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Dimension reduction method for reliability-based robust design optimization

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Abstract

In reliability-based robust design optimization (RBRDO) formulation, the product quality loss function is minimized subject to probabilistic constraints. Since the quality loss function is expressed in terms of the first two statistical moments, mean and variance, three methods have been recently proposed to accurately and efficiently estimate the moments: the univariate dimension reduction method (DRM), performance moment integration (PMI) method, and percentile difference method (PDM). In this paper, a reliability-based robust design optimization method is developed using DRM and compared to PMI and PDM for accuracy and efficiency. The numerical results show that DRM is effective when the number of random variables is small, whereas PMI is more effective when the number of random variables is relatively large.

Keywords: Reliability-based robust design optimization (RBRDO); Dimension reduction method (DRM); Performance moment integration (PMI); Percentile difference method (PDM); Sensitivity analysis; Statistical moment calculation

1. Introduction

In recent years, several approaches to integrate robust design [1,2] and reliability-based design [3–5] have been proposed [6–8]. The reliability-based design optimization (RBDO) is a method to achieve the confidence in product reliability at a given probabilistic level, while the robust design optimization (RDO) is a method to improve the product quality by minimizing variability of the output performance function. Since both design methods make use of uncertainties in design variables and other parameters, it is very natural for the two different methodologies to be integrated to develop a reliability-based robust design optimization (RBRDO) method.

The product quality in robust design can be described by use of the first two statistical moments of a performance function: mean and variance [9]. Thus, it is necessary to develop methods that estimate the first two statistical moments of the performance function and their sensitivities accurately and efficiently. The statistical moments can be analytically expressed using a multi-dimensional integral. However, it is practically impossible to calculate the statistical moments of the performance function using the multi-dimensional integral. Hence, there have been various numerical attempts to estimate the moments more efficiently: experimental design [10], first order Taylor series expansion [1,2,11], Monte Carlo simulation (MCS) [12], importance sampling method [13], and Latin hyper cube sampling method [14].

Monte Carlo simulation could be accurate for the moment estimation, however it requires a very large number of function evaluations. Therefore, in many large-scale engineering applications, it is not practical to use Monte Carlo simulation. The experimental design also needs a

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large amount of computation when the number of design variables is large. The first order Taylor series expansion has been widely used to estimate the first and second statistical moments in robust design. However, the first order Taylor series expansion results in a large error especially when the input random variables have large variations. This is because the first order Taylor series expansion does not use all information of the probability density functions (PDF) of input random variables.

To overcome the shortcomings explained above, three methods have been recently proposed: the univariate dimension reduction method (DRM) [15,16], performance moment integration (PMI) [8], and percentile difference method (PDM) [6,7]. In this paper, RBRDO using the univariate DRM is proposed and the calculation of the statistical moments and their sensitivities using PMI is derived. In addition, the results of RBRDO using the univariate DRM are compared with those designs obtained using PMI and PDM. Both DRM and PMI are directly estimating the statistical moments. On the other hand, in PDM, the robustness is achieved through a design objective in which the variation of the design performance is approximately evaluated through the percentile performance difference between the right and left tails of the performance distribution [6]. Thus, three methods can be compared in terms of how accurately these methods can find an optimum design to minimize the variance of the performance function. Hence, in this paper, three methods are evaluated by comparing the variances at the optimum designs. PMI and DRM are also compared in terms of how accurately and efficiently estimate the statistical moments of the performance function. For the comparisons, several examples including one-dimensional and two-dimensional performance functions, and a large-scale engineering problem are used. These comparisons illustrate that the univariate DRM is the most accurate and efficient method when the number of design variables is small and PMI is a better option when the number of design variables is relatively large.

For the inverse reliability analysis of RBDO, the enriched performance measure approach (PMA+) [5] and its numerical method, the enhanced hybrid mean value (HMV+) [4] are utilized.

2. Fundamental concept of robust design

2.1. Reliability-based robust design

In general, a conventional (deterministic) design optimization problem can be formulated to

$$\begin{aligned} &\text{minimize} && h(\mathbf{d}) \\ &\text{subject to} && G_i(\mathbf{d}) \leqslant 0, \quad i = 1, \ldots, nc \\ &&& \mathbf{d}^L \leqslant \mathbf{d} \leqslant \mathbf{d}^U, \quad \mathbf{d} \in R^{ndv} \end{aligned} \tag{1}$$

where $h(\mathbf{d})$ is the cost function, G_i is the ith constraint, and $\mathbf{d}$ is the design variable vector; and nc and ndv are the number of constraints and design variables, respectively. The optimum design of the conventional optimization problem is the deterministic optimum that could be sensitive to the variation of input design variables and other parameters.

Due to the variation of design variables and other parameters, the performance function $h(\mathbf{d})$ also has variation. Thus, in robust design, the robustness of a design objective can be achieved by simultaneously "optimizing the mean performance μ_H and minimizing the performance variance σ_H^2" [6]. In other words, the goal of robust design is to find the most insensitive design to the variation of the design variables and other parameters. Since robust design is fundamentally considering the variations of the design variables and other parameters, it is very natural to integrate robust design and reliability-based design in one formulation. This design optimization is called reliability-based robust design optimization (RBRDO) and can be formulated to

$$\begin{aligned} &\text{minimize} && f(\mu_H, \sigma_H^2) \\ &\text{subject to} && P(G_i(\mathbf{X};\mathbf{d}) > 0) \leqslant \Phi(-\beta_{t_i}), \quad i = 1, \ldots, nc \\ &&& \mathbf{d}^L \leqslant \mathbf{d} \leqslant \mathbf{d}^U, \quad \mathbf{d} \in R^{ndv} \quad \text{and } \mathbf{X} \in R^{nrv} \end{aligned} \tag{2}$$

where $f(\mu_H, \sigma_H^2)$ is the cost function, $\mathbf{d} = \boldsymbol{\mu}(\mathbf{X})$ is the design vector, $\mathbf{X}$ is the random vector, and G_i is the ith probabilistic constraint. Quantities nc, ndv, nrv and β_{t_i} are the number of probabilistic constraints, design variables, random variables, and the ith target reliability index, respectively. Detailed explanation about RBDO can be found in [3–5]. In this paper, the enriched performance measure approach (PMA+) [5] is introduced to perform inverse reliability analysis of the constraints.

Fig. 1 compares a conventional design optimization with a RDO for a one-dimensional performance function. With the same variability of a design variable, the robust optimum shows less variation of the performance function $h(\mathbf{d})$ than the conventional design optimum.

2.2. Three types of cost function

Since the cost function in Eq. (2) depends on μ_H and σ_H^2 for robust optimum design in RBRDO, it is a bi-objective

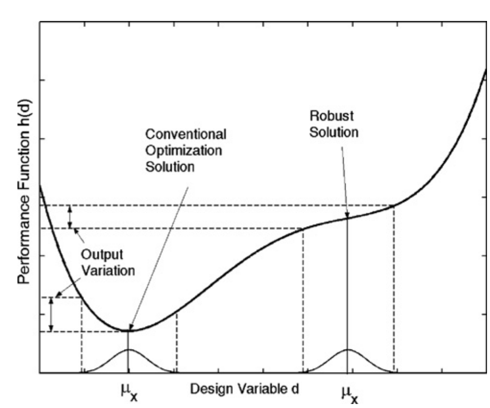


Fig. 1. Comparison of conventional and robust design optimum [1].

optimization problem. The optimum of the bi-objective optimization depends on the weight on each term in the cost function. However, since the main goal of this paper is not focused on determination of the weights, interested readers can refer to [17] for more details.

The cost function $f(\mu_H, \sigma_H^2)$ in Eq. (2) can be formulated in various ways based on engineering application types [8,9]. The following are three important cost function types for reliability based robust design.

(1) Nominal-the-best type

$$f(\mu_H, \sigma_H^2) = w_1\left(\frac{\mu_H - h_t}{\mu_{H_0} - h_{t_0}}\right)^2 + w_2\left(\frac{\sigma_H}{\sigma_{H_0}}\right)^2, \tag{3}$$

where h_t and h_{t_0} are the target nominal value and the initial target nominal value of the performance function $h(\mathbf{X})$ respectively, and w_1 and w_2 are weights to be determined by the designer. To reduce the dimensionality problem of two objectives, each term is normalized by the initial value μ_{H_0} and σ_{H_0}.

(2) Smaller-the-better type

$$f(\mu_H, \sigma_H^2) = w_1 \cdot \mathrm{sgn}(\mu_H) \cdot \left(\frac{\mu_H}{\mu_{H_0}}\right)^2 + w_2\left(\frac{\sigma_H}{\sigma_{H_0}}\right)^2. \tag{4}$$

(3) Larger-the-better type

$$f(\mu_H, \sigma_H^2) = w_1 \cdot \mathrm{sgn}(\mu_H) \cdot \left(\frac{\mu_{H_0}}{\mu_H}\right)^2 + w_2\left(\frac{\sigma_H}{\sigma_{H_0}}\right)^2. \tag{5}$$

3. Three methods for reliability based robust design

As shown in Eqs. (3)–(5), the main concern of RBRDO is how accurately and efficiently the statistical moments and their sensitivities of the performance function $h(\mathbf{X})$ can be estimated. Analytically, the kth statistical moment of the performance function can be obtained using the following integration:

$$E(\{h(\mathbf{X})\}^k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \{h(\mathbf{X})\}^k f_{\mathbf{X}}(\mathbf{x})\mathrm{d}\mathbf{x}, \tag{6}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is a joint probability density function (PDF) of the random parameter $\mathbf{X}$. As stated before, it is practically impossible to calculate the statistical moments of the performance function using Eq. (6) especially when the dimension of the problem is relatively large. For numerical evaluation of Eq. (6), three methods have been recently proposed. These methods are briefly introduced in the following sections and compared. More importantly, sensitivity analysis of statistical moments is derived and evaluated for accuracy. It should be noted that these three methods assume that input variables are statistically independent of each other.

3.1. Dimension reduction method

The dimension reduction method [15,16,18] is a newly developed technique to calculate statistical moments of the output performance function. There are several DRMs depending on the level of dimension reduction: (1) univariate dimension reduction, which is an additive decomposition of N-dimensional performance function into one-dimensional functions; (2) bivariate dimension reduction, which is an additive decomposition of N-dimensional performance function into at most two-dimensional functions; (3) multivariate dimension reduction, which is an additive decomposition of N-dimensional performance function into at most S-dimensional functions, where $S \leqslant N$. In this paper, the univariate DRM is used for computation of statistical moments and their sensitivities. Computational efficiency of DRM is discussed in Section 3.1.2.

3.1.1. Basic concept of univariate dimension reduction method

In the univariate DRM, any N-dimensional performance function $h(\mathbf{X})$ can be additively decomposed into one-dimensional functions as

$$\begin{aligned} h(\mathbf{X}) &\cong \hat{h}(\mathbf{X}) \\ &\equiv \sum_{i=1}^{N} h(\mu_1, \ldots, \mu_{i-1}, x_i, \mu_{i+1}, \ldots, \mu_N) \\ &\quad - (N-1)h(\mu_1, \ldots, \mu_N) \end{aligned} \tag{7}$$

where μ_i is the mean value of a random variable X_i and N is the number of design variables. For example, if $h(\mathbf{X}) = h(x_1, x_2)$, i.e., $N = 2$, then the univariate additive decomposition of $h(\mathbf{X})$ is

$$h(\mathbf{X}) \cong \hat{h}(\mathbf{X}) \equiv h(x_1, \mu_2) + h(\mu_1, x_2) - h(\mu_1, \mu_2). \tag{8}$$

Using the univariate DRM, one N-dimensional integration in Eq. (6) becomes N one-dimensional integrations, which will reduce the number of function evaluations significantly when the number of design variables is large. This reduction of the number of function evaluations is explained in Section 3.1.2. The one-dimensional numerical integration can be calculated using the moment-based integration rule (MBIR) [19], which is similar to Gaussian quadrature [20]. According to MBIR, the kth statistical moment of a one-dimensional function can be obtained as

$$E(\{h(\mathbf{X})\}^k) = \sum_{i=1}^{n} w_i h^k(x_i), \tag{9}$$

where w_i are weights, x_i are quadrature points (realizations of a random variable X) and n is the number of weights and quadrature points. If PDF of the design variables is given, then these weights w_i and quadrature points x_i can be obtained using MBIR. For the standard normal input random variable with three quadrature points, the weights and quadrature points are shown in Table 1 [19].

Using Eqs. (7) and (9), the mean value and variance of the performance function $h(\mathbf{X})$ can be obtained as

Table 1
Weights and quadrature points for standard normal

Quadrature points			Weights		
x_1	x_2	x_3	w_1	w_2	w_3
$-\sqrt{3}$	0	$\sqrt{3}$	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$

$$\begin{aligned}\mu_H &\equiv E[h(\mathbf{X})]\\ &\cong E\left\{\sum_{i=1}^{N} h(\mu_1,\ldots,\mu_{i-1},X_i,\mu_{i+1},\ldots,\mu_N)-(N-1)h(\mu_1,\ldots,\mu_N)\right\}\\ &\cong \sum_{j=1}^{n}\sum_{i=1}^{N} w_i^j h(\mu_1,\ldots,\mu_{i-1},x_i^j,\mu_{i+1},\ldots,\mu_N)-(N-1)h(\mu_1,\ldots,\mu_N)\end{aligned} \tag{10}$$

$$\begin{aligned}\sigma_H^2 &\equiv E[(h(\mathbf{X})-\mu_H)^2]=E[h^2(\mathbf{X})]-\mu_H^2\\ &\cong E\left\{\sum_{i=1}^{N} h^2(\mu_1,\ldots,X_i,\ldots,\mu_N)-(N-1)h^2(\mu_1,\ldots,\mu_N)\right\}-\mu_H^2\\ &\cong \sum_{j=1}^{n}\sum_{i=1}^{N} w_i^j h^2(\mu_1,\ldots,x_i^j,\ldots,\mu_N)-(N-1)h^2(\mu_1,\ldots,\mu_N)-\mu_H^2\end{aligned} \tag{11}$$

The estimation of statistical moments using the univariate DRM involves two approximations. As shown in Eqs. (10) and (11), the univariate DRM approximates the performance function $h(\mathbf{X})$ using the sum of one-dimensional functions. If $h(\mathbf{X})=\sum_{i=1}^{N}h_i(x_i)$ where $h_i(x_i)$ is any function of x_i only, then the approximation is exact. However, if there are off-diagonal or mixed terms, then there is some error that results from approximating off-diagonal terms using sum of one-dimensional functions. To reduce this error, the bivariate DRM or multivariate DRM can be used. The second approximation involves the numerical integration using the weights and quadrature points. Based on Gaussian quadrature theory [20], n quadrature points and weights give a degree of precision of $2n-1$. Hence, if three quadrature points and weights for each variable are used, the numerical integration error for a quadratic performance function will disappear. If the performance function is highly nonlinear, then three quadrature points may not be sufficient to estimate the moments of the performance function. In this case, the error can be reduced if the number of quadrature points is increased.

3.1.2. Computational efficiency

Even though the accuracy is the most important concern, it is also important to efficiently estimate statistical moments of the performance function for large-scale problems. In general, when the output moments are estimated using the univariate DRM and MBIR, the number of function evaluations required is

$$FE = n\times N+1, \tag{12}$$

where n is the number of quadrature points and N is the number of design variables. If the distributions of all input design variables are symmetric, e.g., normal distribution or uniform distribution, and the number of design variables is odd, then the required number of function evaluations is reduces to

$$FE=(n-1)\times N+1. \tag{13}$$

Therefore, when the number of design variables is large, the reduction becomes significant compared to the number of function evaluation in directly integrating Eq. (6), which is n^N. However, although the reduction becomes significant when N is large, the number of function evaluations is still increasing proportionally to the number of design variables as shown in Eq. (13).

If bivariate DRM is used to estimate the first and second output moments, then the number of function evaluations will increase exponentially to

$$FE=\frac{N(N-1)}{2}\times n^2+n\times N+1. \tag{14}$$

For example, if the number of design variables is 5 and the number of quadrature points is 3, then the number of function evaluations by the univariate DRM is 16 from Eq. (12) and the number of function evaluations by bivariate DRM is 106 from Eq. (14). Both of the numbers are less than $3^5=243$, which is the required number of function evaluations for the numerical integration of Eq. (6) by including the mixed variable terms. However, the number of function evaluations by the univariate DRM is significantly less than the number of function evaluations for bivariate DRM. For this reason, the univariate DRM is used to estimate statistical moments in this paper.

3.1.3. Sensitivity of statistical moments

To obtain a robust design, not only the values of the first and second statistical moments but also the sensitivities of these moments are needed. Using Eq. (6) and Rosenblatt transformation [21] from the design space (x-space) to the standard Gaussian space (u-space), which can be described as $F_X(x)=\Phi(u)$, sensitivities of the mean and variance of the performance function with respect to the design variable μ_i can be derived as

$$\begin{aligned}\frac{\partial\mu_H(\boldsymbol{\mu})}{\partial\mu_i} &= \frac{\partial}{\partial\mu_i}\left[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h(\mathbf{x})f_{\mathbf{X}}(\mathbf{x})\,d\mathbf{x}\right]\\ &= \frac{\partial}{\partial\mu_i}\left[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h(\mathbf{x}(\mathbf{u};\mu))\phi_U(\mathbf{u})\,d\mathbf{u}\right]\\ &\quad \text{(Rosenblatt transformation)}\\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{\partial h(\mathbf{x}(\mathbf{u};\boldsymbol{\mu}))}{\partial\mu_i}\phi_U(\mathbf{u})\,d\mathbf{u}\end{aligned} \tag{15}$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\partial h(\mathbf{x}(\mathbf{u};\mu))}{\partial x_i} \frac{\partial x_i(u_i;\mu_i)}{\partial \mu_i} \phi_U(\mathbf{u})\,\mathrm{d}\mathbf{u}$$

(Independency)

$$\frac{\partial \sigma_H^2(\mu)}{\partial \mu_i} = \frac{\partial}{\partial \mu_i}\left[\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h^2(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\,\mathrm{d}\mathbf{x}\right] - \frac{\partial \mu_H^2}{\partial \mu_i}$$
$$= \frac{\partial}{\partial \mu_i}\left[\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h^2(\mathbf{x}(\mathbf{u};\mu)) \phi_U(\mathbf{u})\,\mathrm{d}\mathbf{u}\right] - \frac{\partial \mu_H^2}{\partial \mu_i}$$

(Rosenblatt transformation)

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\partial h^2(\mathbf{x}(\mathbf{u};\mu))}{\partial \mu_i} \phi_U(\mathbf{u})\,\mathrm{d}\mathbf{u} - \frac{\partial \mu_H^2}{\partial \mu_i} \tag{16}$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\partial h^2(\mathbf{x}(\mathbf{u};\mu))}{\partial x_i} \frac{\partial x_i(u_i;\mu_i)}{\partial \mu_i} \phi_U(\mathbf{u})\,\mathrm{d}\mathbf{u} - \frac{\partial \mu_H^2}{\partial \mu_i}$$

(Independency)

where $\mathbf{u}$ is the standard normal variable. The input variables are assumed to be independent for the derivations of Eqs. (15) and (16).

To calculate $\frac{\partial x_i(u_i;\mu_i)}{\partial \mu_i}$ in Eqs. (15) and (16), Rosenblatt transformation shown in Table 2 is used. For example, if the input variable is normally distributed, then Table 2 shows that x_i can be expressed as $x_i = \mu_i + \sigma_i u_i$. Since σ_i is fixed and u_i is independent of an input mean μ_i, $\frac{\partial x_i(u_i;\mu_i)}{\partial \mu_i} = 1$ is obtained. For Gumbel and uniform distribution, the same result $\frac{\partial x_i(u_i;\mu_i)}{\partial \mu_i} = 1$ is obtained from Rosenblatt transformation. For the Lognormal and Weibull distribution, $\frac{\partial x_i(u_i;\mu_i)}{\partial \mu_i}$ can be approximated to be 1.

By using the inverse transformation from u-space to x-space, the assumption $\frac{\partial x_i(u_i;\mu_i)}{\partial \mu_i} \cong 1$, and Eqs. (10) and (11), (15) and (16) can be further approximated by

$$\frac{\partial \mu_H(\boldsymbol{\mu})}{\partial \mu_k} \cong \sum_{j=1}^{n}\sum_{i=1}^{N} w_i^j \cdot \left.\frac{\partial h(\mathbf{x})}{\partial x_k}\right|_{\mathbf{x}=(\mu_1,\ldots,x_i^j,\ldots,\mu_N)} - (N-1)\left.\frac{\partial h(\mathbf{x})}{\partial x_k}\right|_{\mathbf{x}=\boldsymbol{\mu}}, \tag{17}$$

$$\frac{\partial \sigma_H^2(\boldsymbol{\mu})}{\partial \mu_k} \cong \sum_{j=1}^{n}\sum_{i=1}^{N} w_i^j \cdot \left.\frac{\partial h^2(\mathbf{x})}{\partial x_k}\right|_{\mathbf{x}=(\mu_1,\ldots,x_i^j,\ldots,\mu_N)} - (N-1)\left.\frac{\partial h^2(\mathbf{x})}{\partial x_k}\right|_{\mathbf{x}=\boldsymbol{\mu}} - \frac{\partial \mu_H^2}{\partial \mu_k}. \tag{18}$$

Since the univariate DRM does not use sensitivities of the performance function evaluated at the quadrature points to estimate the moments, additional function evaluations are needed for the sensitivity analysis using Eqs. (17) and (18).

3.2. Performance moment integration (PMI)

3.2.1. Derivation of performance moment integration

The multi-dimensional integral in Eq. (6) for statistical moments can be rewritten using Rosenblatt transformation as

$$E(h^k(\mathbf{X})) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h^k(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu})\,\mathrm{d}\mathbf{x}$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h^k(\mathbf{x}(\mathbf{u};\boldsymbol{\mu})) \phi_U(\mathbf{u})\,\mathrm{d}\mathbf{u} \tag{19}$$

which can also be written in terms of the output distribution as

$$E(h^k(\mathbf{X})) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h^k(\mathbf{x}(\mathbf{u};\boldsymbol{\mu})) \phi_U(\mathbf{u})\,\mathrm{d}\mathbf{u}$$
$$= \int_{-\infty}^{\infty} h^k f_H(h;\boldsymbol{\mu})\mathrm{d}h, \tag{20}$$

where $f_H(h)$ is PDF of a performance function $h(\mathbf{X})$. Since the cumulative distribution function (CDF) of the performance function can be expressed in terms of the standard normal CDF using the following transformation $F_H(h) = \Phi(t)$, Eq. (20) becomes

$$E(h^k(\mathbf{X})) = \int_{-\infty}^{\infty} h^k f_H(h;\boldsymbol{\mu})\mathrm{d}h = \int_{-\infty}^{\infty} h^k(t;\boldsymbol{\mu})\phi(t)\mathrm{d}t \tag{21}$$

where the parametric variable t is the distance from the origin in u-space to the most probable point (MPP) as shown in Fig. 2.

Hence, the multi-dimensional integral can be rewritten by a one-dimensional integral. Similar to the univariate DRM, the performance moment integration (PMI) makes use of three quadrature points and weights to approximate the one-dimensional integration in Eq. (21). A difference between the two methods is that quadrature points of the univariate DRM lie on the x_i-axis, whereas quadrature points of PMI lie on the MPP locus [3,22]. Therefore, the number of quadrature points in the univariate DRM

Table 2
Probability distribution and its transformation between x and u-space

	Parameters	PDF	Transformation
Normal	μ = mean; σ = standard deviation	$f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-0.5\left[\frac{x-\mu}{\sigma}\right]^2}$	$X = \mu + \sigma U$
Log-normal	$\bar{\sigma}^2 = \ln\left[1+\left(\frac{\sigma}{\mu}\right)^2\right],\ \bar{\mu} = \ln(\mu) - 0.5\bar{\sigma}^2$	$f(x) = \frac{1}{\sqrt{2\pi}x\bar{\sigma}}e^{-0.5\left[\frac{\ln x-\bar{\mu}}{\bar{\sigma}}\right]^2}$	$X = \exp(\bar{\mu} + \bar{\sigma}U)$
Weibull	$\mu = v\Gamma\left(1+\frac{1}{k}\right), \sigma^2 = v^2\left[\Gamma\left(1+\frac{2}{k}\right) - \Gamma^2\left(1+\frac{1}{k}\right)\right]$	$f(x) = \frac{k}{v}\left(\frac{x}{v}\right)^{k-1}e^{-\left(\frac{x}{v}\right)^k}$	$X = v[-\ln(\Phi(-U)^{a})]^{\frac{1}{k}}$
Gumbel	$\mu = v + \frac{0.577}{\alpha}, \sigma = \frac{\pi}{\sqrt{6}\alpha}$	$f(x) = \alpha e^{-\alpha(x-v)-e^{-\alpha(x-v)}}$	$X = v - \frac{1}{\alpha}\ln[-\ln(\Phi(U))]$
Uniform	$\mu = \frac{a+b}{2}, \sigma = \frac{b-a}{\sqrt{12}}$	$f(x) = \frac{1}{b-a}, a \leqslant x \leqslant b$	$X = a + (b-a)\Phi(U)$

[a] $\Phi(U) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{U} e^{-\frac{u^2}{2}}\mathrm{d}u$.

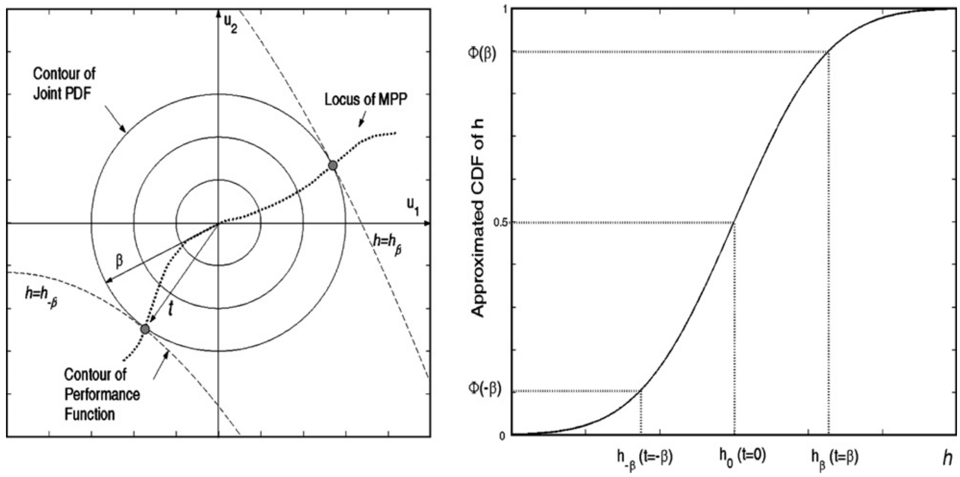


Fig. 2. Approximation of CDF Using MPP Locus [3].

increases as the number of design variables increases as shown in Eq. (12), whereas the number of quadrature points in PMI does not change since the integration is performed in the output space.

Since t follows the standard normal distribution, the weights and quadrature points in Table 1 can be used to discretize Eq. (21) as

$$\begin{aligned}E(h^k(\mathbf{X})) &= \int_{-\infty}^{\infty} h^k(t;\boldsymbol{\mu})\phi(t)\mathrm{d}t \\ &\cong \frac{1}{6}\cdot h^k(t;\boldsymbol{\mu})\big|_{t=-\sqrt{3}} + \frac{4}{6}\cdot h^k(t;\boldsymbol{\mu})\big|_{t=0} \\ &\quad + \frac{1}{6}\cdot h^k(t;\boldsymbol{\mu})\big|_{t=\sqrt{3}}.\end{aligned} \tag{22}$$

By changing the order of calculation, Eq. (22) becomes

$$\begin{aligned}E(h^k(\mathbf{X})) &= \int_{-\infty}^{\infty} h^k(t;\boldsymbol{\mu})\phi(t)\mathrm{d}t \\ &\cong \frac{1}{6}\cdot \{h(t;\boldsymbol{\mu})\}^k\big|_{t=-\sqrt{3}} + \frac{4}{6}\cdot \{h(t;\boldsymbol{\mu})\}^k\big|_{t=0} \\ &\quad + \frac{1}{6}\cdot \{h(t;\boldsymbol{\mu})\}^k\big|_{t=\sqrt{3}} \\ &= \frac{1}{6}\cdot h^k(-\sqrt{3};\boldsymbol{\mu}) + \frac{4}{6}\cdot h^k(0;\boldsymbol{\mu}) + \frac{1}{6}\cdot h^k(\sqrt{3};\boldsymbol{\mu}).\end{aligned} \tag{23}$$

Using the first order reliability method (FORM) [23,24] and MPP locus illustrated in Fig. 2, each term in Eq. (23) can be approximated as two function values at two MPPs and a function value at the design point. The function values at MPPs can be obtained using the inverse reliability analysis PMA to

$$\begin{aligned}&\text{maximize} \quad h(\mathbf{U}),\\ &\text{subject to} \quad \|\mathbf{U}\| = \sqrt{3}\end{aligned} \tag{24}$$

The optimum result of Eq. (24) is denoted as $h^{\max}_{\beta_t=\sqrt{3}}$, which can be used to approximate $h(\sqrt{3};\boldsymbol{\mu})$ in Eq. (23). The term $h(-\sqrt{3};\boldsymbol{\mu})$ in Eq. (23) can be approximated by the optimum result obtained by minimizing $h(\mathbf{U})$ in Eq. (24) and denoted as $h^{\min}_{\beta_t=\sqrt{3}}$. The term $h(0;\boldsymbol{\mu})$ in Eq. (23) can be approximated by $h(\boldsymbol{\mu}_\mathbf{X})$, which is the performance function value at the design point. Hence, using these function values and Eq. (23), the statistical moments of a performance function can be calculated as

$$E(h^k(\mathbf{X})) \cong \frac{1}{6}\cdot\left(h^{\min}_{\beta_t=\sqrt{3}}\right)^k + \frac{4}{6}\cdot h^k(\boldsymbol{\mu}_\mathbf{X}) + \frac{1}{6}\cdot\left(h^{\max}_{\beta_t=\sqrt{3}}\right)^k. \tag{25}$$

Consequently, the mean value and variance can be estimated by

$$\begin{aligned}\mu_H &\cong \frac{1}{6}h^{\min}_{\beta_t=\sqrt{3}} + \frac{4}{6}h(\boldsymbol{\mu}_\mathbf{X}) + \frac{1}{6}h^{\max}_{\beta_t=\sqrt{3}},\\ \sigma_H^2 &\cong \frac{1}{6}(h^{\min}_{\beta_t=\sqrt{3}})^2 + \frac{4}{6}h^2(\boldsymbol{\mu}_\mathbf{X}) + \frac{1}{6}(h^{\max}_{\beta_t=\sqrt{3}})^2 - \mu_H^2.\end{aligned} \tag{26}$$

Thus, PMI is very efficient when the number of design variables is relatively large.

3.2.2. Sensitivity of statistical moments

Similar to the sensitivity calculation in DRM, from Eqs. (21), (23) and (25), the sensitivities of the mean and variance of the performance function with respect to a the design variable μ_i can be derived as

$$\begin{aligned}\frac{\partial\mu_H}{\partial\mu_i} &= \frac{\partial}{\partial\mu_i}\int_{-\infty}^{\infty} h(t;\boldsymbol{\mu})\phi(t)\,\mathrm{d}t = \int_{-\infty}^{\infty}\frac{\partial h(t;\boldsymbol{\mu})}{\partial\mu_i}\phi(t)\,\mathrm{d}t \\ &\cong \frac{1}{6}\frac{\partial h^{\min}_{\beta_t=\sqrt{3}}}{\partial\mu_i} + \frac{4}{6}\frac{\partial h(\boldsymbol{\mu}_\mathbf{X})}{\partial\mu_i} + \frac{1}{6}\frac{\partial h^{\max}_{\beta_t=\sqrt{3}}}{\partial\mu_i} \\ &\cong \frac{1}{6}\frac{\partial h^{\min}_{\beta_t=\sqrt{3}}}{\partial x_i^{\mathrm{MPP}}}\cdot\frac{\partial x_i^{\mathrm{MPP}}}{\partial\mu_i} + \frac{4}{6}\frac{\partial h(\boldsymbol{\mu}_\mathbf{X})}{\partial\mu_i} + \frac{1}{6}\frac{\partial h^{\max}_{\beta_t=\sqrt{3}}}{\partial x_i^{\mathrm{MPP}}}\cdot\frac{\partial x_i^{\mathrm{MPP}}}{\partial\mu_i} \\ &= \frac{1}{6}\frac{\partial h(\mathbf{x})}{\partial x_i}\bigg|_{\mathbf{x}=\mathbf{x}^{\min}_{\mathbf{MPP}}}\cdot\frac{\partial x_i^{\mathrm{MPP}}}{\partial\mu_i} + \frac{4}{6}\frac{\partial h(\mathbf{x})}{\partial x_i}\bigg|_{\mathbf{x}=\boldsymbol{\mu}_\mathbf{X}} \\ &\quad + \frac{1}{6}\frac{\partial h(\mathbf{x})}{\partial x_i}\bigg|_{\mathbf{x}=\mathbf{x}^{\max}_{\mathbf{MPP}}}\cdot\frac{\partial x_i^{\mathrm{MPP}}}{\partial\mu_i}\end{aligned} \tag{27}$$

$$
\begin{aligned}
\frac{\partial\sigma_H^2}{\partial\mu_i} &= \frac{\partial}{\partial\mu_i}\int_{-\infty}^{\infty} h^2(t;\boldsymbol{\mu})\phi(t)\,\mathrm{d}t - \frac{\partial\mu_H^2}{\partial\mu_i} = \int_{-\infty}^{\infty}\frac{\partial h^2(t;\boldsymbol{\mu})}{\partial\mu_i}\phi(t)\,\mathrm{d}t - \frac{\partial\mu_H^2}{\partial\mu_i}\\
&\cong \frac{1}{6}\frac{\partial(h_{\beta_t=\sqrt{3}}^{\min})^2}{\partial\mu_i} + \frac{4}{6}\frac{\partial h^2(\boldsymbol{\mu}_{\mathbf{X}})}{\partial\mu_i} + \frac{1}{6}\frac{\partial\left(h_{\beta_t=\sqrt{3}}^{\max}\right)^2}{\partial\mu_i} - \frac{\partial\mu_H^2}{\partial\mu_i}\\
&\cong \frac{1}{6}\frac{\partial\left(h_{\beta_t=\sqrt{3}}^{\min}\right)^2}{\partial x_i^{\mathrm{MPP}}}\cdot\frac{\partial x_i^{\mathrm{MPP}}}{\partial\mu_i} + \frac{4}{6}\frac{\partial h^2(\boldsymbol{\mu}_{\mathbf{X}})}{\partial\mu_i}\\
&\quad + \frac{1}{6}\frac{\partial\left(h_{\beta_t=\sqrt{3}}^{\max}\right)^2}{\partial x_i^{\mathrm{MPP}}}\cdot\frac{\partial x_i^{\mathrm{MPP}}}{\partial\mu_i} - \frac{\partial\mu_H^2}{\partial\mu_i}\\
&= \frac{1}{6}\frac{\partial h^2(\mathbf{x})}{\partial x_i}\bigg|_{\mathbf{x}=\mathbf{x}_{\mathrm{MPP}}^{\min}}\cdot\frac{\partial x_i^{\mathrm{MPP}}}{\partial\mu_i} + \frac{4}{6}\frac{\partial h^2(\mathbf{x})}{\partial x_i}\bigg|_{\mathbf{x}=\boldsymbol{\mu}_{\mathbf{X}}}\\
&\quad + \frac{1}{6}\frac{\partial h^2(\mathbf{x})}{\partial x_i}\bigg|_{\mathbf{x}=\mathbf{x}_{\mathrm{MPP}}^{\max}}\cdot\frac{\partial x_i^{\mathrm{MPP}}}{\partial\mu_i} - \frac{\partial\mu_H^2}{\partial\mu_i}
\end{aligned} \tag{28}
$$

Since no explicit equation is available for x_i^{MPP}, it is not possible to analytically obtain $\frac{\partial x_i^{\mathrm{MPP}}}{\partial\mu_i}$. However, the term $\frac{\partial x_i^{\mathrm{MPP}}}{\partial\mu_i}$ can be approximated as following. Using Rosenblatt transformation in Table 2, it is clear that x_i^{MPP} is a function of u_i^{MPP} and μ_i written as $x_i^{\mathrm{MPP}} = T^{-1}(u_i^{\mathrm{MPP}};\mu_i)$ where $T: x \to u$ is the transformation. Using $\frac{\partial u_i^{\mathrm{MPP}}}{\partial\mu_i} \cong 0$, $\frac{\partial x_i^{\mathrm{MPP}}}{\partial\mu_i}$ can be approximated by $\frac{\partial x_i^{\mathrm{MPP}}}{\partial\mu_i} \cong 1$. The verification of the approximation $\frac{\partial x_i^{\mathrm{MPP}}}{\partial\mu_i} \cong 1$ for various distributions using the following example $h(\mathbf{X}) = 1 - \frac{80}{X_1^2+8X_2+5}$ with the target reliability $\beta_t = 3$ and finite difference method with 1% perturbation is given in Table 3.

By using the approximation $\frac{\partial x_i^{\mathrm{MPP}}}{\partial\mu_i} \cong 1$, which is a similar to $\frac{\partial x_i(u_i;\mu_i)}{\partial\mu_i} \cong 1$ in DRM, sensitivities of the mean and variance of the performance function with respect to μ_i can be obtained as

$$
\begin{aligned}
\frac{\partial\mu_H}{\partial\mu_i} &\cong \frac{1}{6}\frac{\partial h(\mathbf{x})}{\partial x_i}\bigg|_{\mathbf{x}=\mathbf{x}_{\mathrm{MPP}}^{\min}} + \frac{4}{6}\frac{\partial h(\mathbf{x})}{\partial x_i}\bigg|_{\mathbf{x}=\boldsymbol{\mu}_{\mathbf{X}}} + \frac{1}{6}\frac{\partial h(\mathbf{x})}{\partial x_i}\bigg|_{\mathbf{x}=\mathbf{x}_{\mathrm{MPP}}^{\max}},\\
\frac{\partial\sigma_H^2}{\partial\mu_i} &\cong \frac{1}{6}\frac{\partial h^2(\mathbf{x})}{\partial x_i}\bigg|_{\mathbf{x}=\mathbf{x}_{\mathrm{MPP}}^{\min}} + \frac{4}{6}\frac{\partial h^2(\mathbf{x})}{\partial x_i}\bigg|_{\mathbf{x}=\boldsymbol{\mu}_{\mathbf{X}}} + \frac{1}{6}\frac{\partial h^2(\mathbf{x})}{\partial x_i}\bigg|_{\mathbf{x}=\mathbf{x}_{\mathrm{MPP}}^{\max}} - \frac{\partial\mu_H^2}{\partial\mu_i}
\end{aligned} \tag{29}
$$

Table 3
Verification of assumption $\frac{\partial x_1^{\mathrm{MPP}}}{\partial\mu_1} \cong 1$ using $h(\mathbf{X}) = 1 - \frac{80}{X_1^2+8X_2+5}$

Distribution	$\mathbf{x}^{\mathrm{MPP}}$	$\hat{\mathbf{x}}^{\mathrm{MPP}a}$	$\frac{\hat{x}_1^{\mathrm{MPP}} - x_1^{\mathrm{MPP}}}{\hat{\mu}_1 - \mu_1}$
N(5, 0.3)	(5.7368, 5.5168)	(5.7890, 5.5136)	1.044
LN(5, 0.3)	(5.7923, 5.5207)	(5.8443, 5.5172)	1.040
Weibull(5, 0.3)	(5.5289, 5.4501)	(5.5799, 5.4482)	1.020
Gumbel(5, 0.3)	(6.2823, 5.2749)	(6.3362, 5.2689)	1.078
U(5, 0.3)	(5.5044, 5.4972)	(5.5546, 5.4989)	1.004

[a] $\hat{\mathbf{x}}^{\mathrm{MPP}}$ means $\mathbf{x}^{\mathrm{MPP}}$ obtained from 1% perturbation of μ_1.

Since the sensitivities of the performance function on the right hand side of Eq. (29) are used during the inverse reliability analysis described in Eq. (24), no additional function evaluations are required to calculate sensitivities using Eq. (29).

3.3. Percentile difference method (PDM)

Like PMI, PDM uses the results of the inverse reliability analysis [6,7]. PMI utilizes the function values at two MPPs ($h_{\beta_t=\sqrt{3}}^{\max}$ and $h_{\beta_t=\sqrt{3}}^{\min}$) obtained from the inverse reliability analysis and the function value at the mean $\mu_{\mathbf{X}}$ to approximate the multidimensional integration in Eq. (6), whereas PDM "uses the difference between the function values at two MPPs to represent the variation of the performance function" [6]. Hence, the RBRDO formulation using PDM is to

$$
\begin{aligned}
&\text{minimize} && f(h(\mu_{\mathbf{X}}), h_{p_1} - h_{p_2})\\
&\text{subject to} && P(G_i(\mathbf{X};\mathbf{d}) > 0) \leqslant \Phi(-\beta_{t_i}),\ i = 1,\ldots,nc\\
& && \mathbf{d}^L \leqslant \mathbf{d} \leqslant \mathbf{d}^U,\quad \mathbf{d} \in R^{ndv}\quad \text{and}\ \mathbf{X} \in R^{nrv}
\end{aligned} \tag{30}
$$

where p_1 is a right-tail percentile, p_2 is a left-tail percentile and, in general, $p_1 + p_2 = 1$. When $p_1 = 0.95$ and $p_2 = 0.05$ [6,7], h_{p_1} and h_{p_2} in Eq. (30) are calculated from the inverse reliability analysis with a target reliability index 1.645 ($\beta_t = \Phi^{-1}(p_1) = 1.645$), that is, $h_{p_1} = h_{\beta_t=1.645}^{\max}$ and $h_{p_2} = h_{\beta_t=1.645}^{\min}$.

As shown in Fig. 3, the idea of PDM is simple and could be viewed as meaningful, but it has rather serious shortcomings. If the performance function is not monotonic, it may not be possible to use $h_{p_1} - h_{p_2}$ as a measurement of robustness. In a non-monotonic performance function case, two MPPs obtained from the inverse reliability analysis may not approximate the left-tail and right-tail percentile accurately because the inverse reliability analysis searches MPPs on the surface of the hyper-sphere in

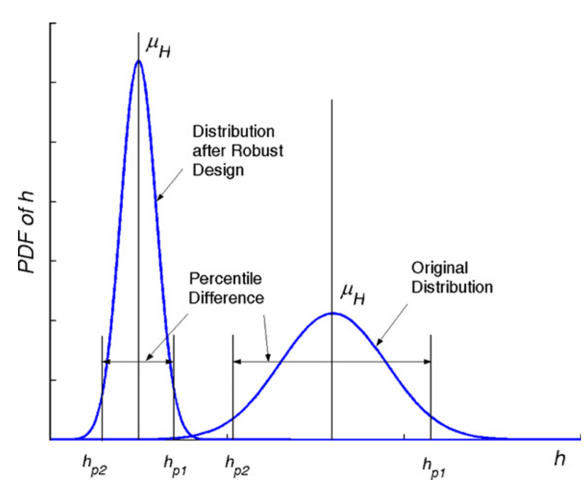


Fig. 3. Basic concept of robust design using percentile difference method [6].

u-space. For example, if $h(X) = X^2$ and $X \sim N(0,1)$ and the target reliability β_t is 1.645, then two MPPs become 1.645 and −1.645. Thus, two percentile performances h_{p_1} and h_{p_2} are identical. In contrast to PDM, PMI and the univariate DRM show the correct moment estimation of the performance function $h(X) = X^2$. Thus, PDM-based RBRDO may identify a wrong global minimum when there are several local minima, as demonstrated in Section 4.2. More significantly, there is no one percentile that can be used in PDM to identify all local optima correctly as shown in Section 4.2.

Sensitivity of the cost function with respect to a design variable μ_i can be calculated using a similar procedure as PMI

$$\frac{\partial h(\boldsymbol{\mu}_{\mathbf{X}})}{\partial \mu_i} = \left.\frac{\partial h(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\boldsymbol{\mu}_{\mathbf{X}}}$$

$$\begin{aligned}\frac{\partial h_{p_1}}{\partial \mu_i} - \frac{\partial h_{p_2}}{\partial \mu_i} &\cong \frac{\partial h_{p_1}}{\partial x_i^{\mathrm{MPP}}} \cdot \frac{\partial x_i^{\mathrm{MPP}}}{\partial \mu_i} - \frac{\partial h_{p_2}}{\partial x_i^{\mathrm{MPP}}} \cdot \frac{\partial x_i^{\mathrm{MPP}}}{\partial \mu_i} \\ &\cong \left.\frac{\partial h(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\mathbf{x}_{\mathbf{p}_1}^{\mathrm{MPP}}} - \left.\frac{\partial h(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\mathbf{x}_{\mathbf{p}_2}^{\mathrm{MPP}}}.\end{aligned} \tag{31}$$

3.4. Comparison

Two criteria to identify which method is effective for robust design optimization (RDO) are computational efficiency and accuracy of the moment estimation. In terms of computational efficiency, both PMI and PDM will require the same number of function evaluations if the same inverse reliability analysis method is used. In general, if the number of design variables is large, Eq. (12) shows that DRM requires more function evaluations than PMI and PDM. However, an advantage of using the univariate DRM is that the univariate DRM does not require sensitivity information (i.e., no MPP search) in estimating the moments. Hence, the univariate DRM can reduce the number of function evaluations during line searches.

The objective of the univariate DRM and PMI is to approximate the multi-dimensional integration in Eq. (6). That is, both methods attempt to transform the multi-dimensional integration into a readily computable numerical integration. However, PDM does not use any numerical integration, instead it uses the difference of percentile performances. Thus, PDM may yield wrong results when the performance function is non-monotonic. Both PMI and PDM may have a difficulty to find MPPs when the performance function is non-monotonic and highly nonlinear. On the other hand, DRM may accurately estimate the moments of the performance function regardless of the performance function type.

In terms of accuracy of the moment estimation, the univariate DRM yields better results in most cases than PMI. If the performance function is highly nonlinear, then the univariate DRM with three quadrature points may not accurately estimate the second moment. In this case, the error can be reduced if more quadrature points are used in the univariate DRM. However, PMI with more quadrature points than 3 may not necessarily yield more accurate results. This is because function values at quadrature points, which are obtained using FORM and MPP search, are approximations. More details of comparison with numerical examples are given in the following section.

4. Numerical examples

In this section, four cases of comparisons are carried out using numerical examples. In Section 4.1, the univariate DRM and PMI are compared in terms of accuracy and efficiency in estimation of the moments and their sensitivities of a performance function. PDM is excluded in Section 4.1 since it cannot estimate the moments of the performance function. In Section 4.2, DRM, PMI, and PDM are compared using a one-dimensional fourth order polynomial for identification of correct robust optimum design. In this one-dimensional problem, PMI and the univariate DRM with three quadrature points can be considered to be the same method. In Section 4.3, comparison of three methods is carried out using a two-dimensional fourth order polynomial for design optimization. In Section 4.4, a side impact crashworthiness example is used for the comparison of DRM and PMI in terms of the number of the function evaluations in a large-scale engineering problem.

4.1. Comparison of PMI and DRM for computation of moments and sensitivities

For the first example, the performance function is

$$h_1(\mathbf{X}) = 1 - \frac{X_1^2 X_2}{20}, \tag{32}$$

where $X_i \sim N(5,1)$ for $i = 1, 2$. As shown in Table 4, both DRM and PMI provide good estimation of the mean value

Table 4
Comparison of the first and second moments of Eq. (32)

	Mean (μ_H)			Variance (σ_H^2)		
	PMI	DRM	NI[a]	PMI	DRM	NI
h_1	−5.6500	−5.5000	−5.5000	8.4623	7.9355	8.3175
Error (%)	2.73	0		1.74	4.58	
No. of F.E.	7 + 7[b]	2 × 2 + 1		7 + 7	2 × 2 + 1	

[a] NI means numerical integration.
[b] 7 + 7 means 7 function evaluations and 7 sensitivity calculations.

Table 5
Sensitivity of mean value using PMI and DRM for Eq. (32)

	PMI		DRM		Analytic	
	$\frac{\partial \mu_H(\mathbf{d})}{\partial d_1}$	$\frac{\partial \mu_H(\mathbf{d})}{\partial d_2}$	$\frac{\partial \mu_H(\mathbf{d})}{\partial d_1}$	$\frac{\partial \mu_H(\mathbf{d})}{\partial d_2}$	$\frac{\partial \mu_H(\mathbf{d})}{\partial d_1}$	$\frac{\partial \mu_H(\mathbf{d})}{\partial d_2}$
Sensitivity	−2.5475	−1.2823	−2.5000	−1.3000	−2.5000	−1.3000
Error (%)	1.90	1.36	0.00	0.00		
Additional no. of F.E.	0		$2 \times 2 + 1$			

and standard deviation in comparison with the exact numerical integration results. The reason DRM has a larger error in estimation of standard deviation is because the performance function in Eq. (32) has an off-diagonal term only. As mentioned in Section 3.1.1, if the performance function has off diagonal terms only, then the univariate additive decompositions of the moments in Eqs. (10) and (11) may contain significant errors.

For this example, PMI yields reasonable estimation of the moments because the design variables are normally distributed, which means that the inverse reliability analysis does not require non-linear transformations from x-space to u-space, and the performance function is monotonic at the given design. In the same token, the sensitivities in Tables 5 and 6 have similar errors as Table 4.

The total number of function evaluations for PMI to evaluate the mean and standard deviation is $7 + 7$ as shown in Table 4, where the first 7 is the number of function evaluation for MPP search and the second 7 is the number of sensitivity calculation for MPP search. The number of function evaluations for DRM is 5. Since the design variables are normally distributed and the number of quadrature points is odd, Eq. (13) is used for the total number of function evaluations.

PMI does not require additional function evaluations for the sensitivity analysis of moments because PMI uses the sensitivity information in MPP search. However, DRM does require additional function evaluations for sensitivity analysis, thus the total number of function evaluations needs to be doubled in DRM as shown in Table 5.

Since the first example contains an off-diagonal term only and the design variables are normally distributed, the second example is modeled as

$$h_2(\mathbf{X}) = 1 - \frac{(X_1 + X_2 - 5)^2}{30} - \frac{(X_1 - X_2 - 12)^2}{120}, \qquad (33)$$

where $X_i \sim \text{Gumbel}(5, 1)$ for $i = 1, 2$. The performance function in Eq. (33) contains both off-diagonal terms and diagonal terms, and the degree of the polynomial performance function is 2. Therefore, it can be expected that

Table 6
Sensitivity of variance using PMI and DRM for Eq. (32)

	PMI		DRM		Analytic	
	$\frac{\partial \sigma_H^2(\mathbf{d})}{\partial d_1}$	$\frac{\partial \sigma_H^2(\mathbf{d})}{\partial d_2}$	$\frac{\partial \sigma_H^2(\mathbf{d})}{\partial d_1}$	$\frac{\partial \sigma_H^2(\mathbf{d})}{\partial d_2}$	$\frac{\partial \sigma_H^2(\mathbf{d})}{\partial d_1}$	$\frac{\partial \sigma_H^2(\mathbf{d})}{\partial d_2}$
Sensitivity	3.9211	2.5894	3.7500	2.5500	3.9000	2.5500
Error (%)	0.54	1.54	3.85	0.00		

Table 9
Sensitivity of variance using PMI and DRM for Eq. (33)

	PMI		DRM		Analytic	
	$\frac{\partial \sigma_H^2(\mathbf{d})}{\partial d_1}$	$\frac{\partial \sigma_H^2(\mathbf{d})}{\partial d_2}$	$\frac{\partial \sigma_H^2(\mathbf{d})}{\partial d_1}$	$\frac{\partial \sigma_H^2(\mathbf{d})}{\partial d_2}$	$\frac{\partial \sigma_H^2(\mathbf{d})}{\partial d_1}$	$\frac{\partial \sigma_H^2(\mathbf{d})}{\partial d_2}$
Sensitivity	0.0724	0.1033	0.0883	0.1149	0.0884	0.1162
Error (%)	18.10	11.10	0.11	1.12		

Table 7
Comparison of the first and second moments of Eq. (33)

	Mean (μ_H)			Variance (σ_H^2)		
	PMI	DRM	NI	PMI	DRM	NI
h_2	−1.0594	−1.1167	−1.1167	0.3357	0.3774	0.3833
Error (%)	5.13	0		12.42	1.54	
No. of F.E.	$5 + 5$	$3 \times 2 + 1$		$5 + 5$	$3 \times 2 + 1$	

Table 8
Sensitivity of mean value using PMI and DRM for Eq. (33)

	PMI		DRM		Analytic	
	$\frac{\partial \mu_H(\mathbf{d})}{\partial d_1}$	$\frac{\partial \mu_H(\mathbf{d})}{\partial d_2}$	$\frac{\partial \mu_H(\mathbf{d})}{\partial d_1}$	$\frac{\partial \mu_H(\mathbf{d})}{\partial d_2}$	$\frac{\partial \mu_H(\mathbf{d})}{\partial d_1}$	$\frac{\partial \mu_H(\mathbf{d})}{\partial d_2}$
Sensitivity	−0.1209	−0.5254	−0.1333	−0.5333	−0.1333	−0.5333
Error (%)	9.30	1.49	0.00	0.00		
Additional no. of F.E.	0		$3 \times 2 + 1$			

DRM may yield better results for this example. As expected, Tables 7–9 illustrate that DRM is accurate in estimation of the moments and their sensitivities.

On the other hand, PMI yields somewhat larger errors in estimation of the moments and their sensitivities. This is because the design variables follow Gumbel distribution. In such a case, the inverse reliability analysis requires non-linear transformations from x-space to u-space, which makes the performance function become highly non-linear and the FORM error become larger.

Since the Gumbel distribution is not symmetric, Eq. (12) is used for the total number of function evaluations for DRM.

4.2. Comparison of PMI, DRM and PDM for identification of robust optimum design

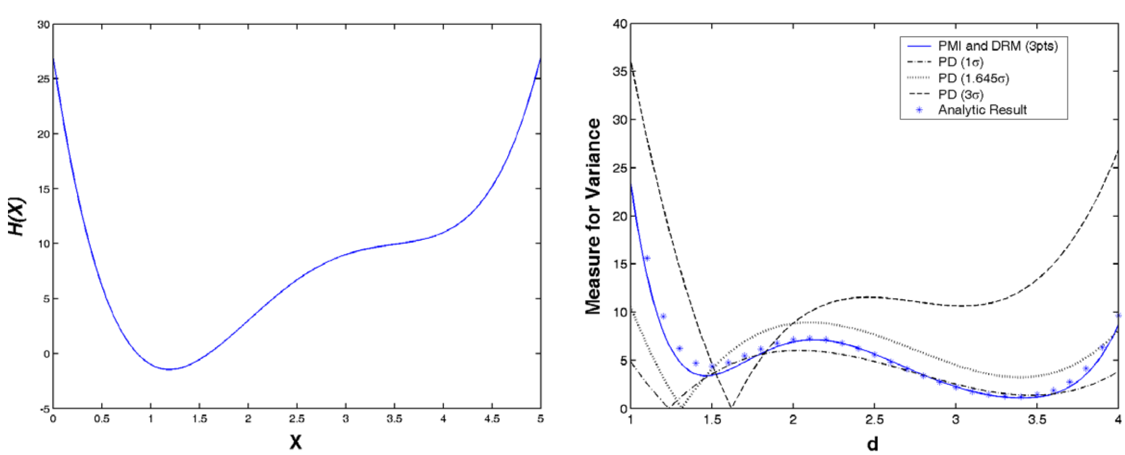
In this section, three methods are compared in detail for proper identification of robust optimum design, using a one-dimensional example. RDO can be formulated to

$$\begin{array}{ll} \text{minimize} & \sigma_H^2 \\ \text{subject to} & 0 \leqslant X \leqslant 5 \end{array} \tag{34}$$

where $h_3(X) = (X-4)^3 + (X-3)^4 + 10$ and $X \sim N(\mu, 0.4)$. Again note that in one-dimensional problem, PMI and the univariate DRM with three quadrature points can be considered to be the same method. Fig. 4a illustrates the shape of the performance function and Fig. 4b illustrates the variances obtained from DRM and PMI and percentile differences from PDM.

As mentioned before, in this example, PMI and DRM with three quadrature points can be considered to be the same method since the design variable is normally distributed and there is no FORM error in a one-dimensional function. As shown in Fig. 4b, PMI and DRM with three quadrature points can approximate the variance of the performance function very well. On the other hand, PDM with various percentiles cannot estimate the moments. More significantly, the location of the optimum point changes depending on the percentiles used. In fact, there is no one percentile that can be used to accurately identify the location of both local minima simultaneously in Fig. 4b. Table 10 shows that the best percentile should be located between 2σ and 3σ for the left local minimum and the best percentile should be located between 1.645σ and 2σ for the right local minimum. In Fig. 4b, 'Measure for variance' is used instead of variance. It is because PDM cannot estimate the variance of the performance function and uses percentile differences as the measure for the variance.

Another problem of using PDM for a highly non-linear performance function such as Eq. (34) is that PDM might not be able to identify which local minimum is the global minimum when there is more than one local minimum. As shown in Table 10, the results of PDM with three different percentiles indicate that the value of the cost function at the left minimum in Fig. 4b is less than the value at the right minimum, which is wrong.



(a) Performance Function ($h_3(X)$) (b) Measure for Variance (σ_H^2)

Fig. 4. Shape and variance of performance function.

Table 10
Position and value of optimum using three methods for Eq. (34)

		PMI and DRM		PDM				NI
		3 pts	5 pts	1σ	1.645σ	2σ	3σ	
Left min.	x_{min}	1.463	1.483	1.236	1.315	1.376	1.622	1.485
	σ_H^2 or $h_{p_1} - h_{p_2}$	3.397	4.361	0.000	0.000	0.000	0.000	4.403
Right min.	x_{min}	3.405	3.359	3.464	3.397	3.341	3.037	3.358
	σ_H^2 or $h_{p_1} - h_{p_2}$	1.075	1.220	1.375	3.239	4.759	10.645	1.234

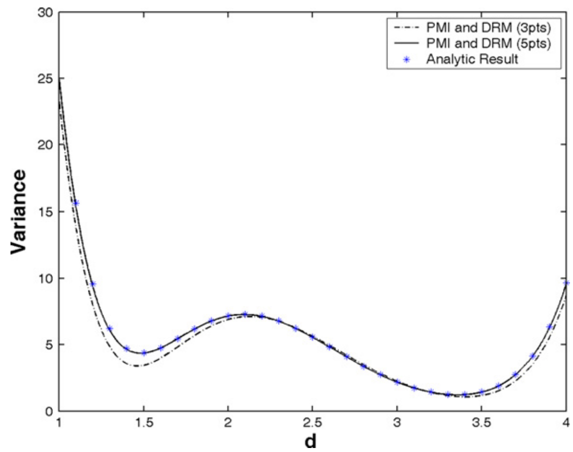


Fig. 5. Accuracy of PMI and DRM with five quadrature points.

Table 10 also shows that PMI and DRM with three quadrature points yields some errors in finding the location of the optimum and estimating the value of the optimum. This is because the performance function is a polynomial of degree 4, thus three quadrature points may not be sufficient. In this case, DRM and PMI with five quadrature points are a good option to achieve accuracy. The accuracy of DRM and PMI with five quadrature points is illustrated in Table 10 and Fig. 5.

4.3. Comparison of PMI, DRM, and PDM for RBRDO

For the purpose of comparison among the three methods, the cost function of the smaller-the-better type in Section 2.2 is used and weights are given as $w_1 = 0$ and $w_2 = 1$. Then, RBRDO can be formulated to [7]:

$$\begin{aligned} \text{minimize} \quad & \sigma_H^2 \\ \text{subject to} \quad & P(-x_1 - x_2 + 6.45 > 0) \leqslant \Phi(-\beta_t) \\ & P(1 \leqslant x_i \leqslant 10) \geqslant \Phi(\beta_t), \; i = 1,2 \end{aligned} \tag{35}$$

where $h(\mathbf{x}) = (x_1 - 4)^3 + (x_1 - 3)^4 + (x_2 - 5)^2 + 10$, $X_i \sim N(\mu_i, 0.4)$ for $i = 1,2$ and $\beta_t = 3$.

Fig. 6 illustrates the contour of the performance function $h(\mathbf{X})$ in the formulation (35) and Table 11 shows the properties of the random variables.

As shown in Table 12, DRM with 5 quadrature points shows the best result in terms of locating the minimum variance and the estimation of the variance has the smallest error. DRM with 3 quadrature points and PMI show error in estimation of the variance since the performance function is fourth order polynomial as explained in Section 3.1.1. PDM with 1.645σ shows better result than DRM with 3 points and PMI. However, as shown in Table 13, the optimum design varies depending on the percentiles used. For this problem, a percentile close to 2.0σ shows the smallest variance of the performance function, which does not mean that the percentile (2.0σ) is the best for all problems as shown in the previous example.

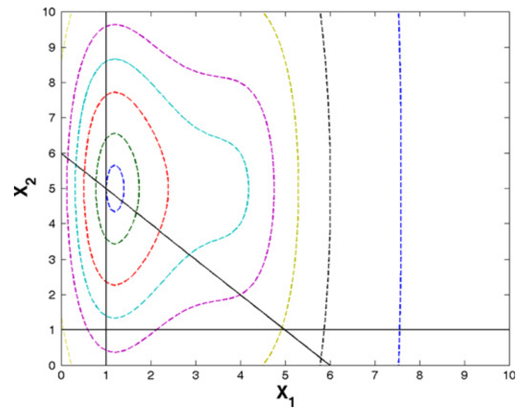


Fig. 6. Contour of performance function $h(\mathbf{X})$.

Table 11
Properties of random variables of Eq. (35)

Random variable	Std. dev.	Distr. type	$\mathbf{d}^L$	$\mathbf{d}$	$\mathbf{d}^U$
x_1	0.400	Normal	1.000	8.000	10.000
x_2	0.400	Normal	1.000	8.000	10.000

Table 12
Optimum design and cost comparison for Eq. (35)

	d_1	d_2	σ_H^2 or $h_{p_1} - h_{p_2}$	Analytic variance
DRM (3pts)	3.423	5.004	1.138	1.326
DRM (5pts)	3.378	4.997	1.316	1.289
PMI	3.407	5.010	1.075	1.308
PDM (1.645σ)	3.396	5.003	3.239	1.299

Table 13
Optimum design and cost of Eq. (35) with various percentiles

Percentile	d_1	d_2	Percentile difference	Analytic variance
0.5σ	3.491	4.998	0.548	1.465
1.0σ	3.465	4.994	1.375	1.400
1.5σ	3.413	4.996	2.724	1.315
1.645σ	3.396	5.003	3.239	1.299
2.0σ	3.341	4.995	4.759	1.286
2.5σ	3.227	5.005	7.495	1.426
3.0σ	3.046	5.099	10.652	2.024

4.4. RBRDO for side impact crashworthiness

The RBRDO model of crashworthiness for vehicle side impact shown in Fig. 7 is formulated to

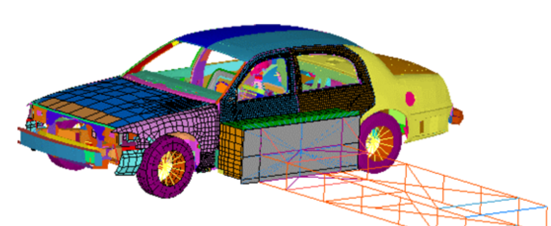


Fig. 7. Vehicle side impact problem.

$$
\begin{aligned}
\text{minimize} \quad & w_1 \frac{M}{M_0} + w_2 \left(\frac{\sigma_H}{\sigma_{H_0}} \right)^2 \\
\text{subject to} \quad & P(\text{abdomen load} > 1.0\,\text{kN}) \leqslant \Phi(-\beta_t) \\
& P(\text{upper/mid/lowerVC} > 0.32\,\text{m/s}) \leqslant \Phi(-\beta_t) \\
& P(\text{upper/mid/lower rib deflection} > 32\,\text{mm}) \leqslant \Phi(-\beta_t) \\
& P(\text{pubic symphysis force}, F > 4.0\,\text{kN}) \leqslant \Phi(-\beta_t) \\
& P(\text{velocity of B-pillar at mid-point} > 9.9\,\text{mm/ms}) \leqslant \Phi(-\beta_t) \\
& P(\text{velocity of front door at B-pillar} > 15.7\,\text{mm/ms}) \leqslant \Phi(-\beta_t) \\
& \mathbf{d}^L \leqslant \mathbf{d} \leqslant \mathbf{d}^U, \quad \mathbf{d} \in R^9 \quad \text{and } \mathbf{X} \in R^{11}, \beta_t = 2
\end{aligned} \tag{36}
$$

where M is the mass of the vehicle door and the performance function $h(\mathbf{x})$is the lower rib deflection. The detailed equations for the mass of vehicle door and constraints can be found in [25].

There are eleven random parameters and nine parameters out of the eleven random parameters are design parameters. The design parameters are the thickness ($d_1 - d_7$) and material properties of critical parts (d_8, d_9) as shown in Table 14.

Tables 15 and 16 show RBRDO results using DRM with 3 points and PMI where equal weights ($w_1 = w_2 = 0.5$) are used. Both methods show significant reduction in the robust objective and very good accuracy in estimation of the variance. However, a total number of function evaluation (54 + 54) in PMI to estimate the variance is much less than (209 + 95) in DRM with 3 points. When equal weights ($w_1 = w_2 = 0.5$) are used, two optimum designs seem to be a little bit different, but this difference is due to the error of variance estimation and characteristic of a bi-objective optimization. If the objective is changed to minimize the variance only, that is, $w_1 = 0.0$ and $w_2 = 1.0$, then two optimum results are almost identical as shown in Table 17.

Table 14
Properties of design and random parameters of Eq. (36)

Random variable	Std. dev.	Distr. type	$\mathbf{d}^L$	$\mathbf{d}$	$\mathbf{d}^U$
1. B-pillar inner (mm)	0.100	Normal	0.500	1.000	1.500
2. B-pillar reinforce (mm)	0.100	Normal	0.450	1.000	1.350
3. Floor side inner (mm)	0.100	Normal	0.500	1.000	1.500
4. Cross member (mm)	0.100	Normal	0.500	1.000	1.500
5. Door beam (mm)	0.100	Normal	0.875	2.000	2.625
6. Door belt line (mm)	0.100	Normal	0.400	1.000	1.200
7. Roof rail (mm)	0.100	Normal	0.400	1.000	1.200
8. Mat. B-pillar inner (GPa)	0.006	Normal	0.192	0.300	0.345
9. Mat. floor side inner (GPa)	0.006	Normal	0.192	0.300	0.345
10. Barrier height (mm)	10.000	Normal	10th and 11th random variables are not regarded as design variables		
11. Barrier hitting (mm)	10.000	Normal			

Table 15
RBRDO results using DRM for side impact problem

Initial design			Optimum design			
Mass	Var.	Analytic variance	Mass	Var.	Analytic variance	No. of F.E.
30.83	2.361	2.374	28.17	1.458	1.471	209 + 95

Table 16
RBRDO results using PMI for side impact problem

Initial design			Optimum design			
Mass	Var.	Analytic variance	Mass	Var.	Analytic variance	No. of F.E.
30.83	2.380	2.374	27.77	1.488	1.492	54 + 54

Table 17
Optimum design comparison for side impact problem

	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9
Initial	1.000	1.000	1.000	1.000	2.000	1.000	1.000	0.300	0.300
DRM[a]	0.949	1.350	0.529	1.500	0.975	1.200	0.400	0.280	0.192
PMI[a]	0.876	1.350	0.527	1.500	0.961	1.200	0.400	0.305	0.192
DRM[b]	1.359	1.350	0.529	1.500	2.000	1.200	1.000	0.192	0.300
PMI[b]	1.360	1.350	0.529	1.500	2.000	1.200	1.000	0.192	0.300

[a] For both DRM and PMI, $w_1 = w_2 = 0.5$ is used.
[b] For both DRM and PMI, $w_1 = 0.0$, $w_2 = 1.0$ is used.

5. Discussion and conclusion

Three methods (PMI, PDM, and univariate DRM) are compared in terms of efficiency and accuracy for computation of the statistical moments and their sensitivities. To compare the accuracy in estimation of the statistical moments of the performance function, two polynomial performance functions with two design variables are employed. In this comparison, PDM is excluded since PDM cannot estimate the moments of the performance function. The comparison shows that DRM can accurately estimate the statistical moments of the performance function for the design variables with both non-normal and normal distributions. On the other hand, PMI can accurately estimate the statistical moments of the performance function for the design variables with normal distributions. For non-normally distributed design variables, PMI shows some errors since non-linear transformations make the performance function become highly non-linear. For RBRDO, a highly nonlinear performance function was used for comparison purposes. Both the one-dimensional and the two-dimensional examples show that, in most cases, PMI and DRM can identify the optimum design and estimate the cost function accurately, whereas the optimum design of PDM varies depending on the percentile used, and PDM has identified a wrong global minimum. To achieve better accuracy, DRM with five quadrature points can be used.

PMI and PDM yield the same efficiency if the same inverse reliability analysis is used to find MPPs. Non-linearity of the performance function affects the total number of function evaluations most significantly in RBRDO using PMI and PDM. In estimation of the statistical moments using DRM, the number of design variables affects the total number of function evaluations most significantly. Hence, if the number of design variables is large, it is recommended to use PMI, compared to DRM, for RBRDO.

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