Utility Theory vs. Robust Optimization

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Utility Maximization vs. Robust Design

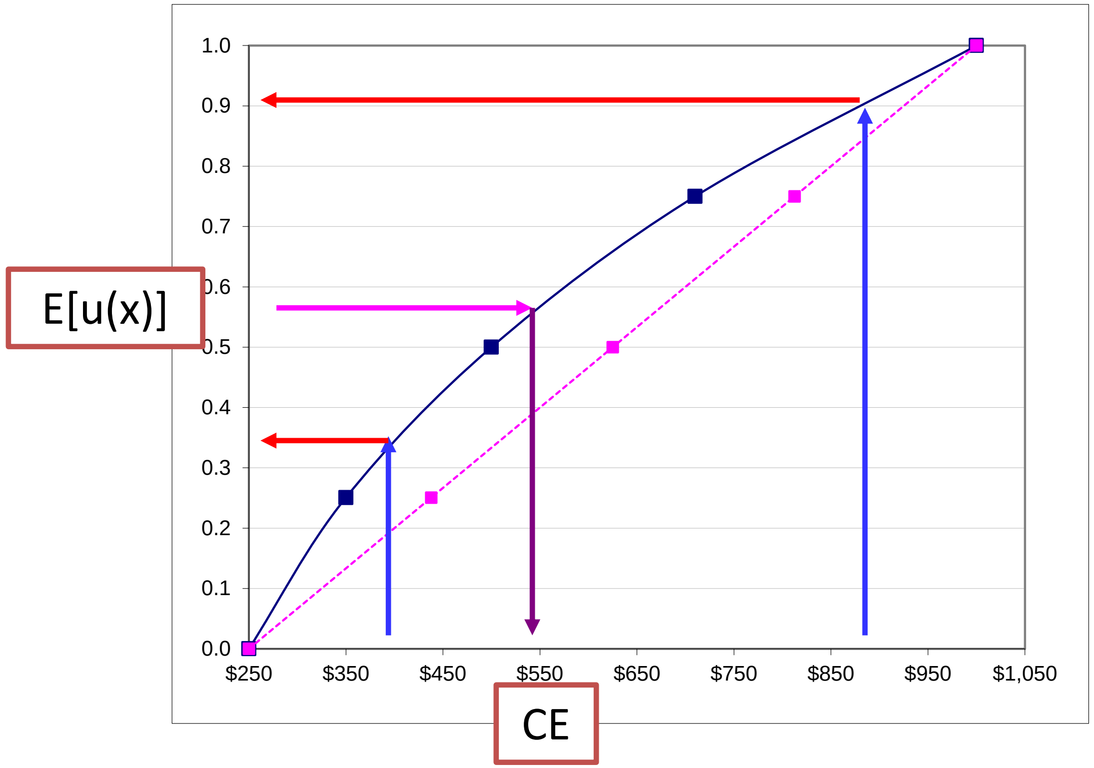
Oregon State University College of Engineering

- Expected Utility Maximization:
  - Objective:
    - maximize $\mathrm{E}[u(x)]$ Expected Utility Objective
  - Subject to:
    - $P\{g(\mathbf{x},\mathbf{p}) \le c\} \ge R_0$ Reliability Constraints
- Robust Optimization
  - Objective:
    - minimize: $w_1 \dfrac{\mu}{\mu_{utop}} + w_2 \dfrac{\sigma}{\sigma_{utop}}$ Robust Objective
  - Subject to:
    - $P\{g(\mathbf{x},\mathbf{p}) \le c\} \ge R_0$ Reliability Constraints

How are these two approaches similar or different?

Expected Utility and Certainty Equivalent

- With our utility function determined, we can find both the Utility and Certainty Equivalent for lotteries not considered previously:
  - For example, what is the utility for a lottery as follows
    0.4 probability of gaining $900 vs. 0.6 prob of gaining $400



$$u(s) = p_0 \cdot u(x_H) + (1 - p_0) \cdot u(x_L)$$

Example:

$$u(s) = 0.4 \cdot u(\$900) + 0.6 \cdot u(\$400)$$

$$u(s) = 0.4 \cdot (0.91) + 0.6 \cdot (0.35) = 0.57$$

Certainty Equivalent:

$$\boxed{u^{-1}(0.57) \approx \$540}$$

$$E(Lottery) = \$600$$

Krishnamurti (2007)

Deriving Robust Design from Utility Theory

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- By the definition of Certainty Equivalent, the following is true:

$$u(\mathrm{CE}) = \mathrm{E}[u(x)]$$ Eq. 1

- Define two new variables:
  - Risk Penalty (Premium): $\pi = \bar{x} - \mathrm{CE}$ (or $\pi = \mathrm{CE} - \bar{x}$) $\bar{x} = \mu_x$
  - $z = x - \bar{x}$
- We can rewrite the 1st equation as follows:
  - $u(\bar{x} - \pi) = \mathrm{E}[u(\bar{x} + z)]$
- Perform 2nd order Taylor Series expansion of **both sides** and doing a lot of math and we get:

$$\pi = -(1/2)\frac{d^2u(\bar{x})/dx^2}{du(\bar{x})/dx}\sigma^2$$

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- A utility function with constant risk aversion has the following property (i.e. the negative of the ratio of 2nd to 1st derivatives is constant):

$$r(x) = -\frac{u''(x)}{u'(x)} = c$$

- In our formulation of the exponential utility function, we used:

$$c = \frac{1}{\rho}$$

- Therefore:

$$\pi = -(1/2)\frac{d^2u(\bar{x})/dx^2}{du(\bar{x})/dx}\sigma^2 = (1/2)c\sigma^2 = \frac{1}{2\rho}\sigma^2$$

- Replace π with $\pi = \bar{x} - \mathrm{CE}$:

$$\mathbf{CE} = \boldsymbol{\mu} - \frac{\boldsymbol{\sigma}^2}{2\boldsymbol{\rho}}$$

Deriving Robust Design from Utility Theory

- In terms of utility theory what does robust design mean?
  - For increasing preferences we have:

$$\mathrm{CE} = \mu - \frac{\sigma^2}{2\rho}$$

  - For decreasing preferences we have:

$$\mathrm{CE} = \mu + \frac{\sigma^2}{2\rho}$$

- This formulation allows us to evaluate design alternatives based upon only their **mean** and **variance**:
  - This is based upon a second order taylor series approximation
  - Throws away info about moments higher than 1^{st} and 2^{nd}.
  - Will be exact for normal distribution

Utility Theory vs. Robust Design

- In terms of utility theory what does robust design mean?
  - For increasing preferences we have:

$$\text{Maximize: } \mathrm{CE} = \mu - \frac{\sigma^2}{2\rho}$$

  - For decreasing preferences we have:

$$\text{Minimize: } \mathrm{CE} = \mu + \frac{\sigma^2}{2\rho}$$

  - In traditional robust optimization, we have:

$$\text{Minimize: } w_1 \frac{\mu}{\mu_{utop}} + w_2 \frac{\sigma}{\sigma_{utop}} = \frac{\mu}{\mu_{utop}} + \frac{w_2}{w_1}\frac{\sigma}{\sigma_{utop}}$$

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- Let's compare the two objectives:
  - $\text{Min}\ \mu + \frac{\sigma^2}{2\rho}\ \ vs\ \text{Min}\frac{\mu}{\mu_{utop}} + \frac{w_2}{w_1}\frac{\sigma}{\sigma_{utop}}$
- Robust design derived from utility theory differs by:
  - Uses **variance**, σ^2, instead of **standard deviation**, σ
  - Uses a weighting on variance of $1/2\rho$ vs w_2/w_1
  - Does **not require normalizing** of μ and σ by the utopia values
- Robust design (RDO) can be seen to approximate utility theory:
  - RDO is based upon the assumption of constant risk aversion, or **exponential utility**.
  - In RDO, the risk aversion coefficient is **positive** (risk averse)

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  - Subject to:
    - $P\{g(\mathbf{x},\mathbf{p}) \le c\} \ge R_0$ Reliability Constraints
- Certainty Equivalent Approximation Optimization:
  - Objective:
    - minimize $\mu + \frac{\sigma^2}{2\rho}$ Expected Utility Approximation Objective
  - Subject to:
    - $P\{g(\mathbf{x},\mathbf{p}) \le c\} \ge R_0$ Reliability Constraints
- Robust Optimization:
  - Objective:
    - minimize: $w_1 \frac{\mu}{\mu_{utop}} + w_2 \frac{\sigma}{\sigma_{utop}}$ Robust Objective
  - Subject to:
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Which Approach to use?

- Traditional Robust Design or the Certainty Equivalent approximation are good for relatively simple problems.
- For large scale system problems, Expected Utility maximization is best:
  - Do not have to take gradients of a variance function.
  - Provides guidance on quantifying robust attitude (i.e. risk attitude)
- For constraints, I recommend:
  - Inverse FORM for relatively simple problems.
  - The SORA method (see paper on Canvas) for large scale system problems.
    - This method uses inverse FORM but reduces number of function calls