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Design Under Uncertainty

ME 615 Spring 2020

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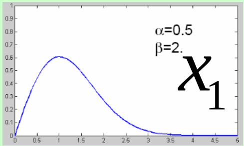
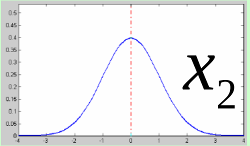
MIME

Different Aspects of Design under Uncertainty

Input uncertainty representation





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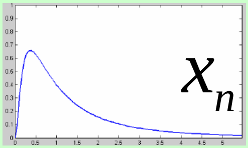

Analysis model

$y = g(x_1, x_2, \cdots, x_n)$ or
$y = f(x_1, x_2, \cdots, x_n)$

Uncertainty propagation and quantification

Moments

(mean, variance,…)

Reliability

Distribution

Design scenario

Reliability assessment

Utility optimization

Challenges for Design under Uncertainty

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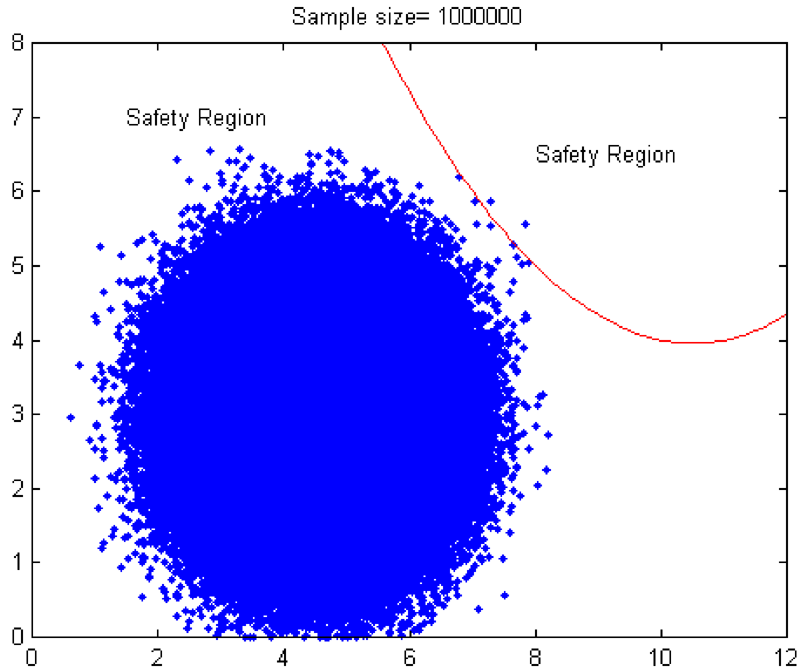
- *Uncertainty Quantification*
- *Efficient Uncertainty Propagation (Utility & Reliability Assessments)*
- *Framework of Decision Making with Multiple Quality Attributes*



For 95% confidence, 10% error bound, $3.8\text{x}10^4$ Monte Carlo samples are needed to catch 0.01 failure rate.

What is uncertainty?

- "There are known knowns. These are things we know that we know. There are known unknowns. That is to say, there are things that we know we don't know. But there are also unknown unknowns. There are things we don't know we don't know."
  – **Donald Rumsfeld 2002**

Ways to Classify Uncertainties:

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- **Classification 1**
  - Model (lack of knowledge)
  - Parametric (lack of knowledge, variability): variability in load, material property, dimensions
  - Numerical
  - Testing data (measurement)
- **Classification 2**
  - Aleatory (random, objective) uncertainty: inherent, irreducible uncertainty. Can also be classified as **risk**
  - Epistemic (subjective) uncertainty: reducible with better knowledge
  - Ontological uncertainty: the possibility of events occurring that we have no knowledge of

– **Classification 3**
  - Noise factors: Come from the environment, processes out of our control (i.e. parameters)
  - Control factors: Come from the factors we control (i.e. design variables)

Classifying Uncertainties

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- "There are known knowns. These are things we know that we know. There are known unknowns. That is to say, there are things that we know we don't know. But there are also unknown unknowns. There are things we don't know we don't know."
  - **Donald Rumsfeld 2002**

  - Aleatory uncertainty: *known knowns*
  - Epistemic uncertainty: *known unknowns*
  - Ontological uncertainty: *unknown unknowns*

Our classification

- Aleatory (random, objective) uncertainty: inherent, irreducible uncertainty. Can also be classified as **risk**
  - Manufacturing tolerances
  - Material property variations
  - Assembly time variations
- Epistemic (subjective) uncertainty: reducible with better knowledge
  - Budget
  - Technology
  - Mission
- Ontological uncertainty: the possibility of events occurring that we have no knowledge of
  - Explosions
  - Fires

Aleatory Uncertainty

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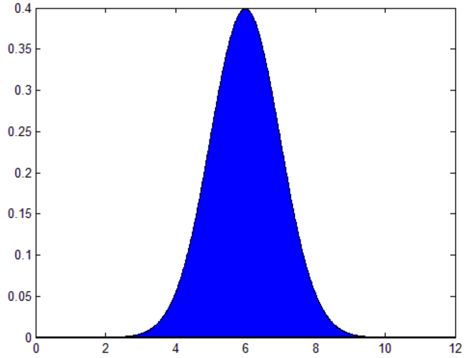
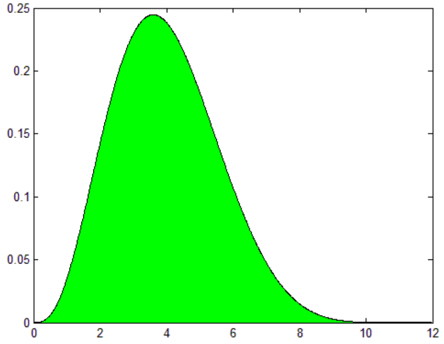
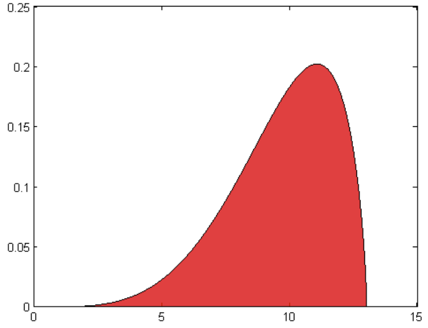
- Aleatory uncertainties
  - Probability Distribution Functions



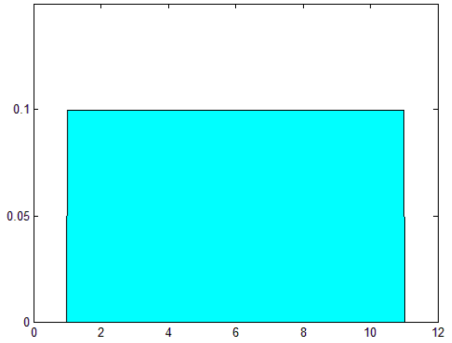
Normal (Gaussian)



Log Normal



Beta



Uniform

Distributions

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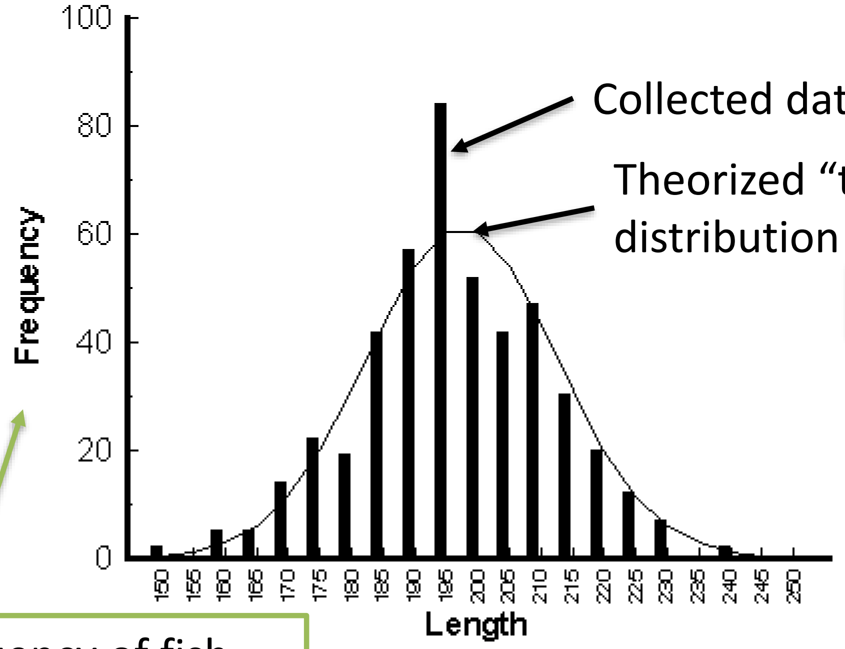
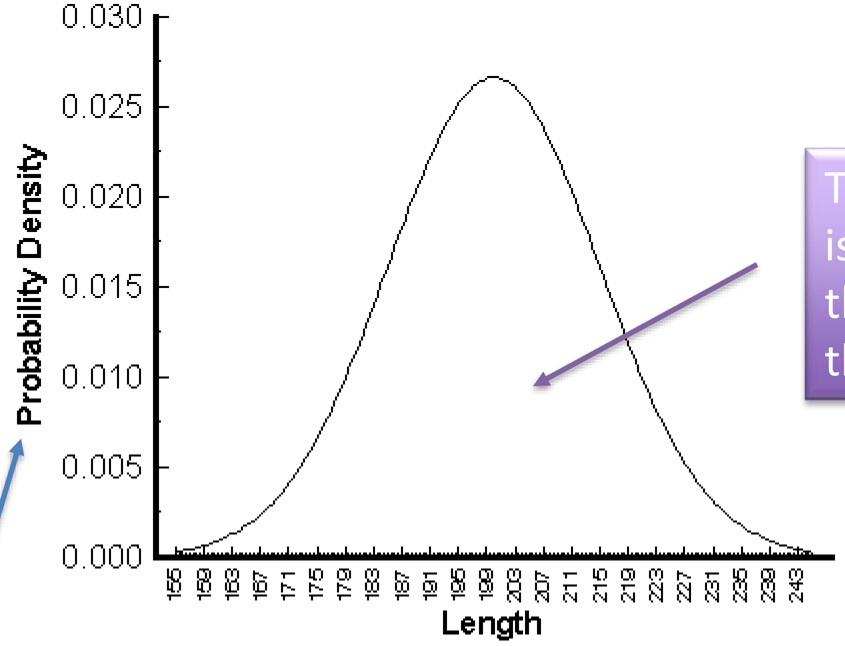
Imagine we created a histogram of lengths of fish caught in a river (in mm) over 1 month



Frequency of fish size caught

We can turn that into a probability distribution function (PDF)



The y axis scale is set to ensure the "area under the curve" is 1

Frequency converted to a probability density

Distributions

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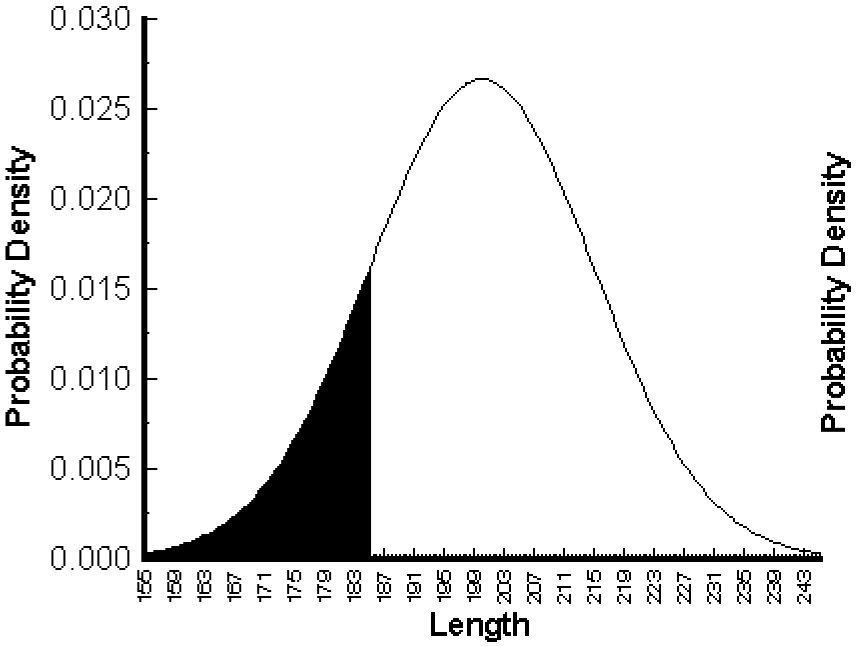
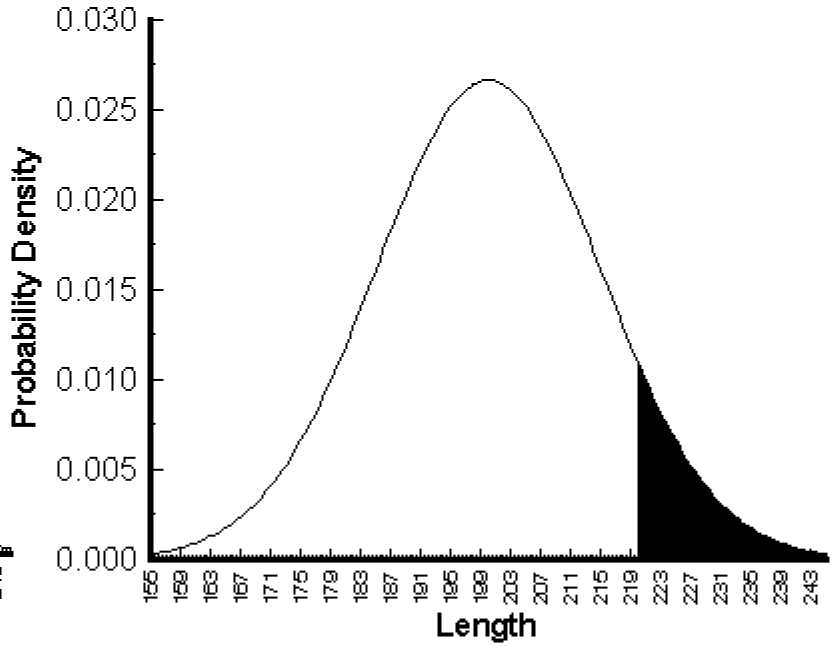
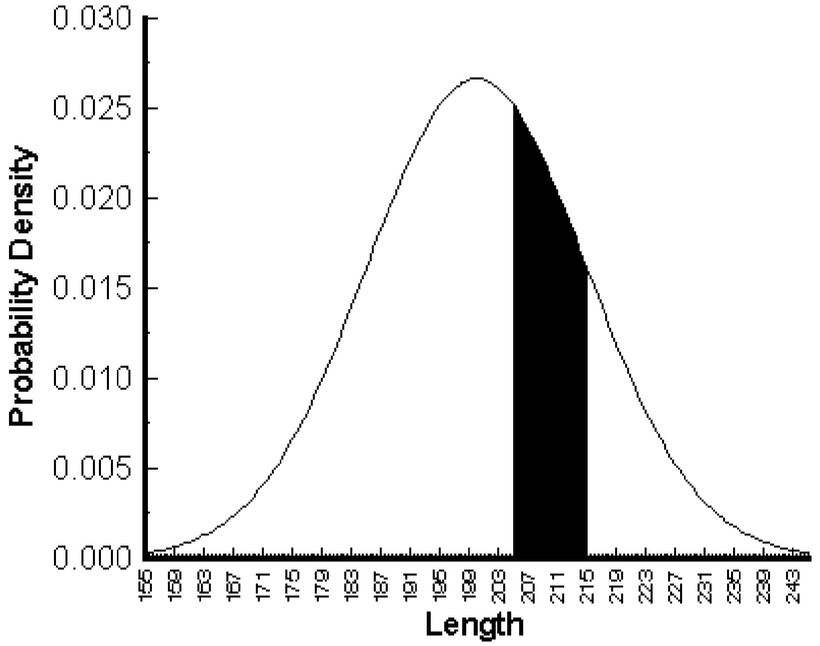
We can now just use integration compute the probability of different events:



What is the probability of catching a fish less than 185 mm?
- It is 0.16



What is the probability of catching a fish greater than 220 mm?
- It is 0.095



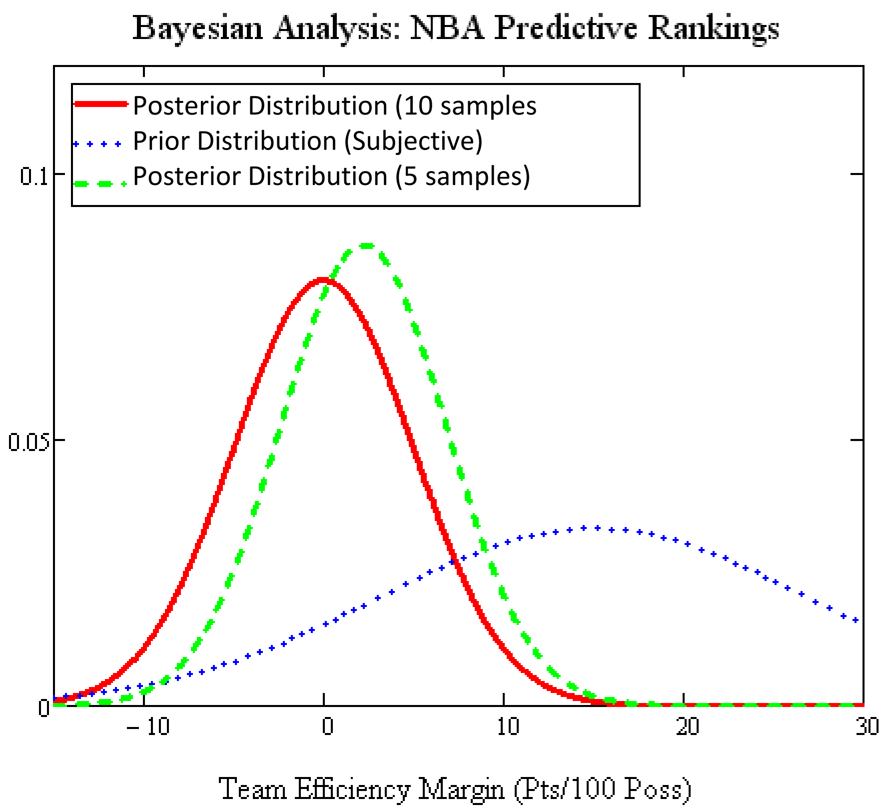
What is the probability of catching a fish greater than 205 and less than 215?
- It is 0.075

Epistemic Uncertainty

- Epistemic Uncertainties
  - Distributions with ranges of Parameters, i.e. a range of mean and variances
  - Ranges only, no distributional assumption
  - Subjective prior distribution updated as information becomes available (Bayesian approach)

Epistemic Uncertainty

- Subjective prior distribution updated as information becomes available (Bayesian approach)
- **We can use probability theory**



Ontological Uncertainty

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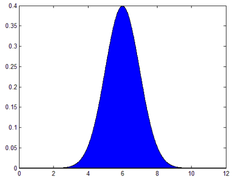
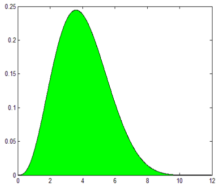
- “Unknown Unknowns”
  - Not much we can do with these since they are unknown.

Representation of Uncertainty

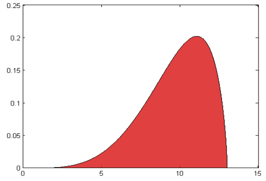
- Use of Probability Theory
  - Use of Probability Density Function (***pdf***) for continuous random variables
  - Use of Probability Mass Function (***pmf***) for discrete random variables
- We will utilize **pdfs** for the course since it will enable use of simplifying mathematics.
- Examples of common distributions
  - Normal distribution



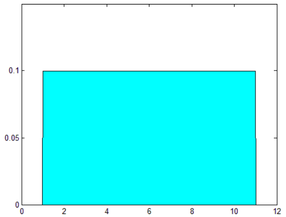
  - Lognormal distribution



  - Beta distribution



  - Uniform distribution



Representation of Uncertainty

- Normal distribution
  - Material property (Young's modulus,…), tolerance, ...
  - *Sum* of random effects, measurement errors
  - Infinite interval [-∞, ∞]
- Lognormal distribution
  - Loading, failure by fatigue…
  - *Product* of random effects, uncertainty expressed as a factor n
    - Example: $P\left(\frac{y_0}{n} < y < y_0 n\right) = 90\%$
  - Semi-Infinite interval [0, ∞]
- Beta distribution
  - Useful for subjective estimates of uncertainty such as costs, schedules.
  - Used to fit a parametric distribution to collected data
  - Bounded Interval [a, b]
- Uniform distribution
  - Useful for subjective estimates of uncertainty such as costs, schedules.
  - Bounded Interval [a, b]

Representation of Uncertainty

- Extreme value distributions
  - Represent maximum or minimum of a number of samples of various distributions
  - Depends on the tail shape of original distribution
  - Gumbel distribution (Type I extreme value distribution for largest value)
    - From distributions with exponential tail (e.g. normal)
    - Wind load, flood levels, …
    - Infinite interval [-∞, ∞]
  - Frechet distribution (Type II extreme value distribution for largest value)
    - From distributions with polynomial tail (e.g. lognormal)
    - Earthquake load, …
    - Semi-Infinite interval [0, ∞]
  - Weibull distribution (Type III extreme value distribution for smallest value)
    - From distributions with bounded ends (e.g. gamma)
    - Breaking strength of material, failure of series of identical components,…
    - Semi-Infinite interval [0, ∞]

Probability Space

- What is a probability space? It is a triple $(\Omega, \mathcal{F}, P)$, with the components defined as:
  - A sample space, Ω, which is the set of all possible outcomes.
  - A set of events, $\mathcal{F}$ where each event is a set containing zero or more outcomes.
  - The assignment of probabilities to the events; that is, a function P from events to probabilities.
- Let's look at an example

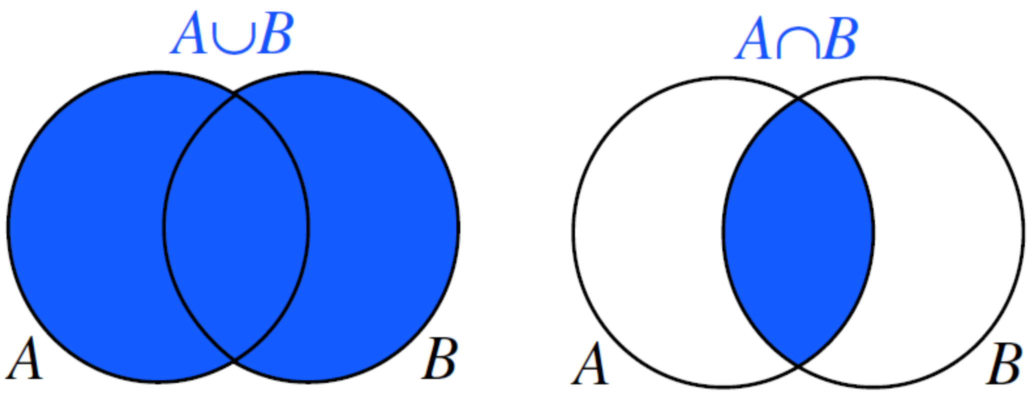
A coin flip

- We are tossing a "fair coin", the sample space is:
  - $\Omega = \{H, T\}$
- The set of events is $\mathcal{F} = 2^{\Omega}$

Or $\mathcal{F} = \{\{\}, \{H\}, \{T\}, \{H, T\}\}$

  - {H} (heads)
  - {T} (tails)
  - {} (neither heads nor tails)
  - {H, T} (either heads or tails)
- The probability measure is using the probability function P:
  - $P(\{\}) = 0$
  - $P(\{T\}) = 0.5$
  - $P(\{H\}) = 0.5$
  - $P(\{H, T\}) = 1$

Basics of Probability



- Notation:
  - **U Union**: indicates the probability of either event *A* or event *B* occurring (OR)
  - **∩ Intersection**: indicates the probability that events *A* and *B* both occur (AND)
- With this notation we can define some axioms…

Axioms of Probability Theory

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- $P(A) \geq 0$ for all events A
- $P(\Omega) = 1$
- $P(A \cap B) = P(A)\,P(B)$ if A and B are independent
- $P(A \cap B) = P(A|B)\,P(B)$ if A and B are dependent
- $P(A \cup B) = P(A) + P(B)$ for disjoint events A and B
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (not disjoint)
- $P(A \cup B) \leq P(A) + P(B)$

Basics of Probability

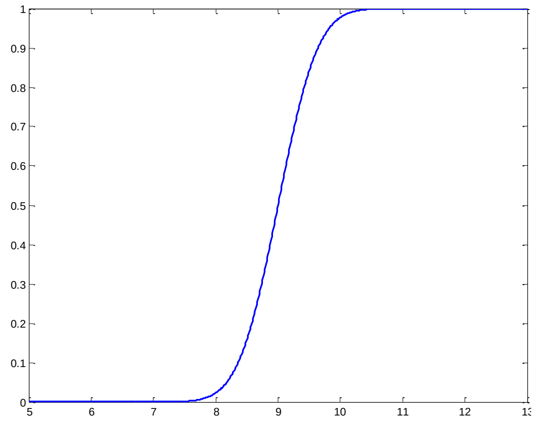
- Random variable
  - A variable whose value is subject to variations due to chance, defined on a sample space $\boldsymbol{\Omega}$ (continuous, discrete)
  - A random variable $X: \Omega \to E$ is a measurable function from a set of possible outcomes Ω to a measurable space E, typically $\mathbb{R}$

$$\boldsymbol{X}: \boldsymbol{\Omega} \to \mathbb{R}$$

  - A random variable does *not* return a probability
- Why is it a Function?
  - ω is the random event, e.g. landing on a head or a tail.
  - $X(\omega)$ is the function that defines a numerical quantity to the random event, e.g., the number of heads in a series of coin flips.
- Types of Random Variables:
  - Discrete Values: takes on value from countable set.
  - Continuous Values: infinite number of possible values.

Basics of Probability

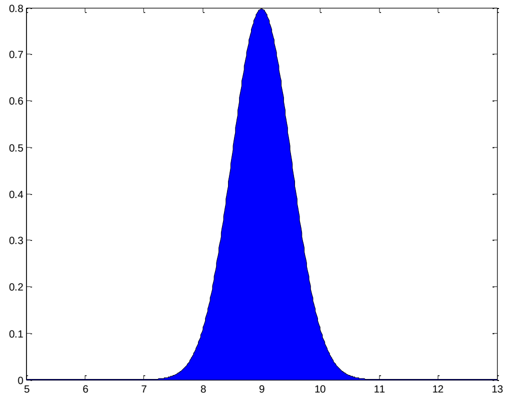
- Realization of X.
  - The value x = X(ω) of a random variable X for outcome w ∊ R is called a realization of X.



- Cumulative distribution function (CDF)

$$F_X(x) = \Pr[X \leq x]$$

$$P\{\, \omega \in \Omega : X(\omega) \leq x \,\}$$



- Probability density function (PDF)

$$\Pr[x \leq X \leq x + dx] = f_X(x)\,dx$$

$$f_X(x) = dF_X(x)/dx, \quad \int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

Basics of Probability

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- Descriptive measures of distribution
  - Expected value

$$E(x) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

Expectation operator is a linear operator!

  - Statistical moments

Raw moments: $\mu_k^{'} = E\left(x^k\right) = \int_{-\infty}^{\infty} x^k f_X(x)\,dx$

Central moments: $\mu_k = E\left[\left(x - \mu_1^{'}\right)^k\right] = \int_{-\infty}^{\infty} \left(x - \mu_1^{'}\right)^k f_X(x)\,dx$

Basics of Probability

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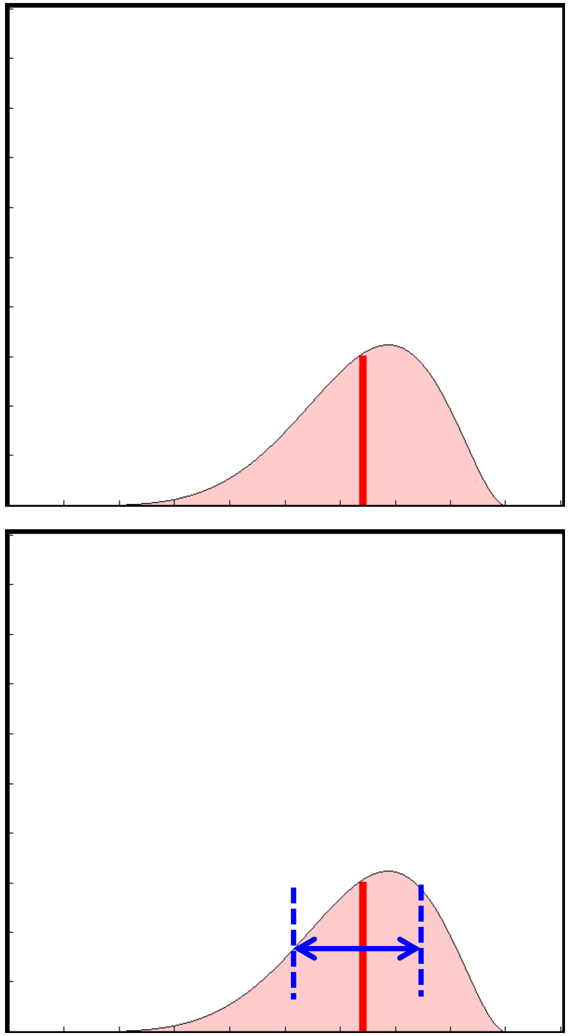
Moments of Interest to us:

- 1st Moment: Mean (expected value)
  - Measure of central location

$$E(x) = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

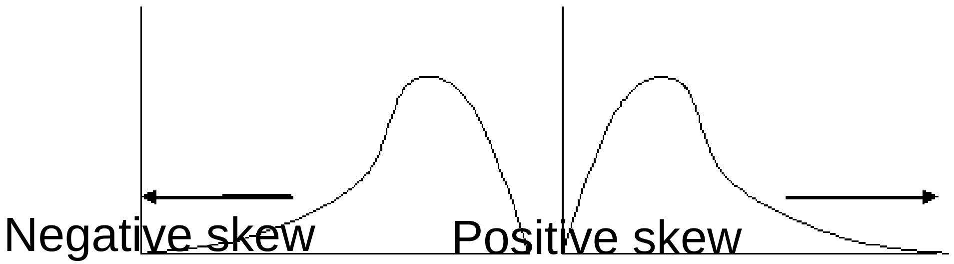
- 2nd Central Moment: Variance
  - Measure of dispersion
    - $Var(x) = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x) dx$



Basics of Probability

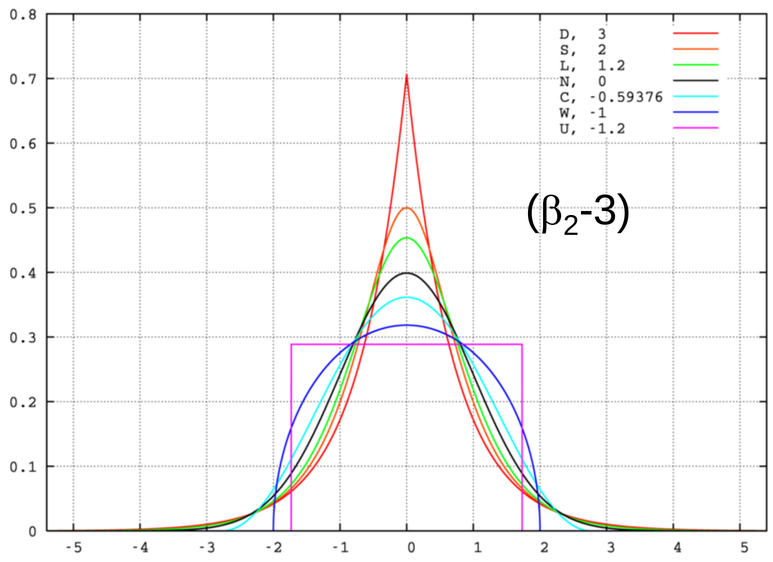
- 3rd Central Moment: Skewness
  - Measure for asymmetry

$$Skw(x) = \int_{-\infty}^{\infty}(x-\mu)^3 f_X(x)dx$$



- 4th Central Moment: Kurtosis
  - Measure for peakedness

$$Kurt(x) = \int_{-\infty}^{\infty}(x-\mu)^4 f_X(x)dx$$



Basics of Probability

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Multiple random variables, i.e. a RANDOM VECTOR: $\vec{X} = [X_1, \ldots, X_N]$:

- Joint probability density function $f_{XY}(x, y)$

$$P[a < X < b, c < Y < d] = \int_c^d \int_a^b f_{XY}(x, y)\, dx\, dy$$

- Marginal density function of x

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy$$

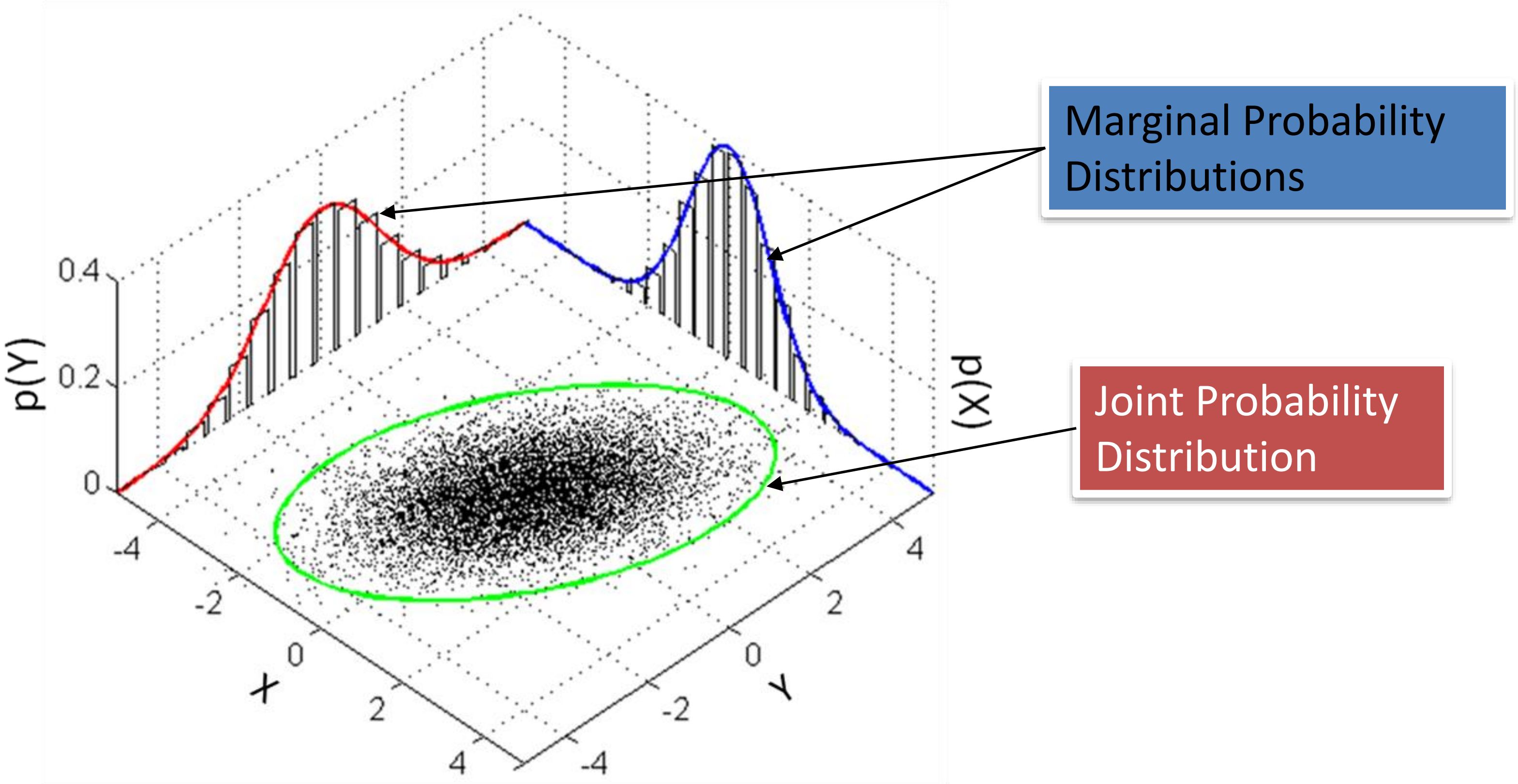
- Conditional probability density of y given x

$$f_{Y|X}(x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

- Independence

$$f_{XY}(x, y) = f_X(x)\, f_Y(y)$$

Two Random Variables (X,Y)



Basics of Probability

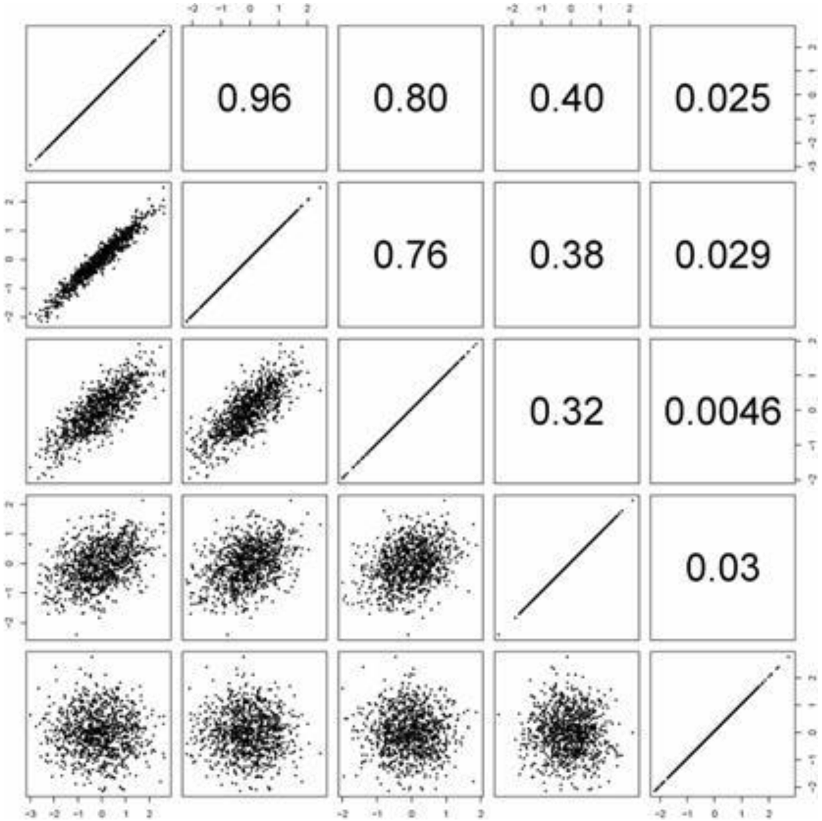
- Covariance

$$COV(X,Y) = E\left[(X-\mu_X)(Y-\mu_Y)\right] = E(XY) - E(X)E(Y)$$

- Correlation coefficient

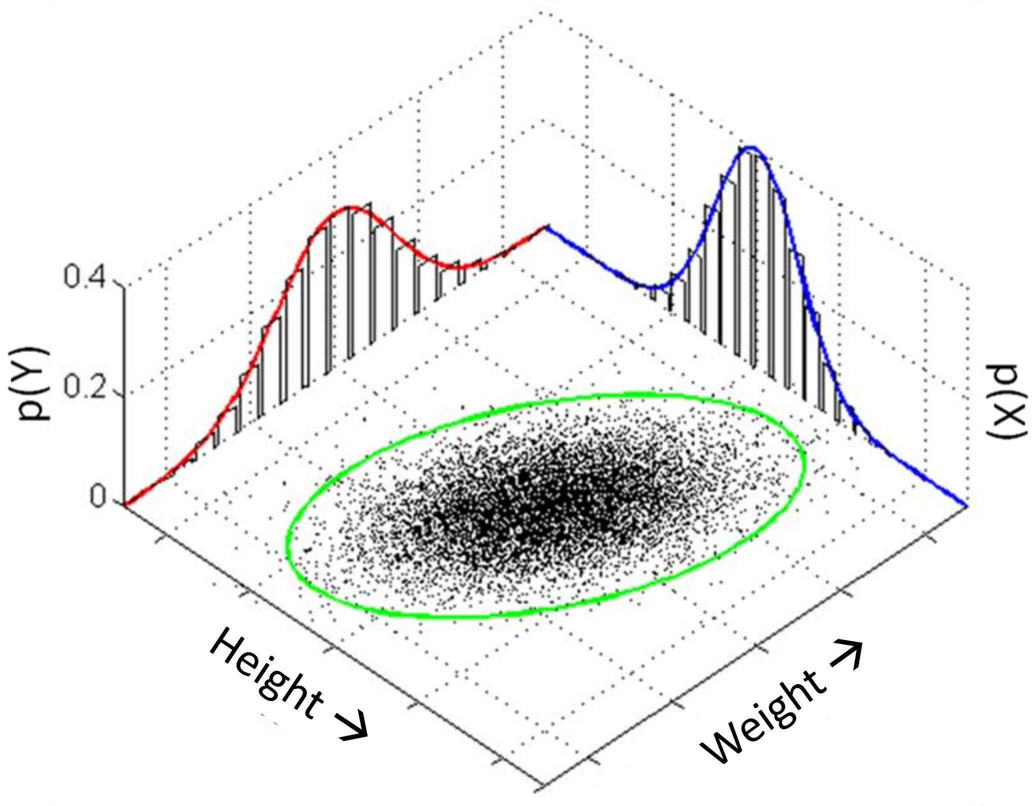
$$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}$$

→ Degree of linear dependence between two random variables

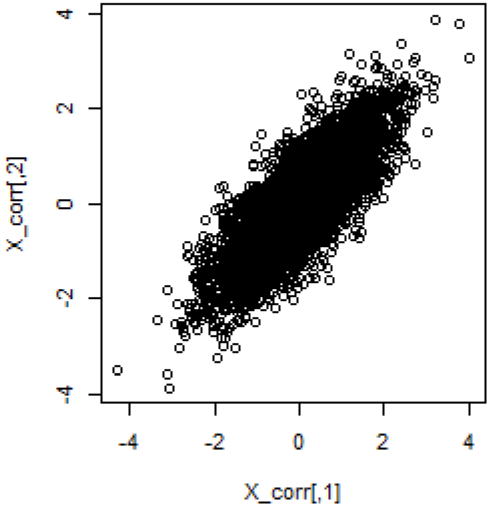


Example of correlation

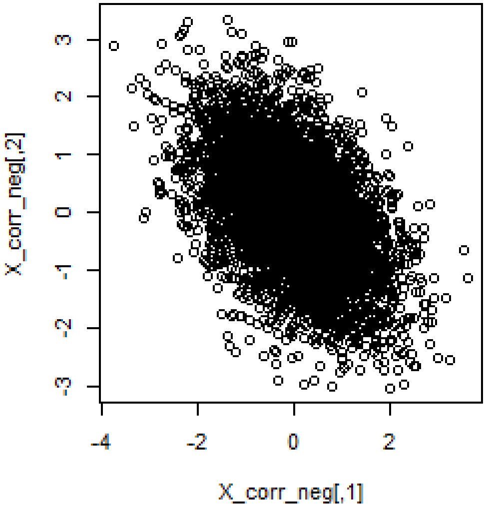
- If we were to measure heights and weights in a certain population, we might see that they are correlated



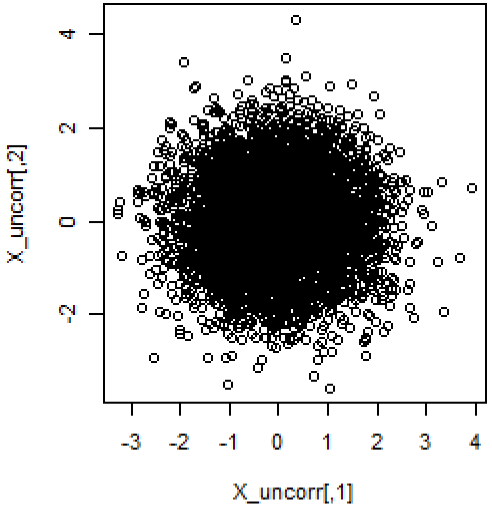
Correlation



Positive Correlation



Negative Correlation



Uncorrelated