6.1

(a)

Recall the formula $H(t,S,q)$

$$H(t,S,q)=\sup_{v\in A_t} E_{t,S,q}\left[\int_t^T (S_u-kv_u)v_u du + Q_T^v(S_T-\alpha Q_T^v)\right]$$

By DPP. $dQ_t^v=-v_t dt$, $dS_t=\sigma dW_t$

$H(t,S,q)+\int_0^t (S_u-kv_u)v_u du$ is

super martingale for all v

is a martingale for some v^*

By Ito's formula, define $H_t=H(t,S,q)+\int_0^t (S_u-kv_u)v_u du$

$$dH=\partial_t H\,dt+\partial_S H(\sigma dW_t)+\tfrac{1}{2}\partial_{SS}H(\sigma^2 dt)$$
$$+\partial_q H(-v_t dt)+(S_v-kv_u)v_u$$

Now, find the drift of H_t, $D=\partial_t H dt+\frac{1}{2}\sigma^2\partial_{SS}H dt - v_t\partial_q H dt+(S_u-kv_u)v_u$

$$D=\begin{cases}\le 0 & \text{for all } v\\ =0 & \text{for some } v^*\end{cases}$$

Now, set $S_t^v=S$, we have

$$\partial_t H+\tfrac{1}{2}\sigma^2\partial_{SS}H+\sup_v\{(S-kv)v - v\partial_q H\}=0$$

take derivative of sup term

$$S-2kv-\partial_q H=0 \qquad v^*=\frac{S-\partial_q H}{2k}$$

then

$$\partial_t H+\tfrac{1}{2}\sigma^2\partial_{SS}H+\frac{1}{4k}(\partial_q H-S)^2=0$$

Simplify

$$-(\partial_q H-S)^2-2k\sigma^2\partial_{SS}H-4k\partial_t H=0$$

(c)

$\alpha\to 0$, In my intuition, I feel like time is not a factor for trading. Moreover, We need to trade at time T. Which makes sense of $H(T)=Q_T S_T$ a little bit

(b)

The corresponding ansatz is

$$H(t,S,q)=h_2(t)q^2+h_1(t)q+h_0(t)+qS$$

$$\frac{\partial H}{\partial q}=2h_2(t)q+h_1(t)+S=\underbrace{S-2\alpha q}_{\text{at time } T}$$

$$\frac{\partial^2 H}{\partial q^2}=2h_2(t)=\underbrace{-2\alpha}_{}\text{ at time } T$$

$$\text{So } S-\frac{\partial H}{\partial q}=(2h_2(t)q+h_1(t))$$

$$\frac{\partial H(t,S,q)}{\partial t}=\partial_t h_2(t)q^2+\partial_t h_1(t)q+\partial_t h_0(t)$$

$$\frac{\partial^2 H(t,S,q)}{\partial S^2}=0$$

then recall $\partial_t H+\frac{1}{2}\sigma^2\partial_{SS}H+\frac{1}{4k}(\partial_q H-S)^2=0$

we need to figure out $v^*=\frac{S-\partial_q H}{2k}=-\frac{2h_2(t)q+h_1(t)}{2k}$

find $h_1(t)$ and $h_2(t)$

$$\partial_t h_2(t)+\frac{h_2^2(t)}{k}=0,\ h_2(T)=-\alpha \quad ①$$

$$\partial_t h_1(t)+\frac{h_2(t)}{k}h_1(t)=0 \quad h_1(T)=0 \quad ②$$

So ① $\Rightarrow \int_t^T \frac{1}{h_2^2(t)}dh_2(t)=-\int_t^T\frac{1}{k}dt$ $\quad \frac{1}{h_2(t)}$

$$\left(\frac{1}{h_2(T)}-\frac{1}{h_2(t)}\right)=\frac{1}{k}(T-t)$$

$$h_2(t)=-\left(\frac{T-t}{k}+\frac{1}{\alpha}\right)^{-1}$$

$$=-\left(\frac{k}{T-t+\frac{k}{\alpha}}\right)$$

for ②, easily get that $h_1(t)=0$ $(\ln h_1(T)\to\infty)$

$$\text{So } v^*=\frac{-2h_2(t)q}{2k}=\frac{q}{T-t+\frac{k}{\alpha}}$$

let us $v^*\to v_t^*$ $q\to Q_t^{v^*}$

get $v_t^*=\frac{Q_t^{v^*}}{T-t+\frac{k}{\alpha}}$

the ansatz is

$$H(t,S,q)=-\alpha q^2+qS$$

$$H(T,S,q)=q(S-\alpha q)$$

6.2

(a) Similar to 6.1.

recall

$$H(t,x,S,q) = \sup_v E[X_T^v + Q_T^v(S_T - \alpha Q_T^v)]$$

$$dX_t^v = (S_t - kv_t)v_t\,dt$$

By DPP, $H(t,x,S,q) + \int_0^t (S_u - kv_u)v_u\,du$

is super martingale for all v

is martingale for some v^*

By Ito formula:

$$dH_t = \partial_t H dt + \partial_x H dX_t + \partial_{SS} H (dS)^2 + \partial_q H dq$$

$$= \partial_t H dt + \frac{1}{2}\sigma^2 \partial_{SS} H dt + (S_t - kv_t)v_t dt - \partial_q H \cdot v_t dt \quad \text{set } S_t \to S$$

Now we find HJB Equation:

$$\partial_t H + \frac{1}{2}\sigma^2\partial_{SS} H + \sup_v \{(S-kv)v\partial_x H - v\partial_q H\} = 0$$

which is the same formula in (6.44) in 6.2.a

Now take the derivative

$$(S-2kv)\partial_x H - \partial_q H = 0 \quad v^* \to v_t^*$$

$$v_t^* = -\frac{\partial_q H - S\partial_x H}{2k\partial_x H}$$

(b) recall $H(t,S,x,q) = x + h(t)q^2 + qS$

when $t = T$

$h(T) = -\alpha$

$$\frac{dH}{dq} = 2qh(t) + S$$

$$\frac{dH}{dx} = 1 \qquad \frac{dH}{dq} - S = 2q\cdot h(t)$$

then

$\partial_{SS} H = 0$

$\partial_t H = \partial_t h(t) \cdot q^2$

$$(S-kv)v\partial_x H - v\partial_q H = \frac{(\partial_q H - S\partial_x H)^2}{4k\partial_x H}$$

recall HJB and we get

$$\partial_t h(t)\cdot q^2 + \frac{(\partial_q H - S\partial_x H)^2}{4k\partial_x H} = 0$$

$$\text{So } \partial_t h(t)\cdot q^2 + \frac{(\partial_q H - S)^2}{4k} = 0$$

$$\partial_t h(t) q^2 + \frac{4h^2(t) q^2}{4k} = 0$$

$$\text{So } \partial_t h(t) + \frac{1}{k}h^2(t) = 0$$

$$\int_t^T \frac{1}{h^2(u)}\,dh(u) = \int_t^T -\frac{1}{k}\,du$$

$$\Rightarrow h(t) = -\left(\frac{k}{T-t+\frac{k}{\alpha}}\right) \quad v^* \to v_t^*$$

$$\text{So } v^* = \frac{2qh(t)}{2k} \quad q \to Q_t^*$$

then

$$\longrightarrow v_t^* = \frac{Q_t^*}{T-t+\frac{k}{\alpha}}$$