shown that the shortest distance point $\mathbf{u}^*$ is the intersection of the perpendicular line drawn from the origin O to the line of the limit state $g(\mathbf{U}) = 0$. The MPP $\mathbf{u}^*$ can be found graphically using the above geometric condition. It can also be solved by the above MPP search algorithm. If the MPP search algorithm is used, only one iteration is needed because of the linear performance function.
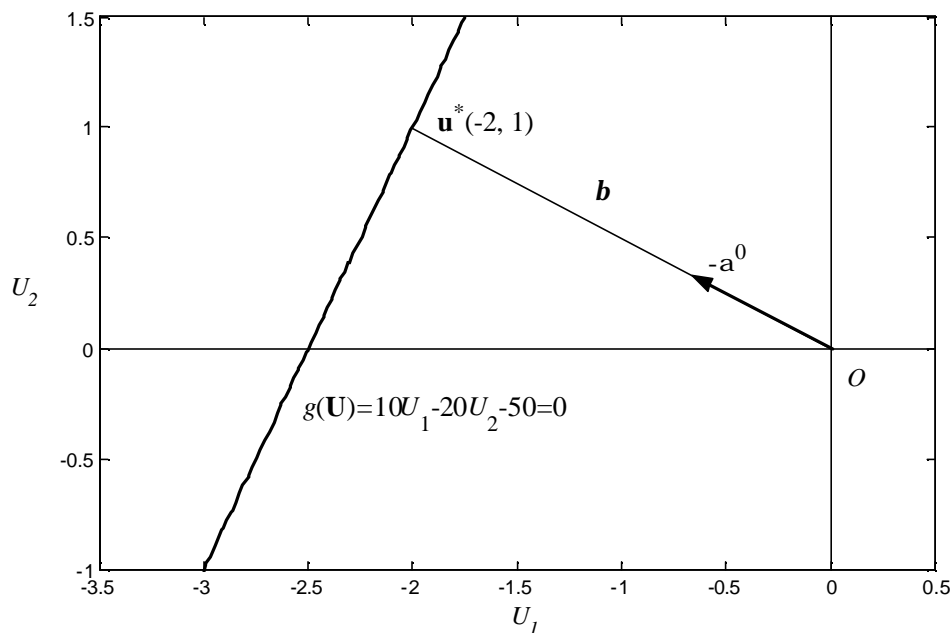

Figure 7.10 The Linear Performance in U-Space

The search process starts from the initial point $\mathbf{u}^0 = (0,0)$. At this starting point,

$$\nabla g(\mathbf{u}^0) = \left(\frac{\partial g}{\partial U_1}, \frac{\partial g}{\partial U_2} \right) = (20, -10), \ g(\mathbf{u}^0) = 50, \ \left\| \nabla g(\mathbf{u}^0) \right\| = \sqrt{(20)^2 + (-20)^2} = 22.3607,$$

$$\mathbf{a}^0 = \frac{\nabla g(\mathbf{u}^0)}{\left\| \nabla g(\mathbf{u}^0) \right\|} = (0.8944, -0.4472), \ \text{and} \ \boldsymbol{b}^0 = \left\| \mathbf{u}^0 \right\| = 0.$$

Using Eq. 7.36, the MPP is found at

$$\mathbf{u}^* = -\mathbf{a}^0 \left\{ \boldsymbol{b}^0 + \frac{g(\mathbf{u}^0)}{\left\| \nabla g(\mathbf{u}^0) \right\|} \right\} = -(0.8944, -0.4472) \frac{50}{22.3607} = (-2.0, 1.0)$$

The reliability index is

$$\boldsymbol{b} = \left\| \mathbf{u}^* \right\| = 2.2361$$

The probability of failure is calculated by

$$p_f = \Phi(-\boldsymbol{b}) = \Phi(-2.2361) = 0.0127$$

An analytical solution to this simple problem exists. Since $g(\mathbf{X})$ is a linear combination of normally distributed random variables X_1 and X_2, $g(\mathbf{X})$ also follows a normal distribution. Its mean and standard deviation are

$$m_g = m_1 - m_2 = 50$$

and

$$\boldsymbol{s}_g = \sqrt{m_1^2 + m_2^2} = 15.6205$$

respectively.

The accurate solution is

$$p_f = P\{g(\mathbf{X}) < 0\} = \Phi\left(\frac{0 - m_g}{\boldsymbol{s}_g} \right) = \Phi\left(\frac{0 - 50}{15.6205} \right) = \Phi(-2.2361) = 0.0127$$

It is noted FORM produces an accurate solution to the probability of failure for this specific problem where the performance function is linear in term of normally distributed variables in U-space.

Example 7.2

A cantilever beam is illustrated in Fig. 7.11. One of the failure modes is that the tip displacement exceeds the allowable value, D_0. The performance function is the difference between D_0 and the tip displacement, and the function is given by

$$g = D_0 - \frac{4L^3}{Ewt}\sqrt{\left(\frac{P_y}{t^2}\right)^2 + \left(\frac{P_x}{w^2}\right)^2}$$

where $D_0 = 3"$, $E = 30 \times 10^6 \text{psi}$ is the modulus of elasticity, $L = 100"$ is the length, w and t are width and height of the cross section, respectively, and $w = 2"$ and $t = 4"$. P_x and P_y are external forces with normal distributions $P_x \sim N(500,100) lb$ and $P_y \sim N(1000,100) lb$.
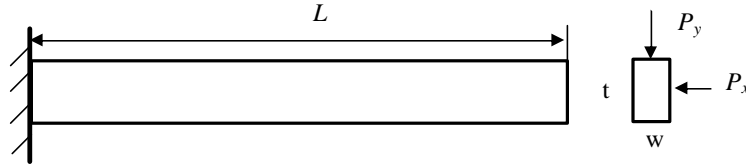

Figure 7.11 A Cantilever Beam

The probability of failure is defined as the probability of the allowable value less than the tip displacement, i.e.

$$p_f = P\left\{ g = D_0 - \frac{4L^3}{Ewt}\sqrt{\left(\frac{P_y}{t^2}\right)^2 + \left(\frac{P_x}{w^2}\right)^2} \leq 0 \right\}$$

First, the normally distributed variables P_x and P_y are transformed into the standard normal variables

$$\mathbf{U} = (U_x, U_y) = \left(\frac{P_x - m_{P_x}}{s_{P_x}}, \frac{P_y - m_{P_y}}{s_{P_x}} \right)$$

Or

$$\mathbf{X} = (P_x, P_y) = \left(m_{P_x} + U_x s_{P_x}, m_{P_y} + U_y s_{P_y} \right)$$

The transformed performance function in U-space becomes

$$g = D_0 - \frac{4L^3}{Ewt}\sqrt{\left(\frac{\boldsymbol{m}_{P_y} + U_y\boldsymbol{s}_{P_y}}{t^2}\right)^2 + \left(\frac{\boldsymbol{m}_{P_x} + U_x\boldsymbol{s}_{P_x}}{w^2}\right)^2}$$

The gradient of $g(\mathbf{U})$ is given by

$$\nabla g(\mathbf{U}) = \frac{4L^3}{Ewt}\left(\frac{(\boldsymbol{m}_x + U_x\boldsymbol{s}_x)\boldsymbol{s}_x}{w^4\sqrt{\left(\frac{\boldsymbol{m}_y + U_y\boldsymbol{s}_y}{t^2}\right)^2 + \left(\frac{\boldsymbol{m}_x + U_x\boldsymbol{s}_x}{w^2}\right)^2}}, \ \frac{(\boldsymbol{m}_y + U_y\boldsymbol{s}_y)\boldsymbol{s}_y}{t^4\sqrt{\left(\frac{\boldsymbol{m}_y + U_y\boldsymbol{s}_y}{t^2}\right)^2 + \left(\frac{\boldsymbol{m}_x + U_x\boldsymbol{s}_x}{w^2}\right)^2}}\right)$$

The starting point of the MPP is set to $\mathbf{u}^0 = (0,0)$.

Iteration 1

At $\mathbf{u}^0 = (0,0)$, $g(\mathbf{u}^0) = 0.67076$, $\nabla g(\mathbf{u}^0) = (-0.37268, \ -0.046585)$,

$\left\|\nabla g(\mathbf{u}^0)\right\| = \sqrt{(-0.37268)^2 + (-0.046585)^2} = 0.3756$,

$\mathbf{a}^0 = \dfrac{\nabla g(\mathbf{u}^0)}{\left\|\nabla g(\mathbf{u}^0)\right\|} = (-0.9923, \ -0.1240)$, and $\boldsymbol{b}^0 = \left\|\mathbf{u}^0\right\| = 0$.

Using Eq. 7.36 produces a new point

$$\mathbf{u}^1 = -\mathbf{a}^0\left\{\boldsymbol{b}^0 + \frac{g(\mathbf{u}^0)}{\left\|\nabla g(\mathbf{u}^0)\right\|}\right\} = -(-0.9923, \ -0.1240)\frac{0.67076}{0.3756} = (1.7722, 0.22152)$$

Iteration 2

At $\mathbf{u}^1 = (1.7722, 0.22152)$, $g(\mathbf{u}^1) = -0.015931$, $\nabla g(\mathbf{u}^1) = (-0.38984, \ -0.036775)$,

$\left\|\nabla g(\mathbf{u}^1)\right\| = \sqrt{(-0.38984)^2 + (-0.036775)^2} = 0.3916$,

$\mathbf{a}^1 = \dfrac{\nabla g(\mathbf{u}^1)}{\left\|\nabla g(\mathbf{u}^1)\right\|} = (-0.9956, \ -0.0939)$, and $\boldsymbol{b}^1 = \left\|\mathbf{u}^1\right\| = 1.7859$.

Using Eq. 7.36 again produces a new point,

$$\mathbf{u}^2 = -\mathbf{a}^1 \left\{ \boldsymbol{b}^1 + \frac{g(\mathbf{u}^1)}{\left\| \nabla g(\mathbf{u}^1) \right\|} \right\} = -(-0.9956, \ -0.0939) \left(1.7859 + \frac{-0.015931}{0.3916} \right)$$

$$= (1.7375, \ 0.16391)$$

The process continues until the solution converges. The search determines after 4 iterations because the solutions in iteration 4 are very close to those in iteration 3. The complete convergence history is shown in Table 7.1 and Fig. 7.12.

Table 7.1 MPP Search History

Iteration	$\boldsymbol{b}$	g	∇g	(U_x, U_y)
0	0	0.67076	(-0.37268, -0.046585)	(1.7722, 0.22152)
1	1.7859	-0.015931	(-0.38984, -0.036775)	(1.7375, 0.16391)
2	1.7453	-0.00032102	(-0.38986, -0.036758)	(1.7367, 0.16375)
3	1.7444	-2.6004e-009	(-0.38986, -0.036761)	(1.7367, 0.16376)
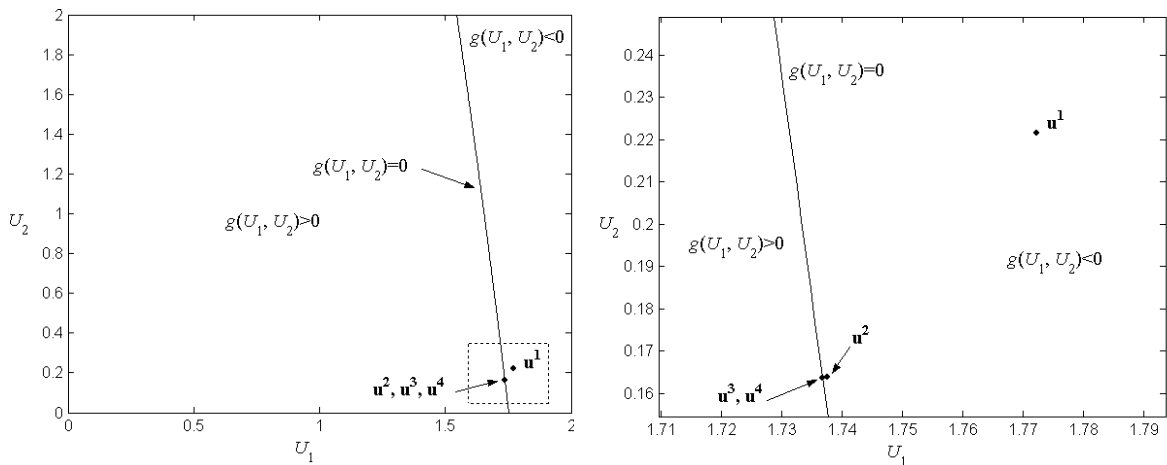
(a) The MPP search history in U-space   (b) Enlarged portion, the dotted box in (a)

Figure 7.12 The MPP Search History in U-Space

The MPP is found at $\mathbf{u}^* = (1.7367, \ 0.16376)$, and the reliability index is $\boldsymbol{b} = 1.7444$. The probability of failure is

$$p_f = \Phi(-1.7444) = 0.04054$$

And the reliability is

$$R = 1 - p_f = 1 - 0.040541 = 0.9595$$

In the above case, all the random variables are normally distributed. FORM can also handle non-normal variables. For example, if both P_x and P_y follows lognormal distributions with the same means and standard deviations, the probability of the failure calculated from FORM is $p_f = 0.0531$.

Example 7.3

A wood beam with Young's modulus E_w and A mm wide by B mm deep by L mm long, has an aluminum plate with Young's modulus E_a and a net section C mm wide by D mm high securely fastened to its bottom face, as shown in Fig. 7.13. Six external vertical forces, P_1, P_2, P_3, P_4, P_5 and P_6 are applied at six different locations along the beam, L_1, L_2, L_3, L_4, L_5, and L_6. The allowable tensile stress is S.
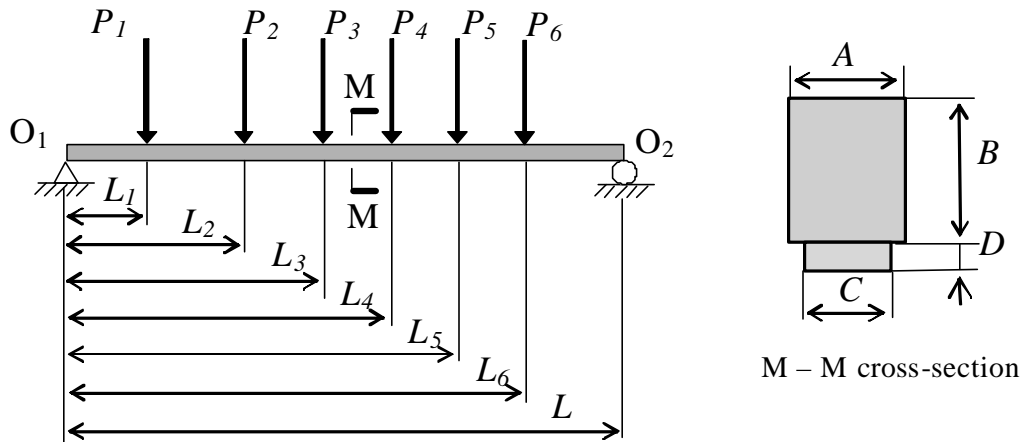

Figure 7.13 A Composite Beam.

In this problem, the twenty random variables are
$$\mathbf{X} = [X_1, \cdots, X_{20}]^{\mathrm{T}} = [A, B, C, D, L_1, L_2, L_3, L_4, L_5, L_6, L, P_1, P_2, P_3, P_4, P_5, P_6, E_a, E_w, S]^{\mathrm{T}}.$$

Details of the random variables are given in Table 7.2.

Table 7.2 Random Variables of the Beam Problem

Variable No.	Variable	Mean value	Standard deviation	Distribution
1	A	100 mm	0.2mm	Normal
2	B	200 mm	0.2 mm	Normal
3	C	80 mm	0.2 mm	Normal
4	D	20 mm	0.2 mm	Normal
5	L_1	200 mm	1 mm	Normal
6	L_2	400 mm	1 mm	Normal

Variable No.	Variable	Mean value	Standard deviation	Distribution
7	L_3	600 mm	1 mm	Normal
8	L_4	800 mm	1 mm	Normal
9	L_5	1000 mm	1 mm	Normal
10	L_6	1200 mm	1 mm	Normal
11	L	1400 mm	2 mm	Normal
12	P_1	15 kN	1.5 kN	Normal
13	P_2	15 kN	1.5 kN	Normal
14	P_3	15 kN	1.5kN	Normal
15	P_4	15 kN	1.5kN	Normal
16	P_5	15 kN	1.5kN	Normal
17	P_6	15 kN	1.5 kN	Normal
18	E_a	70 GPa	7GPa	Normal
19	E_w	8.75 GPa	0.875 GPa	Normal
20	S	25.5MPa	3.825MPa	Normal

The maximum stress occurs in the middle cross-section M-M and is given by

$$\sigma = \frac{\left[\dfrac{\sum\limits_{i=1}^{6} P_i(L-L_i)}{L}L_3 - P_1(L_2-L_1) - P_2(L_3-L_2) \right]\left[\dfrac{0.5AB^2 + \dfrac{E_a}{E_w}DC(B+D)}{AB + \dfrac{E_a}{E_w}DC} \right]}{\dfrac{1}{12}AB^3 + AB\left\{ \left[\dfrac{0.5AB^2 + \dfrac{E_a}{E_w}DC(B+D)}{AB + \dfrac{E_a}{E_w}DC} \right] - 0.5B \right\}^2 + \dfrac{1}{12}\dfrac{E_a}{E_w}CD^3 + \dfrac{E_a}{E_w}CD\left\{ 0.5D + B - \left[\dfrac{0.5AB^2 + \dfrac{E_a}{E_w}DC(B+D)}{AB + \dfrac{E_a}{E_w}DC} \right] \right\}^2}$$

The maximum stress σ should be less than the allowable stress (strength) S. The probability of failure of the beam is defined by

$$P_f = P\{g(\mathbf{X}) < 0\} = P\{S - s < 0\}$$

The MPP found by FORM is shown in Table 7.3. The reliability index $b = 3.1317$ and the probability of failure is

$$p_f = \Phi(-b) = \Phi(-3.1317) = 8.6908 \times 10^{-4}$$

Table 7.3 The MPP in U-Space and X-space

Variable No.	Variable	MPP in U-space	MPP in X-space
1	A	-1.8148E-02	9.9996E-02
2	B	-1.9192E-02	2.0000E-01
3	C	-4.9159E-03	7.9999E-02

Variable No.	Variable	MPP in U-space	MPP in X-space
4	D	-2.8892E-02	1.9994E-02
5	L_1	5.5311E-03	2.0001E-01
6	L_2	-5.5885E-03	3.9999E-01
7	L_3	9.2576E-03	6.0001E-01
8	L_4	-5.4737E-03	7.9999E-01
9	L_5	-5.4163E-03	9.9999E-01
10	L_6	1.9859E-02	1.2000E+00
11	L	-1.0603E-02	1.4000E+00
12	P_1	3.2155E-01	1.5482E+04
13	P_2	2.1435E-01	1.5322E+04
14	P_3	3.2154E-01	1.5482E+04
15	P_4	2.1437E-01	1.5322E+04
16	P_5	1.0719E-01	1.5161E+04
17	P_6	-1.0715E-01	1.4839E+04
18	E_a	-2.0064E-01	6.8596E+10
19	E_w	1.9291E-01	8.9188E+09
20	S	-3.0669E+00	1.3769E+07

7.4   Second Order Reliability Method

As its name implies, the Second Order Reliability Method (SORM) uses the second order Taylor expansion to approximate the performance function at the MPP $\mathbf{u}^*$. The approximation is given by

$$g(\mathbf{U}) \approx q(\mathbf{U}) = g(\mathbf{u}^*) + \nabla(\mathbf{u}^*)(\mathbf{U}-\mathbf{u}^*)^T + \frac{1}{2}(\mathbf{U}-\mathbf{u}^*)\mathbf{H}(\mathbf{u}^*)(\mathbf{U}-\mathbf{u}^*)^T \qquad (7.37)$$

where $\mathbf{H}(\mathbf{u}^*)$ is the Hessian matrix at the MPP, namely,

$$\mathbf{H}(\mathbf{u}^*) = \begin{bmatrix} \dfrac{\partial^2 g}{\partial U_1^2} & \dfrac{\partial^2 g}{\partial U_1 U_2} & \cdots & \dfrac{\partial^2 g}{\partial U_1 U_n} \\[2mm] \dfrac{\partial^2 g}{\partial U_2 U_1} & \dfrac{\partial^2 g}{\partial U_2^2} & \cdots & \dfrac{\partial^2 g}{\partial U_2 U_n} \\[2mm] \cdots & \cdots & \cdots & \cdots \\[2mm] \dfrac{\partial^2 g}{\partial U_n U_1} & \dfrac{\partial^2 g}{\partial U_n U_2} & & \dfrac{\partial^2 g}{\partial U_n^2} \end{bmatrix}_{\mathbf{u}^*} \qquad (7.38)$$

After a set of linear transformation, such as coordinate rotation and orthogonal diagonalization, the performance function is further simplified as

$$q(\mathbf{U}) = \mathbf{U}_n - \left(\boldsymbol{b} + \frac{1}{2}\mathbf{U}^{'\text{T}}\mathbf{D}\mathbf{U}^{'} \right) \tag{7.39}$$

where $\mathbf{D}$ is a $(n-1)\times(n-1)$ diagonal matrix whose elements are determined by the Hessian matrix $\mathbf{H}(\mathbf{u}^{*})$, and $\mathbf{U}^{'} = \{U_1,\ U_2,\ \cdots,\ U_{n-1}\}$.

When $\boldsymbol{b}$ is large enough, an asymptotic solution of the probability of failure can be then derived as

$$p_f = P\{g(\mathbf{X}) < 0\} = \Phi(-\boldsymbol{b})\prod_{i=1}^{n-1}\left(1 + \boldsymbol{b}\,\boldsymbol{k}_i\right)^{1/2} \tag{7.40}$$

in which $\boldsymbol{k}_i$ denotes the i-th main curvature of the performance function $g(\mathbf{U})$ at the MPP.

Since the approximation of the performance function in SORM is better than that in FORM (see Fig. 7.14), SORM is generally more accurate than FORM. However, since SORM requires the second order derivative, it is not as efficient as FORM when the derivatives are evaluated numerically. If we use the number of performance function evaluations to measure the efficiency, SORM needs more function evaluations than FORM.
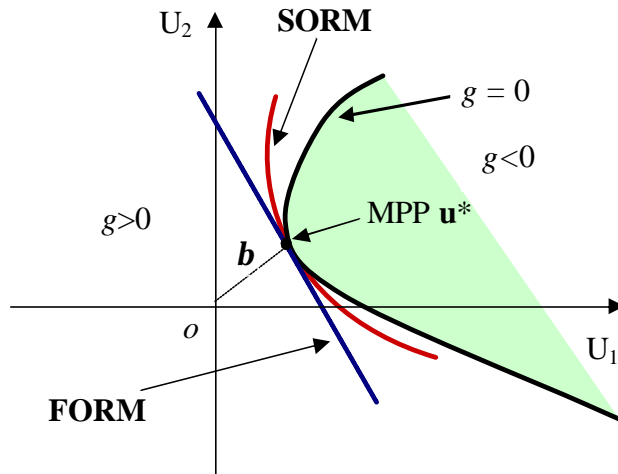

Figure 7.14 Comparison of FORM and SORM

Example 7.4

Use SORM to solve problem 7.2.

The same MPP is used for SORM. The probability of failure is found to be $p_f = 0.04098$. For the nonlinear performance function in this problem, no analytical solution exists. To compare the accuracy of the results, Monte Carlo Simulation (MCS) is used to solve the problem. As we will discuss later in this book, the higher the number of simulations is used, the higher the result is. For this problem, one million simulations are conducted. The result from MCS is considered as the accurate solution for the comparison. The results of the probability of failure are displayed in Table 7.3. The results indicate that SORM is more accurate than FORM.

Table 7.3 The Probability of Failure from Different Methods

Method	FORM	SORM	MCS
p_f	0.04054	0.04098	0.04092

As in Example 7.2, if P_x and P_y follow lognormal distributions with the same mean and standard deviation, the probability of failure calculated from FORM is $p_f = 0.0538$. The comparison between FORM and SORM is given in Table 7.4. MCS with 10^6 simulations is again used as the reference. The result shows that SORM is more accurate than FORM.

Table 7.4 The Probability of Failure When P_x and P_y Follow Lognormal Distributions

Method	FORM	SORM	MCS
p_f	0.0531	0.0538	0.0541

Example 7.5

Use SORM to solve problem 7.3.

The result from SORM is given in Table 5. The results from FORM and MCS are also listed in the table for the comparison of the accuracy. MCS uses 10^6 simulations and its result considered as the reference. For this problem, the first order and second order derivatives of the performance function $g(\mathbf{X})$ are evaluated by difference finite method. The number of evaluating $g(\mathbf{X})$ is used to measure the efficiency.

<div align="center">**Table 7.5 Comparison of Accuracy and Efficiency**</div>

Method	FORM	SORM	MCS
p_f	8.6908×10^{-4}	8.7813×10^{-4}	8.870×10^{-4}
Number of function evaluations	88	550	10^6

The results show that SORM is more accurate than FORM. However, SORM is much less efficient than FORM. SORM calls the performance function 550 times while FORM calls the performance function only 88 times.

7.5   Inverse Reliability Analysis

As we will see in reliability-based design later in this book, the use of the percentile value of a performance function corresponding to a given reliability is more efficient. The evaluation of the percentile value of the performance function is an inverse problem of the reliability analysis. The problem can be stated as: Find the p – percentile value g^p given the probability

$$P\{g(\mathbf{X}) < g^p\} = p \tag{7.41}$$

The above equation indicates that the probability that the performance function is less than the p-percentile value g^p is equal to p. Next, we will discuss how to estimate p – percentile value g^p using FORM.

To make use of FORM we discussed, let a new function be

$$g^{'}(\mathbf{X}) = g(\mathbf{X}) - g^p \tag{7.42}$$

and the MPP for $P\{g^{'}(\mathbf{X}) < 0\} = P\{g(\mathbf{X}) < g^p\}$ be $\mathbf{u}^*$. From FORM, if the probability p is known, the reliability index is given by

$$\boldsymbol{b} = \left| \Phi^{-1}(p) \right| \tag{7.43}$$

Since the reliability index is a distance and is always nonnegative, the absolute value is used in the above equation. As illustrated in Fig. 7.15, the MPP $\mathbf{u}^*$ is a tangent point of the circle with radius $\boldsymbol{b}$ and the performance function $g^{'}(\mathbf{X}) = g(\mathbf{X}) - g^p = 0$ and is also a point that has the minimum value of $g(\mathbf{X})$ on the circle. Therefore, the MPP search for an inverse reliability analysis problem becomes: Find the minimum value of $g(\mathbf{X})$ on the $\boldsymbol{b}$-circle (or $\boldsymbol{b}$-sphere for a 3-D problem or $\boldsymbol{b}$-hyper sphere for a higher dimensional problem).
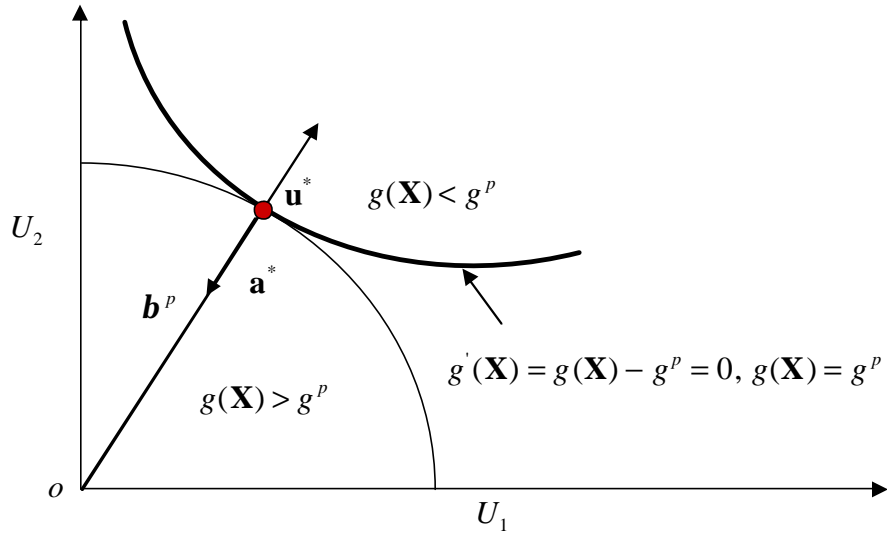
Figure 7.15 The Inverse MPP Search

The mathematical model for the MPP search is then stated as: Find the MPP $\mathbf{u}^*$ where the performance function $g(\mathbf{U})$ is minimized while $\mathbf{u}^*$ remains on the surface of the $\boldsymbol{b}$ - circle, namely

$$\begin{cases} \min_{\mathbf{u}} g(\mathbf{u}) \\ \text{subject to} \quad \|\mathbf{u}\| = \boldsymbol{b} \end{cases} \tag{7.44}$$

Since at the MPP, $\mathbf{u}^* = -\|\mathbf{u}^*\|\mathbf{a}^* = -\boldsymbol{b}\mathbf{a}^*$, in the kth iteration

$$\mathbf{u}^{k+1} = -\boldsymbol{b}\mathbf{a}^k \tag{7.45}$$

where

$$\mathbf{a}^k = \frac{\nabla[g(\mathbf{u}^k)]}{\left\|\nabla[g(\mathbf{u}^k)]\right\|} \tag{7.46}$$

One of the following stopping criteria or the both can be used to terminate the search process.

1) If $\left\|\mathbf{u}^{k+1} - \mathbf{u}^k\right\| \le \boldsymbol{e}_1$, stop.
2) If $\left\|\nabla(\mathbf{u}^{k+1}) - \nabla(\mathbf{u}^k)\right\| \le \boldsymbol{e}_2$, stop.

Eqs. 7.45 and 7.46 give a recursive algorithm for MPP search for an inverse reliability problem. The MPP search algorithm for the inverse reliability problem has the same features as the MPP search algorithm in the last section.

After the MPP $\mathbf{u}^*$ is identified, the p – percentile value g^p is calculated at the MPP as

$$g^p = g(\mathbf{u}^*) \tag{7.47}$$

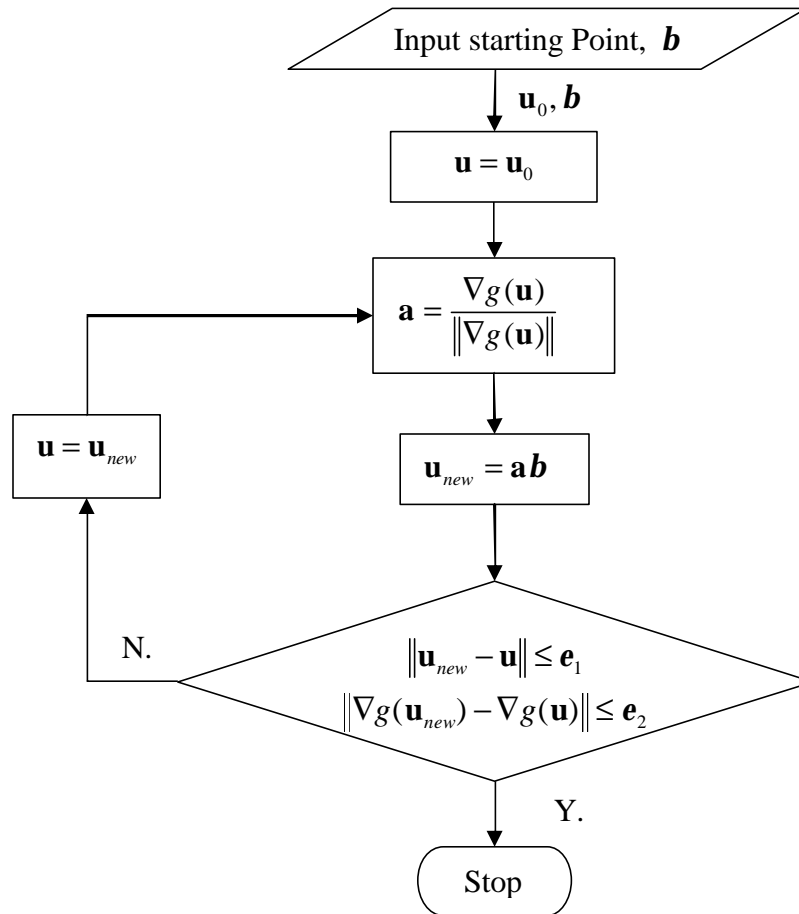The flowchart of the MPP search for the reverse reliability problem is drawn in Fig. 7.16.

Figure 7.16 Flowchart of the MPP Search for an Inverse Reliability Analysis

Example 7.6

If the probability of failure of the cantilever beam in the Example 7.2 is $p_f = 0.001$, what is the corresponding percentile value of the performance function?

Solution:

From Eq. 7.43, the reliability index is

$$b = \left| \Phi^{-1}(0.001) \right| = 3.0902$$

The starting point of the MPP is set to $\mathbf{u}^0 = (0,0)$.

Iteration 1

At $\mathbf{u}^0 = (0,0)$, $g(\mathbf{u}^0) = 0.67076$, $\nabla g(\mathbf{u}^0) = (-0.37268, \ -0.046585)$,

$\left\| \nabla g(\mathbf{u}^0) \right\| = \sqrt{(-0.37268)^2 + (-0.046585)^2} = 0.3756$, and

$\mathbf{a}^0 = \dfrac{\nabla g(\mathbf{u}^0)}{\left\| \nabla g(\mathbf{u}^0) \right\|} = (-0.9923, \ -0.1240)$.

Applying Eq. 7.45 produces a new point,

$$\mathbf{u}^1 = -b\mathbf{a}^0 = -3.0902(-0.9923, \ -0.1240) = (3.0664, 0.3833)$$

Iteration 2

At $\mathbf{u}^1 = (3.0664, 0.3833)$, $g(\mathbf{u}^1) = -0.53073$, $\nabla g(\mathbf{u}^1) = (-0.39663, \ -0.03191)$,

$\left\| \nabla g(\mathbf{u}^1) \right\| = \sqrt{(-0.39663)^2 + (-0.03191)^2} = 0.3979$, and

$\mathbf{a}^1 = \dfrac{\nabla g(\mathbf{u}^1)}{\left\| \nabla g(\mathbf{u}^1) \right\|} = (-0.9968, \ -0.0802)$.

Applying Eq. 7.45 again produces a new point,

$$\mathbf{u}^2 = -b\mathbf{a}^1 = -3.0902(-0.9968, \ -0.0802) = (3.0806, 0.24781)$$

The convergence is achieved after iteration 3. Table 7.6 shows the convergence history.

Table 7.6 MPP Search History

Iteration	g	∇g	(U_x, U_y)
0	0.67076	(-0.37268, -0.046585)	(3.0664, 0.3833)
1	-0.53073	(-0.39663, -0.03191)	(3.0803, 0.24781)
2	-0.53196	(-0.39718, -0.031483)	(3.0806, 0.24418)
3	-0.53196	(-0.39719, -0.031472)	(3.0806, 0.24409)

The MPP is found at $\mathbf{u}^* = (3.0806, 0.24409)$, and the percentile value of the performance function is $g^{0.001} = -0.53196$. The probability of the performance function less than $g^{0.001} = -0.53196$ is equal to 0.001.

7.6   Sensitivity Analysis

Reliability analysis is used to evaluate a given design. If the reliability analysis result shows that the reliability is not satisfactory, i.e., lower than the required reliability, there are several ways to improve the reliability. Some of them are

(1)   To change the mean values of the random variables,
(2)   To reduce the variances of the random variables, and
(3)   To truncate the distributions of the random variables.

When the number of random variables is large, it is difficult to change the distributions of all the random variables. It is also not economic to control all the random variables. To effectively improve the design, a question of interest is: For which random variables we should make changes to order to improve reliability? To answer this question, we need to perform sensitivity analysis. With the information of sensitivity, we will be able to identify the most significant random variables. Only the important variables need to be managed. The sensitivity analysis can give us right directions for the improvement.

Reliability sensitivity analysis is used to find the rate of change in the probability of failure (or reliability) due to the changes in the parameters (usually mean and standard deviation) of distributions. For a distribution parameter p of random variable X_i, the sensitivity is defined by

$$s_p = \frac{\partial p_f}{\partial p} \tag{7.48}$$

s_p can be derived as follows.

$$s_p = \frac{\partial p_f}{\partial p} = \frac{\partial \Phi(-\boldsymbol{b})}{\partial p} = \frac{\partial \Phi(-\boldsymbol{b})}{\partial \boldsymbol{b}} \frac{\partial \boldsymbol{b}}{\partial p} = -\boldsymbol{f}(-\boldsymbol{b}) \frac{\partial \boldsymbol{b}}{\partial p} \tag{7.49}$$

The derivative of the reliability index with respect to the distribution parameter is given by

$$\frac{\partial \boldsymbol{b}}{\partial p} = \frac{\partial \boldsymbol{b}}{\partial u_i^*} \frac{\partial u_i^*}{\partial p} \tag{7.50}$$

where

$$\frac{\partial \boldsymbol{b}}{\partial u_i^*} = \frac{\partial \sqrt{\sum_{j=1}^{n}\left(u_j^*\right)^2}}{\partial u_i^*} = \frac{u_i^*}{\sqrt{\sum_{j=1}^{n}\left(u_j^*\right)^2}} = \frac{u_i^*}{\boldsymbol{b}} \tag{7.51}$$

and from Eq. 7.4

$$u_i^* = \Phi^{-1}\left[F_{X_i}(x_i^*)\right] = w(p) \tag{7.52}$$

where $w(p) = \Phi^{-1}\left[F_{X_i}(x_i^*)\right]$ is a function of the distribution parameter p.

Therefore,

$$\frac{\partial \boldsymbol{b}}{\partial p} = \frac{u_i^*}{\boldsymbol{b}} \frac{\partial w}{\partial p} \tag{7.53}$$

and

$$s_p = \frac{\partial p_f}{\partial p} = -\boldsymbol{f}(-\boldsymbol{b})\frac{u_i^*}{\boldsymbol{b}}\frac{\partial w}{\partial p} \tag{7.54}$$

Using Eq. 7.54, we can calculate the sensitivity of the mean and standard deviation of random variables X_i as follows.

$$s_{m_i} = \frac{\partial p_f}{\partial m_i} = -\boldsymbol{f}(-\boldsymbol{b})\frac{u_i^*}{\boldsymbol{b}}\frac{\partial w}{\partial m_i} \tag{7.55}$$

and

$$s_{s_i} = \frac{\partial p_f}{\partial s_i} = -\boldsymbol{f}(-\boldsymbol{b})\frac{u_i^*}{\boldsymbol{b}}\frac{\partial w}{\partial s_i} \tag{7.56}$$

respectively.

For a normal distributed random variable $X_i \sim N(m_i, s_i)$

$$w(m_i, s_i) = \Phi^{-1}\left[F_{X_i}(x_i^*)\right] = \Phi^{-1}\left[\Phi(\frac{x_i^* - m_i}{s_i})\right] = \frac{x_i^* - m_i}{s_i} \tag{7.57}$$

$$\frac{\partial w}{\partial m_i} = -\frac{1}{s_i} \tag{7.58}$$

And

$$\frac{\partial w}{\partial s_i} = -\frac{x_i^* - m_i}{s_i^2} = -\frac{u_i^*}{s_i} \tag{7.59}$$

Therefore,

$$s_{m_i} = \frac{\partial p_f}{\partial m_i} = f(-b)\frac{u_i^*}{bs_i} \tag{7.60}$$

and

$$s_{s_i} = \frac{\partial p_f}{\partial s_i} = f(-b)\frac{\left(u_i^*\right)^2}{bs_i} \tag{7.61}$$

Example 7.6

Calculate the sensitivity of random variables for Example 7.2.

$$s_{m_{Px}} = \frac{\partial p_f}{\partial m_{Px}} = f(-b)\frac{u_{Px}^*}{bs_{Px}} = f(-1.7444)\frac{1.7367}{1.7444 \times 100} = 8.6735 \times 10^{-4}$$

$$s_{m_{Py}} = \frac{\partial p_f}{\partial m_{Py}} = f(-b)\frac{u_{Py}^*}{bs_{Py}} = f(-1.7444)\frac{0.1638}{1.7444 \times 100} = 8.1786 \times 10^{-5}$$

$$s_{s_{Px}} = \frac{\partial p_f}{\partial s_{Px}} = f(-b)\frac{\left(u_{Px}^*\right)^2}{bs_{Px}} = f(-1.7444)\frac{1.7367^2}{1.7444 \times 100} = 1.5064 \times 10^{-3}$$

$$s_{s_y} = \frac{\partial p_f}{\partial s_{Py}} = f(-b)\frac{\left(u_{Py}^*\right)^2}{bs_{Py}} = f(-1.7444)\frac{0.1638^2}{1.7444 \times 1000} = 1.3394 \times 10^{-5}$$

From the sensitivity results, we can draw the following conclusions.

1) Because the sign of each sensitivity is positive, if we increase each of the means and standard deviations of the external forces P_x and P_y, the probability of failure will increase. Therefore, we need to reduce the means or standard deviations of the two random variables, or their combinations, to improve the reliability.

2) Since the sensitivity of the mean and the standard deviation of P_x is greater than that of P_y, for the same change in the means or standard deviations, P_x will have more significant

impact on the reliability change than P_y. In this sense, P_x is more important than P_y. If the parameters of one random variable would be allowed to change due to cost concern, we would make changes on P_x.

7.7   Conclusion

We have discussed how to estimate the reliability, which is defined as the probability that a performance function (performance) is safe. Two most commonly used reliability analysis methods FORM and SORM have been discussed.

It is noted that the reliability analysis methods are used for evaluating a specific probability at the limit state, and they are not intended for generating a complete distribution of a performance function or its statistical moments. However, if different values that cover the range of distribution of the performance function are used as limit states, FORM or SORM is applicable to generating the whole distribution of the performance function. For example, the *cdf* of a performance function $g(\mathbf{X})$ at a particular value c is calculated by $F_g(c) = P\{g(\mathbf{X}) < c\}$. If a new function is defined by $g^{'}(\mathbf{X}) = g(\mathbf{X}) - c$, the *cdf* estimation becomes $F_g(c) = P\{g^{'}(\mathbf{X}) < 0\}$. Then FORM or SORM is applicable to estimate the *cdf*, which is the probability of failure for function $g^{'}(\mathbf{X}) = g(\mathbf{X}) - c$.

Since both FORM and SORM approximate a performance function at the MPP, the accuracy of the methods depends upon how accurate the approximated performance function is in U-space. If the performance function in U-space is close to a linear function when FORM is used, or close to a quadratic function when SORM is used, both methods will produce accurate reliability estimations. If the performance function is highly nonlinear in U-space, both methods may generate a larger error. Even though a performance function is close to linear in X-space, it may become highly nonlinear in U-space because the normal to non-normal transformation from X-space to U-space is nonlinear. Only under very special cases, for example, the random variables are normally distributed, the transformation is linear. Generally, SORM is more accurate than SORM even though there exist some counter examples where the former is less accurate than the latter.

As for efficiency, FORM is more efficient than SORM since the former uses only the first order derivative and the latter uses both first and second order derivatives. It is noted that a single reliability analysis needs to perform a number of deterministic analyses on the performance function for the MPP search. For many engineering problems, a performance function is expensive to evaluate, and no analytical derivative exists. When the derivative has to be evaluated numerically, the computational effort will be approximated proportional to the number of random variables. Therefore, both FORM and SORM may not be applicable for large scale problems, and we have to resort to Monte Carlo simulation or other approximation methods that will be discussed in what follows.

Reference

[1] Rosenblatt, M., "Remarks on a Multivariate Transformation," Annals of Mathematical Statistic, Vol. 23, 1952, pp. 470-472.