20.4 ROBUST DESIGN APPROACH

What is a robust design?

Taguchi (1987), a pioneer of the robust design approach, defined *robustness* as follows: "robustness is the state where the technology, product, or process performance is minimally sensitive to factors causing variability (either in manufacturing or in the user's environment) and aging at the lowest unit manufacturing cost." This concept of *robustness* has been developed to improve product quality and reliability, and manufacturing processes in industrial engineering. It can account for noise factors, such as environmental variation during a product's use, manufacturing variation, and product deterioration. It has also been extended and applied to all kinds of design situations. This section presents an introduction to the robust design approach. Detailed reviews of the approach have been presented by Park et al. (2006) and Beyer and Sandhoff (2007).

The design process always has some uncertainties in the design variables and/or the problem parameters. The problem parameters are the ones that are considered constants in the design process. Examples of problem parameters in structural design are external forces, material properties, temperature, length of members, dimensions of parts, member support conditions, and so forth. Uncertainties in the final design are introduced from tolerances on the design variables and noise in the problem parameters.

These uncertainties may be known or unknown, and the question is how to treat them in the design process. Designers always want to have steady performance for their design even though the uncertainties noted above exist. In other words, the performance of the designed product should be robust (insensitive) with respect to the uncertainties. The robust design approach attempts to accomplish this objective.

A robust design is relatively insensitive to variations in the problem-related parameters and variables. The procedure to find designs that are insensitive to parameter variations is called the robust design method.

When the robust design approach is related to the optimization process, it is called robust design optimization. The optimization methods and the problem formulations discussed in Chapters 8 through 13 are called deterministic approaches; that is, uncertainties of any parameters are not considered. In robust optimization, the effect of uncertainties in the problem parameters is incorporated into the formulation of the problem using the mean and variance of various data and functions.

Robustness is usually defined for the cost function, although robustness of the constraints can be considered as well. Two main approaches for robust design are presented in this section: (1) robust optimization, which uses conventional optimization algorithms, and (2) the Taguchi method.

20.4.1 Robust Optimization

Mean

Mean is defined as the simple average that is the sum of all values divided by the number of values. It is also called the *average value* or the *expected value*. The standard symbol for the mean is μ. Let a function f be observed (evaluated) at l points as $f_1, f_2, \ldots, f_l$.

The *mean* μ_f of these function values is the average calculated as the sum of all of the observations divided by the number of observations:

$$\mu_f = \frac{\sum_{i=1}^{l} f_i}{l} \tag{20.34}$$

Note that the mean can be positive, negative, or zero.

Variance

Another important statistical property is *variance*, which is defined as the average of the squared difference from the mean (i.e., $(f_i - \mu_f)^2$), usually written as σ^2. Squaring the differences makes them all positive numbers. It also makes the larger ones stand out. For the above example of l observations of function f, σ_f^2 for the function values is calculated as

$$\sigma_f^2 = \frac{\sum_{i=1}^{l} (f_i - \mu_f)^2}{l} \tag{20.35}$$

Variance represents dispersion of the data from its mean value. When data are extracted as samples from a large population, the degree of freedom becomes $(l-1)$. In that case, l in Eq. (20.35) is replaced by $(l-1)$.

Standard Deviation

The standard deviation is defined as the square root of the variance; that is, σ_f. Thus, it is also a measure of the variability or dispersion of the data from the average (mean or expected value). Its standard symbol is σ. Standard deviation is a more meaningful measure of the dispersion of the data and is most commonly used as such. Along with the mean, it gives us a standard way of knowing what is normal—that is, any data falling within the standard deviation of the mean. A smaller standard deviation indicates that the data points tend to be very close to the mean, whereas a larger standard deviation indicates that the data are spread out over a larger band around it.

If X is a random variable with mean value μ, then it is expressed as

$$E[X] = \mu \tag{20.36}$$

where the operator $E[X]$ denotes the average or expected value of X. The standard deviation of X is given as

$$\sigma = \sqrt{E[(X-\mu)^2]} \tag{20.37}$$

The standard deviation is the square root of the variance of X; that is, it is the square root of the average value or expected value of $(X-\mu)^2$.

Probability Density Function

A continuous random variable X takes on various values x within the range $-\infty < x < \infty$. *A random variable is usually expressed by an uppercase letter, while its particular value is denoted by a lowercase letter*. A mathematical function that describes the distribution of a continuous

random variable is called the *probability density function* (PDF) and is designated as $f_X(x)$. That is, it is a function that describes the relative likelihood (probability) for this random variable to occur at a given point x. The probability density function is nonnegative everywhere, and its integral over the entire space is equal to one:

$$f_X(\mathbf{x}) \geq 0 \tag{20.38}$$

$$\int_{-\infty}^{\infty} f_X(x)dx = 1 \tag{20.39}$$

Problem Definition

Since most engineering design problems do not usually involve equality constraints, we consider the inequality-constrained optimization problem:

Minimize

$$f(\mathbf{x}) \tag{20.40}$$

subject to

$$g_j(\mathbf{x}) \leq 0; \quad j = 1 \text{ to } m \tag{20.41}$$

Design variables are changed during the optimization process while any problem parameters are kept fixed. Since there can be uncertainties in the design variables and problem parameters, the problem definition needs to be modified. The cost and constraint functions in Eqs. (20.40) and (20.41) are redefined to include uncertainties:

$$f(\mathbf{x}, \mathbf{y}) \rightarrow f(\mathbf{x} + \mathbf{z}^{\mathbf{x}}, \mathbf{y} + \mathbf{z}^{\mathbf{y}}) \tag{20.42}$$

$$g(\mathbf{x}, \mathbf{y}) \rightarrow g(\mathbf{x} + \mathbf{z}^{\mathbf{x}}, \mathbf{y} + \mathbf{z}^{\mathbf{y}}) \tag{20.43}$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_r)$ is the problem parameter vector, and $\mathbf{z}^{\mathbf{x}} = (z^{x_1}, z^{x_2}, \ldots, z^{x_n})$ and $\mathbf{z}^{\mathbf{y}} = (z^{y_1}, z^{y_2}, \ldots, z^{y_r})$ are uncertainties in the design variable vector and the problem parameter vector, respectively.

The uncertainties can be interpreted as perturbations or noise in the variables and are treated as random variables. In robust optimization, the optimization problem in Eqs. (20.40) and (20.41) is changed to incorporate the perturbations as follows: Find an n-vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of design variables to

minimize a cost function

$$\begin{aligned} F(\mathbf{x}, \mathbf{y}, \mathbf{z}^{\mathbf{x}}, \mathbf{z}^{\mathbf{y}}) &= F(f(\mathbf{x} + \mathbf{z}^{\mathbf{x}}, \mathbf{y} + \mathbf{z}^{\mathbf{y}})) \\ &= F(f(x_1 + z^{x_1}, x_2 + z^{x_2}, \ldots, x_n + z^{x_n};\ y_1 + z^{y_1}, y_2 + z^{y_2}, \ldots, y_r + z^{y_r})) \end{aligned} \tag{20.44}$$

subject to the m inequality constraints

$$\begin{aligned} G_j(\mathbf{x}, \mathbf{y}, \mathbf{z}^{\mathbf{x}}, \mathbf{z}^{\mathbf{y}}) &= G_j(g_j(\mathbf{x} + \mathbf{z}^{\mathbf{x}}, \mathbf{y} + \mathbf{z}^{\mathbf{y}})) \\ &= G_j(g_j(x_1 + z^{x_1}, x_2 + z^{x_2}, \ldots, x_n + z^{x_n}; \\ &\quad y_1 + z^{y_1}, y_2 + z^{y_2}, \ldots, y_r + z^{y_r})) \leq 0 \end{aligned} \tag{20.45}$$

where functions F and G_j are derived by considering noise in the functions f and g_j, respectively. Usually functions F and G_j are defined using the mean and variance of the functions f and g_j, respectively.

When the uncertainties in the variables are given by the probability density functions, the mean μ_f and the variance σ_f^2 of f given in Eq. (20.42) are calculated (Phadke, 1989) as

$$\mu_f = E[f(\mathbf{x},\mathbf{y})] = \int\int\ldots\int f(\mathbf{x}+\mathbf{z}^{\mathbf{x}},\mathbf{y}+\mathbf{z}^{\mathbf{y}}) \times u_1(z^{x_1})\cdots u_n(z^{x_n})v_1(z^{y_1})\cdots v_r(z^{y_r})dz^{x_1}\cdots dz^{x_n}dz^{y_1}\cdots dz^{y_r} \tag{20.46}$$

$$\sigma_f^2 = E[(f(\mathbf{x},\mathbf{y})-\mu_f)^2] = \int\int\ldots\int [f(\mathbf{x}+\mathbf{z}^{\mathbf{x}},\mathbf{y}+\mathbf{z}^{\mathbf{y}})-\mu_f]^2 \times u_1(z^{x_1})\cdots u_n(z^{x_n})v_1(z^{y_1})\cdots v_r(z^{y_r})dz^{x_1}\cdots dz^{x_n}dz^{y_1}\cdots dz^{y_r} \tag{20.47}$$

where $E[b]$ is the expected value of b and $u_i(z^{x_i})$ and $v_i(z^{y_i})$ are the probability density functions of the uncertainties z^{x_i} and z^{y_i}, respectively. In Eqs. (20.46) and (20.47), the uncertainties are assumed to be statistically independent. If they follow the Gaussian (normal) distribution, the probability density function $u_i(z^{x_i})$ is given as

$$u_i(z^{x_i}) = \frac{1}{\sigma_{x_i}\sqrt{2\pi}}\exp\left[\frac{-(x_i-\mu_{x_i})^2}{2\sigma_{x_i}^2}\right] \tag{20.48}$$

where μ_{x_i} and σ_{x_i} are the mean and the standard deviation of the ith design variable x_i. The probability density function $v_i(z^{y_i})$ is defined in the same manner as Eq. (20.48).

The robust optimization problem in Eqs. (20.44) and (20.45) is defined using the mean and the variance in Eqs. (20.46) and (20.47). Robust optimization tries to reduce the dispersion of the cost function with respect to the uncertainties because the dispersion is equivalent to the sensitivity. This implies that standard deviation of the cost function should be minimized. Since the mean of the cost function should be simultaneously minimized, this becomes a two-objective optimization problem. Using a weighted sum method (refer to Section 17.4), the cost function of Eq. (20.44) for robust design optimization is defined as

$$F = w_1\mu_f + w_2\sigma_f \tag{20.49}$$

where w_1 and w_2 are the weighting coefficients. If w_1 is larger, minimization of the cost function is emphasized more than obtaining a robust design and vice versa. If a different method for multi-objective optimization is used, Eq. (20.49) is modified according to that method.

The constraint in Eq. (20.45) should be defined so that the original constraint is satisfied even though uncertainties exist. To sufficiently satisfy the constraint, the constraint in Eq. (20.45) is defined as

$$G_j \equiv \mu_{g_j} + k\sigma_{g_j}^2 \le 0 \tag{20.50}$$

where $k > 0$ is a user-defined constant depending on the design purpose and $\sigma_{g_j}^2$ denotes the dispersion of data for the constraint g_j. If the uncertainties are bounded by some upper and lower limits, the worst case of g_j can be considered as follows:

$$G_j = \mu_{gj} + k^{\mathbf{x}_j} \sum_{i=1}^{n} \left| \frac{\partial g_j}{\partial x_i} \right| |z^{x_i}| + k^{\mathbf{y}_j} \sum_{i=1}^{r} \left| \frac{\partial g_j}{\partial y_i} \right| |z^{y_i}| \tag{20.51}$$

where $|z^{x_i}|$ and $|z^{y_i}|$ denote the maximum values of the uncertainty (tolerance) ranges, and $k^{x_j} > 0$ and $k^{y_j} > 0$ are the user-defined constants. Equation (20.51) is obtained by writing a linear Taylor expansion of G_j and using the absolute values for the quantities in the second and third terms to obtain the worst case.

If we have distribution for the design variables, then we have distribution for the cost function f as well, so we can calculate the mean of f by integration in Eq. (20.46). This calculation, however, is quite costly. Therefore, the mean and the variance of the cost function are approximated as

$$\mu_f \cong f(\boldsymbol{\mu}_{\mathbf{x}}, \ \boldsymbol{\mu}_{\mathbf{y}}) \tag{20.52}$$

$$\sigma_f^2 \cong \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 \sigma_{x_i}^2 + \sum_{i=1}^{r} \left(\frac{\partial f}{\partial y_i} \right)^2 \sigma_{y_i}^2 \tag{20.53}$$

where $\boldsymbol{\mu}_{\mathbf{x}}$ is a vector of means of the design variable vector $\mathbf{x}$ and $\boldsymbol{\mu}_{\mathbf{y}}$ is a vector of means for the problem parameter vector $\mathbf{y}$. These are calculated using their corresponding probability density functions. The mean and the variance for the constraint functions are defined in a similar manner.

Equation (20.53) is derived using a Taylor series for f. The first-order Taylor series expansion of the cost function $f(\mathbf{x}, \mathbf{y})$ at the points $\boldsymbol{\mu}_{\mathbf{x}}$ and $\boldsymbol{\mu}_{\mathbf{y}}$ is defined as follows:

$$f(\mathbf{x}, \mathbf{y}) \cong f(\boldsymbol{\mu}_{\mathbf{x}}, \ \boldsymbol{\mu}_{\mathbf{y}}) + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right) (x_i - \mu_{x_i}) + \sum_{i=1}^{r} \left(\frac{\partial f}{\partial y_i} \right) (y_i - \mu_{y_i}) \tag{20.54}$$

If all random variables and parameters are statistically uncorrelated, the variance of the cost function can be approximated as follows:

$$\begin{aligned}
\mathrm{Var}\left[f(\mathbf{x}, \mathbf{y}) \right] = \sigma_{\mathrm{f}}^2 &\cong \mathrm{Var}\left[f(\boldsymbol{\mu}_{\mathbf{x}}, \ \boldsymbol{\mu}_{\mathbf{y}}) + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_{\mathrm{i}}} \right) (x_i - \mu_{x_i}) + \sum_{i=1}^{r} \left(\frac{\partial f}{\partial y_i} \right) (y_i - \mu_{y_i}) \right] \\
&= \mathrm{Var}\left[f(\boldsymbol{\mu}_{\mathbf{x}}, \ \boldsymbol{\mu}_{\mathbf{y}}) \right] + \mathrm{Var}\left[\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_{\mathrm{i}}} \right) (x_i - \mu_{x_i}) \right] + \mathrm{Var}\left[\sum_{i=1}^{r} \left(\frac{\partial f}{\partial y_i} \right) (y_i - \mu_{y_i}) \right] \\
&= 0 + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 \mathrm{Var}[x_i] + \sum_{i=1}^{r} \left(\frac{\partial f}{\partial y_{\mathrm{i}}} \right)^2 \mathrm{Var}\left[y_i \right] \\
&= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 \sigma_{x_i}^2 + \sum_{i=1}^{r} \left(\frac{\partial f}{\partial y_i} \right)^2 \sigma_{y_i}^2
\end{aligned} \tag{20.55}$$

In robust optimization, first-order derivatives of the cost function f are included in the variances; therefore, we need second-order derivatives of f in the optimization process if a gradient-based method is used. The calculation of the second-order derivatives can be quite expensive, especially for a large-scale problem. Therefore, other methods have been used to avoid this calculation, such as the direct search methods discussed in Chapter 11 or the nature-inspired methods discussed in Chapter 19.

Equation (20.44) or (20.49) is called the robustness index. The robustness index can be defined to have other forms. Also, Eq. (20.45) or (20.50) can be defined differently according to the design purpose.

To transform an optimization problem into a robust optimization problem, probability density functions of uncertainties must be known or assumed. The robust optimization problem is then defined in terms of the mean and variance of the cost and constraint functions.

EXAMPLE 20.7 ROBUST OPTIMIZATION

Solve a robust optimization problem formulated as

Minimize

$$f = x_1 x_2 \cos x_1 + x_1^2 - \frac{1}{4}x_2^2 - e^{x_2} \tag{a}$$

subject to the constraints

$$g_1(\mathbf{x}) = (x_1 - 1)^2 + x_2^2 - x_1 - 6 \le 0 \tag{b}$$

$$g_2(\mathbf{x}) = \tfrac{3}{7}x_1^2 - \frac{1}{10}x_2 + (x_2 - 1)^2 - 5 \le 0 \tag{c}$$

$$-2.0 \le x_1 \le 2.0; \quad -2.0 \le x_2 \le 2.0 \tag{d}$$

The design variables x_1 and x_2 have the normal distribution with $\sigma_{x_1} = \sigma_{x_2} = 0.1$. The maximum tolerances $|z^{x_1}| = |z^{x_2}| = 0.3$. Use Eqs. (20.49) and (20.51) with $w_1 = w_2 = 0.5$ and $k^{x_1} = k^{x_2} = 0.5$. Use (0.4, 0.4) for the initial design.

Solution

For this problem, the standard deviations for design variables x_1 and x_2 are given and the uncertainties are bounded as ± 0.3. Using Eqs. (20.52) and (20.53), the mean μ_f and the standard deviation σ_f of f are calculated as follows:

$$\mu_f = x_1 x_2 \cos x_1 + x_1^2 - \frac{1}{4}x_2^2 - e^{x_2} \tag{e}$$

$$\begin{aligned}\sigma_f &= \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2} \\ &= \sqrt{(-x_1 x_2 \sin x_1 + x_2 \cos x_1 + 2x_1)^2 \sigma_{x_1}^2 + (x_1 \cos x_1 - 0.5x_2 - e^{x_2})^2 \sigma_{x_2}^2}\end{aligned} \tag{f}$$

Due to the approximation in Eq. (20.52), the expression for the mean of the cost function is the same as that for the original cost function. A normalization process is used to define the cost function in Eq. (20.49) because μ_f and σ_f have different orders of magnitude. Let μ_f^* and σ_f^* represent μ_f and σ_f at the initial point (0.4, 0.4), respectively. Therefore, from Eqs. (e) and (f), we have

$$\mu_f^* = (0.4)(0.4)\cos 0.4 + 0.4^2 - 0.25(0.4^2) - e^{0.4} = -1.224 \tag{g}$$

$$\sigma_f^* = \sqrt{\{(-0.4)(0.4)\sin 0.4 + 0.4\cos 0.4 + (2)(0.4)\}^2(0.1^2) + \{0.4\cos 0.4 - (0.5)(0.4) - e^{0.4}\}^2(0.1^2)} = 0.172 \tag{h}$$

Thus the multi-objective function $F = w_1\mu_f + w_2\sigma_f$ is normalized as follows:

$$F = w_1\frac{\mu_f}{|\mu_f^*|} + w_2\frac{\sigma_f}{\sigma_f^*} = (0.5)\frac{x_1x_2\cos x_1 + x_1^2 - \frac{1}{4}x_2^2 - e^{x_2}}{|-1.224|} + (0.5)\frac{\sqrt{(-x_1x_2\sin x_1 + x_2\cos x_1 + 2x_1)^2\sigma_{x_1}^2 + (x_1\cos x_1 - 0.5x_2 - e^{x_2})^2\sigma_{x_2}^2}}{0.172} \tag{i}$$

The constraints considering robustness are given using Eq. (20.51):

$$G_1 = \mu_{g_1} + k^{\mathbf{x}_j}\sum_{i=1}^{2}\left|\frac{\partial g_1}{\partial x_i}\right||z^{x_i}| = (x_1-1)^2 + x_2^2 - x_1 - 6 + (0.5)\{|2(x_1-1)-1|(0.3) + |2x_2|0.3\} \tag{j}$$

$$G_2 = \mu_{g_2} + k^{\mathbf{x}_j}\sum_{i=1}^{2}\left|\frac{\partial g_2}{\partial x_i}\right||z^{x_i}| = \frac{3}{7}x_1{}^2 - \frac{1}{10}x_2 + (x_2-1)^2 - 5 + (0.5)\left\{\left|\frac{6}{7}x_1\right|(0.3) + \left|2(x_2-1) - \frac{1}{10}\right|(0.3)\right\} \tag{k}$$

Equations (i) through (k) are used in the optimization process. The optimization results are shown in Table 20.17. We see that the deterministic optimum has a better μ_f while the robust optimum shows a significantly reduced σ_f.

TABLE 20.17 Robust optimization solution for Example 20.7

	Initial point	Deterministic optimum	Robust optimum
(x_1, x_2)	(0.4,0.4)	(−0.303, 2.0)	(0.265, −0.593)
μ_f	−1.224	−8.875	−0.722
σ_f	0.172	0.875	3.01E-06
f	−1.224	−8.875	−0.722
g_1	−5.88	8.1E-04	−5.373
g_2	−4.611	−4.16	−2.373

20.4.2 The Taguchi method

The Taguchi method (1987) was developed for quality improvement of products and processes. Initially, it was applied to process rather than product design, and it did not use formal optimization methods. Also, robustness of only the cost function was considered. Recently, the method has been extended to product design problems as well. In this section, the Taguchi method is explained from the viewpoint of robust design because quality improvement can be considered equivalent to a robust design.

Taguchi introduced a *quadratic loss function* to represent robustness (insensitiveness) as

$$L(f) = k(f - m_f)^2 \tag{20.56}$$

where L is the loss function, m_f is the target value for the cost function f, and $k > 0$ is a constant. As shown in Eq. (20.42), the cost function is a function of the design variables and problem parameters. Generally, constraints are ignored in the Taguchi method and only the robustness of the cost function is considered. The loss function means the loss when the design does not meet the target value for the cost function due to uncertainty (noise). The higher the loss, the farther away the solution from the target cost function value. Thus, *if the loss function is reduced, quality is enhanced*. This is the objective of the Taguchi method: Minimize the loss function in Eq. (20.56).

When we have various cases due to different noises, the expected value of the loss function in Eq. (20.56) is derived as follows:

$$E[L(f)] = E[k(f^2 - 2m_f f + m_f^2)] = kE[f^2 - 2m_f f + m_f^2] = k\left\{E[f^2] - 2m_f E[f] + E[m_f^2]\right\} \tag{20.57}$$

The variance of f is calculated using the following identity:

$$\text{Var}[f] = E[f^2] - (E[f])^2; \quad \text{or} \quad E[f^2] = \text{Var}[f] + (E[f])^2 = \sigma_f^2 + \mu_f^2 \tag{20.58}$$

Thus, substituting Eq. (20.58) into Eq. (20.57), the expected value of the loss function is given from Eq. (20.57) as

$$E[L(f)] = k(\sigma_f^2 + \mu_f^2 - 2m_f\mu_f + m_f^2) = k(\sigma_f^2 + (\mu_f - m_f)^2) \equiv Q \tag{20.59}$$

where σ_f and μ_f are the standard deviation and the mean value of the cost function f, respectively. It is noted that the loss function Q in Eq. (20.59) is similar to the cost function of robust optimization in Eq. (20.49). This loss function is minimized to obtain a robust design.

Sometimes, the loss function is modified by a scale factor. Suppose we have a scale factor $s = m_f/\mu_f$, which can adjust the current mean μ_f to the target value m_f for the cost function. Since this factor scales σ_f and μ_f as $\frac{m_f}{\mu_f}\sigma_f$ and $\frac{m_f}{\mu_f}\mu_f$, a new loss function Q_a is obtained from Eq. (20.59):

$$Q_a = k\left(\left(\frac{m_f}{\mu_f}\sigma_f\right)^2 + \left(\mu_f\frac{m_f}{\mu_f} - m_f\right)^2\right) = km_f^2\frac{\sigma_f^2}{\mu_f^2} \tag{20.60}$$

The new loss function is the predicted amount of the loss when the current design is changed to the target value. It is noted that σ_f and μ_f are calculated at the current design.

To enhance the additive effect of the design variables, Eq. (20.60) is transformed (see Taguchi, 1987; Phadke, 1989) into:

$$\eta = 10\, log_{10} \frac{\mu_f^2}{\sigma_f^2} \tag{20.61}$$

This equation is obtained using Q_a as $\log_{10}(1/Q_a)$, ignoring the constant km_f^2 and then multiplying the result by the factor 10. Use of a logarithm enhances the additive effect of the design variables. Equation (20.61) is the ratio of the power of the signal μ_f to the power of the noise σ_f. It is called the signal-to-noise (S/N) ratio.

The power of a signal refers to the property the designer wants to improve. In this case, the designer wants to meet the target value m_f. The power of noise is the amount of uncertainty (variance). We try to find the design parameters so that the influence of the noise is minimized; that is, we maximize the S/N ratio. This is equivalent to minimizing the loss function in Eq. (20.60). In other words, maximizing S/N ratio η in Eq. (20.61) results in a robust design.

In the Taguchi method, we try to find values of the design parameters (variables) that minimize the loss function or maximize the S/N ratio.

The response having f at the target value of m_f is referred as "nominal-the-best"; a response with a target value of 0 is referred as "smaller-the-better"; and a response with a target value of infinity is referred as "larger-the-better." These examples of S/N ratios are summarized in Table 20.18, where c is the number of sample points (repetitions). For the "smaller-the-better" case in the table, the S/N ratio is derived as follows.

Since the target value m_f is zero for this case, the loss function in Eq. (20.56) is given as

$$L(f) = kf^2 \tag{20.62}$$

The expected value of this loss function is

$$Q = E\left[L(f)\right] = E\left[kf^2\right] = k\left(\frac{1}{c}\sum_{i=1}^{c} f_i^2\right) \tag{20.63}$$

where c is the total number of sample points (experiments) f_i. Ignoring the factor k, taking the logarithm of $1/Q$, and multiplying it by a factor of 10, we get

$$\eta = -10 \log\left(\frac{1}{c}\sum_{i=1}^{c} f_i^2\right) \tag{20.64}$$

TABLE 20.18 Example requirements for the S/N ratio

Characteristic	S/N ratio
Nominal-the-best	$\eta = 10\, log \frac{\mu_f^2}{\sigma_f^2}$
Smaller-the-better	$\eta = -10\, log\left[\frac{1}{c}\sum_{i=1}^{c} f_i^2\right]$
Larger-the-better	$\eta = -10\, log\left[\frac{1}{c}\sum_{i=1}^{c} \frac{1}{f_i^2}\right]$

A similar procedure can be used to derive the S/N ratio for the third case of "larger-the-better" in Table 20.18.

We need the mean and variance to calculate the loss function or the S/N ratio, and repeated experiments (function evaluation) are required to calculate these quantities. The S/N ratio is usually employed in process design. The loss function is usually directly used in product design although the S/N ratio can also be used (i.e., the one shown in the second row of Table 20.18).

In the Taguchi method, an orthogonal array is used for discrete design, as described in the previous section. Suppose a cost function is to be minimized and there are four design variables with three levels, as shown in Table 20.6. Then we can use the $L_9(3^4)$ orthogonal array shown in Table 20.19. The S/N ratio for each row of the orthogonal array is calculated as shown in the rightmost column of the table. For each row (design point), the experiments (function evaluations) are repeatedly carried out to calculate the S/N ratio. Although the levels of design variables are fixed for each row, the response f can be different because the unknown uncertainties (noises) are included in each design variable. The following loss function is frequently used as well:

$$Q = \frac{1}{c}\sum_{i=1}^{c} f_i^2 \tag{20.65}$$

For the problem of minimizing f, the S/N ratio in Table 20.19 is maximized or the loss function in Eq. (20.65) is minimized.

TABLE 20.19 The $L_9(3^4)$ orthogonal array

Experiment no.	Design variables and levels				Signal-to-noise ratio
	x_1	x_2	x_3	x_4	
1	1	1	1	1	$\eta_1 = -10\,log\left[\frac{1}{c}\sum_{i=1}^{c} f_i^2\right]$
2	1	2	2	2	$\eta_2 = -10\,log\left[\frac{1}{c}\sum_{i=1}^{c} f_i^2\right]$
3	1	3	3	3	$\eta_3 = -10\,log\left[\frac{1}{c}\sum_{i=1}^{c} f_i^2\right]$
4	2	1	2	3	$\eta_4 = -10\,log\left[\frac{1}{c}\sum_{i=1}^{c} f_i^2\right]$
5	2	2	3	1	$\eta_5 = -10\,log\left[\frac{1}{c}\sum_{i=1}^{c} f_i^2\right]$
6	2	3	1	2	$\eta_6 = -10\,log\left[\frac{1}{c}\sum_{i=1}^{c} f_i^2\right]$
7	3	1	3	2	$\eta_7 = -10\,log\left[\frac{1}{c}\sum_{i=1}^{c} f_i^2\right]$
8	3	2	1	3	$\eta_8 = -10\,log\left[\frac{1}{c}\sum_{i=1}^{c} f_i^2\right]$
9	3	3	2	1	$\eta_9 = -10\,log\left[\frac{1}{c}\sum_{i=1}^{c} f_i^2\right]$

When physical experiments are performed for each design point in the orthogonal array, the results are different because the experiments automatically include noises. Therefore, we can calculate the variance of function f using the experimental results. If we conduct a numerical simulation for function evaluation instead of experiments, the same results are always obtained for each row of the orthogonal array (each design point). Thus, artificial noise needs to be introduced into the design variable values to obtain different simulation results. The artificial noise is constructed by perturbing the design variables or the problem parameters. The perturbation can be arbitrarily determined by the user. Alternatively, an outer array is constructed for each row of the orthogonal array, such as the one in Table 20.19.

The *outer array* is constructed as follows: First the noise levels (the values for the noise) are defined. For each row in Table 20.19, each design variable value is then perturbed systematically by different noise levels to generate perturbed design points at which the function is evaluated. The concept and procedure of orthogonal arrays as described previously is used here as well. For example, if three noise levels are selected for each of the four design variables, then each design point (each row of the orthogonal array) will generate nine perturbed points. These nine points define another table called the outer array. This will become clearer in the example problems presented below. The original orthogonal array of design points in Table 20.19 is called the *inner array*.

EXAMPLE 20.8 APPLICATION OF THE TAGUCHI METHOD

Solve a robust optimization problem formulated as

Minimize

$$f = x_1 x_2 \cos x_1 + x_1^2 - \frac{1}{4}x_2^2 - e^{x_2} + 5 \tag{a}$$

Each design variable has the levels -1.0, 0.0, and 1.0, and the disturbance for each design variable is given as $-0.1 \le z_i \le 0.1$. For the repetition of numerical experiments, use an outer array.

Solution

The orthogonal array $L_9(3^4)$ is used as the inner array for the problem. To generate the outer array, three levels are selected for the disturbance z_i in the ith design variable as -0.1, 0.0, and 0.1. Therefore, for each row of the inner array, nine perturbed design points are generated. The inner array and the outer array for the first row of the inner array are shown in Table 20.20 (note that since there are only two design variables, columns 3 an 4 are ignored in the orthogonal array). The outer array is generated by systematically perturbing each design variable by three levels of disturbance -0.1, 0.0, and 0.1. Therefore, we have nine cases for each row of the inner array. In the outer array, we calculate μ_f and σ_f for each row of the inner array, as shown at the top of Table 20.20.

The S/N ratio for the character "smaller-the-better" in Table 20.18 is used for this example because the problem is to minimize f. For the first row of the inner array, the S/N ratio for the "smaller-the-better" problem is calculated from the outer array as

$$\text{S/N ratio} = -10\,log\left[\frac{1}{c}\sum_{i=1}^{c} f_i^2\right] = -10\,log\left[\frac{1}{9}(6.12^2 + 6.09^2 + \cdots + 5.75^2 + 5.70^2)\right] = -15.45 \qquad \text{(b)}$$

In this way, the S/N ratio for each row of the inner array is calculated. The results are shown in Table 20.21. The one-way table described in the previous section is constructed using the data in Table 20.21, as shown in Table 20.22. From this table, the levels (2, 3) for **x** yield a best solution as **x** = (0.0, 1.0) (since we are maximizing the S/N ratio, the levels corresponding to the largest

TABLE 20.20 The inner and outer arrays for the first row of Example 20.8

Inner Array

Sample point	Design variables and levels			
	x_1	x_2	Ignored	Ignored
1	−1.0	−1.0	−1.0	−1.0
2	−1.0	0.0	0.0	0.0
3	−1.0	1.0	1.0	1.0
4	0.0	−1.0	0.0	1.0
5	0.0	0.0	1.0	−1.0
6	0.0	1.0	−1.0	0.0
7	1.0	−1.0	1.0	0.0
8	1.0	0.0	−1.0	1.0
9	1.0	1.0	0.0	−1.0

Outer Array for First Row of Inner Array

Experiment no.	Design variables and levels				
	x_1	x_2	Ignored	Ignored	$f(\mathbf{x})$
1	−1.1	−1.1	−1.1	−1.1	6.12
2	−1.1	−1.0	−1.0	−1.0	6.09
3	−1.1	−0.9	−0.9	−0.9	6.05
4	−1.0	−1.1	−1.0	−0.9	5.96
5	−1.0	−1.0	−0.9	−1.1	5.92
6	−1.0	−0.9	−1.1	−1.0	5.88
7	−0.9	−1.1	−0.9	−1.0	5.79
8	−0.9	−1.0	−1.1	−0.9	5.75
9	−0.9	−0.9	−1.0	−1.1	5.70
				S/N ratio	−15.45

values for each design variable in the one-way table are selected). This solution is compared with the best solution in the rows of the inner array. In Table 20.21, the solution that maximizes the S/N ratio corresponds to the sixth row; this solution is the same as the one from the one-way table. Therefore, the solution $\mathbf{x} = (0.0, 1.0)$ is selected as the final solution.

TABLE 20.21 S/N ratios for Example 20.8

Sample point	Design variables and levels				S/N ratio
	x_1	x_2	Ignored	Ignored	
1	−1.0	−1.0	−1.0	−1.0	−15.45
2	−1.0	0.0	0.0	0.0	−13.99
3	−1.0	1.0	1.0	1.0	−8.03
4	0.0	−1.0	0.0	1.0	−12.84
5	0.0	0.0	1.0	−1.0	−12.05
6	0.0	1.0	−1.0	0.0	−6.22
7	1.0	−1.0	1.0	0.0	−13.73
8	1.0	0.0	−1.0	1.0	−13.99
9	1.0	1.0	0.0	−1.0	−11.05

TABLE 20.22 The one-way table for Example 20.8

Design variable	Level		
	1	2	3
x_1	−12.49	−10.37	−12.92
x_2	−14.01	−13.34	−8.43
Ignored	−11.88	−12.63	−11.27
Ignored	−12.85	−11.31	−11.62

EXAMPLE 20.9 APPLICATION OF THE TAGUCHI METHOD

Solve the constrained optimization problem of Example 20.7 using the Taguchi method. Each design variable has three levels −1.0, 0.0, and 1.0, and the disturbance for each design variable is given as $-0.1 \le z_i \le 0.1$. For repetition of numerical experiments, use an outer array.

Solution

The orthogonal array $L_9(3^4)$ is used as the inner array for the problem. To generate the outer array, three levels are selected for the disturbance z_i in the ith design variable as −0.1, 0.0, and 0.1.

Therefore, for each row of the inner array, nine perturbed design points are generated. The inner array and the outer array for the first row of the inner array are shown in Table 20.23 on the next page (note that since there are only two design variables, columns 3 and 4 are ignored in the orthogonal array). The outer array is generated by systematically perturbing each design variable by three levels of disturbance −0.1, 0.0, and 0.1, so we have nine cases for each row of the inner array. In the outer array, we calculate μ_f and σ_f for each row of the inner array as shown in Table 20.23.

To use the loss function in Eq. (20.59), we need a target; the target of this example is set as $-\infty$. Thus, we cannot use the function in Eq. (20.59). The characteristic "smaller-the-better" in Table 20.18 can be used for a minimization problem; however, the cost function should be positive to use this index. In this case a new index is defined. Since this is a constrained minimization problem, the robustness index F is defined as

$$F = w_1\mu_f + w_2\sigma_f + P(\mathbf{x}) \tag{a}$$

where w_1 and w_2 are weighting factors, and $P(\mathbf{x})$ is a penalty function as defined in Eq. (20.32). That is, when constraints are violated, F is increased.

For each row of the inner array we calculate F using an outer array. The values of μ_f and σ_f are calculated as shown in Table 20.23, and F is evaluated for each row of the inner array. The results for each row of the inner array are shown in Table 20.24. In this problem, the factors are defined as $w_1 = w_2 = 0.5$ and the penalty parameter $R = 100$ for use in Eq. (20.32). The one-way table described in the previous section is constructed using the data in Table 20.24, as shown in Table 20.25. From this table, levels (2, 3) for $\mathbf{x}$ yield the solution as $\mathbf{x} = (0.0, 1.0)$ (since we are minimizing F, the levels corresponding to the smallest values in the one-way table are selected). This solution is compared with the best solution in the rows of the inner array in Table 20.24. The best solution corresponds to the sixth row and it is the same as the one from the one-way table. Therefore, the solution $\mathbf{x} = (0.0, 1.0)$ is selected as the final solution.

20.5 RELIABILITY-BASED DESIGN OPTIMIZATION—DESIGN UNDER UNCERTAINTY

A reliable design is one that satisfies the design criteria even with some uncertainties in the design variables or the problem parameters. Reliability is measured by the probability of satisfying a design criterion. An optimization procedure that incorporates reliability requirements in its calculations is called *reliability-based design optimization* (RBDO). In an RBDO formulation of the problem, a reliability constraint is defined so that the probability of violating the original constraint is less than a specified value. Therefore, reliability is imposed on constraints in RBDO. This is in contrast to the robust design approach (discussed in the previous section), where robustness is imposed on the cost function.

This section presents an introduction to the topic of RBDO. It is noted that considerable work has been done on this subject over the last 30 years. Consult Nikolaidis et al. (2005) and Choi et al. (2007) for more detailed discussion on the subject.

TABLE 20.23 The inner and outer arrays for the first row of Example 20.9

Inner Array

Experiment no.	Design variables and levels			
	x_1	x_2	Ignored	Ignored
1	−1.0	−1.0	−1.0	−1.0
2	−1.0	0.0	0.0	0.0
3	−1.0	1.0	1.0	1.0
4	0.0	−1.0	0.0	1.0
5	0.0	0.0	1.0	−1.0
6	0.0	1.0	−1.0	0.0
7	1.0	−1.0	1.0	0.0
8	1.0	0.0	−1.0	1.0
9	1.0	1.0	0.0	−1.0

Outer Array for First Row of Inner Array

Experiment no.	Design variables and levels				
	x_1	x_2	Ignored	Ignored	$f(\mathbf{x})$
1	−1.1	−1.1	−1.1	−1.1	1.12
2	−1.1	−1.0	−1.0	−1.0	1.09
3	−1.1	−0.9	−0.9	−0.9	1.05
4	−1.0	−1.1	−1.0	−0.9	0.96
5	−1.0	−1.0	−0.9	−1.1	0.92
6	−1.0	−0.9	−1.1	−1.0	0.88
7	−0.9	−1.1	−0.9	−1.0	0.79
8	−0.9	−1.0	−1.1	−0.9	0.75
9	−0.9	−0.9	−1.0	−1.1	0.70
				μ_f	0.92
				σ_f	0.15

20.5.1 Review of Background Material for RBDO

The basic idea of RBDO is to transform the constraints of the optimization problem into reliability-based constraints. This transformation process uses probability and statistics concepts and procedures, some of which are reviewed in this section.

TABLE 20.24 Results of Example 20.9

Experiment no.	Design variables and levels				Constraints				
	x_1	x_2	Ignored	Ignored	g_1	g_2	μ_f	σ_f	F
1	−1.0	−1.0	−1.0	−1.0	0.00	−0.47	0.92	0.15	0.53
2	−1.0	0.0	0.0	0.0	−1.00	−3.57	0.00	0.22	0.11
3	−1.0	1.0	1.0	1.0	0.00	−4.67	−2.51	0.38	−1.06
4	0.0	−1.0	0.0	1.0	−4.00	−0.90	−0.61	0.09	−0.26
5	0.0	0.0	1.0	−1.0	−5.00	−4.00	−1.00	0.09	−0.46
6	0.0	1.0	−1.0	0.0	−4.00	−5.10	−2.97	0.29	−1.34
7	1.0	−1.0	1.0	0.0	−6.00	−0.47	−0.15	0.21	0.03
8	1.0	0.0	−1.0	1.0	−7.00	−3.57	0.00	0.18	0.09
9	1.0	1.0	0.0	−1.0	−6.00	−4.67	−1.44	0.28	−0.58

TABLE 20.25 The one-way table for Example 20.9

	Level		
Design variable	**1**	**2**	**3**
x_1	−0.14	−0.69	−0.15
x_2	0.10	−0.09	−0.99
Ignored	−0.24	−0.25	−0.50
Ignored	−0.17	−0.40	−0.41

Probability Density Function

A continuous random variable X takes on various values x that are within the range $-\infty < x < \infty$. A mathematical function that describes the distribution of a continuous random variable is called the *probability density function* (PDF) and is designated $f_X(x)$. That is, PDF is a function that describes the relative likelihood (probability) of this random variable X occurring at a given point x.

Notation. **A random variable is expressed by an uppercase letter, while its particular value is denoted by a lowercase letter: for example, random variable X and its value x.**

The *probability density function* (also called the *probability distribution function* or *probability mass function*) of one variable is nonnegative everywhere and its integral over the entire

space is equal to one:

$$f_X(x) \geq 0, \quad \int_{-\infty}^{\infty} f_X(x)dx = 1 \tag{20.66}$$

The probability P that the random variable will fall within a particular region is given by the integral of this variable's probability density over the region. For example, the probability that X will lie within a differential interval dx between x and $x + dx$ is given as

$$P[x \leq X \leq x + dx] = f_X(x)dx \tag{20.67}$$

Thus the probability that X will fall between a and b, written as $P[a \leq X \leq b]$, is given as the integral of Eq. (20.67):

$$P[a \leq X \leq b] = \int_a^b f_X(x)dx \tag{20.68}$$

The probability density function having a normal (Gaussian) distribution is shown in Figure 20.4(a).

Cumulative Distribution Function

The *cumulative distribution function* (CDF) $F_X(x)$ describes the probability that a random variable X with a given probability distribution will be found at a value less than or equal to x. This function is given as

$$F_X(x) = P[X \leq x] = \int_{-\infty}^{x} f_X(u)du \tag{20.69}$$

That is, for a given value x, $F_X(x)$ is the probability that the observed value of X is less than or equal to x. If f_X is continuous at x, then the probability density function is the derivative of the cumulative distribution function:

$$f_X(x) = \frac{dF_X(x)}{dx} \tag{20.70}$$

The CDF also has the following properties:

$$\lim_{x \to -\infty} F(x) = 0; \quad \lim_{x \to \infty} F(x) = 1 \tag{20.71}$$

The cumulative distribution function is illustrated in Figure 20.4(b). It shows that the probability of X being less than or equal to x_l is $F_X(x_l)$. This is a point on the $F_X(x)$ versus x curve in Figure 20.4(b) and it is the shaded area in Figure 20.4(a).

Probability of Failure

A reliability-based constraint for the jth inequality constraint $G_j(\mathbf{X}) \geq 0$ is defined as

$$P_f = P[G_j(\mathbf{x} + \mathbf{z}^x, \mathbf{y} + \mathbf{z}^y) \leq \mathbf{0}] \leq P_{j,0}, \quad j = 1, \ldots, m \tag{20.72}$$

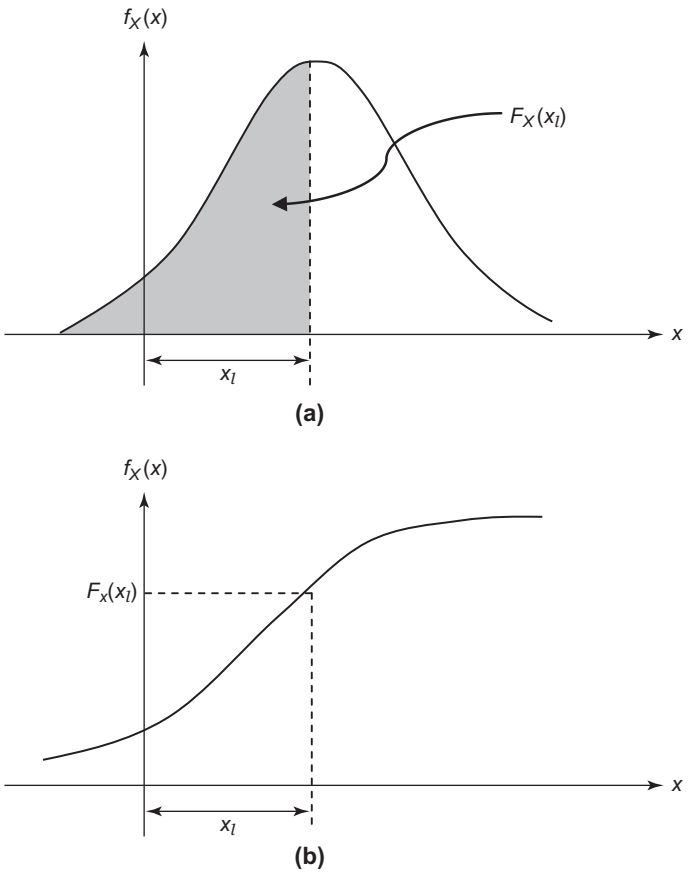

FIGURE 20.4 Graphic of (a) probability density function and (b) cumulative distribution function.

where in the preceding equation P_f is the probability of failure, $P[b]$ denotes the probability of b, $\mathbf{x}$ is the n-dimensional design variable vector, $\mathbf{y}$ is an r-dimensional vector of problem parameters, and $\mathbf{z}^{\mathbf{x}}$ and $\mathbf{z}^{\mathbf{y}}$ are the n-dimensional and r-dimensional vectors containing uncertainties in the design variables and problem parameters, respectively. $P_{j,0}$ represents the limit on the probability of failure for the jth constraint. If the probabilistic distributions of uncertainties are known, the probability of failure in Eq. (20.72) is given as

$$P_f = P[G_j(\mathbf{x} + \mathbf{z}^{\mathbf{x}}, \mathbf{y} + \mathbf{z}^{\mathbf{y}}) \leq \mathbf{0}] = \int_{G_j(\mathbf{x}+\mathbf{z}^{\mathbf{x}}, \mathbf{y}+\mathbf{z}^{\mathbf{y}}) \leq \mathbf{0}} d(\mathbf{z}^{\mathbf{x}}, \mathbf{z}^{\mathbf{y}}) d\mathbf{z}^{\mathbf{x}} d\mathbf{z}^{\mathbf{y}}, \quad j = 1, \ldots, m \tag{20.73}$$

where $d(\mathbf{z}^{\mathbf{x}}, \mathbf{z}^{\mathbf{y}})$ is the joint probability density function of the probabilistic variables $\mathbf{z}^{\mathbf{x}}$ and $\mathbf{z}^{\mathbf{y}}$ and

$$d\mathbf{z}^{\mathbf{x}} d\mathbf{z}^{\mathbf{y}} = (dz^{\mathbf{x}_1} dz^{\mathbf{x}_2} \ldots dz^{\mathbf{x}_n})(dz^{\mathbf{y}_1} dz^{\mathbf{y}_2} \ldots dz^{\mathbf{y}_r}) \tag{20.74}$$

Because the joint probability density function is a density function distributed by multiple variables, it must be known for the random variables in order to calculate the probability of failure.

Expected Value

The *expected value* (or *expectation*, *mean*, or *first moment*) of a random variable is the weighted average of all possible values that this variable can have. The weights used in computing this average correspond to the probabilities in the case of a discrete random variable, or the densities in the case of a continuous random variable. The expected value is the integral of the random variable with respect to its probability measure. It is the sample mean as the sample size goes to infinity.

Let the random variable X take value x_i with probability p_i for $i = 1$ *to* k. X's expected value $E(X)$ is then defined as

$$E[X] = \sum_{i=1}^{k} x_i p_i \tag{20.75}$$

Since all of the probabilities p_i add to one ($\sum_{i=1}^{k} p_i = 1$), the expected value can be viewed as the weighted average of $x_i's$, with $p_i's$ being the weights:

$$E[X] = \frac{\sum_{i=1}^{k} x_i p_i}{\sum_{i=1}^{k} p_i} = \sum_{i=1}^{k} x_i p_i \tag{20.76}$$

If the probability distribution of X admits a probability density function $f_X(x)$, then the expected value is computed as

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx \tag{20.77}$$

It is seen that this is the first moment of X; hence X is also called the first moment.

The expected value of $G(X)$, a function of random variable X, with respect to the probability density function $f_X(x)$ is given as

$$E[G(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx \tag{20.78}$$

The expected value of $G(X) = X^m$ is called the mth moment of X and is given as

$$E[X^m] = \int_{-\infty}^{\infty} x^m f_X(x) dx \tag{20.79}$$

MEAN AND VARIANCE The mean and variance of the random variable X are the first and second moments of X calculated as follows:

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx \tag{20.80}$$

$$\begin{aligned}\text{Var}[X] &= E[(X-\mu_X)^2] = \sigma_X^2 \\ &= \int_{-\infty}^{\infty}(x-\mu_X)^2 f_X(x)dx = \int_{-\infty}^{\infty}(x^2 - 2x\mu_X + \mu_X^2)f_X(x)dx \\ &= \int_{-\infty}^{\infty}x^2 f_X(x)dx - 2\mu_X\int_{-\infty}^{\infty}xf_X(x)dx + \mu_X^2\int_{-\infty}^{\infty}f_X(x)dx \\ &= E[X^2] - 2\mu_X^2 + \mu_X^2 = E[X^2] - \mu_X^2\end{aligned} \tag{20.81}$$

STANDARD DEVIATION The standard deviation σ_X of X is given as

$$\sigma_X = \sqrt{\text{Var}[X]} \tag{20.82}$$

COEFFICIENT OF VARIATION The *coefficient of variation* δ_X indicates the relative amount of uncertainty, defined as the ratio of the standard deviation of X to the mean of X:

$$\delta_X = \frac{\sigma_X}{\mu_X} \tag{20.83}$$

RELIABILITY INDEX The *reliability index* β is defined as the reciprocal of the coefficient of variation δ_X; that is, it is the ratio of the mean of X to the standard deviation of X:

$$\beta = \frac{\mu_X}{\sigma_X} \tag{20.84}$$

COVARIANCE If two random variables X and Y are correlated, the correlation is represented by the covariance σ_{XY}, calculated as follows:

$$\sigma_{XY} = \text{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)d(x,y)dxdy \tag{20.85}$$

where $d(x, y)$ is the joint probability density function of X and Y.

CORRELATION COEFFICIENT The *correlation coefficient* is a nondimensional measure of the correlation, defined as

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} \tag{20.86}$$

GAUSSIAN (NORMAL) DISTRIBUTION The Gaussian (normal) distribution is used in many engineering and science fields and is defined using the mean and standard deviation of X as the probability density function:

$$f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right], \quad -\infty < x < \infty \tag{20.87}$$

It is represented as $N(\mu_X, \sigma_X)$. The Gaussian distribution can be normalized using a transformation of variable X as

$$U = (X - \mu_X)/\sigma_X \tag{20.88}$$

This yields the standard normal distribution $N(0,1)$, and the corresponding probability density function becomes

$$f_U(u) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-u^2}{2}\right), \quad -\infty < u < \infty \tag{20.89}$$

Cumulative distribution with respect to u is then obtained as the cumulative distribution function (CDF) $\Phi(u)$:

$$\Phi(u) = F_U(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} exp\left(\frac{-\xi^2}{2}\right) d\xi \tag{20.90}$$

Numerical values of $\Phi(u)$ can be found in statistics texts. Since the normal distribution in Eq. (20.90) is symmetric with respect to $x = 0$,

$$\Phi(-u) = 1 - \Phi(u) \tag{20.91}$$

INVERSE If the CDF is strictly increasing and continuous, then $\Phi^{-1}(p)$, $p \in [0,1]$ is the unique number u_p such that $\Phi(u_p) = p$; that is, $u_p = \Phi^{-1}(p)$. Also,

$$u_p = \Phi^{-1}(p) = -\Phi^{-1}(1-p) \tag{20.92}$$

where u_p is the standard normalized variable, p is the corresponding cumulative probability, and Φ^{-1} is the inverse of the CDF.

20.5.2 Calculation of the Reliability Index

In this subsection, calculation of the reliability index that is used in the optimization process is explained. Knowing the reliability index, the probability of failure can be calculated, or the index can be used directly in the optimization process.

Limit State Equation

In structural design, the limit state indicates the margin of safety between structural resistance and the structural load. The limit state function ($G(\cdot)$) and the probability of failure (P_f) are defined as

$$G(X) = R(X) - S(X) \tag{20.93}$$

$$P_f = P[G(\mathbf{X}) \leq 0] \tag{20.94}$$

where R is the structural resistance and S is the loading. $G(X) < 0$, $G(X) = 0$, and $G(X) > 0$ indicate the failure region, the failure surface, and the safe region, respectively. They are illustrated in Figure 20.5 (Choi et al., 2007).

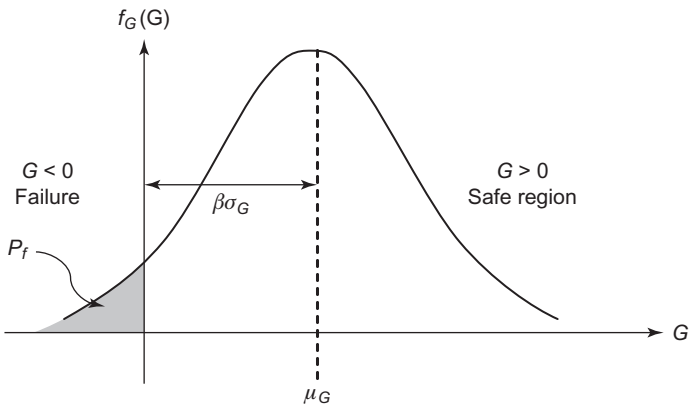

FIGURE 20.5 Probability density function for the limit state $G(X)$.

Using Eq. (20.93), the mean and standard deviation of $G(X)$ are calculated as

$$\mu_G = \mu_R - \mu_S \tag{20.95}$$

$$\sigma_G = \sqrt{\sigma_R^2 + \sigma_S^2 - 2\rho_{RS}\sigma_R\sigma_S} \tag{20.96}$$

where μ_R, μ_S, and ρ_{RS} are the mean of R, the mean of S, and the correlation coefficient between R and S, respectively. The variance of $G(X)$ is calculated as

$$\begin{aligned}\text{Var}[G(X)] &= \text{Var}[R(X) - S(X)] = \text{Var}[R(X)] + \text{Var}[S(X)] - 2\text{Cov}[R(X), S(X)] \\ &= \sigma_R^2 + \sigma_S^2 - 2\sigma_{RS} = \sigma_R^2 + \sigma_S^2 - 2\rho_{RS}\sigma_R\sigma_S \\ \text{Var}[G(X)] &= \sigma_G^2 = \sigma_R^2 + \sigma_S^2 - 2\rho_{RS}\sigma_R\sigma_S\end{aligned} \tag{20.97}$$

Thus the standard deviation is given as in Eq. (20.96).

The reliability index for G is given using Eqs. (20.84), (20.95), and (20.96) as

$$\beta = \frac{\mu_G}{\sigma_G} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2 - 2\rho_{RS}\sigma_R\sigma_S}} \tag{20.98}$$

Suppose that R and S are normally distributed; then the probability density function of the limit state function is

$$f_G(g) = \frac{1}{\sigma_G\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{g-\mu_G}{\sigma_G}\right)^2\right] \tag{20.99}$$

The probability of failure using Eq. (20.69) is given as

$$P_f = P[G(\mathbf{X}) \le 0] = \int_{-\infty}^{0} f_G(g)dg = \int_{-\infty}^{0} \frac{1}{\sigma_G\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{g-\mu_G}{\sigma_G}\right)^2\right]dg \tag{20.100}$$

After introducing the following normalizing transformation for the random variable G, the probability of failure is obtained from Eq. (20.100) as

$$U = (G - \mu_G)/\sigma_G \tag{20.101}$$

$$P_f = \int_{-\infty}^{-\beta} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \tag{20.102}$$

where $\beta = \mu_G/\sigma_G = (\mu_R - \mu_S)/\sqrt{(\sigma_R^2 + \sigma_S^2)}$, which is obtained by assuming that the correlation coefficient in Eq. (20.98) is 0. Using the definition of cumulative distribution function and its property in Eqs. (20.90) and (20.91), we get

$$P_f = \Phi(-\beta) = 1 - \Phi(\beta) \tag{20.103}$$

Linear Limit State Equation

If we suppose that a limit state equation is a linear combination of random variables X_i, $i = 1, \ldots, n$ with normal distributions, the limit state function is given as follows:

$$G = a_0 + \sum_{i=1}^{n} a_i X_i \tag{20.104}$$

Then, assuming a normal distribution for G, the mean and variance of G are calculated as

$$\mu_G = a_0 + \sum_{i=1}^{n} a_i \mu_i \tag{20.105}$$

$$\sigma_G^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \mathrm{Cov}[X_i, X_j] = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \rho_{ij} \sigma_i \sigma_j \tag{20.106}$$

where the mean and variance of X_i are μ_i and σ_i, respectively. The probability of failure for G is

$$P_f = P[G \le 0] = \Phi\left(-\frac{\mu_G}{\sigma_G}\right) = \Phi(-\beta) \tag{20.107}$$

Nonlinear Limit State Equation

When the limit state equation is a nonlinear function of the random variable vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$, it is linearized by the Taylor series around the mean $\boldsymbol{\mu}_X = (\mu_1, \mu_2, \ldots, \mu_n)$ of $\mathbf{X}$ as

$$G(\mathbf{X}) = G(\boldsymbol{\mu}_X) + \sum_{i=1}^{n} \frac{\partial G}{\partial X_i}\Big|_{\mu_X} (X_i - \mu_i) \tag{20.108}$$

The mean and variance of this linearized equation are

$$\mu_G = G(\boldsymbol{\mu}_X) \tag{20.109}$$

$$\sigma_G = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial G}{\partial X_i}\bigg|_{\mu X} \frac{\partial G}{\partial X_j}\bigg|_{\mu_X} Cov[X_i, X_j] = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial G}{\partial X_i}\bigg|_{\mu X} \frac{\partial G}{\partial X_j}\bigg|_{\mu X} \rho_{ij}\sigma_i\sigma_j \tag{20.110}$$

It is noted that $\partial G/\partial X_i\big|_{\mu_X}$ and $\partial G/\partial X_j\big|_{\mu_X}$ correspond to a_i and a_j in Eq. (20.106). Therefore, the reliability index can be calculated using Eq. (20.98). This is called the mean value first-order second-moment method (MVFOSM). MVFOSM has a drawback. Since the limit state equation is linearized around the mean point, the value of the reliability index depends on the equation's form. When the form of the limit-equation is changed by a scale, the reliability index is also changed. That is, the approach lacks the invariance of the reliability index.

Advanced First-Order Second Moment Method

To overcome the lack of invariance, Hasofer and Lind (1974) proposed the advanced first-order second-moment method (AFOSM). First, a random variable for the standard normal distribution $N(0, 1)$ is defined as

$$U_i = \frac{X_i - \mu_i}{\sigma_i}, \quad i = 1 \text{ to } n \tag{20.111}$$

Substituting for X_i from Eq. (20.111), the limit state equation in Eq. (20.104) is transformed as

$$G(\mathbf{U}) = a_0 + \sum_{i=1}^{n} a_i(\mu_i + \sigma_i U_i) \tag{20.112}$$

The mean value μ_G of G(**U**) in Eq. (20.112) is calculated as

$$\begin{aligned}
\mu_G = E[G(\mathbf{U})] &= E\left[a_0 + \sum_{i=1}^{n} a_i(\mu_i + \sigma_i U_i)\right] \\
&= a_0 + \sum_{i=1}^{n} E[a_i(\mu_i + \sigma_i U_i)] \\
&= a_0 + \sum_{i=1}^{n} a_i E[\mu_i + \sigma_i U_i] \\
&= a_0 + \sum_{i=1}^{n} a_i(\mu_i + \sigma_i E[U_i]) \\
&= a_0 + \sum_{i=1}^{n} a_i\mu_i, \quad \text{since } E[U_i] = 0
\end{aligned} \tag{20.113}$$

Therefore, μ_G can be written with all $U_i = 0$ in Eq. (20.112) as

$$\mu_G = a_0 + \sum_{i=1}^{n} a_i\mu_i = \left|G(\text{all } U_i = 0)\right| \tag{20.114}$$

The variance of $G(U)$ in Eq. (20.112) is derived as follows:

$$\begin{aligned}\sigma_G^2 &= \text{Var}[G(\mathbf{U})] = \text{Var}\left[a_0 + \sum_{i=1}^{n} a_i(\mu_i + \sigma_i U_i)\right] \\ &= \text{Var}[a_0] + \text{Var}\left[\sum_{i=1}^{n} a_i\mu_i\right] + \text{Var}\left[\sum_{i=1}^{n} a_i\sigma_i U_i\right] \\ &= 0 + 0 + \sum_{i=1}^{n} a_i^2\sigma_i^2\text{Var}[U_i] \\ &= \sum_{i=1}^{n} a_i^2\sigma_i^2, \quad \text{since} \quad \text{Var}[U_i] = 1\end{aligned} \tag{20.115}$$

Therefore, the standard deviation of $G(U)$ is given as

$$\sigma_G = \sqrt{\sum_{i=1}^{n}(a_i\sigma_i)^2} = \sqrt{\sum_{i=1}^{n}\left(\frac{\partial G}{\partial U_i}\right)^2} \tag{20.116}$$

So the reliability index β is given from Eq. (20.84) as

$$\beta = \frac{\mu_G}{\sigma_G} = \frac{\left|G(\text{all } U_i = 0)\right|}{\sqrt{\sum_{i=1}^{n}\left(\frac{\partial G}{\partial U_i}\right)^2}} \tag{20.117}$$

For the two-variable case, Eq. (20.112) gives

$$G(\mathbf{U}) = a_0 + a_1(\mu_1 + \sigma_1 U_1) + a_2(\mu_2 + \sigma_2 U_2) \tag{20.118}$$

This equation is plotted as a straight line $G(\mathbf{U}) = 0$ in Figure 20.6, designated as line AB. The shortest distance from the origin to this line is given as

$$\frac{\left|a_0 + a_1\mu_1 + a_2\mu_2\right|}{\sqrt{(a_1\sigma_1)^2 + (a_2\sigma_2)^2}} \tag{20.119}$$

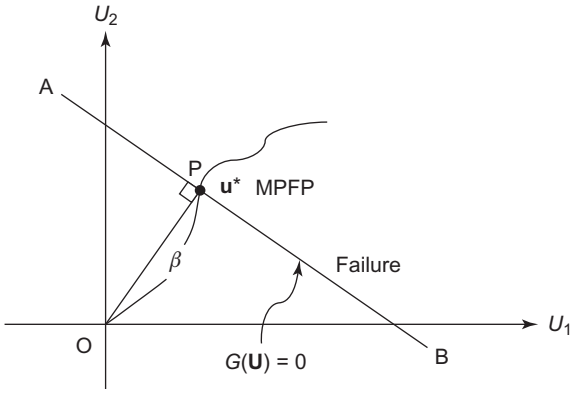

FIGURE 20.6 Geometric representation of the reliability index.

Equation (20.119) is derived by first developing the equation for a line normal to the line AB that also passes through the origin O. Then the coordinates of the point P are determined as the intersection of lines AB and the line normal to it. Knowing the coordinates of the point P, the distance OP can be calculated.

The formula for the shortest distance in Eq. (20.119) is the same as the reliability index in Eq. (20.117). Thus, the geometric meaning of the reliability index is that it is the shortest distance from the origin to the limit state equation. The point on the limit state surface that is closest to the origin is called the most probable failure point (MPFP). In the literature, this has also been called the most probable point (MPP). The MPFP is denoted u^* in Figure 20.6.

In the X space, the coordinates of the MPFP using Eq. (20.111) are

$$x_i^* = \mu_i + u_i^* \sigma_i, \quad i = 1, \ldots, n \tag{20.120}$$

where n is the number of design variables. A nonlinear limit state equation is linearized around the MPFP $\mathbf{x}^*$ as

$$G(\mathbf{X}) \approx \sum_{i=1}^{n} \left. \frac{\partial G}{\partial X_i} \right|_{\mathbf{X}=\mathbf{x}^*} (X_i - x_i^*) \tag{20.121}$$

where $G(\mathbf{x}^*) = 0$ is used.

Using Eqs. (20.109) and (20.110), the mean and variance of $G(\mathbf{X}) = 0$ are

$$\mu_G = \sum_{i=1}^{n} \left. \frac{\partial G}{\partial X_i} \right|_{\mathbf{X}=\mathbf{x}^*} (\mu_i - x_i^*) \tag{20.122}$$

$$\sigma_G^2 = \sum_{i=1}^{n} \left(\left. \frac{\partial G}{\partial X_i} \right|_{\mathbf{X}=\mathbf{x}^*} \right)^2 \sigma_i^2 \tag{20.123}$$

By the chain rule, we get

$$\frac{\partial G}{\partial X_i} = \frac{\partial G}{\partial U_i} \frac{\partial U_i}{\partial X_i} = \frac{1}{\sigma_i} \frac{\partial G}{\partial U_i} \tag{20.124}$$

Substituting Eq. (20.124) into Eqs. (20.122) and (20.123), and using Eq. (20.111), the mean and variance become

$$\mu_G = -\sum_{i=1}^{n} \left. \frac{\partial G}{\partial U_i} \right|_{\mathbf{u}^*} u_i^* \tag{20.125}$$

$$\sigma_G^2 = \sum_{i=1}^{n} \left(\left. \frac{\partial G}{\partial U_i} \right|_{\mathbf{u}^*} \right)^2 \tag{20.126}$$

where U_i is the normalized standard variable vector and u_i^* is the MPFP.

The linearized equation in Eq. (20.121) is in terms of the normalized variables and is given as

$$G(\mathbf{U}) = \sum_{i=1}^{n} \left. \frac{\partial G}{\partial U_i} \right|_{\mathbf{u}^*} (U_i - u_i^*) \tag{20.127}$$

Therefore, if a nonlinear state equation is linearized around the MPFP, the lack of invariance is avoided because the reliability index in Eq. (20.117) is expressed by the normalized variable U_i (Hasofer and Lind, 1974).

In AFOSM, the MPFP is obtained from a nonlinear limit state equation and the limit state equation is linearized around the MPFP. Because the MPFP is the closest point from the origin to the limit state equation, it is obtained typically by solving the following optimization problem:

Minimize

$$\beta = \sqrt{\mathbf{U}^T\mathbf{U}}, \text{ subject to } G(\mathbf{U}) = 0 \tag{20.128}$$

where β is the reliability index.

Using an iterative method (Hasofer and Lind, 1974), the iterative equation for solution to the problem in Eq. (20.128) is given as

$$\mathbf{U}^{(k+1)} = \frac{G_{\mathbf{U}}^{(k)T}\mathbf{U}^{(k)} - G(\mathbf{U}^{(k)})}{G_{\mathbf{U}}^{(k)T}G_{\mathbf{U}}^{(k)}} G_{\mathbf{U}}^{(k)} \tag{20.129}$$

where $G_{\mathbf{U}}^{(k)} = \frac{\partial G}{\partial \mathbf{U}}$ at $\mathbf{U} = \mathbf{U}^{(k)}$ is an n-dimensional vector.

To derive the iterative Eq. (20.129), we proceed as follows: At the kth iteration, we want to update the vector $\mathbf{U}^{(k)}$ to $\mathbf{U}^{(k+1)}$, which is expressed as

$$\mathbf{U}^{(k+1)} = \mathbf{U}^{(k)} + \Delta\mathbf{U} \tag{20.130}$$

At $\mathbf{U}^{(k)}$, we write linear the Taylor expansion of the constraint $G(\mathbf{U}^{(k+1)}) = 0$, as

$$G(\mathbf{U}^{(k)}) + G_{\mathbf{U}}^{(k)T}\Delta\mathbf{U} = 0 \tag{20.131}$$

The question now is how to determine $\Delta\mathbf{U}$. We define a minimization problem for $\Delta\mathbf{U}$ using the original problem defined in Eq. (20.128) as

Minimize

$$(\mathbf{U}^{(k)} + \Delta\mathbf{U})^T(\mathbf{U}^{(k)} + \Delta\mathbf{U}) \text{ subject to } G(\mathbf{U}^{(k)}) + G_{\mathbf{U}}^{(k)T}\Delta\mathbf{U} = 0 \tag{20.132}$$

Note that the cost function in Eq. (20.128) has been replaced by its square, which does not affect the solution to the problem. The problem defined in Eq. (20.132) can be solved by writing the optimality conditions given in Chapter 4 for the equality-constrained problem. For that, we define the Lagrangian function and differentiate it with respect to $\Delta\mathbf{U}$ as

$$L = (\mathbf{U}^{(k)} + \Delta\mathbf{U})^T(\mathbf{U}^{(k)} + \Delta\mathbf{U}) + \lambda(G(\mathbf{U}^{(k)}) + G_{\mathbf{U}}^{(k)T}\Delta\mathbf{U}) \tag{20.133}$$

$$\frac{\partial L}{\partial(\Delta\mathbf{U})} = 2(\mathbf{U}^{(k)} + \Delta\mathbf{U}) + \lambda(G_{\mathbf{U}}^{(k)}) = 0 \tag{20.134}$$

where λ is the Lagrange multiplier for the equality constraint. The equality constraint in Eq. (20.131) and the optimality conditions in Eq. (20.134) provide just the right number of equations to solve for λ and $\Delta\mathbf{U}$.

There are a number of ways to solve for the Lagrange multiplier λ . One is to pre-multiply Eq. (20.134) by $G_{\mathbf{U}}^{(k)T}$, and solve for λ as

$$\lambda = -\frac{2}{G_{\mathbf{U}}^{(k)T} G_{\mathbf{U}}^{(k)}} G_{\mathbf{U}}^{(k)T} (\mathbf{U}^{(k)} + \Delta\mathbf{U}) \tag{20.135}$$

Substituting for λ from Eq. (20.135) into Eq. (20.134) and taking the second term to the right side, we get

$$2(\mathbf{U}^{(k)} + \Delta\mathbf{U}) = \left[\frac{2}{G_{\mathbf{U}}^{(k)T} G_{\mathbf{U}}^{(k)}} G_{\mathbf{U}}^{(k)T} (\mathbf{U}^{(k)} + \Delta\mathbf{U})\right] (G_{\mathbf{U}}^{(k)}) \tag{20.136}$$

Now, replacing the left side of Eq. (20.136) with Eq. (20.130), and substituting the equality constraint from Eq. (20.132) into the right side of Eq. (20.136), we get

$$\mathbf{U}^{(k+1)} = \left[\frac{\{G_{\mathbf{U}}^{(k)T} \mathbf{U}^{(k)} - G(\mathbf{U}^{(k)})\}}{G_{\mathbf{U}}^{(k)T} G_{\mathbf{U}}^{(k)}}\right] G_{\mathbf{U}}^{(k)} \tag{20.137}$$

which is same as Eq. (20.129).

Once the MPFP is determined, the reliability index is obtained as

$$\beta = \sqrt{\mathbf{u}^{*T} \mathbf{u}^{*}} = -\frac{G_{\mathbf{U}}^{*T} \mathbf{u}^{*}}{\sqrt{G_{\mathbf{U}}^{*T} G_{\mathbf{U}}^{*}}} \tag{20.138}$$

In Eq. (20.138), the following vector $\boldsymbol{\alpha}$ is a measure of the sensitivity of the reliability index to each random variable:

$$\boldsymbol{\alpha} = -\frac{G_{\mathbf{U}}^{*}}{\sqrt{G_{\mathbf{U}}^{*T} G_{\mathbf{U}}^{*}}} \tag{20.139}$$

MVFOSM and AFOSM are attributed to the first-order reliability method (FORM). There are many other methods to calculate the reliability index. For details, the reader is referred to Choi et al. (2007).

The preceding method is a mathematical way to calculate the probability of failure of a constraint. Sampling methods are also utilized for this purpose and the most commonly used method in this class is Monte Carlo simulation (MCS). In MCS, many trials are conducted. If N trials are conducted for G_i, the probability of failure is approximately given by

$$P_f = \frac{N_f}{N} \tag{20.140}$$

where N_f is the number of trials for which G_i is violated out of the N trials conducted. For a large-scale problem, the MCS method needs a large number of computations. Thus, MCS is utilized only for small-scale problems or as a reference method for a new approach to calculate the reliability index.

EXAMPLE 20.10 CALCULATION OF THE RELIABILITY INDEX

Evaluate the reliability index β for the function $g_1(\mathbf{x})$ of Example 20.7. The initial design is $(-0.8, 0.8)$ and $\sigma_1 = \sigma_2 = 0.1$.

Solution

The function $G_1(\boldsymbol{X})$ of Example 20.7 is given as follows:

$$G_1(\mathbf{X}) = -(X_1 - 1)^2 - X_2^2 + X_1 + 6 \geq 0 \tag{a}$$

Since the initial design is $(-0.8, 0.8)$, $(\mu_1, \mu_2) = (-0.8, 0.8)$.

The iterative method is used for calculation of β. The termination criterion for the iterative process is defined as

$$\varepsilon = \left| \frac{\beta^{(k)} - \beta^{(k+1)}}{\beta^{(k)}} \right| \leq 0.001 \tag{b}$$

Iteration 1 ($k = 0$) Using Eq. (20.111), $\mathbf{u}^{(0)}$ is obtained at $(x_1^{(0)}, x_2^{(0)}) = (-0.8, 0.8)$ as follows:

$$u_1^{(0)} = \frac{x_1^{(0)} - \mu_1}{\sigma_1} = 0, \quad u_2^{(0)} = \frac{x_2^{(0)} - \mu_2}{\sigma_2} = 0 \tag{c}$$

$$G_1(\mathbf{x}^{(0)}) = 1.32 \tag{d}$$

$G_{\mathbf{U}}$ is calculated using Eq. (20.124) as

$$G_{\mathbf{U}} = \frac{\partial G}{\partial U_i} = \frac{\partial G}{\partial X_i}\sigma_i \tag{e}$$

where

$$\frac{\partial G}{\partial X_1} = -2X_1 + 3, \quad \frac{\partial G}{\partial X_2} = -2X_2 \tag{f}$$

Therefore,

$$G_{\mathbf{U}}^{(0)} = \begin{bmatrix} (-2x_1^{(0)} + 3)\sigma_1 \\ (-2x_2^{(0)})\sigma_2 \end{bmatrix} = \begin{bmatrix} 0.46 \\ -0.16 \end{bmatrix} \tag{g}$$

Using Eq. (20.129)

$$\mathbf{u}^{(1)} = \frac{G_{\mathbf{U}}^{(0)T}\mathbf{u}^{(0)} - G(\mathbf{u}^{(0)})}{G_{\mathbf{U}}^{(0)T}G_{\mathbf{U}}^{(0)}} G_{\mathbf{U}}^{(0)} = \frac{\begin{bmatrix} 0.46 & -0.16 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} - 1.32}{\begin{bmatrix} 0.46 & -0.16 \end{bmatrix}\begin{bmatrix} 0.46 \\ -0.16 \end{bmatrix}} \begin{bmatrix} 0.46 \\ -0.16 \end{bmatrix} = \begin{bmatrix} -2.560 \\ 0.891 \end{bmatrix} \tag{h}$$

From Eq. (20.138), the reliability index is

$$\beta^{(1)} = \sqrt{\mathbf{u}^{(1)T}\mathbf{u}^{(1)}} = 2.710 \tag{i}$$

Iteration 2 ($k = 1$)

$$\begin{aligned} x_1^{(1)} &= \mu_1 + \sigma_1 u_1^{(1)} = (-0.8) + (0.1)(-2.560) = -1.0560 \\ x_2^{(1)} &= \mu_2 + \sigma_2 u_2^{(1)} = 0.8 + (0.1)(0.891) = 0.889 \end{aligned} \tag{j}$$

$$G_1(\mathbf{x}^{(1)}) = -0.0736 \tag{k}$$

$$G_{\mathbf{U}}^{(1)} = \begin{bmatrix} (-2x_1^{(1)} + 3)\sigma_1 \\ (-2x_2^{(1)})\sigma_2 \end{bmatrix} = \begin{bmatrix} 0.511 \\ -0.178 \end{bmatrix} \tag{l}$$

$$\mathbf{u}^{(2)} = \frac{G_{\mathbf{U}}^{(1)T}\mathbf{u}^{(1)} - G(\mathbf{u}^{(1)})}{G_{\mathbf{U}}^{(1)T}G_{\mathbf{U}}^{(1)}} G_{\mathbf{U}}^{(1)} = \frac{\begin{bmatrix} 0.511 & -0.178 \end{bmatrix}\begin{bmatrix} -2.560 \\ 0.891 \end{bmatrix} + 0.0736}{\begin{bmatrix} 0.511 & -0.178 \end{bmatrix}\begin{bmatrix} 0.511 \\ -0.178 \end{bmatrix}} \begin{bmatrix} 0.511 \\ -0.178 \end{bmatrix} = \begin{bmatrix} -2.431 \\ 0.846 \end{bmatrix} \tag{m}$$

$$\beta^{(2)} = \sqrt{\mathbf{u}^{(2)T}\mathbf{u}^{(2)}} = 2.575 \tag{n}$$

$$\varepsilon = \left| \frac{2.710 - 2.575}{2.710} \right| = 0.05 \geq 0.001 \tag{o}$$

The convergence criterion is not satisfied.

Iteration 3 ($k = 2$)

$$\begin{aligned} x_1^{(2)} &= \mu_1 + \sigma_1 u_1^{(2)} = (-0.8) + (0.1)(-2.431) = -1.043 \\ x_2^{(2)} &= \mu_2 + \sigma_2 u_2^{(2)} = 0.8 + (0.1)(0.846) = 0.885 \end{aligned} \tag{p}$$

$$G_1(\mathbf{x}^{(2)}) = -0.0001754 \tag{q}$$

$$G_{\mathbf{U}}^{(2)} = \begin{bmatrix} (-2x_1^{(2)} + 3)\sigma_2 \\ (-2x_2^{(2)})\sigma_2 \end{bmatrix} = \begin{bmatrix} 0.509 \\ -0.177 \end{bmatrix} \tag{r}$$

$$\mathbf{u}^{(3)} = \frac{G_{\mathbf{U}}^{(2)T}\mathbf{u}^{(2)} - G(\mathbf{u}^{(2)})}{G_{\mathbf{U}}^{(2)T}G_{\mathbf{U}}^{(2)}} G_{\mathbf{U}}^{(2)} = \frac{\begin{bmatrix} 0.509 & -0.177 \end{bmatrix}\begin{bmatrix} -2.431 \\ 0.846 \end{bmatrix} + 0.0001754}{\begin{bmatrix} 0.509 & -0.177 \end{bmatrix}\begin{bmatrix} 0.509 \\ -0.177 \end{bmatrix}} \begin{bmatrix} 0.509 \\ -0.177 \end{bmatrix} = \begin{bmatrix} -2.431 \\ 0.846 \end{bmatrix} \tag{s}$$

$$\beta^{(3)} = \sqrt{\mathbf{u}^{(3)T}\mathbf{u}^{(3)}} = 2.574 \tag{t}$$

$$\varepsilon = \left| \frac{2.575 - 2.574}{2.575} \right| = 0.000128 \leq 0.001 \tag{u}$$

The convergence criterion is satisfied.

$$\begin{aligned} x_1^{(3)} &= \mu_1 + \sigma_1 u_1^{(3)} = (-0.8) + (0.1)(-2.431) = -1.043 \\ x_2^{(3)} &= \mu_2 + \sigma_2 u_2^{(3)} = 0.8 + (0.1)(0.846) = 0.885 \end{aligned} \tag{v}$$

Thus the location of MPFP is (−1.043, 0.885). The reliability index is $\beta = 2.574$, and the reliability is $\Phi(\beta) = 0.9947$ from Eq. (20.90). The value of $\Phi(\beta)$ can be read from the table for the normal distribution.

20.5.3 Formulation of Reliability-Based Design Optimization

We now present the formulation for reliability-based design optimization (RBDO). The design variable vector is $\mathbf{X} = (X_1, X_2, \ldots, X_n)$, and the cost function to be minimized is $F(\mathbf{X})$. The reliability constraints are defined as

$$P_f = P\left[G_j(\mathbf{X}) \le 0\right] = \int_{G_j(X) \le 0} f_G(g_j) dg_j = \Phi\left(-\frac{\mu_{G_j}}{\sigma_{G_j}}\right) = \Phi(-\beta) \le P_{j,0}, \quad j = 1, \ldots, m \tag{20.141}$$

Each constraint in Eq. (20.141) is an inner optimization problem in itself. These are the inner optimization problems in the outer reliability-based design optimization problem. Therefore, we need the sensitivity information of the inner problem. This sensitivity is called *optimum sensitivity* and is explained in detail in Choi et al. (2007) and Park (2007).

EXAMPLE 20.11 RELIABILITY-BASED DESIGN OPTIMIZATION

Perform RBDO with Example 20.7. The initial design is $(-0.8, 0.8)$ and $\sigma_1 = \sigma_2 = 0.1$. Compare the solution with that of the one obtained in Example 20.7.

Solution

The optimization formulation for reliability-based design optimization is

Design variables: $\mathbf{X}$ (a)

Cost function: Minimize $F(\mathbf{X})$ (b)

Constraints: $\beta_i(\mathbf{X}) \ge \beta_{i,target} \quad (i = 1, 2)$ (c)

The target reliability index is set as $\beta_{target} = 3.0$ (it is specified by the user). In solving this optimization problem, the inner optimization problems are solved using the procedure described in Example 20.10. The outer problem has inner optimization problems as constraints. Therefore, their derivatives are needed in the outer problem. These are calculated by the sensitivity analysis of the inner problems at their optimum points (Choi et al., 2007; Park, 2007).

The optimization results are shown in Table 20.26. The results of the deterministic optimization approach are also shown. Since RBDO has some reliability margins on the constraints, the optimum cost of RBDO is slightly higher than the cost of the deterministic approach.

TABLE 20.26 Optimum solutions for Example 20.11

		Initial point	Deterministic optimum	RBDO solution
Design variables	(x_1, x_2)	$(-0.8, 0.8)$	$(-0.304, 2.0)$	$(-0.148, 2.0)$
Objective function	f	-2.191	-8.876	-8.660
Constraints	g_1	-1.320	-0.003	-0.534
	g_2	-4.766	-4.161	-4.191