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Design Under Uncertainty: Methods

ME 615 Spring 2020

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MIME

Normal Random Variables

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- The FOSM and upcoming FORM algorithm assume normal variables but note:
  - The FOSM takes as input the mean and variance of input variables and outputs mean and variance of response.
  - We don’t have to assume that the variables and responses are normal, but normal is the only distribution fully defined by just mean and variance.
  - For taking a probability, we assumed a normal distribution in FOSM but we don’t have to assume that for the SOTM method.

FOSM Improvement: SOTM Algorithm

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- *What if we use a second order Taylor series approximation?*

$$g_{\boldsymbol{\mu}} = g(\boldsymbol{\mu}) \qquad g_{,i} = \frac{\partial g(\boldsymbol{\mu})}{\partial x_i} \qquad g_{,ij} = \frac{\partial^2 g(\boldsymbol{\mu})}{\partial x_i \, \partial x_j} \qquad \mu_{i,k} = \int_{-\infty}^{\infty} (x_i - \mu_i)^k f_X(x_i)\, dx_i$$

- Mean:

$$\mu_g \approx \underbrace{g_{\boldsymbol{\mu}}}_{\text{FOSM}} + \underbrace{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} g_{,ij}\, \mu_{ij}}_{\text{SOTM}}$$

- Variance:

$$\sigma_g^2 \approx \underbrace{\sum_{i=1}^{n} g_{,i}^2\, \mu_{i,2}}_{\text{FOSM}} + \underbrace{g_{\boldsymbol{\mu}}^2 - \mu_g^2 + g_{\boldsymbol{\mu}} \sum_{i=1}^{n} g_{,ii}\, \mu_{i,2} + \sum_{i=1}^{n} g_{,i}\, g_{,ii}\, \mu_{i,3}}_{\text{SOTM}}$$

- Skewness:

$$\mu_{g,3} \approx \left.\sum_{i=1}^{n} g_{,i}^3\, \mu_{i,3} \right\} \text{first order approach}$$

$$\left.\begin{aligned} &+ g_{\boldsymbol{\mu}}^3 + \frac{3}{2} g_{\boldsymbol{\mu}}^2 \sum_{i=1}^{n} g_{,ii}\, \mu_{i,2} + 3 g_{\boldsymbol{\mu}} \sum_{i=1}^{n} g_{,i}^2\, \mu_{i,2} \\ &+ 3 g_{\boldsymbol{\mu}} \sum_{i=1}^{n} g_{,i}\, g_{,ii}\, \mu_{i,3} - 3\, \mu_g\, \sigma_g^2 - \mu_g^3 \end{aligned}\right\} \text{SOTM}$$

New Reliability Index

$$\beta_{3M} = \frac{1}{3}\beta_{2M}\left[2 + exp\left(\frac{1}{2}\mu_{g,3}\left(\beta_{2M} - \frac{1}{\beta_{2M}}\right)\right)\right]$$

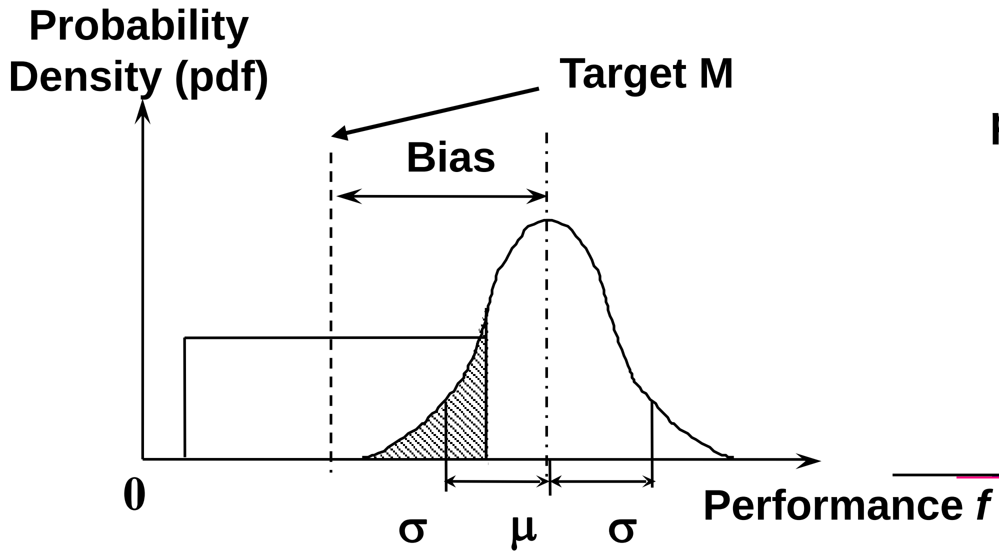
"Old" Reliability Index

3 Moment Index

Issues:
- Requires 2nd derivatives (More function evals)
- Expansion still at mean value

Objectives and Requirements

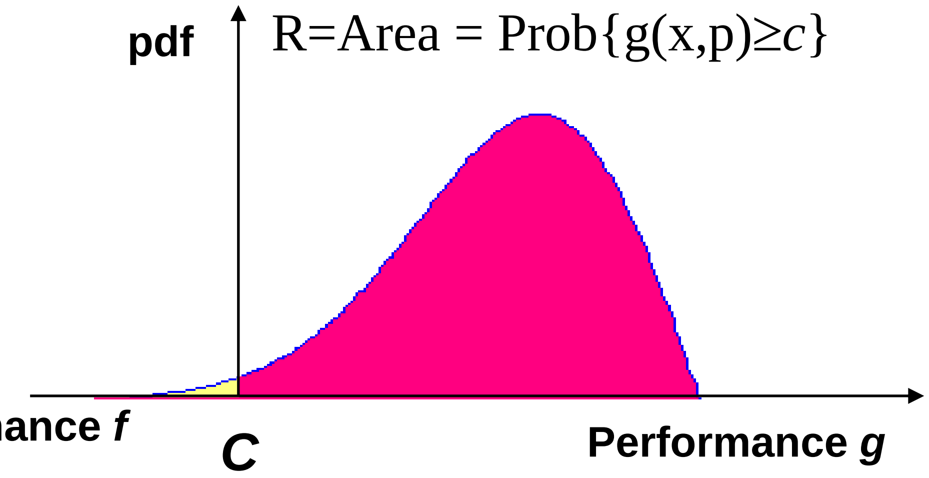
Objectives



> *Look at entire distribution*
> **s.t.** $x \in X$

Considering the effect of variations without eliminating the causes

Requirements



> **Satisfy**
> $R = P\{g(\mathbf{x},\mathbf{p}) \geq c\} \geq R_0$

Limit State

To assure proper levels of "safety" for the system designed

Most Probable Point Methods (MPP)

- Rather than taking the derivative for the Taylor series approximation at the mean point, take the derivative in the area of interest.
  - This corresponds to the limit state function
- **Therefore, the MPP methods can be viewed as a Taylor series approximation for constraints**
- There are two standard formulations for calculating reliability using the MPP methods
  - First Order Reliability Method (FORM): Linear approximation of the limit state.
  - Second Order Reliability Method (SORM): Quadratic approximation of the limit state.

Design Space vs. Criterion Space

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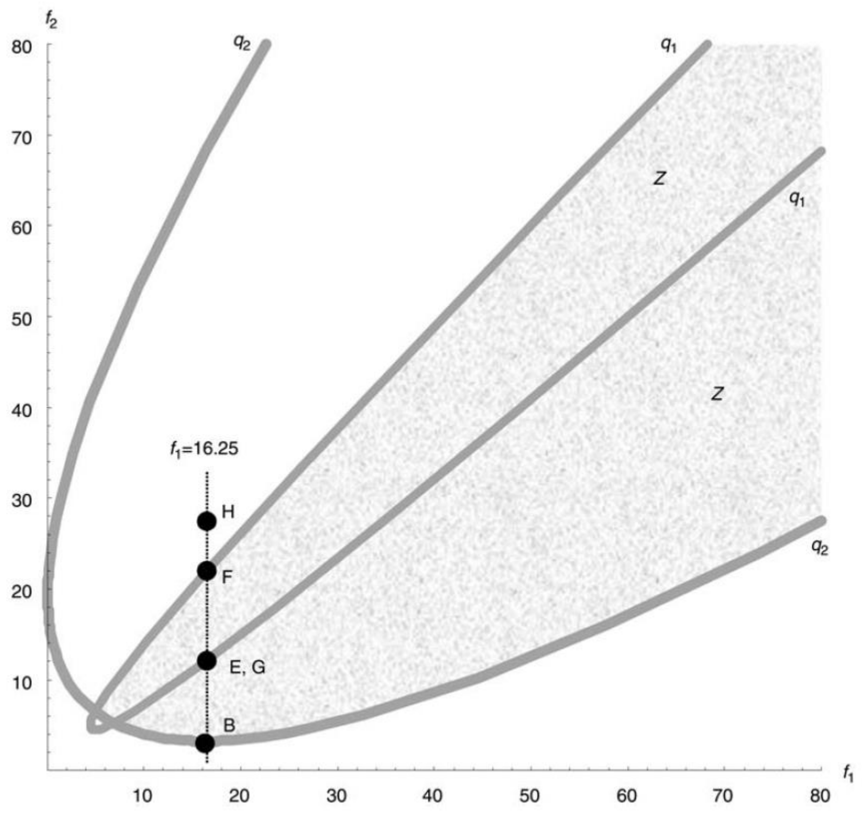

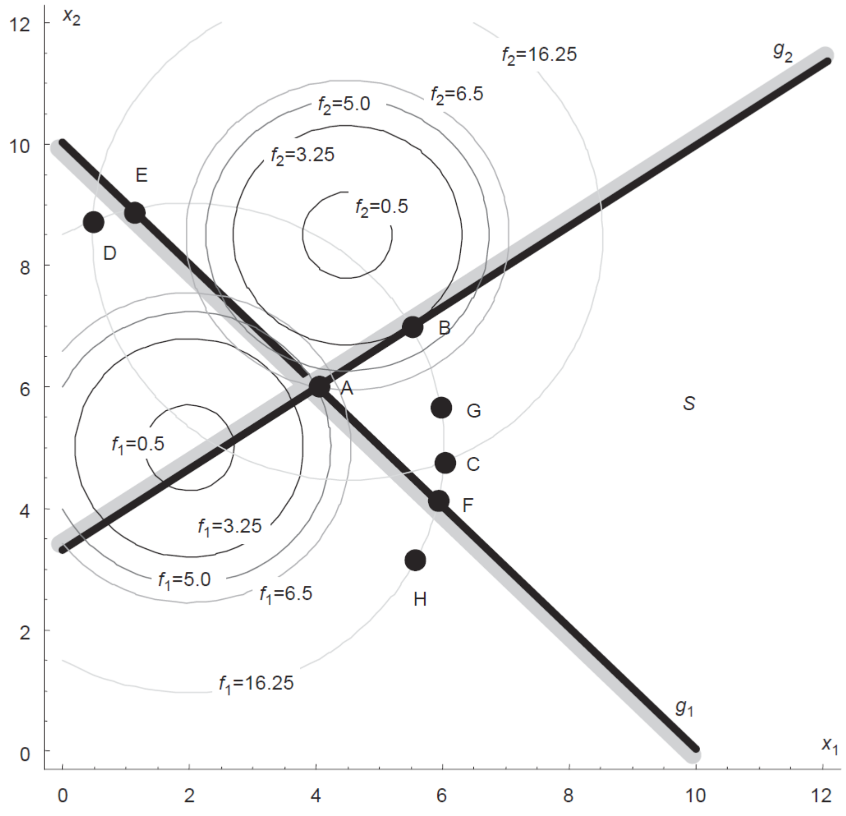

Criterion Space (**f**)

Design Space (**x**)

Comparison Design vs. Criterion Space

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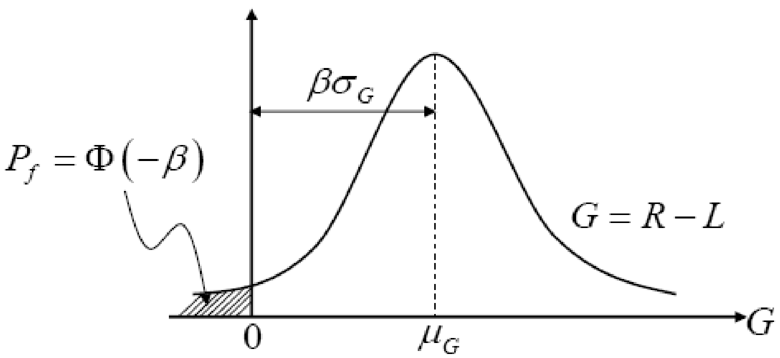
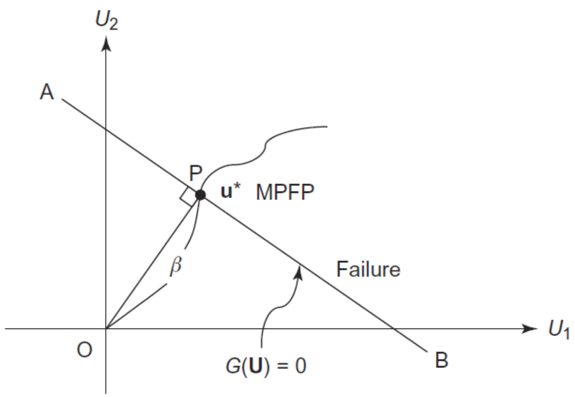


$$P_f = \Phi\left(\frac{0-\mu_g}{\sigma_g}\right) = \Phi(-\beta)$$

Criterion Space (f)



$$\beta = \frac{\mu_G}{\sigma_G} = \frac{\mu_R - \mu_L}{\sqrt{\sigma_R^2 + \sigma_L^2 - 2\rho_{RL}\sigma_R\sigma_L}}$$

Design Space (x)

First Order Reliability Method

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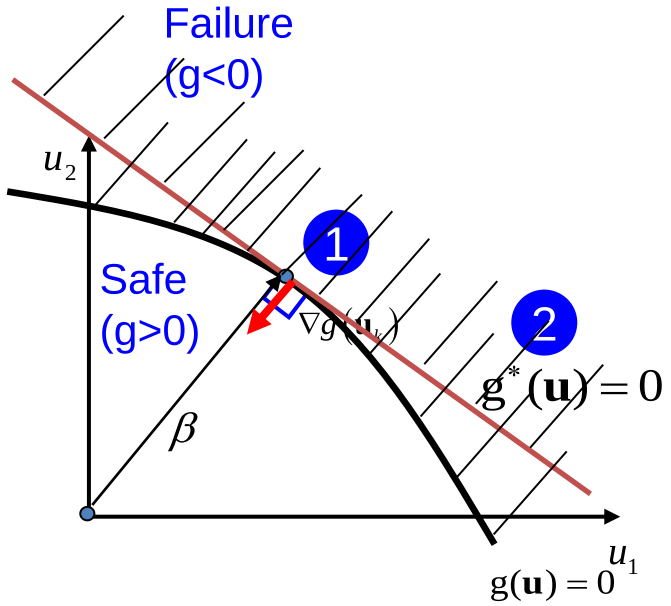
- What exactly does the approximation look like?



The Most Probable Point (MPP) concept

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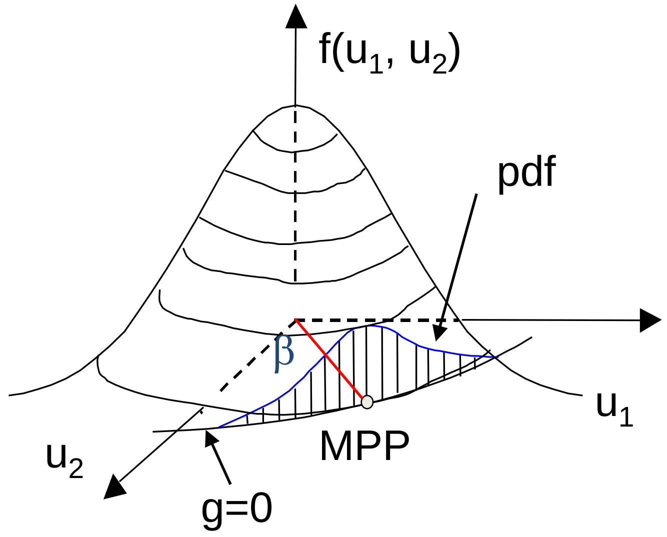
MPP is the point in the **u** space that has the highest probability density function value on the limit state *g*(***u***)=*0* curve and highest contribution to the integral of reliability.

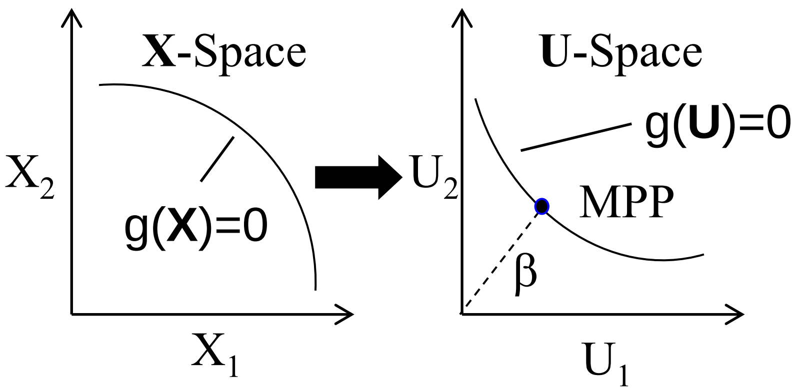


Random Var. X=[D,P] → **Standardized Normal Random Var. u**

$\mathbf{X}=\{X_1,\cdots,X_n\},\ g(\mathbf{X})=0$

$\mathbf{u}=[u_1,\cdots,u_n]^T$

$\boldsymbol{g}(\mathbf{u})=0$



The Most Probable Point (MPP) concept

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- How do you convert from **X** space to **U** space?
  - For Normal variables:
    - $u_i = \frac{x_i - \mu_{x_i}}{\sigma_{x_i}}$ and $x_i = \mu_{x_i} + u_i \sigma_{x_i}$
- How do we calculate Reliability (Probability of Success)?

Reliability

$$\mathrm{Prob}\{\boldsymbol{g}(\mathbf{X}, \mathbf{P}) \geq 0\} \approx \Phi(\beta)$$

$$\begin{cases} \beta = \min\limits_{\mathbf{u}} |\mathbf{u}| \\ \boldsymbol{s.t.} \quad \boldsymbol{g}(\mathbf{u}) = 0 \end{cases}$$

- How do we actually find β?
  - Next slides

First Order Reliability Method Algorithm

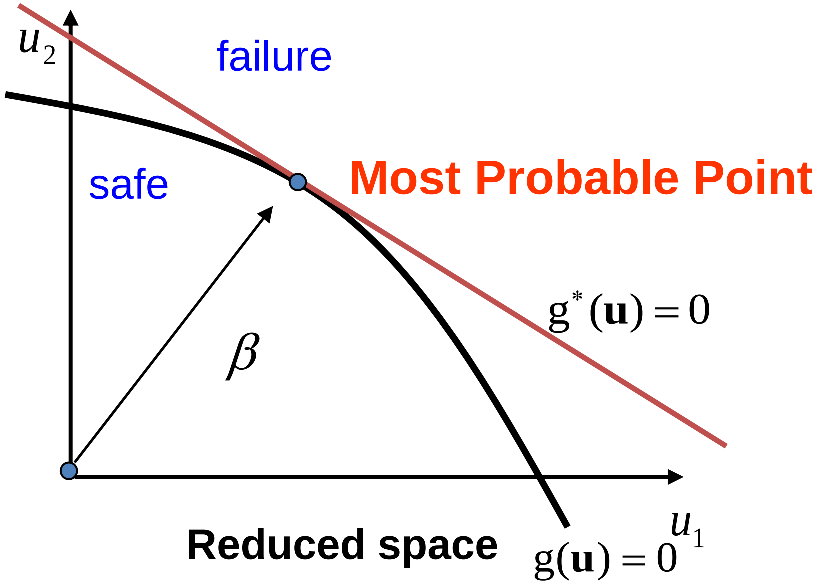
- FORM (First Order Reliability Method)
  - Taylor series expansion at most probable point (MPP)
    - MPP: closest point on g(u)=0 to the origin of u space
  - Transform variables **x** into standard normal variable **u** space
  - Find MPP
    - Reliability index β: shortest distance from origin to g(u)=0 → invariant
    - Linear approximation and probability calculation
  - MPP search

$$\textit{Minimize}\ \mathbf{u}^{\mathrm{T}}\mathbf{u}$$

$$\textit{Subject to}\ G(\mathbf{u}) = 0$$

  - Probability of failure calculation

$$P_f = \Phi(-\beta) = \Phi\left(-\left(\mathbf{u}^{\mathrm{T}}\mathbf{u}\right)^{1/2}\right)$$



Constrained Steepest Descent (CSD)

- QP subproblem:

  Min: $f(x) = \nabla f^T d + \frac{1}{2} d^T d$ or $f_k + \nabla f^T d + d^T d$

 s.t.

  $\nabla G(x^k) d_k + G(x^k) = 0$

First Order Reliability Method

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- MPP search
  - **Specialized Iterative algorithm**: Hasofer-Lind algorithm

- Hasofer-Lind algorithm
  1. Transform *x*, *g(x)* into *u*, *g(u)* (*u*: standard normal variable)

$$u_i = \frac{x_i - \mu_{x_i}}{\sigma_{x_i}}, \quad G\left(x\left(u\right)\right)$$

  2. Find MPP with the following formula (usually $\mathbf{u}_0$ is set to mean point=(**0**))

$$\mathbf{u}_{k+1} = \frac{\nabla g\left(\mathbf{u}_k\right)^T \mathbf{u}_k - g\left(\mathbf{u}_k\right)}{\nabla g\left(\mathbf{u}_k\right)^T \nabla g\left(\mathbf{u}_k\right)} \nabla g\left(\mathbf{u}_k\right)$$

  3. Iterate until $\mathbf{u}_k$ converges, i.e. the difference between $\mathbf{u}_{k+1}$ and $\mathbf{u}_k$ is small.
  4. Calculate probability of failure

$$p_f = \Phi\left(-\beta\right) = \Phi\left(-\left(\mathbf{u}^{\mathrm{T}}\mathbf{u}\right)^{1/2}\right)$$

First Order Reliability Method

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- How to calculate β at the MPP?
  - $\beta = \sqrt{\mathbf{u}^T\mathbf{u}} = -\frac{\nabla g^{*T}\mathbf{u}^*}{\sqrt{\nabla g^{*T}\nabla g^*}} = \frac{\mu_g}{\sigma_g}$
  - where * indicates the quantity is calculated at the MPP
- In this equation, we can define the sensitivity α as:
  - $\alpha = -\frac{\nabla g^*}{\sqrt{\nabla g^{*T}\nabla g^*}}$
- Sensitivity refers to the sensitivity of the reliability index β to each random variable.
  - Could be helpful in deciding where to reduce uncertainty if we can.

FORM: Non Normal Variables

- When **x** follows a non-normal distribution
  - Rackwitz-Fiessler transformation (Rackwitz & Fiessler 1978)
    - Transform non-normal distribution into equivalent normal distribution

$$u_i = \Phi^{-1}[F_{x_i}(x_i)]$$

    - **Matlab example**: u_i=norminv(betacdf (**X**,a,b), 0 , 1)
  - The mean and standard deviation of the equivalent Normal Distribution is:

$$\mu_x^N = x^* - \sigma_x^N \Phi^{-1}\left(F_X(x^*)\right)$$

$$\sigma_x^N = \frac{\phi\left(\Phi^{-1}\left(F_X(x^*)\right)\right)}{f_X(x^*)}$$

PDF
e.g. normpdf

Inverse CDF
e.g. norminv

Original CDF
e.g. betacdf

Issues with Normal Transformation

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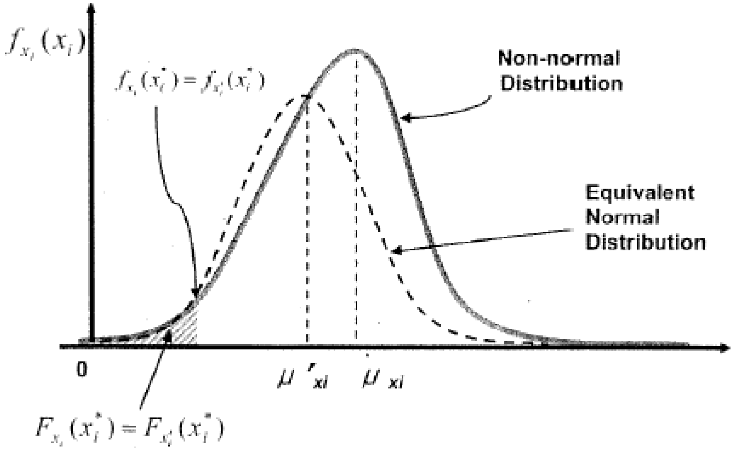
Becomes erroneous when x has large skewness.



- When there are correlated variables (Hohenbichler & Rackwitz 1981)
  - Diagonalize covariance matrix
  - Rosenblatt transformation

Second Order Reliability Method

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- Second order approximation at MPP
  - Curvature fitting (Hessian calculation required)
  - Point fitting (2n+1 g evaluation for curvature approximation)
  - Breitung's asymptotic formula for probability

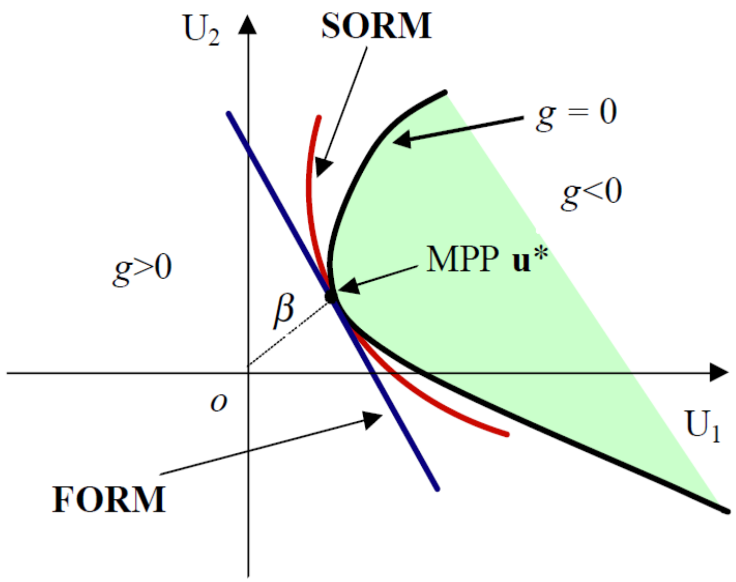
$$P_f = \Phi(-\beta)\prod_{i=1}^{n-1}(1+\beta\kappa_i)^{-1/2} \quad \kappa_i : \text{curvature}$$

correction

→ Accurate for large β

$$g(u) = g(u^*) + \sum_{i=1}^{n} \left.\frac{\partial g}{\partial u_i}\right|_{u^*}(u_i - u_i^*) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left.\frac{\partial^2 g}{\partial u_i \partial u_j}\right|_{u^*}(u_i-u_i^*)(u_j-u_j^*)$$
$$= (\mathbf{u}-\mathbf{u}^*)^{\mathrm{T}}\mathbf{H}(\mathbf{u}-\mathbf{u}^*) + \nabla \mathbf{g}(\mathbf{u}^*)^{\mathrm{T}}(\mathbf{u}-\mathbf{u}^*)$$

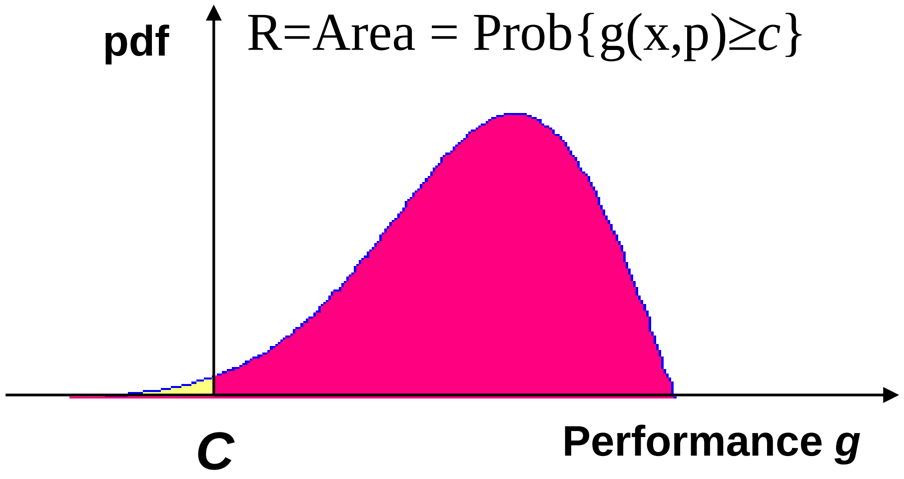
$$\mathbf{H} \rightarrow \kappa_i$$



Concepts in Reliability-based Design

- The problem of determining if constraints are satisfied at a certain reliability level is generally known as **reliability-based design**.

Requirements



Satisfy

$$R = P\{g(\mathbf{x},\mathbf{p}) \geq c\} \geq R_0$$

Two Ways to Look at the Problem

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- **Reliability Index Approach (RIA)**
  - This is the problem of finding the reliability level (i.e. the probability of success) for a specific design.
  - This has been our approach for all methods studied.
  - Can be expensive and MPP can have problems converging if the reliability is very high.
- **Performance Measure Approach (PMA)**
  - This is the problem of finding the constraint violation for a specific reliability value.
  - Better suited to design optimization and more stable.
  - A probability is not calculated in this approach.

Differences with Optimization Framework

- During optimization, gradient (sensitivity) needs to be calculated at each iteration
  - For **RIA**, gradient of reliability index w.r.t. design variable (DV)

    $$\beta_{t_i} - \beta_{s_i} \leq 0$$

  - For **PMA**, gradient of performance function w.r.t. DV

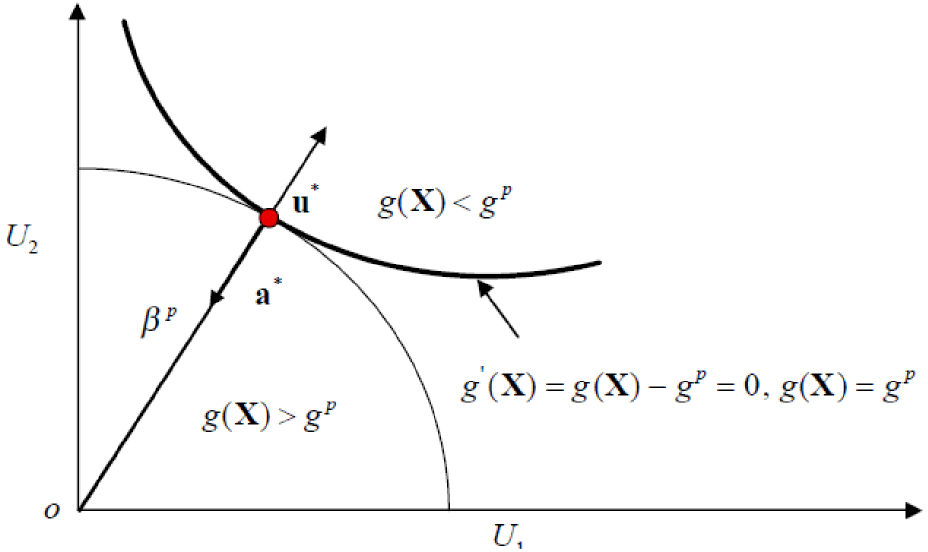
    $$G_{p_i} \leq 0$$

- The PMA method is more consistent with the theory we developed in ME 517.

Inverse Reliability Method Algorithm

- Inverse FORM (First Order Reliability Method)
  - Instead of searching on the *g(u)* = 0 line for the point closest to the origin (i.e. min $\mathbf{u}^T\mathbf{u}$), we will look for the minimum value of the *g(u)* function for a given beta value

$$\begin{aligned}&\textit{Minimize} \quad \mathbf{u}^T\mathbf{u}\\&\textit{Subject to} \quad G(\mathbf{u}) = 0\end{aligned} \Rightarrow \begin{cases}\min\limits_{\mathbf{u}} g(\mathbf{u}) \\ \text{subject to} \quad \|\mathbf{u}\| = \beta\end{cases}$$



  - Now Beta is known:

$$\beta = -\Phi^{-1}(P_f)$$

Inverse Reliability Method

- How to calculate β at the MPP?
  - $\beta = \sqrt{\mathbf{u}^T\mathbf{u}} = -\frac{\nabla g^{*T}\mathbf{u}^*}{\sqrt{\nabla g^{*T}\nabla g^*}} = \frac{\mu_g}{\sigma_g}$
  - where * indicates the quantity is calculated at the MPP
- In this equation, we can define the sensitivity α as:
  - $\alpha = \frac{\nabla g^*}{\sqrt{\nabla g^{*T}\nabla g^*}}$
- Sensitivity refers to the sensitivity of the reliability index b to each random variable.
  - Could be helpful in deciding where to reduce uncertainty if we can.

Inverse Reliability Method

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- We can solve for u in this equation
  - $\beta = \sqrt{\mathbf{u}^T\mathbf{u}} = -\frac{\nabla g^{*T}\mathbf{u}^*}{\sqrt{\nabla g^{*T}\nabla g^*}} = \frac{\mu_g}{\sigma_g}$
  - where * indicates the quantity is calculated at the MPP
- And we get the following update equation:
  - $\mathbf{u} = -\alpha\beta$
- We can create an algorithm similar to the form algorithm.

Inverse FORM Algorithm

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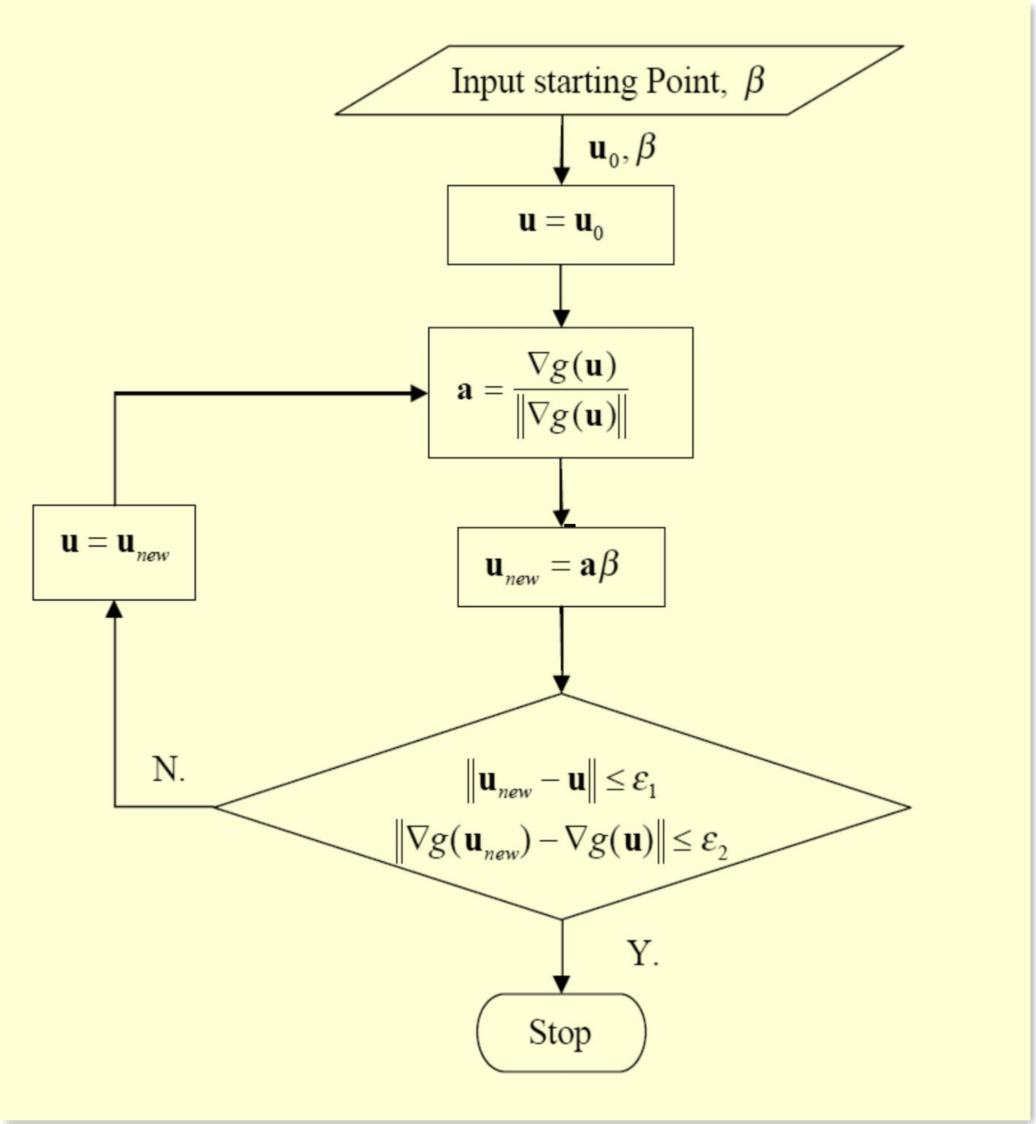

FORM Method Simplification

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- Remember the FORM update equation:

$$\mathbf{u}_{k+1} = \frac{\nabla g(\mathbf{u}_k)^T \mathbf{u}_k - g(\mathbf{u}_k)}{\nabla g(\mathbf{u}_k)^T \nabla g(\mathbf{u}_k)} \nabla g(\mathbf{u}_k)$$

- And recall:
  - $\beta = \sqrt{\mathbf{u}^T\mathbf{u}} = -\frac{\nabla g^{*T}\mathbf{u}^*}{\sqrt{\nabla g^{*T}\nabla g^*}} = \frac{\mu_g}{\sigma_g}$
  - $\alpha = \frac{\nabla g^*}{\sqrt{\nabla g^{*T}\nabla g^*}}$
- We can rewrite the FORM update equation:

$$\mathbf{u}^{k+1} = -\mathbf{a}^k \left\{ \beta^k + \frac{g(\mathbf{u}^k)}{\left\|\nabla g(\mathbf{u}^k)\right\|} \right\}$$

Comparison FORM and INV FORM

- **FORM** update equation:

$$\mathbf{u}^{k+1} = -\mathbf{a}^{k}\left\{\beta^{k} + \frac{g(\mathbf{u}^{k})}{\left\|\nabla g(\mathbf{u}^{k})\right\|}\right\}$$

- **Inverse FORM** update equation:

$$\mathbf{u}^{k+1} = -\beta\mathbf{a}^{k}$$

Comparison FORM and INV FORM

FORM

Iteration	β	g	∇g	(U_x, U_y)
0	0	0.67076	(-0.37268, -0.046585)	(1.7722, 0.22152)
1	1.7859	-0.015931	(-0.38984, -0.036775)	(1.7375, 0.16391)
2	1.7453	-0.00032102	(-0.38986, -0.036758)	(1.7367, 0.16375)
3	1.7444	-2.6004e-009	(-0.38986, -0.036761)	(1.7367, 0.16376)

INVERSE FORM

Iteration	g	∇g	(U_x, U_y)
0	0.67076	(-0.37268, -0.046585)	(3.0664, 0.3833)
1	-0.53073	(-0.39663, -0.03191)	(3.0803, 0.24781)
2	-0.53196	(-0.39718, -0.031483)	(3.0806, 0.24418)
3	-0.53196	(-0.39719, -0.031472)	(3.0806, 0.24409)

Most Probable Point-Based Methods

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- Characteristics of the MPP Method:
  - Scales linearly with the number of inputs, x. (FORM)
  - Gradients can be approximated with Finite Difference estimations for black-box system models, such as Modelica models.
  - The search algorithm is very efficient.
  - Displays invariance to the form of the performance equation.
- Limitations of the MPP Method:
  - Requires that all input variables to be independent normal and that the output distribution is normal (Rackwitz-Fiessler transformation needed for non-normal inputs)
  - Calculates first 2 moments
  - Linear approximation of performance at the MPP (FORM).
  - Does require a search algorithm.
  - Search algorithm has trouble when β is large.

Where we are at…

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- Monte Carlo Simulation
  - Able to recover the full distribution of the model response(s).
  - Requires many model simulations
- Mean Value First Order Second Moment (i.e. Taylor series)
  - Linearizes the model around the mean value of the inputs—very fast
  - Poor approximation in the tail regions
- Most Probable Point method (i.e. FORM)
  - Linearizes at the limit state-better approximation for reliability.
  - Requires a search algorithm, no information on the response distribution
- Where we are going—
  - Numerical Quadrature based methods—approximate the actual multidimensional integral: $\int_{\Omega} \cdots \int f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$