Design Under Uncertainty: Methods

ME 615 Spring 2020

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MIME

Objectives and Requirements

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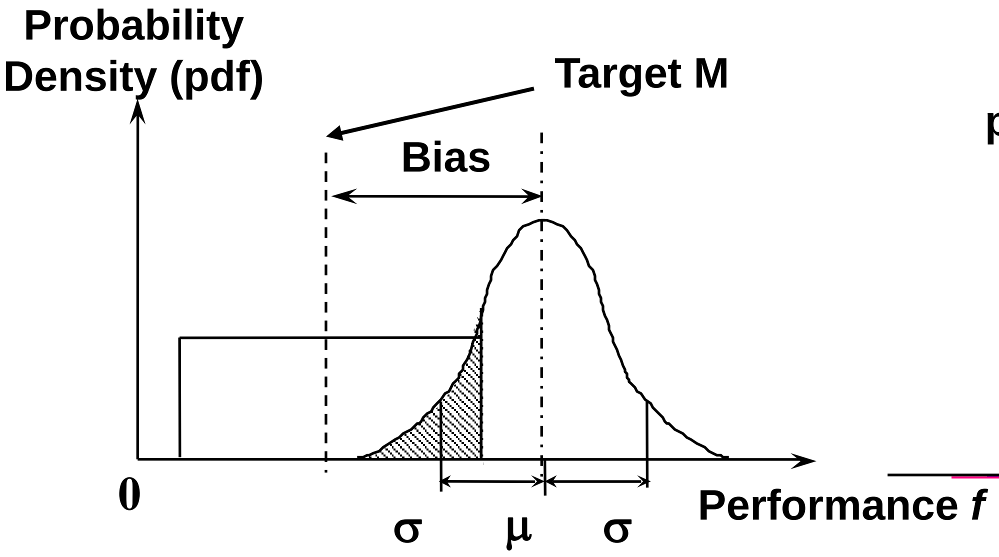
Objectives

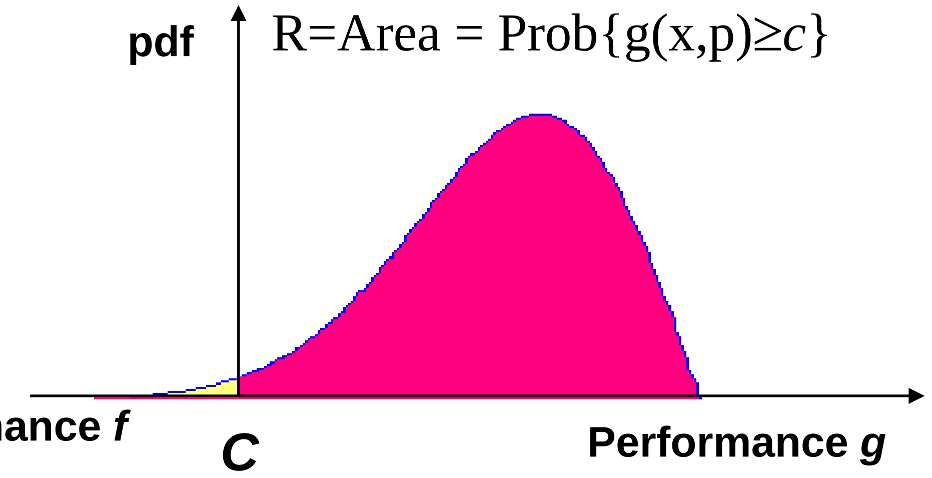


Look at entire distribution

s.t. $x \in X$

Considering the effect of variations without eliminating the causes

Requirements



Satisfy

$R = P\{g(\mathbf{x},\mathbf{p}) \geq c\} \geq R_0$

Limit State

To assure proper levels of “safety” for the system designed

Mean-value First Order Second Moment Method (MVFOSM)

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- This method uses a first order Taylor Series expansion
- Works well if the model is linear or approximately linear
- Has problems as the model deviates from linearity

Terminology

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- We need to differentiate between the *mean* and *variance* of a function:
  - μ_f, σ_f^2 or μ_g, σ_g^2
- versus the *mean* and *variance* of a variable in the model:
  - μ_x, σ_x^2

- Next, we will look at how to get the mean and variance of a **function** (*f* or *g*) as a function of the mean and variance of the **model variables** (*x*).

Taylor series expansion

- Taylor series expansion of *f* at mean point of design variables (I'll assume we are doing this for an objective *f*)
  - $f(x) \approx f(\mu_{x1} \dots \mu_{xn}) + \sum_i \frac{\partial f}{\partial x_i}|_{x=\mu_x}(x_i - \mu_{xi})$
- If we take the Expected Value of the above expression, we get:
  - $E[f(x)] \approx E[f(\mu_{x1} \dots \mu_{xn}) + \sum_i \frac{\partial f}{\partial x_i}|_{x=\mu_x}(x_i - \mu_{xi})]$

    0
  - $E[f(x)] \approx E[f(\mu_{x1} \dots \mu_{xn})] + E[\sum_i \frac{\partial f}{\partial x_i}|_{x=\mu_x}(x_i - \mu_{xi})]$
  - $\boxed{\mu_f = f(\mu_{x1} \dots \mu_{xn})}$

Taylor series expansion (cont)

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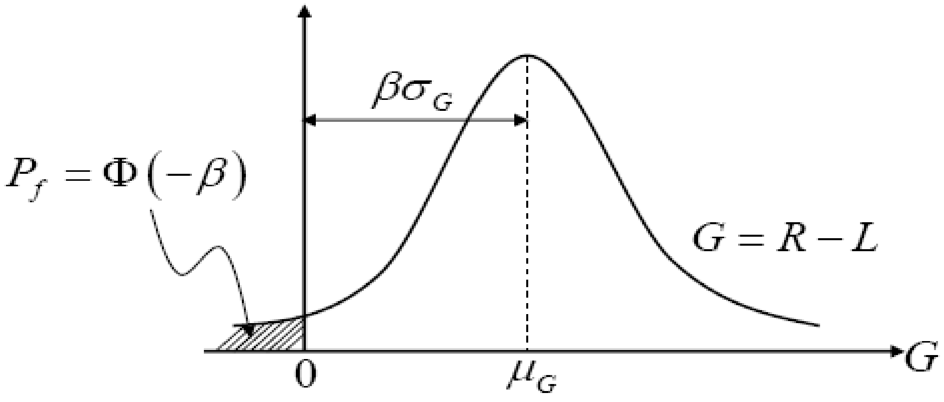
- Variance:
  - $Var[f(x)] \approx Var[f(\mu_{x1} \dots \mu_{xn}) + \sum_i \frac{\partial f}{\partial x_i}|_{x=\mu_x}(x_i - \mu_{xi})]$
  - $Var[f(x)] \approx \underbrace{Var[f(\mu_{x1} \dots \mu_{xn})]}_{0} + Var[\sum_i \frac{\partial f}{\partial x_i}|_{x=\mu_x}(x_i - \mu_{xi})]$
  - $Var[f(x)] \approx 0 + \sum_i \left(\frac{\partial f}{\partial x_i}\right)^2 |_{x=\mu_x} Var[(x_i - \mu_{xi})]$
  - $Var[f(x)] \approx \sum_i \left(\frac{\partial f}{\partial x_i}\right)^2 |_{x=\mu_x} Var[(x_i)]$
  - $\boxed{\sigma_f^2 = Var[f(x)] \approx \sum_i \left(\frac{\partial f}{\partial x_i}\right)^2 |_{x=\mu_x} \sigma_{xi}^2}$

What if there is correlation?

- $\mu_f = f(\mu_{x1} \dots \mu_{xn})$
- $\sigma_f^2 = Var[f(x)] \approx \sum_i \sum_j \left(\frac{\partial f}{\partial x_i}\right)|_{x=\mu_x} \left(\frac{\partial f}{\partial x_j}\right)|_{x=\mu_x} [\rho \sigma_{xi} \sigma_{xj}]$

Mean-value First Order Second Moment Method

- Probability approximation
  - Linear combination of normal variables → follow normal distribution
  - When g<0 implies failure state (positive null form),



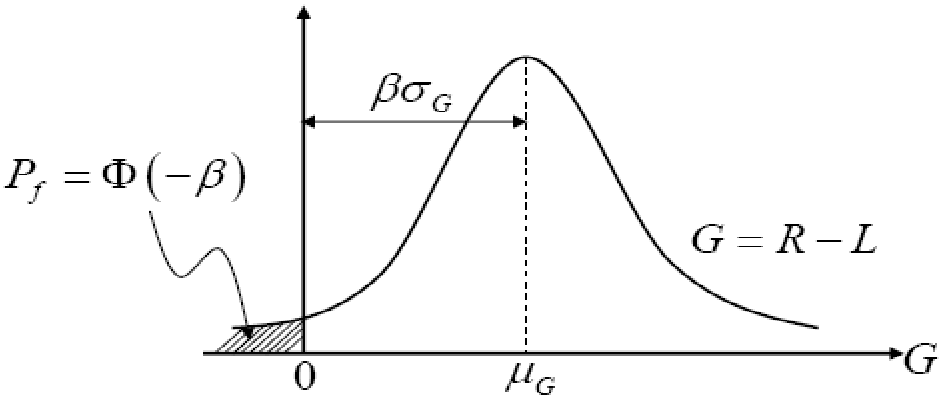
$$Z = \frac{X - \mu}{\sigma}$$

$$\boxed{P_f = \Phi\left(\frac{0 - \mu_g}{\sigma_g}\right) = \Phi(-\beta)}$$

Φ: CDF of standard normal distribition

β: reliability index

Mean-value First Order Second Moment Method



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- Example
  - **g(R,L)=R - L**
    R~N(30000,1500²),L~N(20000,3000²), independent

$$E[g(\mathbf{x})] = g(\mu_{\mathbf{x}})$$

$$\sigma_g^2 = \sum_j \sum_i \left.\frac{\partial g}{\partial x_i}\right|_{\mathbf{x}=\mu_{\mathbf{x}}} \left.\frac{\partial g}{\partial x_j}\right|_{\mathbf{x}=\mu_{\mathbf{x}}} \rho_{ij}\sigma_i\sigma_j$$

$$\mu_g = 30000 - 20000 = 10000$$

$$\sigma_g^2 = 1^2 \cdot 1500^2 + 1^2 \cdot 3000^2 = 3354^2$$

$$\beta = \mu_g / \sigma_g = 10000/3354 = 2.98, \quad P_f = \Phi(-\beta) = 0.00144$$

(MV)FOSM Algorithm

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1. Evaluate the mean by plugging in the mean values from your random inputs/parameters into your system equation or simulation model
2. Compute the 1st derivatives of each random input using a finite difference approximation (μ_x is the mean value)
   - $\frac{f(\mu_x+h)-f(\mu_x-h)}{2h}$; generally h is 10% of σ_x
3. Create a double loop over *i* and *j* to compute the following

$$E\left[g(\mathbf{x})\right] \approx g(\mu_{\mathbf{x}})$$

$$\sigma_g^2 = \sum_j \sum_i \left.\frac{\partial g}{\partial x_i}\right|_{\mathbf{x}=\boldsymbol{\mu}_\mathbf{x}} \left.\frac{\partial g}{\partial x_j}\right|_{\mathbf{x}=\boldsymbol{\mu}_\mathbf{x}} \rho_{ij}\sigma_i\sigma_j$$

- σ_i, σ_j, and ρ_{ij} are all known from your input distributions

MVFOSM inaccuracy

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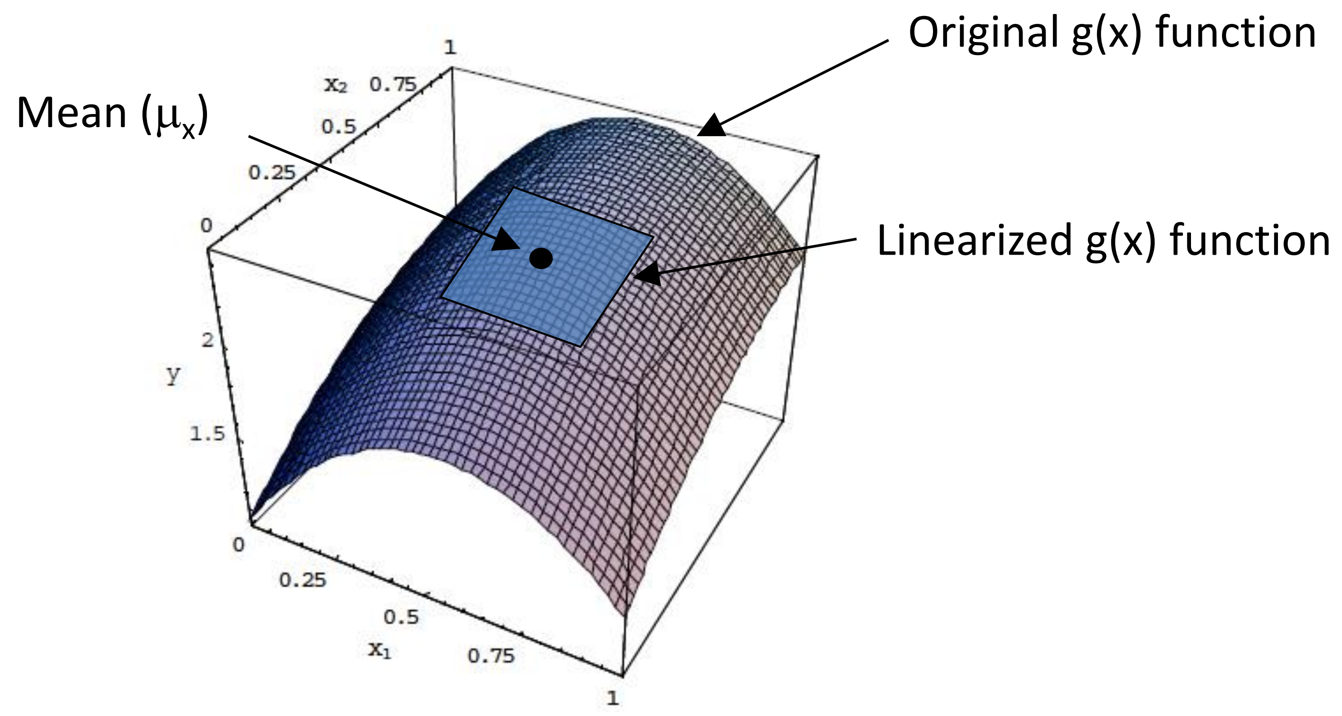
- Inaccurate results when g(x) is nonlinear, especially when β is large.



MVFOSM lack of invariance

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- Lack of invariance to the expression of g(x)
  - Ex. Failure of a rod

R: allowable stress, A: cross section area Q: load

$R \sim N(62, 6.2^2)$ $A \sim N(2.8, 0.14^2)$, $Q=100$

1) G= R-Q/A

$$\mu_G = 62 - 100/2.8 = 26.29$$

$$\sigma_G^2 = \sigma_R^2 + \left(\frac{Q}{A^2}\right)^2 \sigma_A^2 = 6.45^2$$

β=4.07

Should be the same, but different!

2) G=RA-Q

$$\mu_G = 62 \cdot 2.8 - 100 = 73.6$$

$$\sigma_G^2 = A^2\sigma_R^2 + R^2\sigma_A^2 = 19.4^2$$

β=3.79

Mean-value First Order Second Moment Method

- Characteristics of the Taylor Series Approximation:
  - Scales linearly with the size of the variance-covariance matrix of inputs x.
  - Partial derivatives can be approximated with Finite Difference estimations for black-box system models, such as Modelica models.
  - Once derivatives are computed, the mean and variance of the output can be computed directly—additional model parameters are not required.
  - Correlation of inputs (ρ) can be considered.
- Limitations of the Taylor Series Approximation
  - Assumes the distribution type of the performance output is normal:
  - Approximates the performance equations with a 1st order Taylor Series expansion.
  - Does not display invariance to the *form* of the performance equation.
  - **Generally, only use for looking at uncertainty in an objective**