6.1

(a)

Recall the formula $H(t,S,q)$

$$H(t,S,q) = \sup_{v \in A_T} E_{t,S,q}\left[\int_t^T (S_u - kv_u)v_u du + Q_T^v(S_T - \alpha Q_T^v)\right]$$

By DPP,
$$dQ_t^v = -v_t dt$$
$$dS_t = \sigma dW_t$$

$$H(t,S,q) + \int_0^t (S_u - kv_u)v_u du \quad \text{is}$$

super martingale for all v

is a martingale for some v^*

By **Ito's formula**, define $H_t = H(t,S,q) + \int_t^t (S_u - kv_u)v_u du$

$$dH = \partial_t H\, dt + \partial_S H(\sigma dW_t) + \frac{1}{2}\partial_{SS}H(\sigma^2 dt)$$
$$+ \partial_q H(-v_t dt) + (S_u - kv_u)v_u$$

Now, find the drift of H_t, $D = \partial_t H dt + \frac{1}{2}\sigma^2 \partial_{SS}H dt$
$$- v_t \partial_q H dt + (S_u - kv_u)v_u$$

$$D = \begin{cases} \leq 0 & \text{for all } v \\ = 0 & \text{for some } v^* \end{cases}$$

Now, set $S_t^v = S$, we have

$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{SS}H + \sup\{(S - kv)v - v\partial_q H\} = 0$$

take derivative of sup term

$$S - 2kv - \partial_q H = 0 \qquad \boxed{v^* = \frac{S - \partial_q H}{2k}}$$

then

$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{SS}H + \frac{1}{4k}(\partial_q H - S)^2 = 0$$

Simplify

$$-(\partial_q H - S)^2 - 2k\sigma^2 \partial_{SS}H - 4k\partial_t H = 0$$

(c)
$a \to 0$, In my intuition, I feel like time is not a factor for trading. Moreover, We need to trade at time T. Which makes sense of $H(T) = Q_T S_T$ a little bit

(b)

The corresponding ansutz is
$$H(t,S,q) = h_2(t)q^2 + h_1(t)q + h_0(t) + qS$$

$$\frac{\partial H}{\partial q} = 2h_2(t)q + h_1(t) + S = S - 2\alpha q \quad \text{at time } T$$

$$\frac{\partial^2 H}{\partial q^2} = 2h_2(t) = -2\alpha \quad \text{at true } T$$

So $S - \frac{\partial H}{\partial q} = -(2h_2(t)q + h_1(t))$

$$\frac{\partial H(t,S,q)}{\partial t} = \partial_t h_2(t)q^2 + \partial_t h_1(t)q + \partial_t h_0(t)$$

$$\frac{\partial^2 H(t,S,q)}{\partial S^2} = 0$$

then recall $\partial_t H + \frac{1}{2}\sigma^2 \partial_{SS}H + \frac{1}{4k}(\partial_q H - S)^2 = 0$

we need to figure out $v^* = \frac{S - \partial_q H}{2k} = -\frac{2h_2(t)q + h_1(t)}{2k}$
find $h_1(t)$ and $h_2(t)$

$$\partial_t h_2(t) + \frac{h_2^2(t)}{k} = 0, \quad h_2(T) = -\alpha \quad ①$$

$$\partial_t h_1(t) + \frac{h_2(t)}{k}h_1(t) = 0 \quad h_1(T) = 0 \quad ②$$

So $① = \int_t^T \frac{1}{h_2(t)} dh_2(t) = -\int_t^T \frac{1}{k} dt \quad \frac{1}{h_2(t)}$

$$\left(\frac{1}{h_2(T)} - \frac{1}{h_2(t)}\right) = \frac{1}{k}(T-t)$$
$$h_2(t) = -\left(\frac{T-t}{k} + \frac{1}{\alpha}\right)^{-1}$$
$$= -\left(\frac{k}{T-t+\frac{k}{\alpha}}\right)$$

for ②, easily get that $h_1(t) = 0$ $(\ln h_1(T) \to \infty)$

So $v^* = \frac{-2h_2(t)q}{2k} = \frac{q}{T-t+\frac{k}{\alpha}}$

let us $v^* \to v_t^*$ $q \to Q_t^{v^*}$

get $v_t^* = \frac{Q_t^{v^*}}{T-t+\frac{k}{\alpha}}$

6.2

(a) Similar to 6.1.

recall

$$H(t,x,S,q) = \sup_v E[X_T^v + Q_T^v(S_T - \alpha Q_T^v)]$$

$$dX_t^v = (S_t - kv_t)v_t \, dt$$

By DPP, $H(t,x,S,q) + \int_0^t (S_u - kv_u)v_u \, du$

is super martingale for all v

is martingale for some v^*

By Ito formula:

$$dH_t = \partial_t H \, dt + \partial_x H \, dX_t^v + \partial_{SS} H (dS)^2$$
$$+ \partial_q H \, dq$$

$$= \partial_t H \, dt + \frac{1}{2}\sigma^2 \partial_{SS} H \, dt + (S_t - kv_t)v_t \, dt$$

$$- \partial_q H \cdot v_t \, dt \qquad \text{Set } S_t \to S$$

Now we find HJB Equation:

$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H + \sup_v \{(S-kv)v\partial_x H - v\partial_q H\} = 0$$

which is the same formula in (6.44) in 6.2.a

Now take the derivative

$$(S - 2kv)\partial_x H - \partial_q H = 0 \qquad v^* \to v_t^*$$

$$v_t^* = -\frac{\partial_q H - S\partial_x H}{2k\partial_x H}$$

(b)

recall $H(t,S,x,q) = x + h(t)q^2 + qS$

$$\frac{\partial H}{\partial q} = 2qh(t) + S \qquad \begin{matrix} \text{when} \\ t = T \\ h(T) = -\alpha \end{matrix}$$

$$\frac{\partial H}{\partial x} = 1 \qquad \frac{\partial H}{\partial q} - S = 2q \cdot h(t)$$

then

$$\partial_{SS} H = 0$$

$$\partial_t H = \partial_t h(t) \cdot q^2$$

$$(S - kv)v\partial_x H - v\partial_q H = \frac{(\partial_q H - S\partial_x H)^2}{4k\partial_x H} \qquad \qquad v$$

recall HJB and we get

$$\partial_t h(t) \cdot q^2 + \frac{(\partial_q H - S\partial_x H)^2}{4k\partial_x H} = 0$$

So $\partial_t h(t) \cdot q^2 + \frac{(\partial_q H - S)^2}{4k} = 0$

$$\partial_t h(t)q^2 + \frac{4h^2(t)q^2}{4k} = 0$$

So $\partial_t h(t) + \frac{1}{k}h^2(t) = 0$

$$\int_t^T \frac{1}{h^2(u)} \, dh(u) = \int_t^T -\frac{1}{k} \, du$$

$$\Rightarrow h(t) = -\left(\frac{k}{T - t + \frac{k}{\alpha}}\right) \qquad v^* \to v_t^*$$

So $v^* = -\frac{2qh(t)}{2k}$ \qquad $q \to Q_t^*$

then

$$\Rightarrow v^* = \frac{Q_t^*}{T - t + \frac{k}{\alpha}}$$