4

Methods of Structural Reliability

This chapter presents methods for two significant reliability measures: safety index and probability of failure. Because of the iterative nature of calculating these measures, use of limit-state function approximations is a necessary aspect. However, efficient selection of suitable approximations at different stages of reliability analysis makes these tools practical for many large-scale engineering problems. Also, the physical interpretation of sensitivity factors as used in design is discussed. At the end of the chapter, several engineering problems are presented with corresponding results for use as test cases.

4.1 First-order Reliability Method (FORM)

In principle, random variables are characterized by their first moment (mean), second moment (variance), and higher moments. Different ways of approximating the limit-state function form the basis for different reliability analysis algorithms (*i.e.*, FORM, SORM, *etc.*). In this section, we first discuss the first-order second moment method, and then the details of FORM, because the development of FORM can be traced to FOSM.

4.1.1 First-order Second Moment (FOSM) Method

The FOSM method, also referred to as the Mean Value FOSM (MVFOSM), simplifies the functional relationship and alleviates the complexities of the probability-of-failure calculation. The name "first-order" come from the first-order expansion of the function. As implied, inputs and outputs are expressed as the mean and standard deviation. Higher moments, which might describe skew and flatness of the distribution, are ignored.

In the MVFOSM method, the limit-state function is represented as the first-order Taylor series expansion at the mean value point. Assuming that the variables X are statistically independent, the approximate limit-state function at the mean is written as

$$\tilde{g}(X) \approx g(\mu_X) + \nabla g(\mu_X)^T (X_i - \mu_{X_i}) \tag{4.1}$$

where, $\mu_X = \{\mu_{x_1}, \mu_{x_2}, ...\mu_{x_n}\}^T$, and $\nabla g(\mu_X)$ is the gradient of g evaluated at μ_X, $\nabla g(\mu_X) = \left\{ \frac{\partial g(\mu_X)}{\partial x_1}, \frac{\partial g(\mu_X)}{\partial x_2}, ..., \frac{\partial g(\mu_X)}{\partial x_n} \right\}^T$.

The mean value of the approximate limit-state function $\tilde{g}(X)$ is

$$\mu_{\tilde{g}} \approx E[g(\mu_X)] = g(\mu_X) \tag{4.2}$$

Because

$$\mathrm{Var}[\mathrm{g}(\mu_X)] = 0, \quad \mathrm{Var}[\nabla g(\mu_X)] = 0 \tag{4.3}$$

$$\begin{aligned} \mathrm{Var}[\nabla g(\mu_X)^T (X - \mu_X)] &= \mathrm{Var}[\nabla g(\mu_X)^T X] - \mu_X \mathrm{Var}[\nabla g(\mu_X)] \\ &= \mathrm{Var}[(\nabla g(\mu_X)^T X] \\ &= [\nabla g(\mu_X)^T]^2 \mathrm{Var}(X) \end{aligned} \tag{4.4}$$

The variance of the approximate limit-state function $\tilde{g}(X)$ is

$$\mathrm{Var}[\tilde{g}(X)] \approx \mathrm{Var}[g(\mu_X)] + \mathrm{Var}[\nabla g(\mu_X)^T (X - \mu_X)] \tag{4.5}$$

Therefore, the standard deviation of the approximate limit-state function is

$$\begin{aligned} \sigma_{\tilde{g}} &= \sqrt{\mathrm{Var}[\tilde{g}(X)]} = \sqrt{[\nabla g(\mu_X)^T]^2 \mathrm{Var}(X)} \\ &= \left[\sum_{i=1}^{n} \left(\frac{\partial g(\mu_X)}{\partial x_i} \right)^2 \sigma_{x_i}^2 \right]^{\frac{1}{2}} \end{aligned} \tag{4.6}$$

The reliability index β is computed as:

$$\beta = \frac{\mu_{\tilde{g}}}{\sigma_{\tilde{g}}} \tag{4.7}$$

Equation 4.7 is the same as Equation 3.6 if the limit-state function is linear. If the limit-state function is nonlinear, the approximate limit-state surface is obtained by linearizing the original limit-state function at the mean value point. Therefore, this method is called the mean-value method, and the β given in Equation 4.7 is called a MVFOSM reliability index.

In a general case with independent variables of n-dimensional space, the failure surface is a hyperplane and can be defined as a linear-failure function:

$$\tilde{g}(X) = c_0 + \sum_{i=1}^{n} c_i x_i \tag{4.8}$$

The reliability index given in Equation 4.7 can still be used for this n-dimensional case, in which

$$\mu_{\tilde{g}} = c_0 + c_1\mu_{x_1} + c_2\mu_{x_2} + ... + c_n\mu_{x_n} \tag{4.9}$$

$$\sigma_{\tilde{g}} = \sqrt{\sum_{i=1}^{n} c_i^2 \sigma_{x_i}^2} \tag{4.10}$$

The MVFOSM method changes the original complex probability problem into a simple problem. This method directly establishes the relationship between the reliability index and the basic parameters (mean and standard deviation) of the random variables via Equation 4.7. However, there are two serious drawbacks in the MVFOSM method:

1) Evaluation of reliability by linearizing the limit-state function about the mean values leads to erroneous estimates for performance functions with high nonlinearity, or for large coefficients of variation. This can be seen from the following mean value calculation of $\tilde{g}(X)$, which assumes that truncation of the Taylor series expansion for a case of only one random variable at the first three terms is

$$\tilde{g}(X) \approx g(\mu_X) + (X - \mu_X)\nabla g(\mu_X) + \frac{(X - \mu_X)^2}{2}\nabla g^2(\mu_X) \tag{4.11}$$

The mean value of the approximate limit-state function $\tilde{g}(X)$ can be calculated as

$$\mu_{\tilde{g}} \approx E[g(\mu_X)] + E[(X - \mu_X)\nabla g(\mu_X)] + E\left[\frac{(X - \mu_X)^2}{2}\nabla g^2(\mu_X)\right] \tag{4.12}$$

Because

$$E[g(\mu_X)] = g(\mu_X) \tag{4.13a}$$

$$E[(X - \mu_X)\nabla g(\mu_X)] = E[(X\nabla g(\mu_X)] - E[\mu_X \nabla g(\mu_X)]$$

$$= \nabla g(\mu_X)E(X) - \mu_X \nabla g(\mu_X) = 0 \tag{4.13b}$$

$$E\left[\frac{(X-\mu_X)^2}{2}\nabla^2 g(\mu_X)\right] = \frac{1}{2}\nabla^2 g(\mu_X)E[(X-\mu_X)^2]$$

$$= \frac{1}{2}\nabla g^2(\mu_X)Var(X) \tag{4.13c}$$

From Equation 4.13c, it is obvious that the third term on the right side of Equation 4.11 depends on the variance of X and the second-order gradients of the limit-state function. If the variance of X is small or the limit-state function is close to linear, the third term of Equation 4.11 can be ignored and the mean value of $\tilde{g}(X)$ is the same as Equation 4.2. Otherwise, large errors in the mean value estimation will result.

2) The MVFOSM method fails to be invariant with different mathmatically equivalent formulations of the same problem. This is a problem not only for nonlinear forms of $g(\cdot)$, but also for certain linear forms. Example 4.2 shows that two different equivalent formulations of the limit-state function for the same problem result in different safety indices.

Example 4.1

The performance function is

$$g(x_1, x_2) = x_1^3 + x_2^3 - 18$$

in which x_1 and x_2 are the random variables with normal distributions (mean $\mu_{x_1} = \mu_{x_2} = 10$, standard deviation $\sigma_{x_1} = \sigma_{x_2} = 5$). Find the safety-index β by using the mean-value FOSM method, and check the accuracy of the obtained result with the MCS.

Solution:

The mean of the linearized performance function is

$$\mu_{\tilde{g}} = g(\mu_{x_1}, \mu_{x_2}) = 1982.0$$

From Equation 4.6, the standard deviation of the linearized performance function is

$$\sigma_{\tilde{g}} = \sqrt{\left(\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_1}\sigma_{x_1}\right)^2 + \left(\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_2}\sigma_{x_2}\right)^2}$$

$$= \sqrt{(3\times 10^2 \times 5.0)^2 + (3\times 10^2 \times 5.0)^2} = 2121.32$$

From Equation 4.7, the safety-index β is

$$\beta = \frac{\mu_{\tilde{g}}}{\sigma_{\tilde{g}}} = \frac{1982.0}{2121.32} = 0.9343$$

We can expect that the accuracy of the MVFOSM method is not acceptable, since the given limit-state function is highly nonlinear. The result of MCS (1,000,000 runs) yields $P_f = 0.005524$ and $\beta = 2.5412$.

Example 4.2

Consider Example 3.1 with the same structural and statistical properties. Investigate the invariant property of MVFOSM for two different formulations of the same limit-state function. Two different formulations of the limit-state function can be given as:

$$g_1(P, L, W, T) = WT - \frac{PL}{4}$$

$$g_2(P, L, W, T) = T - \frac{PL}{4W}$$

Solution:

The safety-index for the g_1 function is

$$\beta_1 = \frac{\mu_{g_1}}{\sigma_{g_1}}$$

$$= \frac{100\times10^{-6}\times600\times10^3 - \dfrac{10\times8}{4}}{\sqrt{(-2\times2)^2 + (-2.5\times0.1)^2 + (600\times10^3\times2\times10^{-5})^2 + (100\times10^{-6}\times10^5)^2}}$$

$$= 2.48$$

and the safety index for the function g_2 is

$$\beta_2 = \frac{\mu_{g_2}}{\sigma_{g_2}}$$

$$= \frac{600\times10^3 - \dfrac{10\times8}{4\times100\times10^{-6}}}{\sqrt{(-2\times10^4\times2)^2 + (-2.5\times10^3)^2 + (4\times10^4)^2 + (1\times10^5)^2}}$$

$$= 3.48$$

β_1 and β_2 are different even though the above two limit-state equations are equivalent. This lack of invariance was overcome by the Hasofer and Lind method.

4.1.2 Hasofer and Lind (HL) Safety-index

Searching for the MPP on the limit-state surface is a key step in the HL method. The improvement of the HL method compared with the MVFOSM also comes from changing the expansion point from the mean value point to the MPP. In Section 3.1.1, Figure 3.1 shows how the reliability index could be interpreted as the measure of the distance from the origin to the failure surface. In the one-dimensional case, the standard deviation of the safety margin was conveniently used as the scale. To obtain a similar scale in the case of multiple variables, Hasofer and Lind [8] proposed a linear mapping of the basic variables into a set of normalized and independent variables, u_i.

Consider the fundamental case with the independent variables of strength, *R*, and stress, *S*, which are both normally distributed. First, Hasofer and Lind introduced the standard normalized random variables:

$$\hat{R} = \frac{R - \mu_R}{\sigma_R}, \ \hat{S} = \frac{S - \mu_s}{\sigma_s} \tag{4.14}$$

where μ_R and μ_S are the mean values of random variables *R* and *S*, respectively, and σ_R and σ_S are the standard deviations of *R* and *S*, respectively.

Next, transform the limit-state surface $g(R, S) = R - S = 0$ in the original (*R*, *S*) coordinate system into the limit-state surface in the standard normalized $(\hat{R}, \hat{S})$ coordinate system,

$$g(R(\hat{R}), S(\hat{S})) = \hat{g}(\hat{R}, \hat{S}) = \hat{R}\sigma_R - \hat{S}\sigma_S + (\mu_R - \mu_S) = 0 \tag{4.15}$$

Here, the shortest distance from the origin in the $(\hat{R}, \hat{S})$ coordinate system to the failure surface $\hat{g}(\hat{R}, \hat{S}) = 0$ is equal to the safety-index, $\beta = \hat{O}P^* = (\mu_R - \mu_S)/\sqrt{\sigma_R^2 + \sigma_S^2}$, as shown in Figure 4.1. The point $P^*(\hat{R}^*, \hat{S}^*)$ on $\hat{g}(\hat{R}, \hat{S}) = 0$, which corresponds to this shortest distance, is often referred to as the MPP.

In a general case with normally distributed and independent variables of *n*-dimensional space, the failure surface is a nonlinear function:

$$g(X) = g(\{x_1, x_2, \ldots x_n\}^T) \tag{4.16}$$

Transform the variables into their standardized forms:

$$u_i = \frac{x_i - \mu_{x_i}}{\sigma_{x_i}} \tag{4.17}$$

where μ_{x_i} and σ_{x_i} represent the mean and the standard deviation of x_i, respectively. The mean and standard deviation of the standard normally distributed variable, u_i, are zero and unity, respectively.

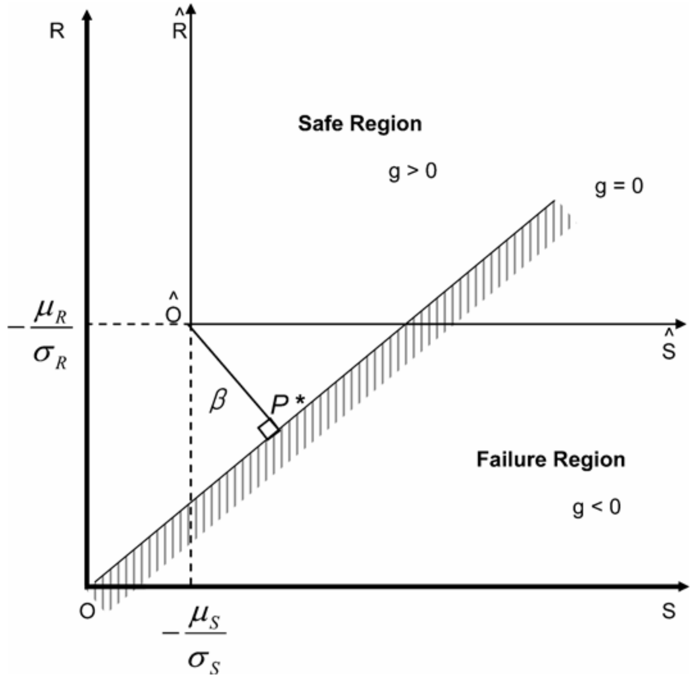


Figure 4.1. Geometrical Illustration of Safety-index

Any orthogonal distribution of standard normally distributed variables $U = \{u_1, u_2, ..., u_n\}^T$ results in a new set of normalized and uncorrelated variables. Therefore, the distributions of U are rotationally symmetric with respect to second moment distribution. Based on the transformation of Equation 4.17, the mean value point in the original space (X-space) is mapped into the origin of the normal space (U-space). The failure surface $g(X)=0$ in X-space is mapped into the corresponding failure surface $g(U)=0$ in U-space, as shown in Figure 3.2 and Figure 4.2. Due to the rotational symmetry of the second-moment representation of U, the geometrical distance from the origin in U-space to any point on $g(U)=0$ is simply the number of standard deviations from the mean value point in X-space to the corresponding point on $g(X)=0$. The distance to the failure surface can then be measured by the safety-index function:

$$\beta(U) = (U^T U)^{1/2} = \|U\|_2 , U \in g(U) = 0 \tag{4.18}$$

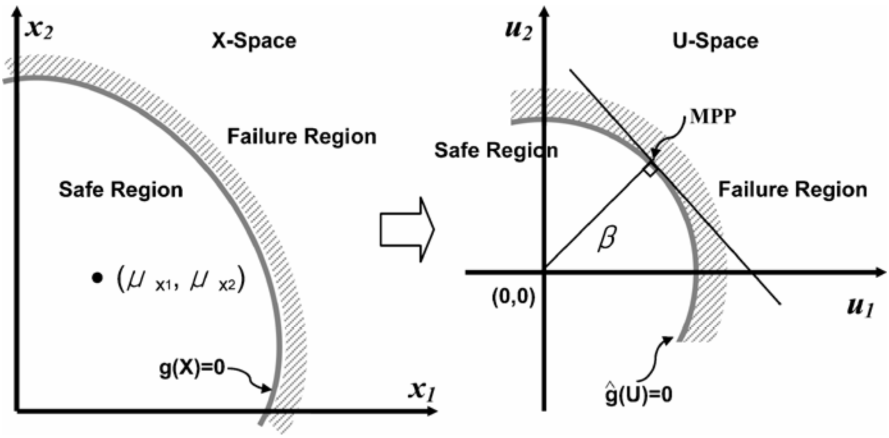


Figure 4.2. Mapping of Failure Surface from X-space to U-space

The safety-index β is the shortest distance from the origin to the failure surface $g(U) = 0$, *i.e.*,

$$\beta = \min_{U \in g(U)=0} (U^T U)^{\frac{1}{2}} \tag{4.19}$$

This safety-index is also called the Hasofer and Lind (HL) safety-index, β_{HL}. The point $U^*(u_1^*, u_2^*, ..., u_n^*)$ on $g(U)=0$ is the design point. The values-of-safety indices given in Equations 4.7 and 4.19 are the same when the failure surface is a hyperplane. The Hasofer and Lind reliability index can also be interpreted as a FOSM reliability index. The value of β_{HL} is the same for the true failure surface as well as for the approximate tangent hyperplane at the design point. The ambiguity in the value of the first-order reliability index is thus resolved when the design point is taken as the linearization point. The resultant reliability index is a sensible measure for the distance to the failure surface.

4.1.3 Hasofer and Lind Iteration Method

Equation 4.19 shows that safety-index, β, is the solution of a constrained optimization problem in the standard normal space.

$$\text{Minimize:} \quad \beta(U) = (U^T U)^{\frac{1}{2}} \tag{4.20a}$$

$$\text{Subject to:} \quad g(U) = 0 \tag{4.20b}$$

There are many algorithms available that can solve this problem, such as mathematical optimization schemes or other iteration algorithms. In [6], several constrained optimization methods were used to solve this optimization problem, including primal methods (feasible directions, gradient, projection, reduced gradient), penalty methods, dual methods, and Lagrange multiplier methods. Each method has its advantages and disadvantages, depending upon the attributes of the method and the nature of the problem. In the following description, the most commonly used recursive algorithms, the HL and HL-RF methods, are introduced to solve the reliability problems.

The HL method was proposed by Hasofer and Lind. Rackwitz and Fiessler extended the HL method to include random variable distribution information, calling their extended method the HL-RF method. Assuming that the limit-state surface with n-dimensional normally distributed and independent random variables X is

$$g(X) = g(\{x_1, x_2, \ldots, x_n\}^T) = 0 \tag{4.21}$$

This limit-state function can be linear or nonlinear. Based on the transformation given in Equation 4.17, the limit-state function given in Equation 4.21 is transformed into

$$g(U) = g(\{\sigma_{x_1} u_1 + \mu_{x_1}, \sigma_{x_2} u_2 + \mu_{x_2}, \ldots., \sigma_{x_n} u_n + \mu_{x_n}\}^T) = 0 \tag{4.22}$$

The normal vector from the origin $\hat{O}$ to the limit-state surface $g(U)$ generates an intersection point P^* as shown in Figure 4.1 and Figure 4.2. The distance from the origin to the MPP is the safety-index β. The first-order Taylor series of expansion of $g(U)$ at the MPP U^* is

$$\tilde{g}(U) \approx g(U^*) + \sum_{i=1}^{n} \frac{\partial g(U^*)}{\partial U_i}(u_i - u_i^*) \tag{4.23}$$

where k denotes the iteration number of the recursive algorithm.

From the transformation of Equation 4.17, we have

$$\frac{\partial \hat{g}(U)}{\partial u_i} = \frac{\partial g(X)}{\partial x_i} \sigma_{x_i} \tag{4.24}$$

The shortest distance from the origin to the above approximate failure surface given in Equation 4.23 is

$$\hat{O}P^* = \beta = \frac{g(U^*) - \sum_{i=1}^{n} \frac{\partial g(U^*)}{\partial x_i} \sigma_{x_i} u_i^*}{\sqrt{\sum_{i=1}^{n} (\frac{\partial g(U^*)}{\partial x_i} \sigma_{x_i})^2}} \tag{4.25}$$

The direction cosine of the unit outward normal vector is given as

$$\cos\theta_{x_i} = \cos\theta_{u_i} = -\frac{\frac{\partial g(U^*)}{\partial u_i}}{|\nabla g(U^*)|} \tag{4.26}$$

$$= -\frac{\frac{\partial g(X^*)}{\partial x_i} \sigma_{x_i}}{\left[\sum_{i=1}^{n} (\frac{\partial g(X^*)}{\partial x_i} \sigma_{x_i})^2\right]^{1/2}} = \alpha_i$$

where α_i expresses the relative effect of the corresponding random variable on the total variation. Thus, it is called the *sensitivity factor*. More details about α_i will be given later in Section 4.1.4.

The coordinates of the point P^* are computed as

$$u_i^* = \frac{x_i^* - \mu_{x_i}}{\sigma_{x_i}} = \hat{O}P^* \cos\theta_{x_i} = \beta \cos\theta_{x_i} \tag{4.27}$$

The coordinates corresponding to P^* in the original space are

$$x_i^* = \mu_{x_i} + \beta\sigma_{x_i} \cos\theta_{x_i}, (i = 1,2,\ldots,n) \tag{4.28}$$

Since P^* is a point on the limit-state surface,

$$g(\{x_1^*, x_2^*, \ldots x_n^*\}^T) = 0 \tag{4.29}$$

The main steps of the HL iteration method are:

1) Define the appropriate limit-state function of Equation 4.21
2) Set the mean value point as an initial design point, *i.e*, $x_{i,k} = \mu_{x_i}$ i=1,2,…,n, and compute the gradients $\nabla g(X_k)$ of the limit-state function at this point. Here, $x_{i,k}$ is the i^{th} element in the vector X_k of the k^{th} iteration

3) Compute the initial β using the mean-value method (Cornell safety-index), *i.e.*, $\beta = \mu_{\tilde{g}} / \sigma_{\tilde{g}}$ and its direction cosine

4) Compute a new design point X_k and U_k (Equations 4.27 and 4.28), function value, and gradients at this new design point

5) Compute the safety-index β using Equation 4.25 and the direction cosine or sensitivity factor from Equation 4.26

6) Repeate steps 4)~6) until the estimate of β converges

7) Compute the coordinates of the design point X_k or most probable failure point (MPP), X^*

In some cases, the failure surface may contain several points corresponding to stationary values of the reliability-index function. Therefore, it may be necessary to use several starting points to find all the stationary values $\beta_1, \beta_2, ..., \beta_m$. This is called a *multiple MPP* problem.

The HL safety-index is

$$\beta_{HL} = \min\{\beta_1, \beta_2, ..., \beta_m\} \tag{4.30}$$

From Equations 4.7 and 4.25, the difference between the MVFOSM method and the HL method is that the HL method approximates the limit-state function using the first-order Taylor expansion at the design point $X^{(k)}$ or $U^{(k)}$ instead of the mean value point μ_X. Also, the MVFOSM method does not require iterations, while the HL method needs several iterations to converge for nonlinear problems. The HL method usually provides better results than the mean-value method for nonlinear problems. How well a linearized limit-state function, $\tilde{g}(U) = 0$, approximates a nonlinear function $g(U)$ in terms of the failure probability P_f depends on the shape of $g(U)=0$. If it is concave towards the origin, P_f is underestimated by the hyperplane approximation. Similarly, a convex function implies overestimation. However, there is no guarantee that the HL algorithm converges in all situations. Furthermore, the HL method only considers normally distributed random variables, so it cannot be used for non-Gaussian random variables.

Example 4.3a

Solve the safety-index β of Example 4.1 by using the HL method with the same performance function, mean values, standard deviations, and distributions of the random variables.

Solution:

(1) Iteration 1:

(a) Set the mean value point as an initial design point and set the required β convergence tolerance to $\varepsilon_r = 0.001$. Compute the limit-state function value and gradients at the mean value point:

$$g(X_1) = g(\mu_{x_1}, \mu_{x_2}) = \mu_{x_1}^3 + \mu_{x_2}^3 - 18$$
$$= 10.0^3 + 10.0^3 - 18 = 1982.0$$
$$\frac{\partial g}{\partial x_1}|_{\mu_X} = 3\mu_{x_1}^2 = 3 \times 10^2 = 300 \ , \ \frac{\partial g}{\partial x_2}|_{\mu_X} = 3\mu_{x_2}^2 = 3 \times 10^2 = 300$$

(b) Compute the initial β using the mean-value method and its direction cosine α_i

$$\beta_1 = \frac{\mu_{\tilde{g}}}{\sigma_{\tilde{g}}}$$
$$= \frac{g(X_1)}{\sqrt{(\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_1}\sigma_{x_1})^2 + (\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_2}\sigma_{x_2})^2}}$$
$$= \frac{1982.00}{\sqrt{(300 \times 5.0)^2 + (300 \times 5.0)^2}}$$
$$= 0.9343$$

$$\alpha_1 = -\frac{\left.\frac{\partial g}{\partial x_1}\right|_{\mu_X} \sigma_{x_1}}{\sqrt{(\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_1}\sigma_{x_1})^2 + (\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_2}\sigma_{x_2})^2}}$$
$$= -\frac{300 \times 5.0}{\sqrt{(300 \times 5.0)^2 + (300 \times 5.0)^2}}$$
$$= -0.7071$$

$$\alpha_2 = -\frac{\left.\frac{\partial g}{\partial x_2}\right|_{\mu_X} \sigma_{x_2}}{\sqrt{(\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_1}\sigma_{x_1})^2 + (\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_2}\sigma_{x_2})^2}}$$
$$= -\frac{300 \times 5.0}{\sqrt{(300 \times 5.0)^2 + (300 \times 5.0)^2}}$$
$$= -0.7071$$

(c) Compute a new design point X_2 from Equation 4.28

$$x_{1,2} = \mu_{x_1} + \beta_1 \sigma_{x_1} \alpha_1 = 10.0 + 0.9343 \times 5.0 \times (-0.7071) = 6.6967$$

$$x_{2,2} = \mu_{x_2} + \beta_1 \sigma_{x_2} \alpha_2 = 10.0 + 0.9343 \times 5.0 \times (-0.7071) = 6.6967$$

$$u_{1,2} = \frac{x_{1,2} - \mu_{x_1}}{\sigma_{x_1}} = \frac{6.6967 - 10.0}{5.0} = -0.6607$$

$$u_{2,2} = \frac{x_{2,2} - \mu_{x_2}}{\sigma_{x_2}} = \frac{6.6967 - 10.0}{5.0} = -0.6607$$

(2) Iteration 2:

(a) Compute the limit-state function and its gradient at X_2

$$g(X_2) = (x_{1,2})^3 + (x_{2,2})^3 - 18 = 6.6967^3 + 6.6967^3 - 18 = 582.63$$

$$\left.\frac{\partial g}{\partial x_1}\right|_{X_2} = 3 \times (x_{1,2})^2 = 3 \times 6.6967^2 = 134.5374$$

$$\left.\frac{\partial g}{\partial x_2}\right|_{X_2} = 3 \times (x_{2,2})^2 = 3 \times 6.6967^2 = 134.5374$$

(b) Compute β using Equation 4.25 and the direction cosine α_i

$$\beta_2 = \frac{g(X_2) - \sum_{i=1}^{2} \frac{\partial g(X_2)}{\partial x_i} \sigma_{x_i} u_{i,2}}{\sqrt{\sum_{i=1}^{2} \left(\frac{\partial g(X_2)}{\partial x_i} \sigma_{x_i}\right)^2}}$$

$$= \frac{582.63 - 134.5374 \times 5.0 \times (-0.6607) - 134.5374 \times 5.0 \times (-0.6607)}{\sqrt{(134.5347 \times 5.0)^2 + (134.5347 \times 5.0)^2}}$$

$$= 1.5468$$

$$\alpha_1 = -\frac{\left.\frac{\partial g}{\partial x_1}\right|_{X_2} \sigma_{x_1}}{\sqrt{\left(\left.\frac{\partial g}{\partial x_1}\right|_{X_2} \sigma_{x_1}\right)^2 + \left(\left.\frac{\partial g}{\partial x_2}\right|_{X_2} \sigma_{x_2}\right)^2}}$$

$$= -\frac{134.5374 \times 5.0}{\sqrt{(134.5374 \times 5.0)^2 + (134.5374 \times 5.0)^2}}$$

$$= -0.7071$$

$$\alpha_1 = \alpha_2 = -0.7071$$

(c) Compute a new design point X_3

$$x_{1,3} = \mu_{x_1} + \beta_2 \sigma_{x_1} \alpha_1 = 10.0 + 1.5468 \times 5.0 \times (-0.7071) = 4.5313$$

$$x_{2,3} = x_{1,3} = 4.5313$$

$$u_{1,3} = \frac{x_{1,3} - \mu_{x_1}}{\sigma_{x_1}} = \frac{4.5313 - 10.0}{5.0} = -1.0937$$

$$u_{2,3} = u_{1,3} = -1.0937$$

(d) Check β convergence

$$\varepsilon = \frac{|\beta_2 - \beta_1|}{\beta_1} = \frac{1.5468 - 0.9343}{0.9343} = 0.6556$$

Since $\varepsilon > \varepsilon_r$, continue the process.

Table 4.1. Iteration Results in the HL Method (Example 4.3a)

Iteration No.	1	2	3	4	5	6	7
$g(X_k)$	1982.0	582.63	168.08	45.529	10.01	1.1451	0.023
$\frac{\partial g}{\partial x_1}\|_{X_k}$	300	134.5374	61.598	30.0897	17.43	13.5252	12.9917
$\frac{\partial g}{\partial x_2}\|_{X_k}$	300	134.5374	61.598	30.0897	17.43	13.5252	12.9917
β	0.9343	1.5468	1.9327	2.1467	2.2279	2.2398	2.2401
α_1	-0.7071	-0.7071	-0.7071	-0.7071	-0.7071	-0.7071	-0.771
α_2	-0.7071	-0.7071	-0.7071	-0.7071	-0.7071	-0.7071	-0.7071
$x_{1,k}$	6.6967	4.5313	3.1670	2.4104	2.1233	2.0810	2.0801
$x_{2,k}$	6.6967	4.5313	3.1670	2.4104	2.1233	2.0810	2.0801
$u_{1,k}$	-0.6607	-1.0937	-1.3666	-1.5179	-1.5753	-1.5838	-1.5840
$u_{2,k}$	-0.6607	-1.0937	-1.3666	-1.5179	-1.5753	-1.5838	-1.5840
ε	-	0.6556	0.2495	0.1107	0.036	0.005	0.0001

The same procedures can be repeated until the stopping criterion ($\varepsilon < \varepsilon_r$) is satisfied. The iteration results are summarized in Table 4.1. The safety-index β is 2.2401. Since the limit-state function value at the MPP, X^*, is close to zero, this safety-index can be considered as the shortest distance from the origin to the limit surface. Compared with the safety-index $\beta = 0.9343$ obtained from the MVFOSM method given in Example 4.1, the safety-index computed from the HL method is much more accurate for this highly nonlinear problem.

Example 4.3b

Solve the safety-index β of Example 4.1 by using the HL method and the mean value of $x_2 = 9.9$ instead of $x_2 = 10.0$. The other properties remain the same.

Solution:

In this example, the performance function, the mean value of x_1, the standard deviations, and the distributions of both random variables are the same as in Example 4.1. The only difference between Example 4.3a and 4.3b is that the mean value of x_2 is 9.9 instead of 10.0.

(1) Iteration 1:

(a) Set the mean value point as an initial design point and set the required β convergence tolerance to $\varepsilon_r = 0.001$. Compute the limit-state function value and gradient at the mean value point.

$$g(X_1) = g(\mu_{x_1}, \mu_{x_2}) = \mu_{x_1}^3 + \mu_{x_2}^3 - 18$$

$$= 10.0^3 + 9.9^3 - 18 = 1952.299$$

$$\frac{\partial g}{\partial x_1}|_{\mu_X} = 3\mu_{x_1}^2 = 3 \times 10^2 = 300$$

$$\frac{\partial g}{\partial x_2}|_{\mu_X} = 3\mu_{x_2}^2 = 3 \times 9.9^2 = 294.03$$

(b) Compute the initial β value using the mean-value method and its direction cosine α_i

$$\beta_1 = \frac{\mu_{\tilde{g}}}{\sigma_{\tilde{g}}}$$

$$= \frac{g(X_1)}{\sqrt{(\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_1}\sigma_{x_1})^2 + (\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_2}\sigma_{x_2})^2}}$$

$$= \frac{1952.299}{\sqrt{(300 \times 5.0)^2 + (294.03 \times 5.0)^2}}$$

$$= 0.9295$$

$$\alpha_1 = -\frac{\frac{\partial g}{\partial x_1}|_{\mu_X}\,\sigma_{x_1}}{\sqrt{(\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_1}\sigma_{x_1})^2 + (\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_2}\sigma_{x_2})^2}}$$

$$= -\frac{300 \times 5.0}{\sqrt{(300 \times 5.0)^2 + (294.03 \times 5.0)^2}}$$

$$= -0.7142$$

$$\alpha_2 = -\frac{\frac{\partial g}{\partial x_2}|_{\mu_X} \sigma_{x_2}}{\sqrt{(\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_1}\sigma_{x_1})^2 + (\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_2}\sigma_{x_2})^2}}$$

$$= -\frac{294.03 \times 5.0}{\sqrt{(300 \times 5.0)^2 + (294.03 \times 5.0)^2}}$$

$$= -0.7000$$

Table 4.2. Iteration Results in the HL Method (Example 4.3b)

Iteration No.	1	2	….	21	22	23
$g(X_k)$	1952.299	573.8398	….	678.9088	676.7346	677.655
$\frac{\partial g}{\partial x_1}\|_{X_k}$	300	133.8982	….	218.0401	56.9049	217.6582
$\frac{\partial g}{\partial x_2}\|_{X_k}$	294.03	132.5409	….	54.4352	216.2786	54.61056
β	0.9295	1.5387	….	1.1636	1.1650	1.1657
α_1	-0.7142	-0.7107	….	-0.9702	-0.2544	-0.9699
α_2	-0.7000	-0.7035	….	-0.2422	-0.9671	-0.2434
$x_{1,k}$	6.6808	4.5323	….	4.3553	8.5178	4.3468
$x_{2,k}$	6.6468	4.4877	….	8.4908	4.2666	8.4816
$u_{1,k}$	-0.6638	-1.0935	….	-1.1289	-0.2964	-1.1306
$u_{2,k}$	-0.6506	-1.0825	….	-0.2818	-1.1267	-0.2837
ε	-	0.6554	….	0.002	0.0012	0.0006

(c) Compute a new design point X_2 from Equation 4.28

$$x_{1,2} = \mu_{x_1} + \beta_1 \sigma_{x_1} \alpha_1 = 10.0 + 0.9295 \times 5.0 \times (-0.7142) = 6.6808$$

$$x_{2,2} = \mu_{x_2} + \beta_1 \sigma_{x_2} \alpha_2 = 9.9 + 0.9295 \times 5.0 \times (-0.7000) = 6.6468$$

$$u_{1,2} = \frac{x_{1,2} - \mu_{x_1}}{\sigma_{x_1}} = \frac{6.6808 - 10.0}{5.0} = -0.6638$$

$$u_{2,2} = \frac{x_{2,2} - \mu_{x_2}}{\sigma_{x_2}} = \frac{6.6468 - 9.9}{5.0} = -0.6506$$

The same procedures can be repeated until the stopping criterion ($\varepsilon < \varepsilon_r$) is satisfied. The iteration results are summarized in Table 4.2. The safety-index converges after 23 iterations; however, the MPP is not on the limit-state surface $(g(X^*) = 677.655)$. Also, beginning with iterations 21, 22, and 23, the design point X^* oscillates. If a convergence check to determine whether or not the MPP is on the surface is added, the process will continue. However, no final MPP on the surface can be found, even after hundreds of iterations, due to the oscillation. From this example, it is clear that the HL method may not converge in some cases due to its linear approximation. A more efficient method will be used to deal with this problem, and the correct safety-index for this example will be given in Example 4.6.

4.1.4 Sensitivity Factors

As mentioned in Section 4.1.3, the direction cosine of the unit outward normal vector of the limit-state function α_i , given in Equation 4.26, is defined as the sensitivity factor. The sensitivity factor shows the relative importance of each random variable to the failure probability. The sensitivity of the failure probability or the safety index to small changes in the random variables can be examined, which usually provides information useful to studying the statistical variation of the response.

In Equation 4.26, the physical meaning of α_i implies the relative contribution of each random variable to the failure probability (Figure 4.3). For example, the larger the α_i value is, the higher the contribution towards the failure probability. This is due to

$$\alpha_1^2 + \alpha_2^2 + ... + \alpha_n^2 = 1 \tag{4.31}$$

In fact, α_i is the sensitivity of the safety-index β at the MPP. From the definition of β as the distance from the origin to the limit-state surface, $g(U) = 0$, it follows that

$$\frac{\partial \beta}{\partial u_i} = \frac{\partial}{\partial u_i}\sqrt{u_1^2 + u_2^2 +,...,+u_n^2} = \frac{u_i}{\beta} = \alpha_i , \ (i = 1,2,\ldots,n) \tag{4.32}$$

The sensitivity factors for the failure probability P_f are

$$\frac{\partial \beta}{\partial u_i} = \frac{\partial}{\partial u_i}\Phi(-\beta) = \phi(-\beta)\frac{\partial \beta}{\partial u_i} \tag{4.33}$$

where $\phi(\cdot)$ represents the standard normal density function.

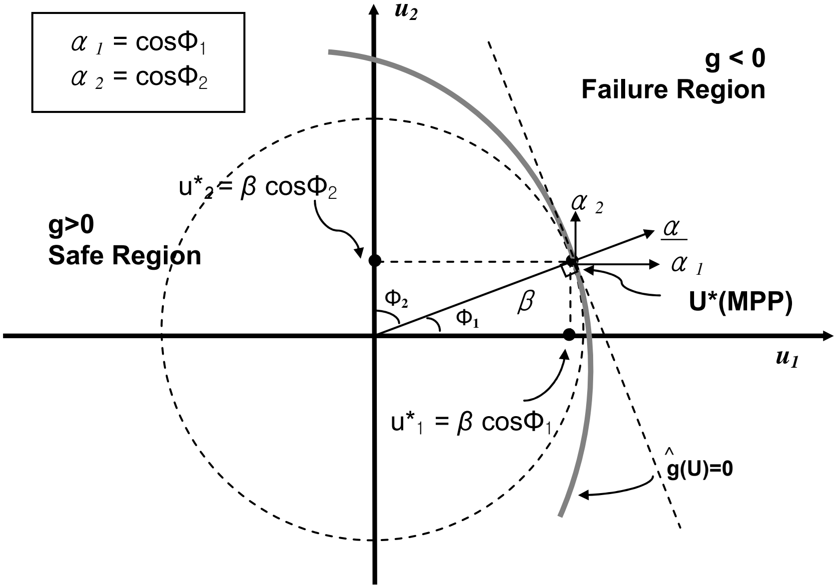


Figure 4.3. Sensitivity Factors

In Equation 4.26, $\partial g(X)/\partial x_i$ represents the sensitivity of the performance function $g(X)$, which measures the change in the performance function resulting from changes in the physical random variables. However, the sensitivity of the safety-index β represents both the change in the safety-index due to the change in random variables and the uncertainty. That is, it depends on the sensitivity of the performance function, $\partial g(X)/\partial x_i$ and on the standard deviation of random variable σ_i (from Equation 4.26).

In summary, computing α_i provides the sensitivity of the safety-index with respect to u_i, which has two major purposes. First, these sensitivity factors show the contributions of the random variables to the safety-index or failure probability. Second, the sign of the sensitivity factor shows the relationship between the performance function and the physical variables. A positive α_i means that the performance function $g(U)$ decreases as the random variable increases, and a negative factor means $g(U)$ increases as the random variable increases.

The sensitivity of the failure probability will have the same direction, but α_i is multiplied by the probability-density function value (Equation 4.33). For example, the limit-state function is

$$\bar{S} - S(X) = g(X) \tag{4.34}$$

where $\bar{S}$ is the allowable stress and $S(X)$ is the structural stress. If α_i is positive, it means $\partial g / \partial x_i$ is negative and $\partial S(X) / \partial x_i$ is positive. In very simplified terms, if the random variable value increases, both $S(X)$ and the failure probability increase.

4.1.5 Hasofer Lind - Rackwitz Fiessler (HL-RF) Method

In the Hasofer-Lind method, the random variables X are assumed to be normally distributed. In non-Gaussian cases, even when the limit-state function $g(X)$ is linear, the structural probability calculation given in Equation 3.9 is inappropriate. However, many structural reliability problems involve non-Gaussian random variables. It is necessary to find a way to solve these non-Gaussian problems. There are many methods available for conducting the transformations, such as Rosenblatt [16], and Hohenbichler and Rackwitz [10]. A simple, approximate transformation called the equivalent normal distribution, or the normal tail approximation, is described below. The main advantages of this transformation are:

1) It does not require the multi-dimensional integration
2) Transformation of non-Gaussian variables into equivalent normal variables has been accomplished prior to the solution of Equations 4.21 – 4.29
3) Equation 3.9 for calculation of the structural probability is retained
4) It often yields excellent agreement with the exact solution of the multi-dimensional integral of probability formula.

When the variables are mutually independent, the transformation is given as

$$u_i = \Phi^{-1}[F_{x_i}(x_i)] \tag{4.35}$$

where $\Phi^{-1}[.]$ is the inverse of $\Phi[.]$

One way to get the equivalent normal distribution is to use the Taylor series expansion of the transformation at the MPP X^*, neglecting nonlinear terms [13],

$$u_i = \Phi^{-1}[F_{x_i}(x_i^*)] + \frac{\partial}{\partial x_i}([\Phi^{-1}F_{x_i}(x_i)])|_{x_i^*}\,(x_i - x_i^*) \tag{4.36}$$

where

$$\frac{\partial}{\partial x_i}\Phi^{-1}[F_{x_i}(x_i)] = \frac{f_{x_i}(x_i)}{\phi(\Phi^{-1}[F_{x_i}(x_i)])} \tag{4.37}$$

Upon substituting (4.37) into (4.36) and rearranging,

$$u_i = \frac{x_i - [x_i^* - \Phi^{-1}[F_{x_i}(x_i^*)]\phi(\Phi^{-1}[F_{x_i}(x_i^*)])/f_{x_i}(x_i^*)]}{\phi(\Phi^{-1}[F_{x_i}(x_i^*)])/f_{x_i}(x_i^*)} \tag{4.38a}$$

which can be written as

$$u_i = \frac{x_i - \mu_{x_i'}}{\sigma_{x_i'}} \tag{4.38b}$$

where $F_{x_i}(x_i)$ is the marginal cumulative distribution function, $f_{x_i}(x_i)$ is the probability density function, and $\mu_{x_i'}$ and $\sigma_{x_i'}$ are the equivalent means and standard deviations of the approximate normal distributions, and which are given as

$$\sigma_{x_i'} = \frac{\phi(\Phi^{-1}[F_{x_i}(x_i^*)])}{f_{x_i}(x_i^*)} \tag{4.39a}$$

$$\mu_{x_i'} = x_i^* - \Phi^{-1}[F_{x_i}(x_i^*)]\sigma_{x_i'} \tag{4.39b}$$

Another way to get equivalent normal distributions is to match the cumulative distribution functions and probability density function of the original, non-normal random variable distribution, and the approximate or equivalent normal random variable distributions at the MPP [16]. Assuming that x_i' is an equivalent normally distributed random variable, the cumulative distribution function values of x_i and x_i' are equal:

$$F_{x_i}(x_i^*) = F_{x_i'}(x_i^*) \tag{4.40}$$

$$\text{or } F_{x_i}(x_i^*) = \Phi(\frac{x_i^* - \mu_{x_i'}}{\sigma_{x_i'}}) \tag{4.41}$$

so

$$\mu_{x_i'} = x_i^* - \Phi^{-1}[F_{x_i}(x_i^*)]\sigma_{x_i'} \tag{4.42}$$

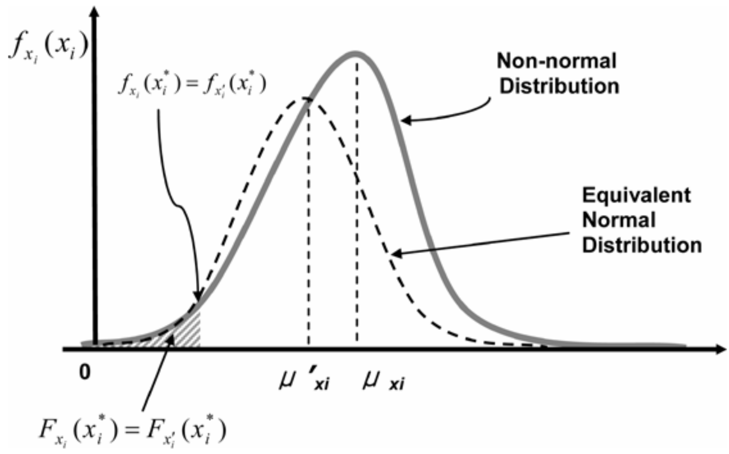


Figure 4.4. Normal Tail Approximation

The probability density function value of x and x_i' at x_i^* are equal:

$$f_{x_i}(x_i^*) = f_{x_i'}(x_i^*) \tag{4.43}$$

$$f_{x_i}(x_i^*) = \frac{1}{\sigma_{x_i'}}\phi\left(\frac{x_i^* - \mu_{x_i'}}{\sigma_{x_i'}}\right) \tag{4.44}$$

From Equations 4.42 and 4.44, the equivalent mean $\mu_{x_i'}$ and standard deviation $\sigma_{x_i'}$ of the approximate normal distributions are derived as Equations 4.39a and 4.39b. This normal-tail approximation is shown in Figure 4.4. Using Equations 4.39a and 4.39b, the transformation of the random variables from the *X*-space to the *U*-space can easily be performed, and the performance function $g(U)$ in *U*-space is approximately obtained.

The RF algorithm is similar to the Hasofer-Lind iteration method shown in Section 4.1.3, except that steps 2) and 4) are necessary to implement the calculation of the mean and standard deviation of the equivalent normal variables based on Equations 4.39a and 4.39b. The RF method is also called the HL-RF method, since the iteration algorithm was originally proposed by Hasofer and Lind and later extended by Rackwitz and Fiessler to include random variable distribution information. A computer program based on the HL-RF method can be readily developed to perform the reliability analysis. A flow chart of the algorithm is given in Figure 4.5. The following examples are given to illustrate the HL-RF method and the transformation of non-Gaussian variables.

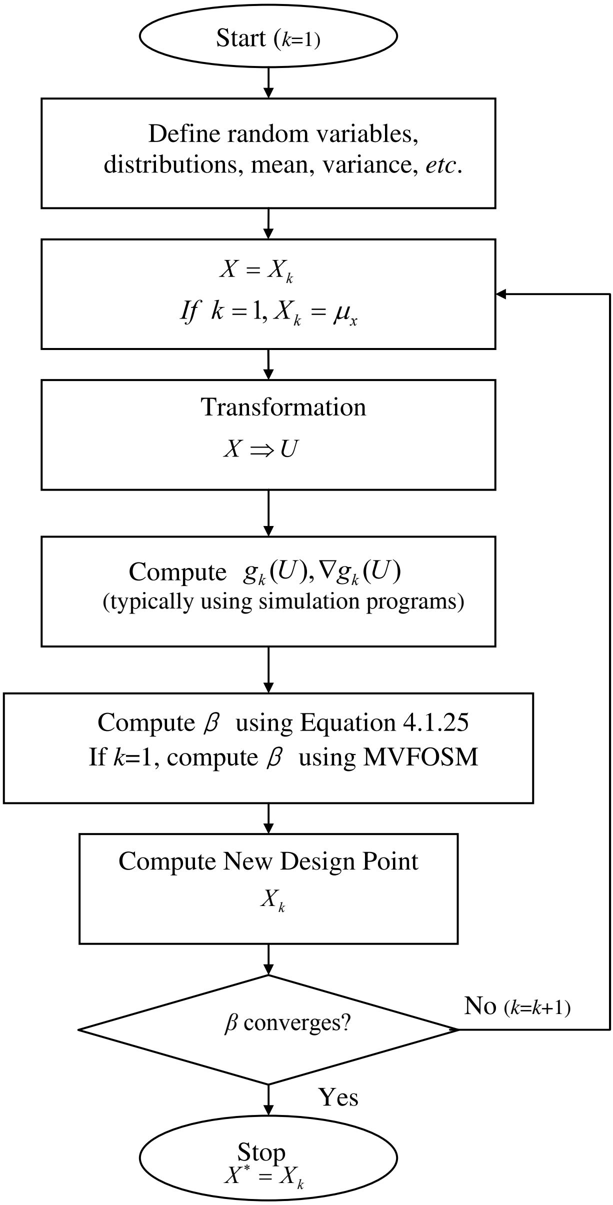


Figure 4.5. HL-RF Method Flow-chart

Example 4.4

Let x be a random variable having a lognormal distribution. Its mean value and standard deviation are $\mu_X = 120$ and $\sigma_X = 12$. Calculate the mean value and standard deviation of the equivalent normal distribution variable x' at $x^* =$ 80.0402.

Solution:

(1) Compute the mean value, μ_y, and standard deviation, σ_y, of normally distributed variable y ($y = ln\, x$) using Equations 2.44 and 2.45:

$$\sigma_y = \sqrt{ln\left[\left(\frac{\sigma_x}{\mu_x}\right)^2 + 1\right]} = \sqrt{ln\left[\left(\frac{12}{120}\right)^2 + 1\right]} = 0.09975$$

$$\mu_y = ln\,\mu_x - \frac{1}{2}\sigma_y^2 = ln\ 120 - \frac{1}{2} \times 0.09975^2 = 4.7825$$

(2) Compute the density function value at x^*:

$$f_x(x^*) = \frac{1}{\sqrt{2\pi}x^*\sigma_y} exp\left[-\frac{1}{2}\left(\frac{ln\,x^* - \mu_y}{\sigma_y}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi} \times 80.0402 \times 0.09975} exp\left[-\frac{1}{2}\left(\frac{\ln 80.0402 - 4.7825}{0.099975}\right)^2\right]$$

$$= 1.6114 \times 10^{-5}$$

(3) Compute the cumulative distribution function value at x^*:

From the density function above, it is obvious that the cumulative distribution function can be given as

$$F_x(x^*) = \Phi\left(\frac{ln\,x^* - \mu_y}{\sigma_y}\right)$$

(4) Compute $\Phi^{-1}[F_x(x^*)]$:

$$\Phi^{-1}[F_x(x^*)] = \frac{ln\ x^* - \mu_y}{\sigma_y} = \frac{ln\ 80.0402 - 4.7835}{0.09975} = -4.0098$$

(5) Compute $\phi(\Phi^{-1}[F_x\ (x^*)])$:

$$\phi(\Phi^{-1}[F_x(x^*)]) = \frac{1}{\sqrt{2\pi}} exp\left[-\frac{1}{2}\left(\frac{ln\, x^* - \mu_y}{\sigma_y}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi}} exp\left[-\frac{1}{2}\left(\frac{ln\, 80.0402 - 4.7825}{0.09975}\right)^2\right]$$

$$= 1.2865 \times 10^{-4}$$

(6) Compute the mean value and standard deviation of the equivalent normal distribution, $\mu_{x'}$ and $\sigma_{x'}$, using Equations 4.39a and 4.39b:

$$\sigma_{x'} = \frac{\phi(\Phi^{-1}[F_x(x^*)])}{f_x(x^*)} = \frac{1.2865\times10^{-4}}{1.6114\times10^{-5}}$$

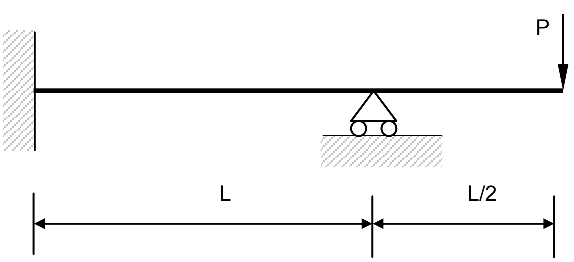
$$= 7.9841$$

$$\mu_{x'} = x^* - \Phi^{-1}[F_x(x^*)]\sigma_{x'}$$

$$= 80.042 + 4.0098\times7.9841 = 112.0553$$

Using $\mu_{x'}$ and $\sigma_{x'}$, the standard normal variable of x can be computed from Equation 4.38b.

Example 4.5



Consider the plane frame structure shown above. Evaluate the safety-index β and the coordinates x_i^* of the MPP. The limit state on the displacement is

$$d = \frac{5PL^3}{48EI} \le \frac{L}{30} = d_{\max}$$

where $d_{\max}$ is an allowable maximum displacement, E is Young's modulus, and I is the area moment of the cross-section. There are three random variables in this example. E and I are normally distributed, and the CDF and other parameters of the random variable P are defined by

$$F_P(x) = \exp[-[\exp\{-\alpha(x-\delta)\}]] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

$$\mu_P = \delta + \frac{0.577}{\alpha}, \quad \sigma_P = \frac{1.283}{\alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(B)}$$

(*Note*: this distribution is known as the type-I extreme value distribution)

The mean values, μ_P, μ_L, μ_E, μ_I, of the load *P*, the beam length *L*, the Young's modulus *E*, and the area moment *I* are 4 kN, 5 m, 2.0×10^7 kN/m^2, and 10^{-4} m^4, respectively. The corresponding standard deviations, σ_P, σ_L, σ_E, σ_I, are 1 kN, 0 m, 0.5×10^7 kN/m^2, and 0.2×10^{-4} m^4. Here, *L* is a deterministic parameter (*L* = 5 m) because $\sigma_L = 0$.

Solution:

The limit-state function is given as

EI-78.12P = 0

(1) Compute the scale and location parameters of the type-I extreme-value distribution using Equation B for the variable *P*.

$$\mu_P = \delta + 0.5772/\alpha, \qquad \sigma_P = 1.2825/\alpha$$

Substituting $\mu_P = 4$ kN and $\sigma_P = 1$ kN into the above formulas, the scale and location parameters are obtained as

$$\delta = 3.5499, \ \alpha = 1.2825$$

(2) Iteration 1:

(a) Compute the mean and standard deviation of the equivalent normal distribution for *P*:

First, assuming the design point, $X_1 = \{E_1, I_1, P_1\}^T$, as the mean value point, the coordinates of the initial design point are

$$E_1 = \mu_E = 2\times10^7, \ I_1 = \mu_I = 10^{-4}, \ P_1 = \mu_P = 4$$

The density function value at P_1 is

$$\begin{aligned} f_P(P_1) &= \alpha \exp\{-(P_1-\delta)\alpha - \exp[-(P_1-\delta)\alpha]\} \\ &= 1.2825\exp\{-(4-3.5499)\times1.2825 - \exp[-(-4-3.5499)\times1.2825]\} \\ &= 0.4107 \end{aligned}$$

The cumulative distribution value at P_1 is

$$\begin{aligned} F_P(P_1) &= \exp\{-\exp[-(P_1-\delta)\alpha]\} \\ &= \exp\{-\exp[-(-4-3.5499)1.2825\,\alpha]\} \\ &= 0.5704 \end{aligned}$$

Therefore, the standard deviation and mean value of the equivalent normal variable at P_1 from Equations 4.44a and 4.44b are

$$\sigma_{P'} = \frac{\phi(\Phi^{-1}[F_P(P_1)])}{f_P(P_1)} = \frac{\phi(\Phi^{-1}[0.5703])}{0.4107} = \frac{0.3927}{0.4107} = 0.9561$$

where $\Phi^{-1}[0.5704] = 0.177$ and $\phi[0.177] = 0.3927$.

$$\mu_{p'} = P_1 - \Phi^{-1}[F_P(P_1)]\sigma_{P'} = 4 - \Phi^{-1}[0.5704]\times 0.9561 = 3.8304$$

(b) Compute the function value and gradients of the limit-state function at the mean value point:

$$g(E_1, I_1, P_1) = EI - 78.12P = 2\times 10^7 \times 10^{-4} - 78.12\times 4 = 1687.52$$

$$\frac{\partial g(X_1)}{\partial E} = 10^{-4}, \quad \frac{\partial g(X_1)}{\partial I} = 2\times 10^7, \quad \frac{\partial g(X_1)}{\partial P} = -78.12$$

(c) Compute the initial β using the mean-value method and its direction cosine α_i:

$$\begin{aligned} \beta_1 &= \frac{\mu_{\tilde{g}}}{\sigma_{\tilde{g}}} \\ &= \frac{g(E_1, I_1, P_1)}{\sqrt{(\frac{\partial g(E_1,I_1,P)}{\partial E}\sigma_E)^2 + (\frac{\partial g(E_1,I_1,P)}{\partial I}\sigma_I)^2 + (\frac{\partial g(E_1,I_1,P)}{\partial P}\sigma_P)^2}} \\ &= \frac{1687.52}{\sqrt{(10^{-4}\times 0.5\times 10^7)^2 + (2\times 10^7\times 0.2\times 10^{-4})^2 + (-78.12\times 0.9561)^2}} \\ &= 2.6383 \end{aligned}$$

$$\begin{aligned} \alpha_E &= -\frac{\frac{\partial g}{\partial E}|_{\mu_X}\ \sigma_E}{\sqrt{(\frac{\partial g(E_1,I_1,P)}{\partial E}\sigma_E)^2 + (\frac{\partial g(E_1,I_1,P)}{\partial I}\sigma_I)^2 + (\frac{\partial g(E_1,I_1,P)}{\partial P}\sigma_P)^2}} \\ &= -\frac{10^{-4}\times 0.5\times 10^7}{\sqrt{(10^{-4}\times 0.5\times 10^7)^2 + (2\times 10^7\times 0.2\times 10^{-4})^2 + (-78.12\times 0.9561)^2}} \\ &= -0.7756 \end{aligned}$$

$$\alpha_I = -\frac{\frac{\partial g}{\partial I}|_{\mu_X}\ \sigma_I}{\sqrt{(\frac{\partial g(E_1,I_1,P)}{\partial E}\sigma_E)^2 + (\frac{\partial g(E_1,I_1,P)}{\partial I}\sigma_I)^2 + (\frac{\partial g(E_1,I_1,P)}{\partial P}\sigma_P)^2}}$$

$$= -\frac{2\times10^7\times0.2\times10^{-4}}{\sqrt{(10^{-4}\times0.5\times10^7)^2+(2\times10^7\times0.2\times10^{-4})^2+(-78.12\times0.9561)^2}}$$

$$= -0.6205$$

$$\alpha_P = -\frac{\frac{\partial g}{\partial P}|_{\mu_X}\ \sigma_P}{\sqrt{(\frac{\partial g(E_1,I_1,P)}{\partial E}\sigma_E)^2 + (\frac{\partial g(E_1,I_1,P)}{\partial I}\sigma_I)^2 + (\frac{\partial g(E_1,I_1,P)}{\partial P}\sigma_P)^2}}$$

$$= -\frac{-78.122\times0.9561}{\sqrt{(10^{-4}\times0.5\times10^7)^2+(2\times10^7\times0.2\times10^{-4})^2+(-78.12\times0.9561)^2}}$$

$$= 0.1159$$

(d) Compute the coordinates of the new design point using Equation 4.28:

$$E_2 = \mu_E + \beta_1\sigma_E\alpha_E$$
$$= 2\times10^7 + 2.6383\times0.5\times10^7\times(-0.7756)$$
$$= 9.7687\times10^6$$

$$I_2 = \mu_I + \beta_1\sigma_I\alpha_I$$
$$= 10^{-4} + 2.6383\times0.2\times10^{-4}\times(-0.6205)$$
$$= 0.6726\times10^{-4}$$

$$P_2 = \mu_{P'} + \beta_1\sigma_{P'}\alpha_P = 3.8304 + 2.6383\times0.9561\times0.1159$$
$$= 4.1227$$

$$u_{E,2} = \frac{E_2-\mu_E}{\sigma_E} = \frac{9.7687\times10^6 - 2\times10^7}{0.5\times10^7} = -2.0463$$

$$u_{I,2} = \frac{I_2-\mu_I}{\sigma_I} = \frac{0.6726\times10^{-4} - 10^{-4}}{0.2\times10^{-4}} = -1.6370$$

$$u_{P,2} = \frac{P_2-\mu_P}{\sigma_P} = \frac{4.1227-3.8304}{0.9561} = 0.3057$$

(3) Iteration 2:

(a) Compute the mean and standard deviation of the equivalent normal distribution at P_2. The density function value at P_2 is

$$f_P(P_2) = \alpha\ \exp\{-P_2-\delta)\alpha - exp[-(P_2-\delta)\alpha]\}$$

$$= 1.2825 exp\{-(4.1227 - 3.5499)\times 1.2825 - exp[-(4.1227 - 3.5499)\times 1.2825]\}$$
$$= 0.3808$$

The cumulative distribution function value at P_2 is

$$F_P(P_2) = \exp\{-\exp[-(P_2 - \delta)\alpha]\}$$
$$= \exp\{-\exp[-(4.1227 - 3.5499)\times 1.2825]\} = 0.6189$$

The standard deviation and mean value of the equivalent normal variable at P_2 are

$$\sigma_{P'} = \frac{\phi(\Phi^{-1}[F_P(P_2)])}{f_P(P_2)} = \frac{\phi(\Phi^{-1}[0.6189])}{0.3808} = \frac{0.3811}{0.3808}$$
$$= 1.0007$$

where $\Phi^{-1}[0.6189] = 0.3028$ and $\phi[0.6189] = 0.3811$

$$\mu_{P'} = P_2 - \Phi^{-1}[F_P(P_2)]\sigma_{P'}$$
$$= 4.1227 - \Phi^{-1}[0.6189]\times 1.007 = 3.8197$$

(b) Compute the function value and gradients of the limit-state function at $X_2(E_2, I_2, P_2)$

$$g(E_2, I_2, P_2) = EI - 78.12P$$
$$= 97.6871\times 10^5 \times 0.6726\times 10^{-4} - 78.12\times 4.1227$$
$$= 334.9737$$

$$\frac{\partial g(X_2)}{\partial E} = 6.726\times 10^{-5}, \quad \frac{\partial g(X_2)}{\partial I} = 97.6871\times 10^5,$$
$$\frac{\partial g(X_2)}{\partial P} = -78.12$$

(c) Compute the initial β using Equation 4.25 and the direction cosine α_i:

$$\beta_2 = \frac{g(X_2) - \frac{\partial g(X_2)}{\partial E}\sigma_E u_{E,2} - \frac{\partial g(X_2)}{\partial I}\sigma_I u_{I,2} - \frac{\partial g(X_2)}{\partial P}\sigma_P u_{P,2}}{\sqrt{(\frac{\partial g(X_2)}{\partial E}\sigma_E)^2 + (\frac{\partial g(X_2)}{\partial I}\sigma_I)^2 + (\frac{\partial g(X_2)}{\partial P}\sigma_P)^2}}$$

$$= \frac{334.9737 + 672.6\times 0.5\times 2.0463 + 976.87\times 0.2\times 1.6370 + 78.12\times 1.0007\times 0.3057}{\sqrt{(6.726\times 10^{-5}\times 0.5\times 10^7)^2 + (9.7687\times 10^6\times 0.2\times 10^{-4})^2 + (-78.12\times 1.0007)^2}}$$

$$= 3.4449$$

$$\alpha_E = -\frac{\frac{\partial g}{\partial E}|_{\mu_X}\ \sigma_E}{\sqrt{(\frac{\partial g(E_2, I_2, P_2)}{\partial E}\sigma_E)^2 + (\frac{\partial g(E_2, I_2, P_2)}{\partial I}\sigma_I)^2 + (\frac{\partial g(E_2, I_2, P_2)}{\partial P}\sigma_P)^2}}$$

$$= -\frac{6.726\times 10^{-5}\times 0.5\times 10^7}{\sqrt{(6.726\times 10^{-5}\times 0.5\times 10^7)^2 + (9.7687\times 10^6\times 0.2\times 10^{-4})^2 + (-78.12\times 1.0007)^2}}$$

$$= -0.8477$$

$$\alpha_I = -\frac{\frac{\partial g}{\partial I}|_{\mu_X}\ \sigma_I}{\sqrt{(\frac{\partial g(E_2, I_2, P_2)}{\partial E}\sigma_E)^2 + (\frac{\partial g(E_2, I_2, P_2)}{\partial I}\sigma_I)^2 + (\frac{\partial g(E_2, I_2, P_2)}{\partial P}\sigma_P)^2}}$$

$$= -\frac{9.7687\times 10^6\times 0.2\times 10^{-4}}{\sqrt{(6.726\times 10^{-5}\times 0.5\times 10^7)^2 + (9.7687\times 10^6\times 0.2\times 10^{-4})^2 + (-78.12\times 1.0007)^2}}$$

$$= -0.4925$$

$$\alpha_P = -\frac{\frac{\partial g}{\partial P}|_{\mu_X}\ \sigma_P}{\sqrt{(\frac{\partial g(E_2, I_2, P_2)}{\partial E}\sigma_E)^2 + (\frac{\partial g(E_2, I_2, P_2)}{\partial I}\sigma_I)^2 + (\frac{\partial g(E_2, I_2, P_2)}{\partial P}\sigma_P)^2}}$$

$$= -\frac{-78.12\times 1.0007}{\sqrt{(6.726\times 10^{-5}\times 0.5\times 10^7)^2 + (9.7687\times 10^6\times 0.2\times 10^{-4})^2 + (-78.12\times 1.0007)^2}}$$

$$= 0.1971$$

(d) Compute the coordinates of the new design point using Equation 4.28:

$$E_3 = \mu_E + \beta_2\sigma_E\alpha_E$$
$$= 2\times 10^7 + 3.4449\times 0.5\times 10^7\times(-0.8477) = 5.3985\times 10^6$$
$$I_3 = \mu_I + \beta_2\sigma_I\alpha_I$$
$$= 10^{-4} + 3.4449\times 0.2\times 10^{-4}\times(-0.4925) = 0.6607\times 10^{-4}$$
$$P_3 = \mu_{P'} + \beta_2\sigma_{P'}\alpha_P$$
$$= 3.8197 + 3.4449\times 1.0007\times 0.1971 = 4.4990$$
$$u_{E,3} = \frac{E_3 - \mu_E}{\sigma_E} = \frac{5.3985\times 10^6 - 2\times 10^7}{0.5\times 10^7} = -2.9203$$
$$u_{I,3} = \frac{I_3 - \mu_I}{\sigma_I} = \frac{1.3393\times 10^{-4} - 10^{-4}}{0.2\times 10^{-4}} = -1.6966$$
$$u_{P,3} = \frac{P_3 - \mu_{P'}}{\sigma_{P'}} = \frac{4.4990 - 3.8197}{1.0007} = 0.6788$$

(e) Check β convergence:

$$\varepsilon = \frac{|\beta_2 - \beta_1|}{\beta_1} = \frac{3.4449 - 2.6383}{2.6383} = 0.3057$$

Since $\varepsilon > \varepsilon_r(0.001)$, continue the process.

Table 4.3. Iteration Results in the HL-RF Method (Example 4.5)

Iteration No.	1	2	3	4	5	6
$f_P(P_k)$	0.4107	0.3808	0.2824	0.2316	0.2535	0.2732
$F_P(P_k)$	0.5704	0.6189	0.7438	0.7973	0.7748	0.7538
$\sigma_{P'}$	0.9561	1.0007	1.13998	1.2184	1.1835	1.1535
$\mu_{P'}$	3.8304	3.8197	3.7524	3.6939	3.7217	3.7433
$g(E_k,I_k,P_k)$	1687.52	334.9737	5.2055	-12.8426	-1.6961	-0.2352
$\frac{\partial g(X_k)}{\partial E}$	1.0×10^{-4}	6.726×10^{-5}	6.6069×10^{-5}	7.9676×10^{-5}	8.5848×10^{-5}	8.7569×10^{-5}
$\frac{\partial g(X_k)}{\partial I}$	2×10^{7}	9.7687×10^{6}	5.3985×10^{6}	4.4547×10^{6}	4.1799×10^{6}	4.0432×10^{6}
$\frac{\partial g(X_k)}{\partial P}$	-78.12	-78.12	-78.12	-78.12	-78.12	-78.12
β	2.6383	3.4449	3.3766	3.3292	3.3232	3.3222
α_E	-0.7756	-0.8477	-0.9208	-0.9504	-0.9603	-0.9638
α_I	-0.6205	-0.4925	-0.3009	-0.2125	-0.1870	-0.1780
α_P	0.1159	0.1971	0.2482	0.2271	0.2068	0.1984
E_k	9.7687×10^{6}	5.3985×10^{6}	4.4547×10^{6}	4.1799×10^{6}	4.0432×10^{6}	3.9897×10^{6}
I_k	0.6726×10^{-4}	1.3393×10^{-4}	0.7968×10^{-4}	0.8585×10^{-4}	0.8757×10^{-4}	0.8817×10^{-4}
P_k	4.1227	4.4990	4.7079	4.6151	4.5352	4.5035
$u_{E,k}$	-2.0463	-2.9203	-3.1091	-3.1640	-3.1914	-3.2021
$u_{I,k}$	-1.6370	-1.6966	-1.0162	-0.7076	-0.6215	-0.5914
$u_{P,k}$	0.3057	0.6788	0.8381	0.7560	0.6874	0.6590
ε	-	0.3057	0.0198	0.094	0.10	0.0003

The same procedures can be repeated until the stopping criterion ($\varepsilon < \varepsilon_r$) is satisfied. The iteration results are summarized in Table 4.3. The safety-index β is 3.3222. Since the limit-state function value at MPP, X^*, is close to zero compared to the starting value, this safety-index can be considered as the shortest distance from the origin to the limit-state surface.

4.1.6 FORM with Adaptive Approximations

In the previous algorithm, the limit-state function, $g(U)$, was approximated by the first-order Taylor expansion at the MPP. For nonlinear problems, this approach is only an approximation, and several iterations are usually required. How fast the algorithm converges depends on how well the linearized limit-state function approximates the nonlinear function *g(U)*. The limit-state function could be approximated by other functions, such as the *Two-point Adaptive Nonlinear*

Approximations (TANA), including TANA and TANA2 (the origin of function approximations and detailed developments are presented in Appendix A). This new class of approximations is constructed by using the Taylor series expansion in terms of adaptive intervening variables. The nonlinearity of the adaptive approximations is automatically changed by using the known information generated during the iteration process. TANA2 also has a correction term for second-order terms.

To compute the approximate U-space limit-state surface $\tilde{g}(U)$ using TANA, we must first obtain the adaptive approximate limit-state surface in X-space. Two possible methods, TANA and TANA2, for performing this step are given below.

TANA:

$$\tilde{g}(X) = g(X_k) + \frac{1}{r}\sum_{i=1}^{n} x_{i,k}^{1-r}\frac{\partial g(X_k)}{\partial x_i}(x_i^r - x_{i,k}^r) \tag{4.45}$$

where $x_{i,k}$ is the i^{th} element in the vector X_k of the k^{th} point/iteration. The comma notation does not signify differentiation.

The nonlinear index r in Equation 4.45 can be determined from

$$g(X_{k-1}) - \{g(X_k) + \frac{1}{r}\sum_{i=1}^{n} x_{i,k}^{1-r}\frac{\partial g(X_k)}{\partial x_i}(x_{i,k-1}^r - x_{i,k}^r)\} = 0 \tag{4.46}$$

TANA2:

$$\tilde{g}(X) = g(X_k) + \sum_{i=1}^{2}\frac{\partial g(X_k)}{\partial x_i}\frac{(x_{i,k})^{1-p_i}}{p_i}(x_i^{p_i} - (x_{i,k})^{p_i}) + \frac{1}{2}\varepsilon_2\sum_{i=1}^{2}(x_i^{p_i} - (x_{i,k})^{p_i})^2 \tag{4.47}$$

p_i and ε in Equation 4.46 can be determined from

$$\frac{\partial g(X_{k-1})}{\partial x_i} = (\frac{x_{i,k-1}}{x_{i,k}})^{p_{i-1}}\frac{\partial g(X_k)}{\partial g x_i} + \varepsilon_2(x_{i,k-1}^{p_i} - x_{i,k}^{p_i})x_{i,k-1}^{p_{i-1}}p_i \tag{4.48}$$

$$g(X_{k-1}) = g(X_k) + \sum_{i=1}^{n}\frac{\partial g(X_k)}{\partial x_i}\frac{x_{i,k}^{1-p_i}}{p_i}(x_i^{p_i} - x_{i,k}^{p_i}) + \frac{1}{2}\varepsilon_2\sum_{i=1}^{n}(x_{i,k-1}^{p_i} - x_{i,k}^{p_i})^2 \tag{4.49}$$

$$(i = 1,2,\ldots,n)$$

The next step is to map $\tilde{g}(X)$ into $\tilde{g}(U)$ by using the standard normal or equivalent normal transformations:

$$\tilde{g}(X) = \tilde{g}(\sigma_{x_1'}u_1 + \mu_{x_1'}, \sigma_{x_2'}u_2 + \mu_{x_1'}, \ldots., \sigma_{x_n'}u_n + \mu_{x_n'}) \tag{4.50}$$

The nonlinear index, r, is numerically calculated by minimizing the difference between the exact and the approximate limit-state functions at the previous point X_{k-1}. In theory, r can be any positive or negative real number (not equal to 0). In practice, r can be restricted from, say, -5 to 5, for the X-space iterations to avoid numerical difficulties associated with higher order polynomials [20]. r can be solved using any nonlinear equation solver, but here a simple iterative scheme is used. The iteration searching for r starts from r = 1. When r is increased or decreased a step length (0, 1), the difference ε between the exact and approximate function is calculated. If ε is smaller than the initial error (e.g., corresponding to r = 1), the above iteration is repeated until the allowable error, ε =0.001, or limitation of r is reached and the nonlinear index r is determined. Otherwise, r is decreased by one half and the above iteration process is repeated until the final r is obtained. This search is computationally inexpensive because Equation 4.46 is available in a closed form equation and is very easy to implement.

Usually, the adaptive safety-index algorithm is better than the HL-RF method, because the nonlinear index r is determined by comparing linear approximations (starting from 1) and minimizing the difference between exact and approximate limit-state functions. In the process of searching for r, the nonlinear index will automatically become 1 if other values of r cannot provide any improvement over the linear approximation.

The main steps of this algorithm are summarized as follows:

1) In the first iteration, compute the mean and standard deviation of the equivalent normal distribution at the mean value point for non-Gaussian distribution variables. Construct a linear approximation of Equation 4.23 by using the first-order Taylor series expansion at an initial point (if the initial point is selected as the mean value point, μ , the linear approximation is expanded at μ) and compute the limit-state function value and gradients at the initial point.
2) Compute the initial safety-index β_1 using the HL-RF method and its direction cosine, α_i (if the initial point is the mean value point, the mean-value method is used.).
3) Compute the new design point X_k using Equation 4.28.
4) Compute the mean and standard distribution of the equivalent normal distribution at X_k for non-normal distribution variables. Calculate the limit-state function value gradients at the new design point, X_k.
5) Determine the nonlinear index r by solving Equation 4.46 for TANA; or determine the nonlinear index p_k by solving Equation 4.48 and Equation 4.49 for TANA2; based on the information of the current and previous points (when k is equal to 2, previous design point is the mean value $\overline{X}$).
6) Obtain the adaptive nonlinear approximation, Equation 4.45 for TANA or Equation 4.47 for TANA2.
7) Transform the X-space approximate limit-state function into the U-space function using Equation 4.50.

8) Find the most probable failure point X_{k+1} of the approximate safety-index model given in Equation 4.20 using the HL-RF method or by solving the optimization problem using any commercial software. Compute the safety-index β_{k+1}.

9) Check the convergence: $\varepsilon = \dfrac{|\beta_{k+1} - \beta_k|}{\beta_k}$

10) Stop the process if ε satisfies the required convergence tolerance limit (e.g., 0.001); otherwise, continue.

11) Compute the exact limit-state function value and approximate gradients at X_{k+1} and estimate the approximate safety-index $\tilde{\beta}_{k+1}$ using the HL-RF method.

12) Approximate β convergence check, $\varepsilon = \dfrac{|\tilde{\beta}_{k+1} - \beta_k|}{\beta_k}$

13) Continue the process if ε satisfies the required convergence tolerance (0.001); otherwise, stop.

14) Compute the exact gradients of the limit-state function at X_{k+1} and go to step 5); repeat the process until β converges.

In Step 8, the safety-index β of the approximate model given in Equation 4.20 can be obtained easily by computing the explicit function $\tilde{g}(U)$, in which any optimization scheme or iteration algorithm can be used. Computation of the exact performance function $g(X)$ is not required; therefore, computer time is greatly reduced for problems involving complex and implicit performance functions, particularly with finite element models for structural response simulation.

Example 4.6

For Example 4.3b, solve the safety-index β using the TANA algorithm (Equation 4.45).

Solution:

(1) Iteration 1:

(a) Set the mean value point as an initial design point and the required β convergence tolerance as $\varepsilon_r = 0.001$. Compute the limit-state function value and gradients at the mean value point.

$$
\begin{aligned}
g(X^*) &= g(\mu_{x_1}, \mu_{x_2}) = \mu_{x_1}^3 + \mu_{x_2}^3 - 18 \\
&= 10.0^3 + 9.9^3 - 18 = 1952.299
\end{aligned}
$$

$$
\frac{\partial g}{\partial x_1}|_\mu = 3\mu_{x_1}^2 = 3 \times 10^2 = 300 , \quad \frac{\partial g}{\partial x_2}|_\mu = 3\mu_{x_2}^2 = 3 \times 9.9^2 = 294.03
$$

(b) Compute the initial β using the mean-value method and its direction cosine α_i

$$\beta_1 = \frac{\mu_{\tilde{g}}}{\sigma_{\tilde{g}}} = \frac{g(X^*)}{\sqrt{(\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_1}\sigma_{x_1})^2 + (\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_2}\sigma_{x_2})^2}}$$

$$= \frac{1952.299}{\sqrt{(300 \times 5.0)^2 + (294.03 \times 5.0)^2}} = 0.9295$$

$$\alpha_1 = -\frac{\frac{\partial g}{\partial x_1}|_{\mu}\ \sigma_{x_1}}{\sqrt{(\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_1}\sigma_{x_1})^2 + (\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_2}\sigma_{x_2})^2}}$$

$$= -\frac{300 \times 5.0}{\sqrt{(300 \times 5.0)^2 + (294.03 \times 5.0)^2}} = \text{-}0.7142$$

$$\alpha_2 = -\frac{\frac{\partial g}{\partial x_2}|_{\mu}\ \sigma_{x_2}}{\sqrt{(\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_1}\sigma_{x_1})^2 + (\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_2}\sigma_{x_2})^2}}$$

$$= -\frac{294.03 \times 5.0}{\sqrt{(300 \times 5.0)^2 + (294.03 \times 5.0)^2}} = -0.6999$$

(c) Compute a new design point X^* using Equation 4.28

$$x_1^* = \mu_{x_1} + \beta_1 \sigma_{x_1} \alpha_1 = 10.0 + 0.9295 \times 5.0 \times (-0.7142) = 6.6808$$

$$x_2^* = \mu_{x_2} + \beta_1 \sigma_{x_2} \alpha_2 = 9.9 + 0.9295 \times 5.0 \times (-0.6999) = 6.6468$$

$$u_1^* = \frac{x_1^* - \mu_{x_1}}{\sigma_{x_1}} = \frac{6.6808 - 10.0}{5.0} = -0.6638$$

$$u_2^* = \frac{x_2^* - \mu_{x_2}}{\sigma_{x_2}} = \frac{6.6468 - 9.9}{5.0} = -0.6506$$

(2) Iteration 2:

(a) Compute the limit-state function value and gradients at X^*:

$$g(X^*) = g(x_1^*, x_2^*) = x_1^{*3} + x_2^{*3} - 18$$

$$= 6.6808^3 + 6.6468^3 - 18 = 573.8398$$

$$\frac{\partial g}{\partial x_1}|_{\mu} = 3x_1^{*2} = 3 \times 6.6808^2 = 133.8982$$

$$\frac{\partial g}{\partial x_2}|_{\mu} = 3x_2^{*2} = 3 \times 6.6468^2 = 132.5409$$

(b) Compute the nonlinearity index r based on the function values and gradients of the two points, $\mu(10.0,9.9)$ and $X^*(6.6808,6.6468)$, using Equation 4.46:

$$g(X_{k-1}) - \{ g(X_k) + \frac{1}{r}\sum_{i=1}^{2} x_{i,k}^{1-r} \frac{\partial g(X_k)}{\partial x_i}(x_{i,k-1}^r - x_{i,k}^r)\}$$
$$= 1952.299 - \{573.8398 + \frac{1}{r}[6.6808^{1-r} \times 133.8982 \times (10^r - 6.6808^r) +$$
$$6.66468^{1-r} \times 132.5409 \times (9.9^r - 6.6468^r)] \le 0.001$$

where $X_{k-1} = \mu(10.0,9.9)$ and $X_k = X^*(6.6808,6.6468)$

Using the adaptive search procedure mentioned before, r can be solved as r = 3.0.

(c) Construct the two-point adaptive nonlinear approximation (TANA) using Equation 4.45:

$$\tilde{g}(X) = g(X_k) + \frac{1}{r}\sum_{i=1}^{2} x_{i,k}^{1-r} \frac{\partial g(X_k)}{\partial x_i}(x_i^r - x_{i,k}^r)$$
$$= 573.8398 + \frac{1}{3}[6.6808^{-2} \times 133.8982 \times (x_1^3 - 6.6808^3) +$$
$$6.6468^{-2} \times 132.5409 \times (x_2^3 - 6.6468^3)] = x_1^3 + x_2^3 - 18.0$$

The approximate function is same as the exact problem given, because it is separable function and TANA predicted the nonlinearity index precisely.

(d) Transfer the above X-space approximate limit-state function into the U-space function using Equation 4.50:

$$\tilde{g}(U) = \tilde{g}(\sigma_{x_1}\mu_1 + \mu_{x_1}, \sigma_{x_2}\mu_2 + \mu_{x_2})$$
$$=(5u_1 + 10)^3 + (5u_2 + 9.9)^3 - 18$$

(e) Find the most probable failure point X^* of the approximate safety-index model given in Equation 4.20 using an optimization algorithm.

After four iterations, the MPP point is found as

$$x_1^* = 2.0718,\ \ x_2^* = 2.088\ ,\ u_1^* = -1.5856,\ u_2^* = -1.5623$$

(f) Compute the safety-index β_2 :

$$\beta_2 = \sqrt{u_1^{*2} + u_2^{*2}} = \sqrt{(-1.5856)^2 + (-1.5623)^2} = 2.2260$$

(g) Convergence check:

$$\varepsilon = \frac{|\beta_2 - \beta_1|}{\beta_1} = \frac{|2.2260 - 0.9295|}{0.9295} = 1.3948$$

Since $\varepsilon > \varepsilon_r (0.001)$, continue the process.

(3) Iteration 3:

(a) Compute the limit-state function value and gradients at X^* :

$$\begin{aligned} g(X^*) &= g(x_1^*, x_2^*) = x_1^{*3} + x_2^{*3} - 18 \\ &= 2.0718^3 + 2.0883^3 - 18 \\ &= -0.1276 \times 10^{-5} \end{aligned}$$

(b) Compute approximate gradients using the approximate limit-state function:

$$\frac{\partial \tilde{g}}{\partial x_1}|_{\mu} = 3x_{1_1}^{*2} = 3 \times 2.0718^2 = 12.8769$$

$$\frac{\partial \tilde{g}}{\partial x_2}|_{\mu} = 3x_2^{*2} = 3 \times 2.0883^2 = 13.0832$$

(c) Compute approximate safety-index $\tilde{\beta}$ using the HL-RF method (Equation 4.25) and the direction cosine α_i :

$$\tilde{\beta}_3 = \frac{g(X^*) - \dfrac{\partial \tilde{g}(X^*)}{\partial x_1} \sigma_{x_1} u_{x_1}^* - \dfrac{\partial \tilde{g}(X^*)}{\partial x_2} \sigma_{x_2} u_{x_2}^*}{\sqrt{(\dfrac{\partial \tilde{g}(X^*)}{\partial x_1} \sigma_{x_1})^2 + (\dfrac{\partial \tilde{g}(X^*)}{\partial x_2} \sigma_{x_2})^2}}$$

$$= \frac{-0.1276 \times 10^{-5} - 12.8769 \times 5 \times -1.5856 - 13.0832 \times 5 \times -1.5623}{\sqrt{(12.8769 \times 5)^2 + (13.0832 \times 5)^2}}$$

$$= 2.2258$$

(d) Approximate convergence check

$$\varepsilon = \frac{|\tilde{\beta}_3 - \beta_2|}{\beta_2} = \frac{|2.2258 - 2.2260|}{2.2260} = 0.00009$$

Since $\varepsilon < \varepsilon_r (0.001)$, stop the process. The final safety-index is 2.2258. Compared with the result of Example 4.3b ($\beta = 1.1657$), the safety-index algorithm using TANA is much more efficient for this example. It needs only 3 *g*-function and 2 gradient calculations to reach the convergent point. Since the *g*-function value is very small, the final MPP is on the limit-state surface.

Example 4.7

The performance function is

$$g(x_1, x_2) = x_1\ x_2\ - 1400$$

in which x_1 and x_2 are the random variables with lognormal distributions. The mean values and standard deviations of two variables are $\mu_{x_1} = 40.0$, $\mu_{x_2} = 50.0$, $\sigma_{x_1} = 5.0$, and $\sigma_{x_2} = 2.5$. Solve the safety-index β by using the TANA2 algorithm (Equation 4.47).

Solution:

(1) Compute the mean values and standard deviations of normally distributed variables y_1 and y_2 $(y_1 = ln\, x_1, y_2 = ln\, x_2)$ using Equations 2.44 and 2.45:

$$\sigma_{y_1} = \sqrt{\ln\left[\left(\frac{\sigma_{x_1}}{\mu_{x_1}}\right)^2 + 1\right]} = \sqrt{\ln\left[\left(\frac{5.0}{40.0}\right)^2 + 1\right]} = 0.1245$$

$$\sigma_{y_2} = \sqrt{\ln\left[\left(\frac{\sigma_{x_2}}{\mu_{x_2}}\right)^2 + 1\right]} = \sqrt{\ln\left[\left(\frac{2.5}{50.0}\right)^2 + 1\right]} = 4.9969 \times 10^{-2}$$

$$\mu_{y_1} = ln\, \mu_{x_1} - \frac{1}{2}\sigma_{y_1}^2 = ln\, 40 - \frac{1}{2} \times 0.1245^2 = 3.6811$$

$$\mu_{y_2} = ln\, \mu_{x_2} - \frac{1}{2}\sigma_{y_2}^2 = ln\, 50 - \frac{1}{2} \times (4.9969 \times 10^{-2})^2 = 3.9108$$

(2) Iteration 1:

(a) Compute the mean value and standard deviations of the equivalent normal distributions for x_1 and x_2:

First, assuming the design point $X^* = \{x_1^*, x_2^*\}^T$ as the mean value point, the coordinates of the initial design point are

$$x_1^* = \mu_{x_1} = 2\times 40.0,\ \ x_2^* = \mu_{x_2} = 50.0$$

The density function values at x_1^* and x_2^* are

$$f_{x_1}(x_1^*) = \frac{1}{\sqrt{2\pi}\, x_1^*\, \sigma_{y_1}} \exp[-\frac{1}{2}(\frac{\ln x_1^* - \mu_{y_1}}{\sigma_{y_1}})^2]$$

$$= \frac{1}{\sqrt{2\pi}\times 40\times 0.1245} \exp[-\frac{1}{2}(\frac{\ln 40 - 3.6811}{0.1245})^2] = 7.9944\times 10^{-2}$$

$$f_{x_2}(x_2^*) = \frac{1}{\sqrt{2\pi}\, x_2^*\, \sigma_{y_2}} \exp[-\frac{1}{2}(\frac{\ln x_2^* - \mu_{y_2}}{\sigma_{y_2}})^2]$$

$$= \frac{1}{\sqrt{2\pi}\times 50\times 4.9969\times 10^{-2}} \exp[-\frac{1}{2}(\frac{\ln 50 - 3.9108}{4.9969\times 10^{-2}})^2] = 0.1596$$

$$\phi(\Phi^{-1}[F_{x_1}(x_1^*)]) = \frac{1}{\sqrt{2\pi}} \exp[-\frac{1}{2}(\frac{\ln x_1^* - \mu_{y_1}}{\sigma_{y_1}})^2]$$

$$= \frac{1}{\sqrt{2\pi}} \exp[-\frac{1}{2}(\frac{\ln 40 - 3.6811}{0.1245})^2] = 0.3982$$

$$\phi(\Phi^{-1}[F_{x_2}(x_2^*)]) = \frac{1}{\sqrt{2\pi}} \exp[-\frac{1}{2}(\frac{\ln x_2^* - \mu_{y_2}}{\sigma_{y_2}})^2]$$

$$= \frac{1}{\sqrt{2\pi}} \exp[-\frac{1}{2}(\frac{\ln 50 - 3.9108}{4.9969\times 10^{-2}})^2] = 0.3988$$

Therefore, the standard deviation and mean value of the equivalent normal variable at P^* using Equations 4.44a and 4.44b are

$$\sigma_{x_1'} = \frac{\phi(\Phi^{-1}[F_{x_1}(x_1^*)])}{f_{x_1}(x_1^*)} = \frac{0.3982}{7.9944\times 10^{-2}} = 4.9806$$

$$\sigma_{x_2'} = \frac{\phi(\Phi^{-1}[F_{x_2}(x_2^*)])}{f_{x_2}(x_2^*)} = \frac{0.3988}{0.1596} = 2.4984$$

$$\mu_{x_1'} = x_1^* - \Phi^{-1}[F_{x_1}(x_1^*)]\sigma_{x_1'} = 40 - 6.2258\times 10^{-2}\times 4.9806 = 39.6899$$

$$\mu_{x_2'} = x_2^* - \Phi^{-1}[F_{x_2}(x_2^*)]\sigma_{x_2'} = 50 - 2.4984\times 10^{-2}\times 2.4984 = 49.9376$$

(b) Set the mean value point as an initial design point and the required β convergence tolerance as $\varepsilon_r = 0.001$. Compute the limit-state function value and the gradients at the mean value point:

$$g(X^*) = g(\mu_{x_1}, \mu_{x_2}) = \mu_{x_1}\mu_{x_2} - 1400 = 40 \times 50 - 1400 = 600$$

$$\left.\frac{\partial g}{\partial x_1}\right|_{\mu} = \mu_{x_2} = 50, \quad \left.\frac{\partial g}{\partial x_2}\right|_{\mu} = \mu_{x_1} = 40$$

(c) Compute the initial β using the mean-value method and its direction cosine, α_i:

$$\beta_1 = \frac{\mu_{\tilde{g}}}{\sigma_{\tilde{g}}} = \frac{g(X^*)}{\sqrt{(\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_1}\sigma_{x_1'})^2 + (\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_2}\sigma_{x_2'})^2}}$$

$$= \frac{600}{\sqrt{(50 \times 4.9806)^2 + (40 \times 2.4984)^2}} = 2.1689$$

$$\alpha_1 = -\frac{\left.\frac{\partial g}{\partial x_1}\right|_{\mu}\sigma_{x_1'}}{\sqrt{(\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_1}\sigma_{x_1'})^2 + (\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_2}\sigma_{x_2'})^2}}$$

$$= -\frac{50 \times 4.9806}{\sqrt{(50 \times 4.9806)^2 + (40 \times 2.4984)^2}} = -0.9281$$

$$\alpha_2 = -\frac{\left.\frac{\partial g}{\partial x_2}\right|_{\mu}\sigma_{x_2'}}{\sqrt{(\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_1}\sigma_{x_1'})^2 + (\frac{\partial g(\mu_{x_1}, \mu_{x_2})}{\partial x_2}\sigma_{x_2'})^2}}$$

$$= -\frac{40 \times 2.4984}{\sqrt{(50 \times 4.9806)^2 + (40 \times 2.4984)^2}} = -0.3724$$

(d) Compute a new design point X^* from Equation 4.28:

$$x_1^* = \mu_{x_1'} + \beta_1\sigma_{x_1'}\alpha_1 = 39.6899 + 2.1689 \times 4.9806 \times (-0.9281) = 29.6645$$

$$x_2^* = \mu_{x_2'} + \beta_1\sigma_{x_2'}\alpha_2 = 49.9376 + 2.1689 \times 2.4984 \times (-0.3724) = 47.9194$$

$$u_1^* = \frac{x_1^* - \mu_{x_1'}}{\sigma_{x_1'}} = -2.0129, \quad u_2^* = \frac{x_2^* - \mu_{x_2'}}{\sigma_{x_2'}} = -0.8078$$

(3) Iteration 2:

(a) Compute the mean values and standard deviations of the equivalent normal distributions for x_1^* and x_2^*:

The density function values at x_1^* and x_2^* are

$$f_{x_1}(x_1^*) = \frac{1}{\sqrt{2\pi}\, x_1^*\, \sigma_{y_1}} \exp\left[-\frac{1}{2}\left(\frac{\ln x_1^* - \mu_{y_1}}{\sigma_{y_1}}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi} \times 29.6645 \times 0.1245} \exp\left[-\frac{1}{2}\left(\frac{\ln 29.6645 - 3.6811}{0.1245}\right)^2\right] = 7.0144 \times 10^{-3}$$

$$f_{x_2}(x_2^*) = \frac{1}{\sqrt{2\pi}\, x_2^*\, \sigma_{y_2}} \exp\left[-\frac{1}{2}\left(\frac{\ln x_2^* - \mu_{y_2}}{\sigma_{y_2}}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi} \times 47.9194 \times 4.9969 \times 10^{-2}} \exp\left[-\frac{1}{2}\left(\frac{\ln 47.9194 - 3.9108}{4.9969 \times 10^{-2}}\right)^2\right] = 0.1185$$

$$\phi(\Phi^{-1}[F_{x_1}(x_1^*)]) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln x_1^* - \mu_{y_1}}{\sigma_{y_1}}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln 29.6645 - 3.6811}{0.1245}\right)^2\right] = 2.5909 \times 10^{-2}$$

$$\phi(\Phi^{-1}[F_{x_2}(x_2^*)]) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln x_2^* - \mu_{y_2}}{\sigma_{y_2}}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln 47.9194 - 3.9108}{4.9969 \times 10^{-2}}\right)^2\right] = 0.2837$$

Therefore, the standard deviation and mean value of the equivalent normal variable at P^* using Equations 4.44a and 4.44b are

$$\sigma_{x_1'} = \frac{\phi(\Phi^{-1}[F_{x_1}(x_1^*)])}{f_{x_1}(x_1^*)} = \frac{2.5909 \times 10^{-2}}{7.0144 \times 10^{-3}} = 3.6937$$

$$\sigma_{x_2'} = \frac{\phi(\Phi^{-1}[F_{x_2}(x_2^*)])}{f_{x_2}(x_2^*)} = \frac{0.2837}{0.1185} = 2.3945$$

$$\mu_{x_1'} = x_1^* - \Phi^{-1}[F_{x_1}(x_1^*)]\sigma_{x_1'} = 29.6645 - (-2.3384) \times 3.6937 = 38.3021$$

$$\mu_{x_2'} = x_2^* - \Phi^{-1}[F_{x_2}(x_2^*)]\sigma_{x_2'} = 47.9194 - (-0.8256) \times 2.3945 = 49.8963$$

(b) Compute the limit-state function value and gradients at X^*:

$$g(X^*) = g(x_1^*, x_2^*) = x_1^* \, x_2^* - 1400 = 29.6645 \times 47.9194 - 1400 = 21.5041$$

$$\left.\frac{\partial g}{\partial x_1}\right|_{X^*} = x_2^* = 47.9194\,, \quad \left.\frac{\partial g}{\partial x_2}\right|_{X^*} = x_1^* = 29.6645$$

(c) Compute the nonlinearity indices p_1 and p_2 based on the function values and gradients of the two points μ (40,50) and X^* (29.6645, 47.9194) using Equations 4.48 and 4.49:

$$\frac{\partial g(\mu)}{\partial x_1} = \left(\frac{\mu_1}{x_1^*}\right)^{p_1-1} \frac{\partial g(X^*)}{\partial x_1} + \varepsilon_2\,(\mu_1^{p_1} - (x_1^*)^{p_1})\mu_1^{p_1-1} p_1$$

$$50 = \left(\frac{40}{29.6645}\right)^{p_1-1} 47.9194 + \varepsilon_2\,(40^{p_1} - (47.9194)^{p_1})40^{p_1-1} p_1$$

$$\frac{\partial g(\mu)}{\partial x_2} = \left(\frac{\mu_2}{x_2^*}\right)^{p_2-1} \frac{\partial g(X^*)}{\partial x_2} + \varepsilon_2\,(\mu_2^{p_2} - (x_2^*)^{p_2})\mu_2^{p_2-1} p_2$$

$$40 = \left(\frac{50}{47.9194}\right)^{p_2-1} 29.6645 + \varepsilon_2(50^{p_2} - (47.9194)^{p_2})50^{p_2-1} p_2$$

$$g(\mu) = g(X^*) + \sum_{i=1}^{2} \frac{\partial g(X^*)}{\partial x_i} \frac{(x_i^*)^{1-p_i}}{p_i} (\mu_i^{p_i} - (x_i^*)^{p_i}) + 0.5\varepsilon_2 \sum_{i=1}^{2} (\mu_i^{p_i} - (x_i^*)^{p_i})^2$$

$$\begin{aligned} 600 = 21.5041 &+ 47.9194 \times \frac{29.6645^{1-p_1}}{p_1}(40^{p_1} - 29.6645^{p_1}) \\ &+ 29.6645 \times \frac{47.9194^{1-p_2}}{p_2}(50^{p_1} - 47.9194^{p_2}) \\ &+ 0.5\varepsilon_2\,[(\mu_1{}^{p_1} - (x_1^*)^{p_1})^2 + (\mu_2{}^{p_2} - (x_2^*)^{p_2})^2] \end{aligned}$$

Based on the above three equations, p_1, p_2 and ε_2 are solved using the adaptive search procedure.

$$p_1 = 1.0375\,, \quad p_2 = 1.4125\,, \quad \varepsilon = 0.1$$

(d) Construct TANA2 model using Equation 4.47:

$$\begin{aligned} \tilde{g}(X) &= g(X^*) + \sum_{i=1}^{2} \frac{\partial g(X^*)}{\partial x_i} \frac{(x_i^*)^{1-p_i}}{p_i} (x_i^{p_i} - (x_i^*)^{p_i}) + 0.5\varepsilon_2 \sum_{i=1}^{2} (x_i^{p_i} - (x_i^*)^{p_i})^2 \\ &= 21.5041 + 47.9194 \times \frac{29.6645^{1-1.0375}}{1.0375}(x_1^{1.0375} - 29.6645^{1.0375}) \\ &\quad + 29.6645 \times \frac{47.9194^{1-1.4125}}{1.4125}(x_2^{1.4125} - 47.9194^{1.4125}) \end{aligned}$$

$$+\frac{0.1}{2}[(x_1^{1.0375}-29.6645^{1.0375})^2+(x_2^{1.4125}-47.9194^{1.4125})^2]$$

$$=21.5041+40.6738(x_1^{1.0375}-29.6645^{1.0375})+4.2564(x_2^{1.4125}-47.9194^{1.4125})$$
$$+\frac{0.1}{2}[(x_1^{1.0375}-29.6645^{1.0375})^2+(x_2^{1.4125}-47.9194^{1.4125})^2]$$

(e) Transfer the above *X*-space approximate limit-state function into the *U*-space function using Equation 4.50:

$$\tilde{g}(U)=\tilde{g}(\sigma_{x_1'}u_1+\mu_{x_1'},\ \sigma_{x_2'}u_2+\mu_{x_2'})$$
$$=21.5041+40.6738[(3.6937\,\mu_1+38.3021)^{1.0375}-29.6645^{1.0375}]$$
$$+4.2564[(2.3945\,\mu_2+49.8963)^{1.4125}-47.9194^{1.4125}]$$
$$+\frac{0.1}{2}[((3.6937\,\mu_1+38.3021)^{1.0375}-29.6645^{1.0375})^2$$
$$+((2.3945\,\mu_2+49.8963)^{1.4125}-47.9194^{1.4125})^2]$$

(f) Find the most probable failure point X^* of the approximate safety-index model given in Equation 4.20 using an optimization algorithm.

After two iterations, the MPP point is found as

$$x_1^*=29.3517,\ x_2^*=47.6961,\ u_1^*=-2.4236,\ u_2^*=-0.9191$$

At each iteration, the mean value and standard deviation of the equivalent normal distributions at the new design point X^* must be calculated.

(g) Compute the safety-index β_2:

$$\beta_2=\sqrt{(u_1^*)^2+(u_2^*)^2}=\sqrt{(-2.4236)^2+(-0.9191)^2}=2.5920$$

(h) Convergence Check:

$$\varepsilon=\frac{|\beta_2-\beta_1|}{\beta_1}=\frac{2.5920-2.1689}{2.1689}=0.1951$$

Since $\varepsilon>\varepsilon_r\,(0.001)$, continue the process

(4) Iteration 3:

(a) Compute the mean values and standard deviations of the equivalent normal distributions for x_1^* and x_2^*:

The density function values at x_1^* and x_2^* are

$$f_{x_1}(x_1^*) = \frac{1}{\sqrt{2\pi}\, x_1^*\, \sigma_{y_1}} \exp\left[-\frac{1}{2}\left(\frac{\ln x_1^* - \mu_{y_1}}{\sigma_{y_1}}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi} \times 29.3517 \times 0.1245} \exp\left[-\frac{1}{2}\left(\frac{ln\, 29.3517 - 3.6811}{0.1245}\right)^2\right]$$

$$= 5.7886 \times 10^{-3}$$

$$f_{x_2}(x_2^*) = \frac{1}{\sqrt{2\pi}\, x_2^*\, \sigma_{y_2}} \exp\left[-\frac{1}{2}\left(\frac{\ln x_2^* - \mu_{y_2}}{\sigma_{y_2}}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi} \times 47.6961 \times 4.9969 \times 10^{-2}}\, exp\left[-\frac{1}{2}\left(\frac{ln\, 47.6961 - 3.9108}{4.9969 \times 10^{-2}}\right)^2\right]$$

$$= 0.1097$$

$$\phi(\Phi^{-1}[F_{x_1}(x_1^*)]) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln x_1^* - \mu_{y_1}}{\sigma_{y_1}}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi}}\, exp\left[-\frac{1}{2}\left(\frac{ln\, 29.3517 - 3.6811}{0.1245}\right)^2\right] = 2.1156 \times 10^{-2}$$

$$\phi(\Phi^{-1}[F_{x_2}(x_2^*)]) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln x_2^* - \mu_{y_2}}{\sigma_{y_2}}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi}}\, exp\left[-\frac{1}{2}\left(\frac{ln\, 47.6961 - 3.9108}{4.9969 \times 10^{-2}}\right)\right] = 0.2615$$

Therefore, the standard deviation and mean value of the equivalent normal variable at P^* using Equations 4.44a and 4.44b are

$$\sigma_{x_1'} = \frac{\phi(\Phi^{-1}[F_{x_1}(x_1^*)])}{f_{x_1}(x_1^*)} = \frac{2.1156 \times 10^{-2}}{5.7886 \times 10^{-3}} = 3.6548$$

$$\sigma_{x_2'} = \frac{\phi(\Phi^{-1}[F_{x_2}(x_2^*)])}{f_{x_2}(x_2^*)} = \frac{0.2615}{0.1097} = 2.3833$$

$$\mu_{x_1'} = x_1^* - \Phi^{-1}[F_{x_1}(x_1^*)]\sigma_{x_1'} = 29.3517 - (-2.4236) \times 3.6548 = 38.2094$$

$$\mu_{x_2'} = x_2^* - \Phi^{-1}[F_{x_2}(x_2^*)]\sigma_{x_2'} = 47.6961 - (-0.9191) \times 2.3833 = 49.8865$$

(b) Compute the limit-state function value at X^*:

$$g(X^*) = g(x_1^*, x_2^*) = x_1^* x_2^* - 1400 = 29.3517 \times 47.6961 - 1400 = -0.0359$$

(c) Compute approximate gradients using the approximate limit-state function:

$$\left.\frac{\partial \tilde{g}}{\partial x_1}\right|_{\mu} = 47.8569, \quad \left.\frac{\partial \tilde{g}}{\partial x_2}\right|_{\mu} = 28.5261$$

(d) Compute the approximate safety-index $\tilde{\beta}$ using the HL-RF method (Equation 4.25) and the direction cosine, α_i :

$$\tilde{\beta}_3 = \frac{g(X^*) - \frac{\partial \tilde{g}(X^*)}{\partial x_1}\sigma'_{x_1} u^*_{x_1} - \frac{\partial \tilde{g}(X^*)}{\partial x_2}\sigma'_{x_2} u^*_{x_2}}{\sqrt{(\frac{\partial \tilde{g}(X^*)}{\partial x_1}\sigma'_{x_1})^2 + (\frac{\partial \tilde{g}(X^*)}{\partial x_2}\sigma'_{x_2})^2}}$$

$$= \frac{-0.0359 - 47.8569 \times 3.6548 \times (-2.4236) - 28.5261 \times 2.3833 \times (-0.9191)}{\sqrt{(47.8569 \times 3.6548)^2 + (28.5261 \times 2.3833)^2}}$$

$$= 2.5917$$

(e) Approximate convergence check:

$$\varepsilon = \frac{\left|\tilde{\beta}_3 - \beta_2\right|}{\beta_2} = \frac{|2.5917 - 2.5920|}{2.5920} = 0.0001$$

Since $\varepsilon < \varepsilon_r (0.001)$, stop the process. The final safety-index is 2.5917.

4.2 Second-order Reliability Method (SORM)

FORM usually works well when the limit-state surface has only one minimal distance point and the function is nearly linear in the neighborhood of the design point. However, if the failure surface has large curvatures (high nonlinearity), the failure probability estimated by FORM using the safety-index β may give unreasonable and inaccurate results [13]. To resolve this problem, the second-order Taylor series (or other polynomials) is considered. Various nonlinear approximate methods have been proposed in the literature.

Breitung [3], Tvedt [18], [19], Hohenbichler and Rackwitz [9], Koyluoglu and Nielsen [11], and Cai and Elishakoff [4] have developed SORM using the second-order approximation to replace the original surfaces. Wang and Grandhi [20] and Der Kiureghian, *et al.* [5] calculated second-order failure probabilities using approximate curvatures to avoid exact second-order derivatives calculations of the limit-state surface.

First, in Section 4.2.1, we present the fundamentals of the second-order approximation of the response surface with orthogonal transformation. Then, Breitung's and Tvedt's formulations are introduced in Sections 4.2.2 and 4.2.3,