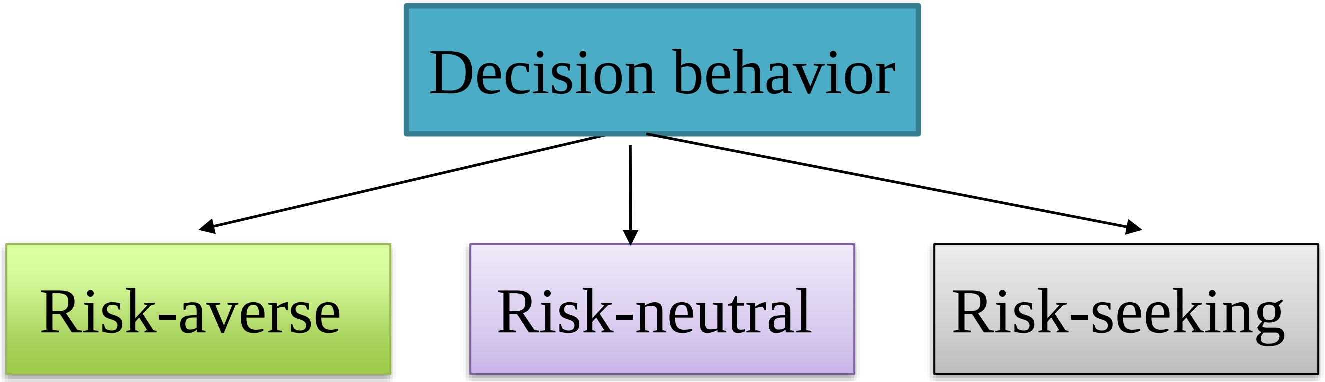
Decision-based Design: Utility Theory

Prof Chris Hoyle

ME 615 Spring 2020

What are we going to measure: Risk-Preference Attitude

- Not all people select the same action when faced with the same decision situation.
  - One important reason for making different choices is that the decision makers have different attitudes about taking a risk.
  - Their risk preferences influence their view on the potential outcomes.



Risk-preference Attitude (contd.)

Oregon State University
College of Engineering

- Decision makers who base their **decision solely on the highest expected payoff** or the lowest expected cost, i.e. on expected-value, are **risk-neutral**.

- Decision makers who are willing to take on additional **risk for higher payoffs** are **risk-seeking (risk prone)**.

- Decision makers who are **not willing to take on additional risk** for higher payoffs are **risk-averse**.

Lottery Method for Utility Function Formulation

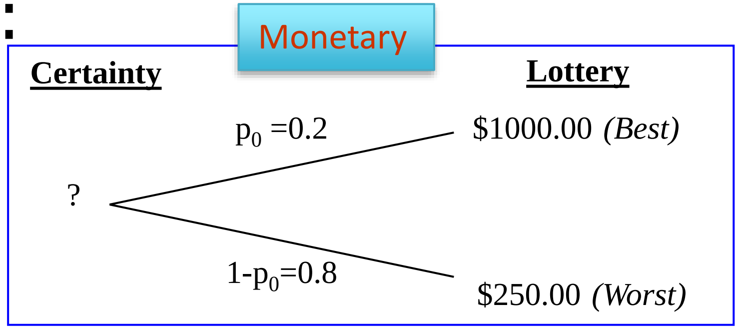
- Using properties 1-3, we can now create a methodology for estimating utility functions.
- We will use the "lottery" method for measuring risk preference attitude.

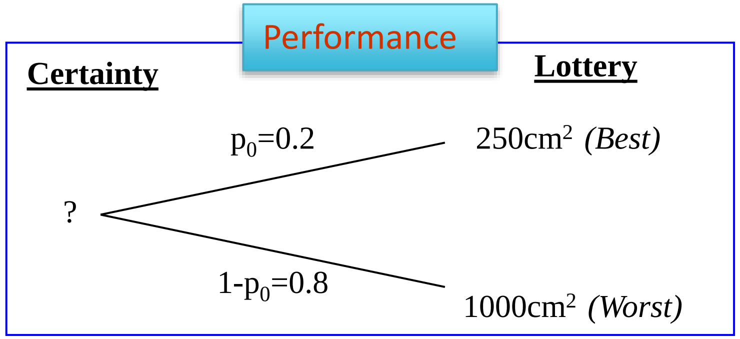
1. Ask a series of questions in which the user expresses his/her "indifference" point for a lottery vs. a certain sum of money. *The indifference point is a measure of utility (property 1)*
2. From the answers, we can create a plot of utility vs. value. We can choose the scale of the utility (*property 3*)
3. With this plot, we can compute the utility for lotteries not assessed in the original interview (*property 2*)

Lottery Method for Utility Function

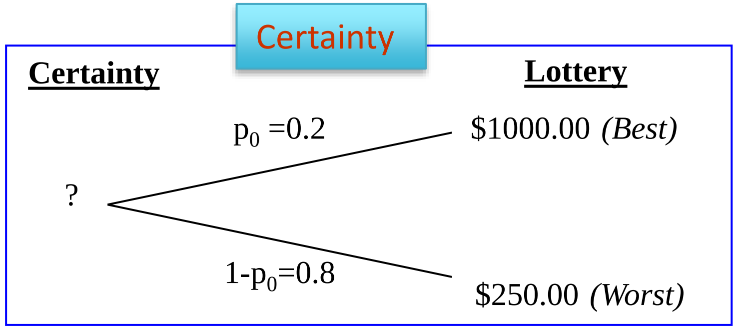
Oregon State University
College of Engineering

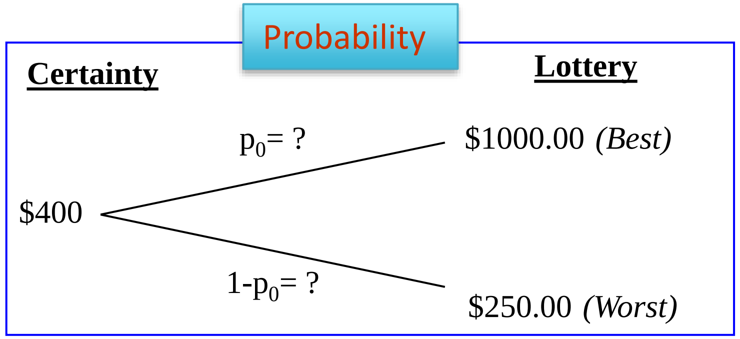
- Can conduct the lottery for both monetary and performance measures:





- Can ask the lottery questions both in terms of certain equivalent or probability



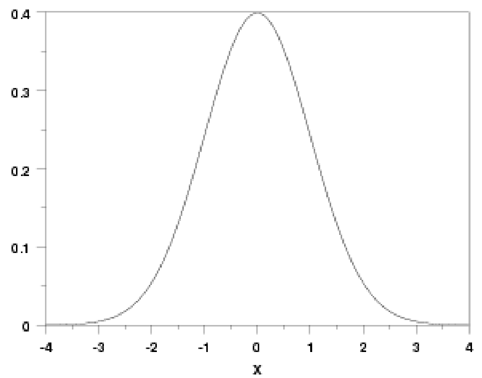


Krishnamurty (2007)

Lottery vs. Uncertainty

Oregon State University
College of Engineering

- Note that in developing the utility curve, we utilize **discrete probabilities:**
  - Outcome A with probability p
  - Outcome C with probability (1-p)
- In our discussion of uncertainty, we focused upon **probability distribution functions**:



- We will develop utility theory with discrete probabilities and then apply to continuous pdfs.

Steps in Creating a Utility Curve

Oregon State University
College of Engineering

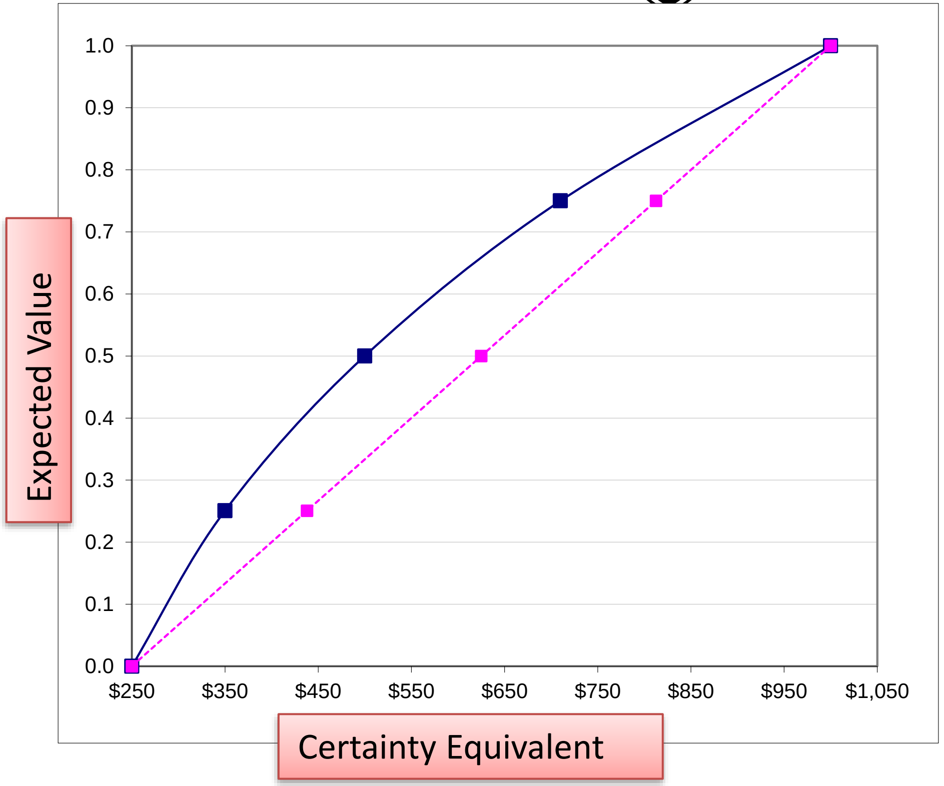
1. Define the "Best" and "Worst" outcomes.
2. Determine approximately even spaced lotteries.
3. Compute the expected **values** of the lotteries:
   - $\boldsymbol{E[x] = \sum_{i=1}^{n} x_i p_i}$
4. Normalize the E[x] to create $E[x]_N$
   - $\frac{E[x] - worst}{(best - worst)}$
5. Ask the respondent for the "Certainty Equivalent" (CE) of the lottery.
6. Plot Normalized Expected Value $E[x]_N$ vs. the Certainty Equivalent CE.
7. Fit a parametric curve to the experimental curve.

Lottery Method for Utility Function

Oregon State University
College of Engineering

1. Define the "Best" and "Worst" outcomes.
   - Best = $1000
   - Worst = $250
2. Determine approximately even spaced lotteries
3. Compute the expected values of the lotteries: $E[x] = \sum_{i=1}^{n} x_i p_i$
4. Normalize the E[x] to create $E[x]_{N}$: $\frac{E[x]-worst}{(best-worst)}$
5. Ask the respondent for the "Certainty Equiv." (CV) of the lottery
6. Plot Normalized Expected Value $E[x]_N$ vs. the Certainty Equiv. CE.

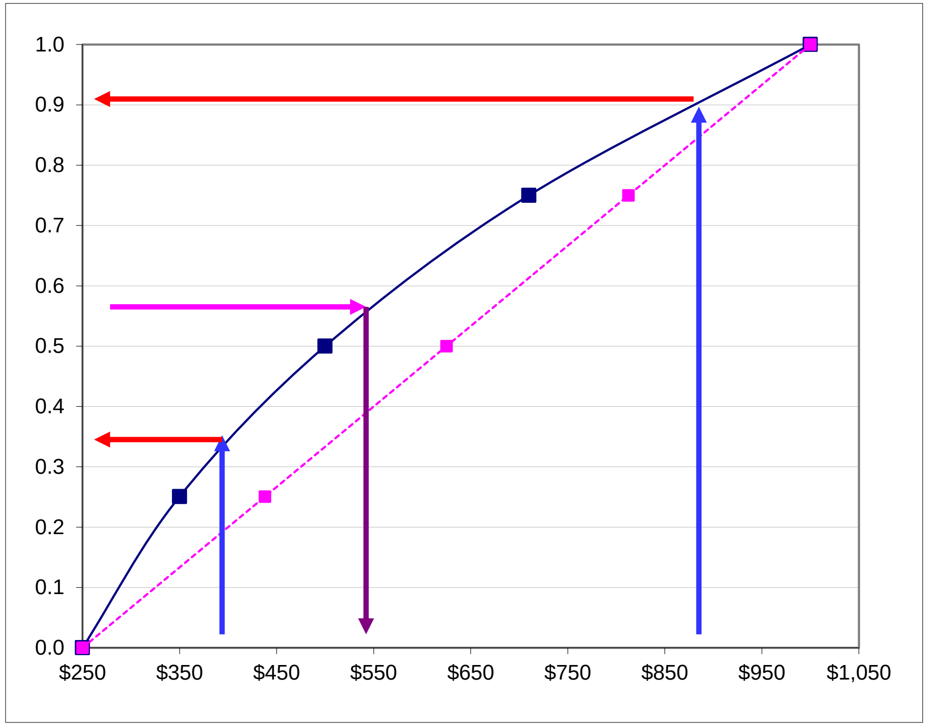
Krishnamurti (2007)



Best	Worst	E(x)	Normalized	Certainty
0	(1)*250	250	0.00	250
(.5) *625	(.5)*250	438	0.25	350
(.5)*1000	(.5)*250	625	0.50	500
(.5)*1000	(.5)*625	813	0.75	710
(1)*1000	0	1000	1.00	1000

Finding the utility and certainty equivalent

- With our utility function determined, we can find both the Utility and Certainty Equivalent for lotteries not considered previously:
  - For example, what is the utility for a lottery as follows
    0.4 probability of gaining \$900 vs. 0.6 prob of gaining \$400



$$u(s) = p_0 \cdot u(x_H) + (1 - p_0) \cdot u(x_L)$$

Example:

$$u(s) = 0.4 \cdot u(\$900) + 0.6 \cdot u(\$400)$$

$$u(s) = 0.4 \cdot (0.91) + 0.6 \cdot (0.35) = 0.57$$

Certainty Equivalent:

$$u^{-1}(0.57) \approx \$540$$

$$E(Lottery) = \$600$$

Krishnamurti (2007)

Utility for a state lottery

Oregon State University
College of Engineering

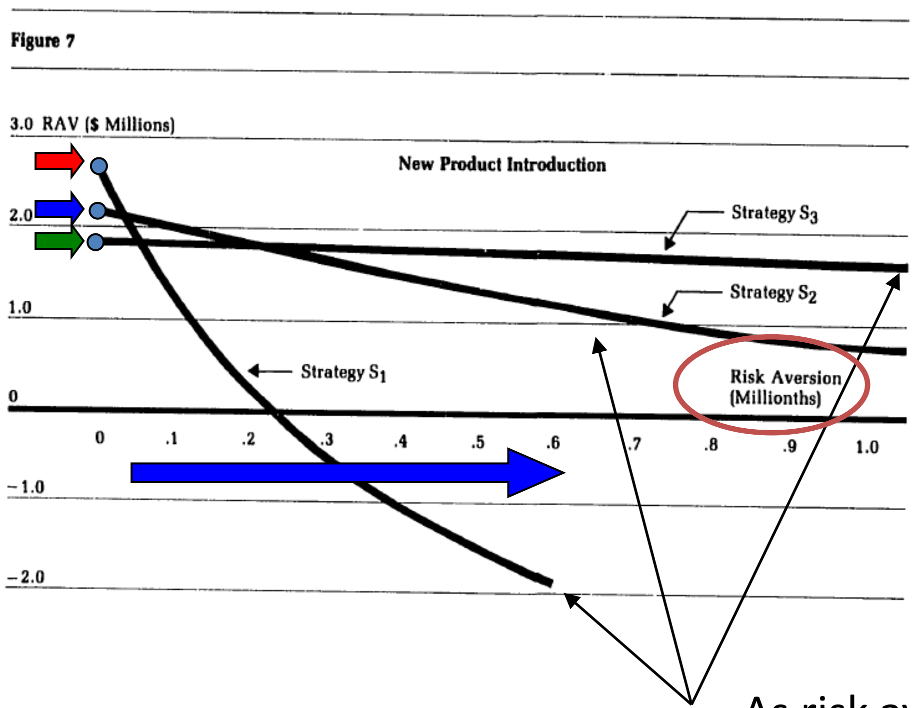
- *Washington State Lotto:*
  - *Odds of winning:* 1 in 6,991,908
  - *Jackpot:* $1.6 MM
  - *Expected Return:* [Jackpot * Prob (winning Jackpot)] + 0 = $0.22
  - *Cost of ticket:* $1 for 2 plays or $0.50/play

Best	Worst	E(x)	Certainty
(1.43e-7)*1.6 MM	0	$0.22	**$0.50**

  - Why does anyone play the lottery?

Effect of Risk Attitude on Selection

Oregon State University
College of Engineering



Strategy S_1: High Uncertainty

Strategy S_2: Medium Uncertainty

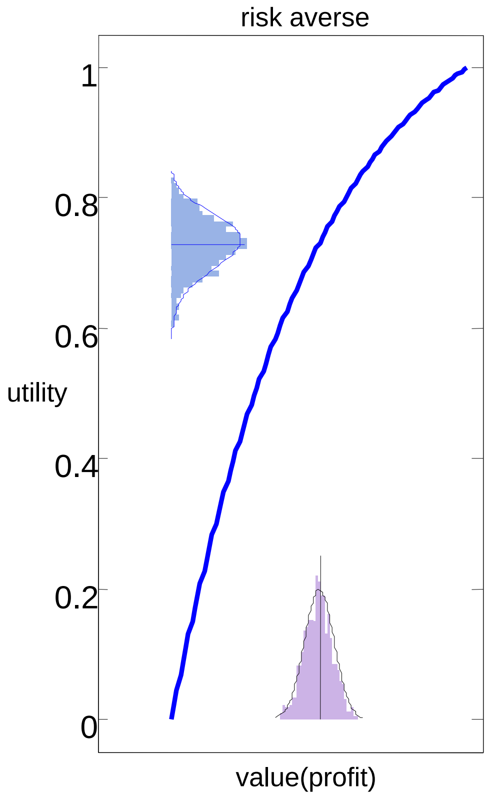
Strategy S_3: Low Uncertainty

As risk aversion (seeking) increases, the Risk Adjusted Value of the attribute approaches the “worst” (“best”) outcome.

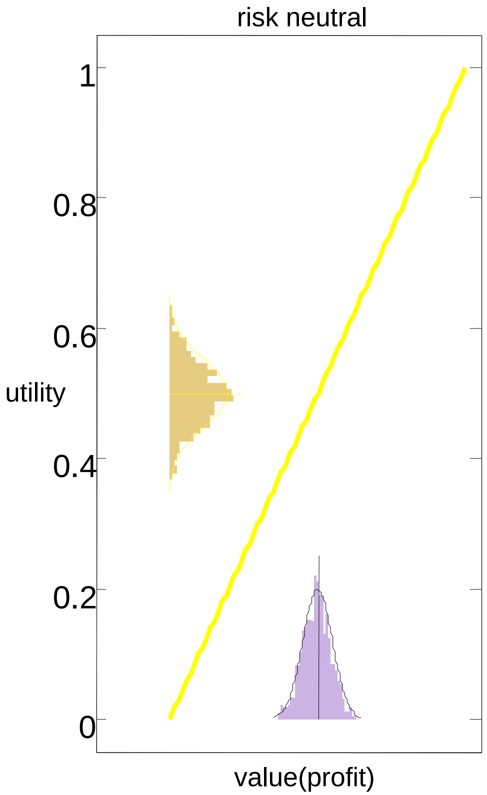
Cozzolino (1979)

Risk Averse, Risk Neutral & Risk Seeking

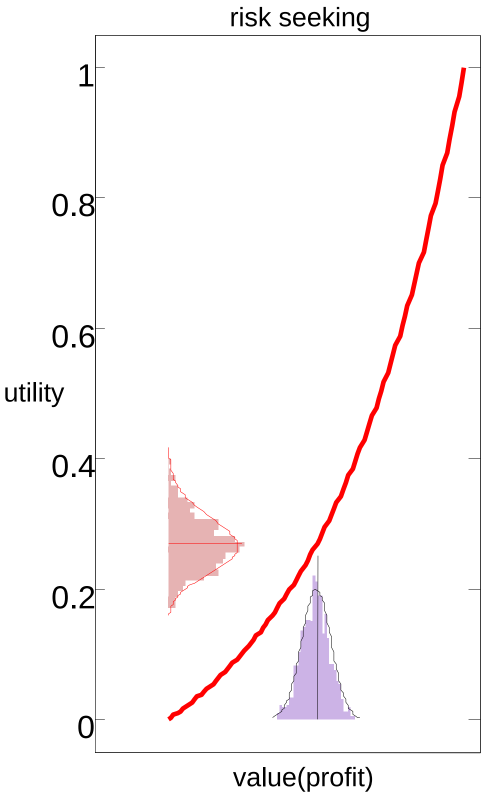
Oregon State University
College of Engineering



Risk Averse



Risk Neutral



Risk Seeking

Some Issues to consider

- The “best” and “worst” outcome:
  - Use of reasonable or current best and worst outcomes
  - Use of constrained best and worst outcomes
  - *It seems that the use of the constrained best/worst outcomes is the most reasonable approach*.
- Constraints:
  - Could create a utility function for constraints, but the utility curve will have a strange shape and the lottery will be hard to conduct.
  - Preserve constraints but cast them as reliability constraints

Alternatives to lottery evaluations for Risk Attitude

Oregon State University College of Engineering

- Determine risk aversion coefficient (e.g. c) from previous decisions (Cozzolino, 1979).
  - Back out the risk aversion coefficient from previous decisions made in similar situations, or
  - "Calibrate" the utility curve to match previous decisions.
  - *Assumes the new situation is sufficiently similar to the previous situations.*
- Use psychometric tests (Van Bouysset, 2012):
  - Give decision makers a psychometric test and determine risk attitude by the answers given on the test.
  - *A standard method for converting test results to ρ does not exist.*
- Use rules of thumb (Howard, 1988)
  - Values of ρ which are proportional to ~ 6% of net sales, or
  - Approximately 100% to 150% of net income, or
  - Approximately 1/6 of equity.
  - *There do not appear to be rules of thumb for engineering design risk.*

Fitting the curve: Standard Utility Function Forms

- Standard Utility Function Forms
  - Quadratic: $u(x) = a + bx - cx^2$
  - Logarithmic: $u(x) = x + c \cdot \ln(x + b)$
  - Exponential: $u(x) = a - b \cdot e^{-c \cdot x}$
- Only the Exponential Form (except for linear) has constant risk aversion property (constant local risk aversion) (Pratt, 1964).

$$r(x) = -\frac{u''(x)}{u'(x)} = c$$

- For this reason, the exponential form is a good "default"

Fitting Utility functions using the Exponential Form

- Convert *x* to *u*(*x*) using the exponential form:
  - For preferences monotonically increasing over x with risk aversion ρ:

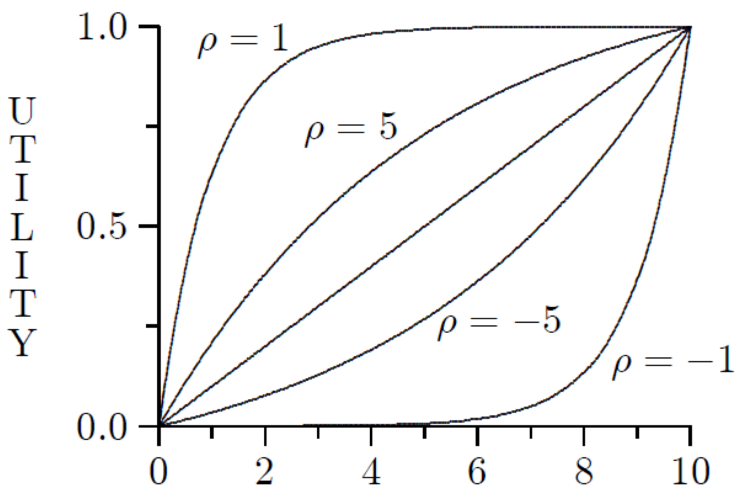
$$u(x) = \begin{cases} \dfrac{\exp\left[-(x - \text{Low})/\rho\right] - 1}{\exp\left[-(\text{High} - \text{Low})/\rho\right] - 1}, & \rho \neq \text{Infinity} \\ \dfrac{x - \text{Low}}{\text{High} - \text{Low}}, & \text{otherwise}, \end{cases}$$

  - For preferences monotonically decreasing over x with risk aversion ρ:

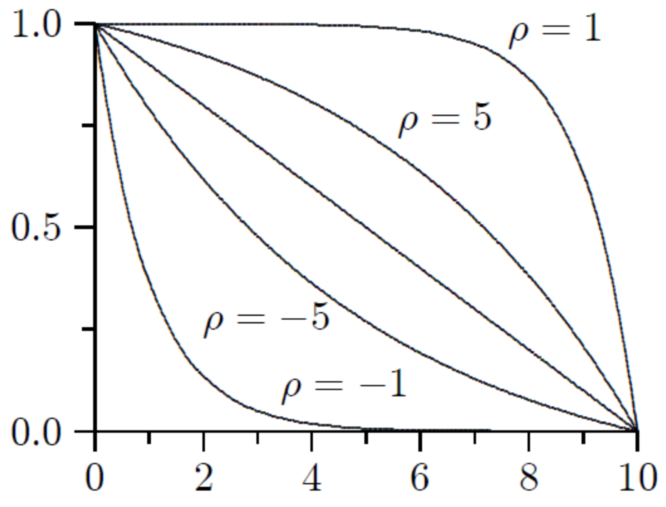
$$u(x) = \begin{cases} \dfrac{\exp\left[-(\text{High} - x)/\rho\right] - 1}{\exp\left[-(\text{High} - \text{Low})/\rho\right] - 1}, & \rho \neq \text{Infinity} \\ \dfrac{\text{High} - x}{\text{High} - \text{Low}}, & \text{otherwise}, \end{cases}$$

Fitting Utility functions using the Exponential Form

- The shape of the utility function will vary based upon the risk aversion coefficient ρ:
  - As ρ approaches infinity, the decision maker is risk neutral
  - As ρ approaches -1, the decision maker becomes more risk seeking
  - As ρ approaches 1, the decision maker becomes more risk averse



a. Increasing Preferences



b. Decreasing Preferences

Fitting Utility functions using the Exponential Form

Oregon State University
College of Engineering

- Taking the expected utility $E[u(x)]$:
  - The expected value of x:
    - $E[x] = \sum_{i=1}^{n} x_i p_i$ *discrete*
    - $E[x] = \int_{-\infty}^{\infty} x \cdot pdf(x)dx$ *continuous*
  - The expected utility of x (by Property 2):
    - $E[u(x)] = \sum_{i=1}^{n} u(x_i) p_i$ *discrete*
    - $E[u(x)] = \int_{-\infty}^{\infty} u(x) \cdot pdf(x)dx$ *continuous*

Fitting Utility functions using the Exponential Form

- Computing the certainty equivalent CE:
  - For preferences monotonically increasing over x:

$$\mathrm{CE} = \begin{cases} -\rho \ln \mathrm{E}[\exp(-x/\rho)], & \rho \neq \text{Infinity} \\ \mathrm{E}(x), & \text{otherwise} \end{cases}$$

  - For preferences monotonically decreasing over x:

$$\mathrm{CE} = \begin{cases} \rho \ln \mathrm{E}[\exp(x/\rho)], & \rho \neq \text{Infinity} \\ \mathrm{E}(x), & \text{otherwise} \end{cases}$$

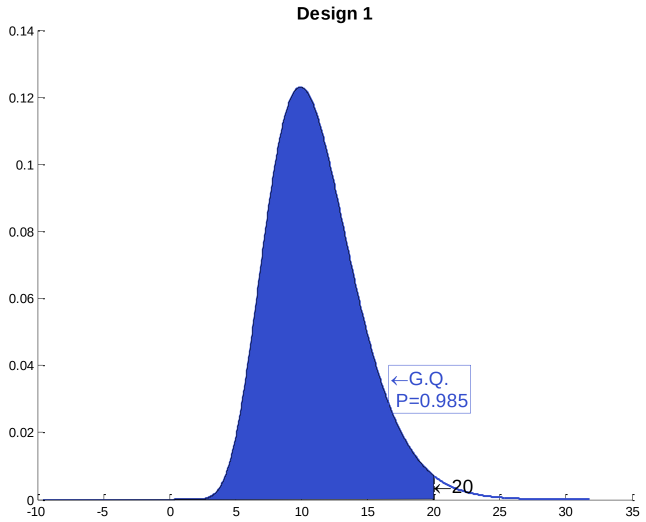
What is the utility function doing?

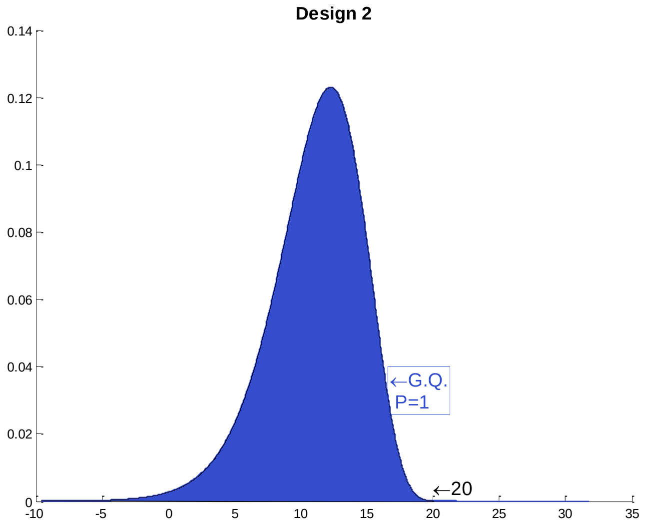
- In effect, the utility function is weighting outcomes
- The expected utility is then the mean of utility, just like the expected value is the mean of the outcomes.
- The certainty equivalent is then a transformation from the utility scale to the original scale.
  - *View the certainty equivalent as the "penalized mean" or "risk adjusted value".*
  - If "more is better", then the penalty for risk will result in a CE less than the mean value if attitude is risk averse.
  - If "less is better", then the penalty for risk will result in a CE more than the mean value if attitude is risk averse.

Examples

Oregon State University
College of Engineering

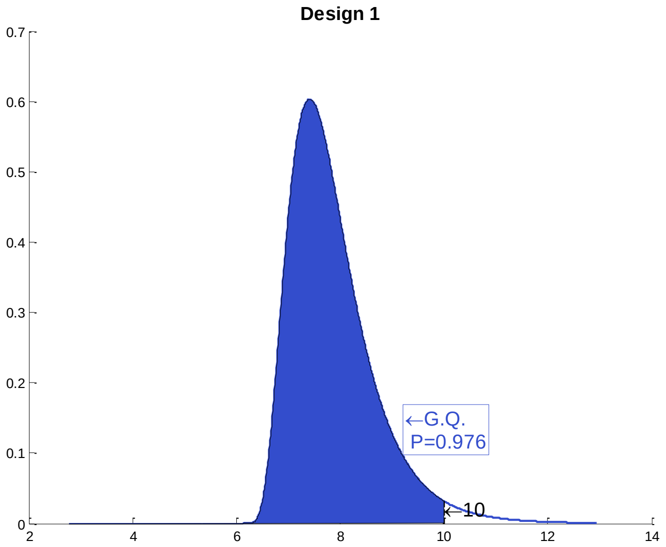
- Motor Design weight: *Less is better*
  - *Expected Value of Weight = 11.0855*
    - D 1 Expected Utility/CE Risk averse attitude (r = 20): **0.9623/11.7316**
    - D 2 Expected Utility/CE Risk averse attitude (r = 20): **0.9629/ 11.6351**
    - D 1 Expected Utility Risk neutral attitude (r = ∞): **0.7609/11.0855**
    - D 2 Expected Utility Risk neutral attitude (r = ∞): **0.7609/ 11.0855**
    - D 1 Expected Utility Risk Seeking attitude (r = -20): **0.4128/10.5358**
    - D 2 Expected Utility Risk Seeking attitude (r = -20): **0.4170/10.4377**

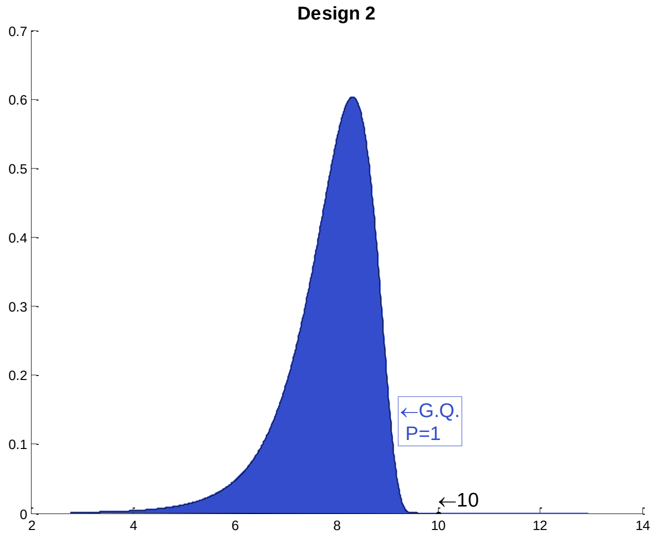




Examples

- MotorDesign HP: *More is better*
  - *Expected Value of HP = 7.8652*
  - D 1 Expected Utility/CE Risk averse attitude (r = 20): 0.6443/7.8476
  - D 2 Expected Utility/CE Risk averse attitude (r = 20): 0.6442/7.8468
  - D 1 Expected Utility Risk neutral attitude (r = ∞): 0.5865/ 7.8652
  - D 2 Expected Utility Risk neutral attitude (r = ∞): 0.5865/ 7.8652
  - D 1 Expected Utility Risk Seeking attitude (r = -20): 0.5272/ 7.8836
  - D 2 Expected Utility Risk Seeking attitude (r = -20): 0.5271/7.8828





Stochastic Dominance

Oregon State University
College of Engineering

- First Order Dominance
  - Gamble A has first-order stochastic dominance over gamble B if for any good outcome *x*:
    - A gives at least as high a probability of receiving at least *x* as does B,
    - For some *x*, A gives a higher probability of receiving at least *x*.

$$P[A \geq x] \geq P[B \geq x] \text{ and for some } x,\ P[A \geq x] > P[B \geq x]$$

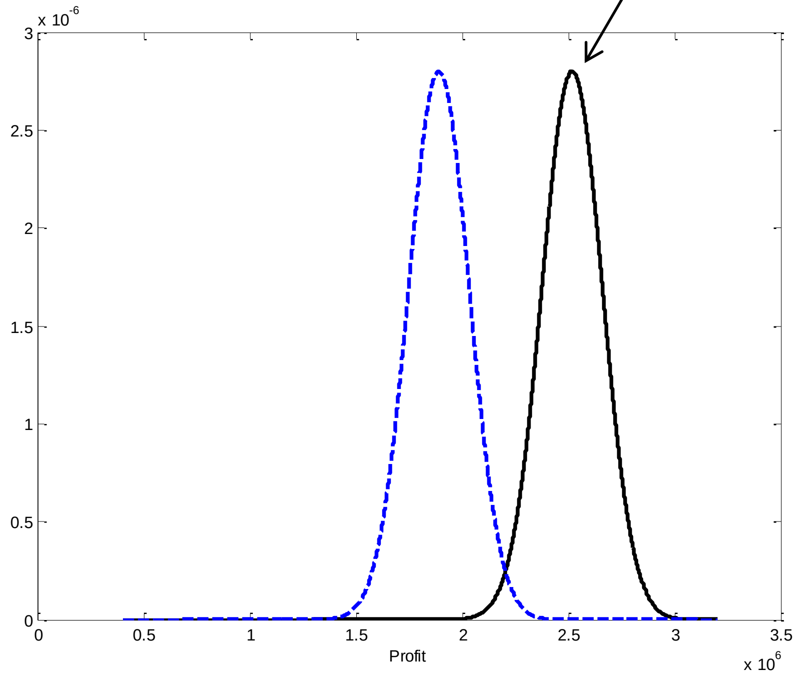
  - Decision makers will prefer A to B regardless of risk attitude.
- Second Order Dominance
  - For two gambles A and B, gamble A has second-order stochastic dominance over gamble B if:
    - A is more predictable (i.e. involves less risk) than B,
    - A has at least as high a mean as B

$$E[(A)] \geq E[(B)]$$

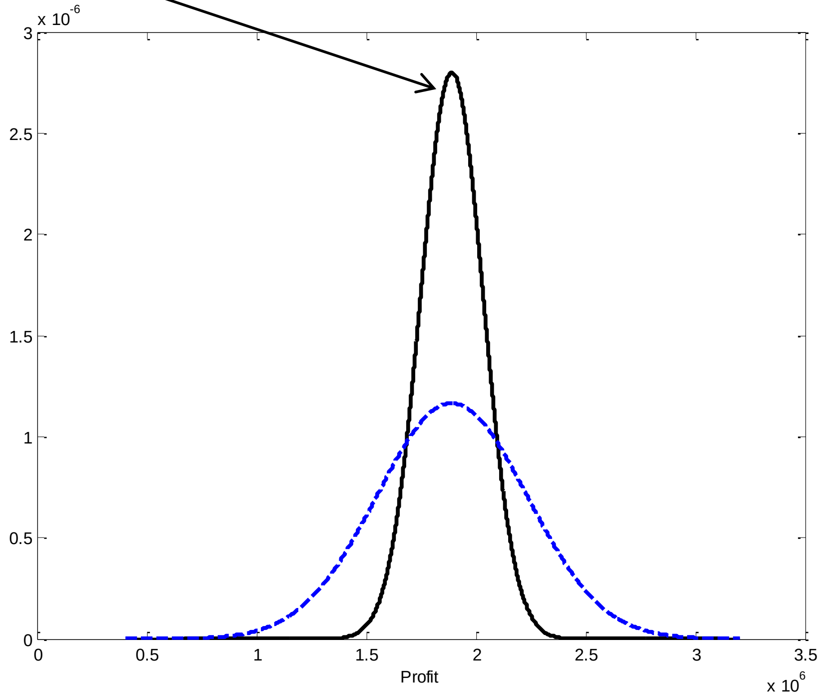
  - Risk averse decision makers will prefer A to B

Stochastic Dominance Examples (More is better)

Preferred Design



First Order Stochastic Dominance



Second Order Stochastic Dominance

Criticisms of Expected Utility theory

Oregon State University
College of Engineering

- The axioms are too strong: Do rational decision makers really need to obey these axioms?
  - Axioms seem to hold better for engineering design situations than personal decision making.
- No action guidance: To create the utility function, one needs to have a defined set of preferences. The output of the expected utility method is a mathematical formulation of the preferences you provide to the method.
  - This isn't an issue for us since we are seeking to automate the design selection process.
- Has no meaning without risk: The utility function requires risky outcomes and has no means for measurement without risk.
  - This isn't an issue for us since we are specifically interested in decisions under risk.

From: An Introduction to Decision Theory, M. Peterson, Cambridge University Press, 2009

Criticisms of Expected Utility theory

- Uncertain probabilities: Do we know the probabilities for all outcomes?
  - If we are dealing with epistemic uncertainties, probabilities are subjective. For aleatory uncertainties, probabilities are quantified.
- Lotteries are based on monetary outcomes: The lotteries were developed considering small sums of money. In engineering design, we have lotteries of design attributes
  - We can bypass the lottery method and base the risk attitude on previous decisions made in similar situations.
- Assumption of trade-off: It is assumed that we can be indifferent between a certain outcome and a lottery.
  - Only use utility theory for metrics which represent goals or preferences.

What if there are multiple decision makers?

Paradox due to Aggregating Multiple Decision Makers

Remember Transitive Preference Axiom?

$A \succ B$ and $B \succ C$ implies $A \succ C$

Voters	Elections		
	A vs. B	B vs. C	A vs. C
I (A≻B≻C)	A	B	A
II (B≻C≻A)	B	B	C
III (C≻A≻B)	A	C	C
Result	$A \succ B$	$B \succ C$	$C \succ A$

I = Designer
II = Boss
III = Customer

Group preference can be intransitive

Conclusion: ***Cannot aggregate group preferences***

What if there are multiple criteria?

Paradox due to Aggregating Multiple Criteria

Remember Transitive Preference Axiom?

$A \succ B$ and $B \succ C$ implies $A \succ C$

Criterion	"Elections" A vs. B	B vs. C	A vs. C
I $(A \succ B \succ C)$	A	B	A
II $(B \succ C \succ A)$	B	B	C
III $(C \succ A \succ B)$	A	C	C
Result	$A \succ B$	$B \succ C$	$C \succ A$

I = Weight
II = Horse Power
III = Cost

Multi-criteria preference can be intransitive

Conclusion: ***Cannot aggregate preferences of multiple criteria***

Social Choice Theory

Oregon State University
College of Engineering

- The problems of multiple decision makers and criteria are generally referred to as social choice theory.
- Kenneth Arrow studied the problem and created Arrow's General Possibility Theorem or Arrow's Impossibility Theorem:
- There are four properties for social choice as he defined.

Arrow's Impossibility Theorem Properties

Oregon State University College of Engineering

- Unrestricted Domain states that each criterion or preference i.e., the measure of value that facilitates rank ordering of alternatives, should be unrestricted.
- Pareto Optimality states that if every criterion or person ranks alternative A before alternative B, then the set of criteria/people as a whole should rank alternative A before alternative B.
- Independence of Irrelevant Alternatives (IIA) states that the rank order of an alternative should not depend on the alternative set:
  - If alternative A is ranked before alternative B then A should still be ranked before B when alternative C is added (or removed from consideration).
- Non-Dictatorship states that the results cannot simply mirror that of any ONE single person's preferences (or one criterion) without consideration of the other voters (or criteria).

Arrow showed that a function does not exist to satisfy all four properties

What does this mean?

Oregon State University College of Engineering

- Group Preference:
  - Cannot create a “group” utility function
  - Can create individual utility functions
  - In the “random utility” method, we can use individual utility functions to predict choices made, and aggregate the *choices*.
- Multi-criteria Preference:
  - There are multiple ways to formulate the multi-criteria or multi-attribute problem, but they have each have some issues to address.

Limitations of Aggregation of Design Preference

- Weighted Sum Method $u(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_i$
  - *Straightforward way to create a utility function as a function of multiple criteria.*
  - Could suffer from the voting paradox covered earlier.
  - Weights are generally subjective.
- Multi-Attribute Ranking (Borda Count)
  - *The alternatives are ranked on each attribute according to some ranking and point scheme.*
  - Could suffer from the voting paradox covered earlier.
  - Outcome is a function of the ranking/point scheme

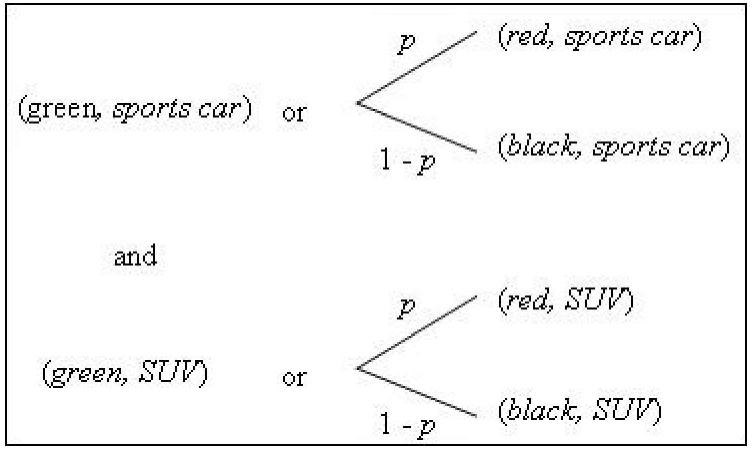
Limitations of Aggregation of Design Preference (cont)

- The multi-attribute utility (MAU) function
  - A specific utility function formulated for multiple criteria:

$$U(X)=\frac{1}{K}\left[\left[\prod_{i=1}^{n}(Kk_i u_i(x_i)+1)\right]-1\right]$$

the k_i are single attribute scaling constants, and K is normalizing constant

  - For each criteria, **independence is required** regarding preference, which is different from functional independence.
  - The k_i and K are determined using multi-attribute lotteries:



Recommendations

- Arrow's theorem states that we cannot create a group utility function, unless we relax one of the properties.
- Group preference
  - Operate as one (i.e. relax the non-dictatorship property) and decide upon a single utility function.
- Multiple Criteria
  - Use the multi-attribute utility (MAU) function with its challenges.
  - Use a single criterion (i.e. relax the non-dictatorship property) that is broad enough to cover your selection criterions:
    - Profit
    - Winning a race
    - Maximizing performance

  and treat the remainder of criteria as constraints.