2
Topics in Mechanics

2.1 Background

This book is not about mechanics, but those with a serious interest in vehicle dynamics must understand mechanics. Classical mechanics is a subject that has been studied for well over three hundred years, and over that time it has evolved into a broad and sophisticated subject that is now covered by hundreds of books and many thousands of research papers. We will not attempt to review this literature. Instead, we will point the reader in the direction of some of our preferred texts. In this way one should be able to discover any literature required.

A good undergraduate-level treatment of vector mechanics can be found in Synge and Griffith [40]. This book also provides an introductory treatment of generalized coordinates and constraints (chapter 10), and analytical mechanics (chapters 15 and 16). The chapter on the theory of vibrations (chapter 17) may also be of interest. We would recommend this book to those with an engineering background, especially if mechanics has not been a focal point of their prior education.

A more advanced set of books comprise those by Goldstein and his colleagues [41, 42], which are aimed primarily at graduate-level physicists. These books have more of an analytical mechanics flavour, and therefore focus on variational principles. Given the present context, the reader should be familiar with the material on classical mechanics (not relativity or quantum mechanics), with the treatment of rigid body rotations standing out as being particularly relevant.

The course notes on mechanics by Landau and Lifshitz [43] are a masterpiece of clarity and succinctness, but are probably not for those in the early stages of learning the subject. The focus is on analytical mechanics (rather than vector mechanics) and the treatment is from the perspective of the theoretical physicist; not dissimilar to Goldstein [41]. While there is certainly an overlap with other books, seeing the central ideas (the conservation of energy, momentum, and angular momentum) treated in a different way, and with different insights, is helpful. In the context of vehicle dynamics, chapters I, II, and VI are particularly relevant.

Lanczos' book [44] on variational principles in mechanics is a favourite of ours. This book provides an excellent overview of mechanics, and shows the relationships between vector mechanics ideas and d'Alembert's principle, virtual work, and the energy-based methods of Euler, Lagrange, and Hamilton. Lanczos' treatment of constraints (and the physical significance of the Lagrange multiplier) is excellent (chapter 5). To readers who feel they know mechanics already, we would respectfully suggest: 'read Lanczos anyway'.

Dynamics and Optimal Control of Road Vehicles. D. J. N. Limebeer and M. Massaro.

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The connections between classical mechanics and mathematical physics are many and varied. Arnold's book [45] treats mechanics from a theoretical physicist's point of view with an emphasis on variational principles and analytical dynamics. All the basic problems in dynamics, including the theory of oscillations, the theory of rigid body motion, and the Hamiltonian formalism are examined. This book starts from the beginning, assuming only standard courses in differential and integral calculus, differential equations, geometry (vector spaces, vectors), and linear algebra (linear operators, quadratic forms). This is not an easy book for readers with an engineering background, but it offers a great deal for those willing to put in the effort. Many of the basic concepts are illustrated with beautiful sketches.

The main emphasis of Milne's book [46] is the power and utility of vector-based methods. The techniques of analytical mechanics were deliberately excluded in order to focus on elementary mathematical methods. However, this book also shows how difficult problems can be solved, with vectorial methods providing a clear insight into 'how systems move'. The book highlights the tremendous power of scalar and vector triple products. One theme of the book is the vector product $\mathbf{a}\times\mathbf{b}$ and its applications, with the identity $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{a}(\mathbf{b}\cdot\mathbf{c})$ playing a fundamental role. Several interesting examples are provided.

An introduction to the geometric approach to classical mechanics is given in [47]. This book provides the unifying viewpoint of Lagrangian and Hamiltonian mechanics in the language of differential geometry and fills a gap between traditional classical mechanics texts and advanced modern mathematical treatments of the subject. This book provides a comprehensive treatment of the ideas and methods of geometric mechanics, and is aimed at the graduate student, or the specialist who wants to refresh their knowledge of the formulations of Lagangrian and Hamiltonian mechanics.

All road vehicles have rolling wheels. As a result, nonholonomic constraints have a significant role to play in any study of vehicle dynamics, even though force-generating tyres are predominant in the modern literature. If one accepts this point of view, Neĭmark and Fufaev's book [48] on nonholonimc systems is certainly helpful. Apart from a masterful treatment of the general theory, it contains a detailed analysis of several examples of direct relevance. These include: Čhaplygin's sleigh, a disc rolling on a horizontal plane, the bicycle, and aircraft and road vehicle shimmy phenomena. Books with a more applied flavour include [49–52].

The chapter begins (Section 2.2) with the formulation of the equations of motion using the methods of Newton and Lagrange, which are techniques we employ throughout the book. Topics such as generalized coordinates, constraints, and kinetic and potential energy are discussed briefly. Some of the basics of conservation laws are discussed in Section 2.3. Hamilton's equations and Poisson brackets, and their relationship to conservation laws, are reviewed in Section 2.4.

A brief review of rotating reference frames using matrix-theoretic arguments and linear transformations is given in Section 2.5. This approach has found favour in the modern robotics literature [53], because it is well suited to computer-based computations. In our view this approach also clarifies several aspects of the classical theory. One of the key ideas is the use of skew-symmetric matrix representations of the vector cross product. Important also is the fact that rotations and cross products commute, by which we mean $\mathcal{R}(\mathbf{a} \times \mathbf{b}) = \mathcal{R}\mathbf{a} \times \mathcal{R}\mathbf{b}$ in which $\mathcal{R}$ represents a rotation matrix.

Equilibria, stability, and linearization are discussed briefly in Section 2.6; our simple treatment follows that given in [45].

Time-reversal symmetry, which is a property associated with many conservative systems, is discussed in Section 2.7.

The chapter contains numerous examples of varying levels of difficulty that serve two purposes. First, they illustrate several of the key ideas from mechanics that will be required in the later chapters. Second, they will provide a brief comparison between the ideas of vectorial and analytical mechanics. In some cases we will solve the problem using both vector mechanics ideas and Lagrange's equations, thereby providing this comparison. These examples have also been chosen to illustrate, in a simple context, some of the fundamental ideas required in the study of mechanics and vehicle dynamics.

The chapter concludes with three substantial examples. The first is a detailed analysis of the classical Čhaplygin sleigh [48], which is given in Section 2.8. This system is conservative and nonholonomic, as well as being time-reversible. An intriguing feature of this system is that linearizations around stable equilibria are neither conservative nor time-reversible. An interesting question relates to the effect of linearization on properties such as energy conservation and time-reversibility.

In the second example we study a ball rolling on an inclined surface and then on an inclined turntable. This example is elegantly solved using vectorial methods and serves as a warning: 'Do not underestimate the subtleties that can arise in systems containing multiple rotating components.' The rolling ball is dealt with in Section 2.9.

The third example (given in Section 2.10) is the rolling disc, which is ubiquitous in the context of literature on nonholonomic dynamic systems. Nonholonomic rolling constraints are a feature of early bicycle models such as those introduced by Whipple [2] and Carvallo [54]; more will be said about this later. This example also illustrates the use of Euler angles and the use of linearization techniques in the study of stability. A facility with Euler angles is required, because bicycles yaw and roll through large angles, as well as pitch through smaller ones. In common with the bicycle, we will show that the rolling disc has speed-dependent stability properties.

While computers obviate the burden of performing tedious and often complicated calculations, they do not remove the need to understand mechanics, or to make decisions relating to analysis frameworks, reference frames, problem formulations, and context-dependent approximations.

We consider Sections 2.2, 2.3, 2.5, and 2.6 essential background that will be required to understand the remainder of the book. Section 2.4 on Hamiltonian mechanics can be skipped on a first reading. Sections 2.8, 2.9, and 2.10 will be of interest to those wishing to gain an understanding of rolling-contact mechanics and nonholonomic constraints. Section 2.7 looks at some of the deeper issues at the foundations of mechanics—this material can be skipped by those not interested in the more esoteric issues in classical mechanics.

2.2 Equations of motion

The equations of motion of a mechanical system can be derived using 'vectorial mechanics', or 'variational techniques (analytical mechanics)'. 'Vectorial mechanics' is based on forces and moments, and builds on Newton's three laws (published in 1687 in the *Principia Mathematica*). 'Analytical mechanics' is based on kinetic energy and work (sometimes replaceable by potential energy), and builds on the 'principle of stationary action', which was developed by Euler (1707–83), Lagrange (1736–13), Jacobi (1804–51), and Hamilton (1805–65). These analytical principles can all be derived from d'Alembert's (1717–83) principle, which is the dynamic analogue of the principle of virtual displacement for forces applied to a static system [44]. D'Alembert's principle itself makes use of Newton's second law.

We will now briefly review the basics of analytical and vector mechanics, since both are used to obtain the equations of motion of mechanical systems.

2.2.1 Inertial reference frame

In order to describe and analyse mechanical systems one requires a frame of reference to which positions can be referred. In general, the laws of motion take different forms in different reference frames. A coordinate system in which a free particle that is instantaneously at rest remains at rest, for all time, is called *inertial*. The laws of mechanics are the same at all moments in time in all inertial coordinate systems. Any coordinate system in uniform rectilinear motion relative to an inertial coordinate system is itself inertial. In the context of vehicle dynamics problems, any earth-fixed frame of reference will be treated as inertial, although this is not strictly true.

2.2.2 Newton's equations

Newton's laws of motion laid the foundations of classical mechanics. They describe the relationship between a body, the forces acting on the body, and the body's subsequent motion. They have been expressed in several different ways, but can be summarized as follows:

1. When viewed in an inertial reference frame, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.
2. The vector sum $\sum \boldsymbol{F}$ of all the external forces acting on an object is equal to the mass m of that object multiplied by the acceleration $\boldsymbol{a}$ of the object (bold characters are used to denote vector quantities).
3. When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

A particle of mass m, moving with velocity $\boldsymbol{v}$, is deemed to have *linear momentum* $\boldsymbol{P} = m\boldsymbol{v}$. It follows from the second law that

$$\frac{d\boldsymbol{P}}{dt} = \boldsymbol{F}, \tag{2.1}$$

in which $\boldsymbol{F}$ is the force acting on the particle. If $\boldsymbol{r}$ is the position of m relative to some (fixed) point O, then the particle's *angular momentum* is given by $\boldsymbol{H}_O = \boldsymbol{r} \times m\boldsymbol{v}$. Premultiplication of (2.1) by $\boldsymbol{r}$ gives

$$\frac{d\boldsymbol{H}_O}{dt} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{M}_O, \tag{2.2}$$

where $\boldsymbol{M}_O$ is the *moment* of $\boldsymbol{F}$ around O.

In the case of a constellation of particles, we have

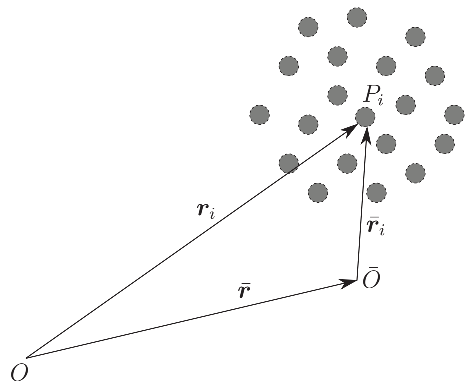
$$\begin{aligned}\boldsymbol{P} &= \sum \boldsymbol{P}_i \\ &= \sum m_i \dot{\boldsymbol{r}}_i \\ &= (\sum m_i)\frac{d}{dt}\left(\frac{\sum m_i \boldsymbol{r}_i}{\sum m_i}\right) \\ &= m\dot{\boldsymbol{r}}_G,\end{aligned}$$

where m is the constellation's mass and $\boldsymbol{r}_G$ is the position of its mass centre. Thus

$$\begin{aligned}\boldsymbol{F} &= \sum \boldsymbol{F}_i = \frac{d\boldsymbol{P}}{dt} \\ &= m\ddot{\boldsymbol{r}}_G; \end{aligned} \tag{2.3}$$

for the constellation.

Now suppose that $\boldsymbol{r}_i = \bar{\boldsymbol{r}} + \bar{\boldsymbol{r}}_i$:



If $\boldsymbol{v}_i$ is the absolute velocity of the ith particle, there holds

$$\begin{aligned}\boldsymbol{H}_O &= \sum \boldsymbol{r}_i \times m_i \boldsymbol{v}_i \\ &= \sum (\bar{\boldsymbol{r}} + \bar{\boldsymbol{r}}_i) \times m_i \boldsymbol{v}_i \\ &= \bar{\boldsymbol{r}} \times \sum m_i \boldsymbol{v}_i + \sum \bar{\boldsymbol{r}}_i \times m_i \boldsymbol{v}_i \\ &= \bar{\boldsymbol{r}} \times \boldsymbol{P} + \boldsymbol{H}_{\bar{O}}, \end{aligned} \tag{2.4}$$

which is the constellation's angular momentum (about O). If $\bar{O}$ is the constellation's mass centre, (2.4) becomes

$$\boldsymbol{H}_O = \boldsymbol{r}_G \times \boldsymbol{P} + \boldsymbol{H}_G. \tag{2.5}$$

We now suppose that the particles are the constituents of a *rigid body*. Suppose also that $\bar{O}$ is the origin of a moving reference frame which has translational velocity $\bar{\boldsymbol{v}}$ and angular velocity $\boldsymbol{\omega}$. In this case the absolute velocity of each particle is given by

$$\boldsymbol{v}_i = \bar{\boldsymbol{v}} + \boldsymbol{\omega} \times \bar{\boldsymbol{r}}_i.$$

This gives

$$\begin{aligned}
\boldsymbol{H}_{\bar{O}} &= \sum m_i \bar{\boldsymbol{r}}_i \times \boldsymbol{v}_i \\
&= \sum m_i \bar{\boldsymbol{r}}_i \times (\bar{\boldsymbol{v}} + \boldsymbol{\omega} \times \bar{\boldsymbol{r}}_i) \\
&= \sum m_i \bar{\boldsymbol{r}}_i \times \bar{\boldsymbol{v}} + \sum m_i (\bar{\boldsymbol{r}}_i \times (\boldsymbol{\omega} \times \bar{\boldsymbol{r}}_i)) \\
&= m\bar{\boldsymbol{r}}_G \times \bar{\boldsymbol{v}} + \sum m_i (\boldsymbol{\omega}(\bar{\boldsymbol{r}}_i \cdot \bar{\boldsymbol{r}}_i) - \bar{\boldsymbol{r}}_i(\boldsymbol{\omega} \cdot \bar{\boldsymbol{r}}_i)) \\
&= m\bar{\boldsymbol{r}}_G \times \bar{\boldsymbol{v}} + \sum m_i \left((\bar{\boldsymbol{r}}_i \cdot \bar{\boldsymbol{r}}_i) I - \bar{\boldsymbol{r}}_i \bar{\boldsymbol{r}}_i^T \right) \boldsymbol{\omega} \\
&= m\bar{\boldsymbol{r}}_G \times \bar{\boldsymbol{v}} + J_{\bar{O}} \boldsymbol{\omega},
\end{aligned} \tag{2.6}$$

where $J_{\bar{O}}$ is the *moment of inertia tensor* of the rigid body around $\bar{O}$. If the summation in (2.6) is replaced by a continuous integral the inertia tensor can be represented by a symmetric matrix of the form

$$\boldsymbol{J}_{\bar{O}} = \begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{xy} & J_{yy} & J_{yz} \\ J_{xz} & J_{yz} & J_{zz} \end{bmatrix} \quad J_{kk} = \int (i^2 + j^2) dm \quad J_{ij} = -\int ij dm \quad i \neq j. \tag{2.7}$$

If $\bar{O}$ is a stationary point there holds

$$\boldsymbol{H}_{\bar{O}} = J_{\bar{O}} \boldsymbol{\omega}. \tag{2.8}$$

When $\bar{\boldsymbol{r}}$ is coincident with the body's mass centre (i.e. $\bar{O} = G$), (2.6) becomes

$$\begin{aligned}
\boldsymbol{H}_{\bar{O}} &= m\bar{\boldsymbol{r}}_G \times \bar{\boldsymbol{v}} + J_G \boldsymbol{\omega} \\
&= \bar{\boldsymbol{r}}_G \times \boldsymbol{P} + J_G \boldsymbol{\omega}.
\end{aligned} \tag{2.9}$$

When $\bar{\boldsymbol{r}}_G = 0$ the mass centre coincides with O, and

$$\boldsymbol{H}_O = J_G \boldsymbol{\omega}. \tag{2.10}$$

We conclude this this brief summary of vector mechanics by finding the rate of change of angular momentum about a moving point that coincides instantaneously with a fixed point. Consider (2.4) in which point O is fixed, while point $\bar{O}$ is moving. Differentiating this equation with respect to time gives

$$\frac{d\boldsymbol{H}_O}{dt} = \frac{d\boldsymbol{H}_{\bar{O}}}{dt} + \dot{\bar{\boldsymbol{r}}} \times \boldsymbol{P} + \bar{\boldsymbol{r}} \times \frac{d\boldsymbol{P}}{dt}.$$

At the moment of coincidence, $\bar{\boldsymbol{r}} = 0$, and $\dot{\bar{\boldsymbol{r}}} = \bar{\boldsymbol{v}}$ at the same instant. That is

$$\boldsymbol{M}_O = \frac{d\boldsymbol{H}_O}{dt} = \left. \frac{d\boldsymbol{H}_{\bar{O}}}{dt} \right|_{O=\bar{O}} + \bar{\boldsymbol{v}} \times \boldsymbol{P}. \tag{2.11}$$

The $\bar{\boldsymbol{v}} \times \boldsymbol{P}$ term is zero when point $\bar{O}$ is either fixed, or it coincides with G, or $\bar{\boldsymbol{v}}$ is parallel to $\boldsymbol{v}_G$, where $\boldsymbol{v}_G$ is the velocity of the mass centre. Thus the two rates of change of angular momentum in (2.11) are equal if the moving point is instantaneously at rest, or it is moving parallel to the mass centre, or when the mass centre is at rest.

In sum, the dynamics of a rigid body are governed by the translational equation (2.1) and the rotational equations given in (2.11). A comprehensive treatment of vector mechanics can be found in [46].

2.2.3 Properties of the inertia tensor

The inertia tensor (matrix) has some useful properties that we will now review briefly.

The *perpendicular axis theorem* applies to laminatae, that is, thin flat bodies lying in the plane. If J_x and J_y are the moments of inertia about the x- and y-axes, respectively, which lie in the laminate, then the moment of inertia about the z-axis is[1]

$$J_z = J_x + J_y. \tag{2.12}$$

The *parallel axis theorem* applies to general rigid bodies. Let J_G be the moment of inertia of a rigid body, with mass m, about an axis through its mass centre G. Then the moment of inertia around a parallel axis through a point O is given by

$$J_O = J_G + m\left(\boldsymbol{r}^T\boldsymbol{r}I - \boldsymbol{r}\boldsymbol{r}^T\right), \tag{2.13}$$

where I is the identity matrix and $\boldsymbol{r}$ is the position of G relative to O.

Since the inertia matrix is symmetric and non-negative, it has real non-negative eigenvalues and orthogonal eigenvectors. The eigenvalues are the principal moments of inertia and the (orthogonal) eigenvectors are the principal axes (of the inertia matrix). To reference the inertia tensor to alternative axes, one uses

$$J_F = \mathcal{R}^T J_I \mathcal{R}, \tag{2.14}$$

in which J_I and J_F are referenced to the initial and final coordinate systems, respectively, and $\mathcal{R}$ is the transformation between the two (see Section 2.5).

In most vehicle-related problems a vertical plane of symmetry can be assumed, with the pitch axis a principal axis that is perpendicular to the plane of symmetry. The other two axes lie in the plane of symmetry itself, with their locations fixed by experiment.

Suppose that in Figure 2.1 the roll moment of inertia is measured as J_ϕ, that the yaw moment of inertia is measured as J_ψ, and that the product of inertia is $J_{\phi\psi}$. The corresponding principal moments of inertia J_1 and J_2, and the inclination angle μ of the principal axis system, are given by

$$J_{1,2} = \frac{J_\phi + J_\psi \mp \sqrt{(J_\phi - J_\psi)^2 + 4J_{\phi\psi}^2}}{2} \qquad \mu = \frac{1}{2}\arctan\left(\frac{-2J_{\phi\psi}}{J_\phi - J_\psi}\right). \tag{2.15}$$

In the case that $J_{\phi\psi} > 0$ and $J_\phi < J_\psi$, a typical motorcycle scenario, μ is positive as shown in Figure 2.1.

[1] $J_z \leq J_x + J_y$ in the case of a thick laminate.

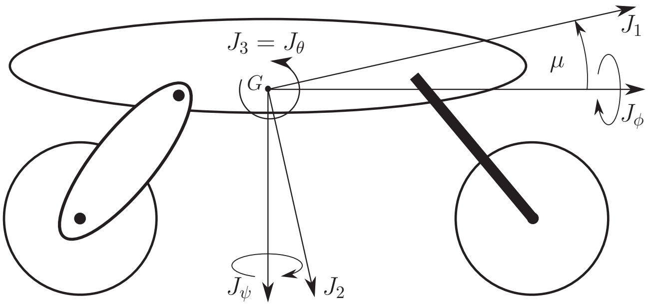


Figure 2.1: Inertia axis systems for a symmetric road vehicle.

Example 2.1 Suppose a motorcycle has a mass $m^M = 200\,\text{kg}$, with its centre of mass a distance $b^M = 0.75\,\text{m}$ from the rear ground-contact point and at a height $h^M = 0.55\,\text{m}$ above the ground. When a rider of mass $m^R = 80\,\text{kg}$ sits on the saddle, his centre of mass is located at $b^R = 0.60\,\text{m}$ from the rear contact point and $h^R = 0.90\,\text{m}$ from the ground. Supposing the motorcycle has inertia matrix J^M and the rider has inertia matrix J^R with respect to their centres of mass (the x-axis points forwards, the z-axis points downwards, and the y-axis points to the rider's right)

$$J^M = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 45 & 0 \\ 2 & 0 & 35 \end{bmatrix} \qquad J^R = \begin{bmatrix} 6.5 & 0 & 0 \\ 0 & 6.5 & 0 \\ 0 & 0 & 4.0 \end{bmatrix}.$$

Compute the position of the centre of mass, and the inertia matrix of the motorcycle–rider combination. Using the definition of the centre of mass and (2.13), one obtains

$$m^{M+R} = 280 \qquad b^{M+R} = 0.71 \qquad h^{M+R} = 0.65 \qquad J^{M+R} = \begin{bmatrix} 23.5 & 0 & -1 \\ 0 & 59.8 & 0 \\ -1 & 0 & 40.3 \end{bmatrix},$$

where J^{M+R} is defined with respect to the whole-system mass centre. The principal moments of inertia in Figure 2.1 of the motorcycle–rider assembly are $J_1 = 23.4\,\text{kg}\,\text{m}^2$ and $J_2 = 40.3\,\text{kg}\,\text{m}^2$, while the inclination angle is $\mu = -3.4$ degrees.

2.2.4 Generalized coordinates

In order to describe the configuration of a system of N rigid bodies as many as $6N$ coordinates might be needed; this number will reduce if the motions of these bodies is restricted in some way. Any $n \leq 6N$ quantities $q_1, q_2, \cdots, q_n$ that completely determine the configuration of the system are called *generalized coordinates*, with the *generalized velocities* given by $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$ in which the 'dot' denotes the derivative with respect to time. Taken together, the generalized coordinates and the generalized velocities can be used to determine the *state* of the system; a system of N rigid bodies may require as many as $12N$ states to describe its motion. The relationships between the generalized coordinates, the generalized velocities, and the generalized accelerations are called the *equations of motion*. The solution of the mechanical problem can be thought of as a curve described by a point C with coordinates $q_1, q_2, \cdots, q_n$ moving on an n-dimensional manifold in the configuration space. Given the n initial generalized coordinates $q_1, q_2, \cdots, q_n$, and the n initial generalized velocities $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$, the solution is obtained by integration.

The $2n$-dimensional space with coordinates $q_1, q_2, \cdots, q_n$ and $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$ is called the *phase space*. When time is included, the resulting $2n+1$ dimensional space is called the *state space*. If the system's motion starts at some point P in the state space, its subsequent motion is determined by the *equations of motion*. The equations of motion provide a set transformations

$$\begin{aligned} q_i(t) &= f_i(q_1, q_2, \cdots, q_n; \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n; t) \\ \dot{q}_i(t) &= g_i(q_1, q_2, \cdots, q_n; \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n; t), \end{aligned}$$

called the *phase flow*; this name comes from the analogy with fluid flows [44].

2.2.5 Constraints

Constraints on the generalized coordinates are called *holonomic* constraints. Constraints that can only be given as relationships between differentials of the coordinates (and not as finite relationships between the coordinates themselves) are called *nonholonomic* constraints (after Hertz). In addition, constraints can be independent of time or time-dependent. These constraints are called *scleronomic* and *rheonomic* respectively (after Bolzmann). The four possibilities can be summarized as follows

$$\begin{array}{lll} & \text{holonomic} & \text{nonholonomic} \\ \text{scleronomic} & \left\{\begin{array}{c} f_1(q_1, \cdots, q_n) = 0 \\ \vdots \\ f_m(q_1, \cdots, q_n) = 0 \end{array}\right. & \left\{\begin{array}{c} F_{11}dq_1 + \cdots + F_{1n}dq_n = 0 \\ \vdots \\ F_{m1}dq_1 + \cdots + F_{mn}dq_n = 0 \end{array}\right. \\ \\ \text{rheonomic} & \left\{\begin{array}{c} f_1(q_1, \cdots, q_n, t) = 0 \\ \vdots \\ f_m(q_1, \cdots, q_n, t) = 0 \end{array}\right. & \left\{\begin{array}{c} F_{11}dq_1 + \cdots + F_{1n}dq_n + F_{1t}dt = 0 \\ \vdots \\ F_{m1}dq_1 + \cdots + F_{mn}dq_n + F_{mt}dt = 0. \end{array}\right. \end{array}$$

If the motion is restricted by m constraints, these constraints must be appended to the equations of motion, thus increasing by m the number of equations (and variables) governing the system's behaviour. The additional variables are the constraint forces in the case of the vectorial mechanics framework and Lagrange multipliers in the case of the variational mechanics framework. In the case of holonomic constraints only $n - m$ of the q s and $n - m$ of the $\dot{q}$ s can be defined arbitrarily, with the remainder determined by the constraint equations and their derivatives. In the case of nonholonomic constraints, n of q s and $n - m$ of $\dot{q}$ s can be defined arbitrarily, with the remainder determined by the constraints equations.

Holonomic constraints of the form

$$f(q_1, \cdots, q_n) = 0 \tag{2.16}$$

can be 'disguised' as nonholonomic by writing

$$\begin{aligned} 0 &= \frac{\partial f}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial f}{\partial q_n}\dot{q}_n \\ &= F_1 dq_1 + \cdots + F_n dq_n. \end{aligned} \tag{2.17}$$

The converse, however, is not necessarily true, because the kinematic constraint may not be integrable. Constraint (2.17) can only be integrated if there exists an *integrating factor* $\beta(q)$ such that

$$\beta(q)F_j(q) = \frac{\partial h(q)}{\partial q_j}, \qquad j = 1, \cdots, n, \tag{2.18}$$

for some function $h(q)$. Conversely, if $\beta(q)F(q)$ is the gradient of $h(q)$, then (2.17) is integrable. Schwartz's theorem allows (2.18) to be replaced by

$$\frac{\partial(\beta F_k)}{\partial q_j} = \frac{\partial(\beta F_j)}{\partial q_k} \qquad j, k = 1, \cdots, n, \tag{2.19}$$

which does not involve $h(q)$.

Example 2.2 Consider the following differential constraint [44]:

$$\left(q_2^2 - q_1^2 - q_3\right)\dot{q}_1 + \left(q_3 - q_2^2 - q_1 q_2\right)\dot{q}_2 + q_1\dot{q}_3 = 0. \tag{2.20}$$

When written in matrix form, the integrability conditions are

$$\begin{bmatrix} q_1 q_2 + q_2^2 - q_3 & -q_1^2 + q_2^2 - q_3 & 0 & 3\,q_2 \\ -q_1 & 0 & -q_1^2 + q_2^2 - q_3 & -2 \\ 0 & -q_1 & -q_1 q_2 - q_2^2 + q_3 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial\beta}{\partial q_1} \\ \frac{\partial\beta}{\partial q_2} \\ \frac{\partial\beta}{\partial q_3} \\ \beta \end{bmatrix} = 0.$$

After Gaussian elimination it is found that

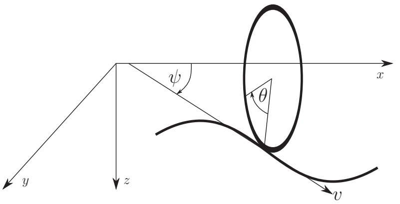
$$\begin{bmatrix} q_1 q_2 + q_2^2 - q_3 & -q_1^2 + q_2^2 - q_3 & 0 & 3\,q_2 \\ 0 & -\frac{q_1\left(q_1^2 - q_2^2 + q_3\right)}{q_1 q_2 + q_2^2 - q_3} & -q_1^2 + q_2^2 - q_3 & \frac{q_1 q_2 - 2\,q_2^2 + 2\,q_3}{q_1 q_2 + q_2^2 - q_3} \\ 0 & 0 & 0 & \frac{q_1^2 - q_1 q_2 + q_2^2 - q_3}{q_1^2 - q_2^2 + q_3} \end{bmatrix} \begin{bmatrix} \frac{\partial\beta}{\partial q_1} \\ \frac{\partial\beta}{\partial q_2} \\ \frac{\partial\beta}{\partial q_3} \\ \beta \end{bmatrix} = 0.$$

The fourth row shows that β is free when

$$q_1^2 - q_1 q_2 + q_2^2 - q_3 = 0, \tag{2.21}$$

which is the holonomic equivalent to (2.20). This can be checked by substituting (2.21) into (2.20).

Example 2.3 Consider the classical example of a disc that rolls vertically in the plane without slipping [41].



The coordinates used to describe the motion of the disc are the x and y coordinates of the disc's centre, the angular position θ of the disc, and its orientation ψ relative to the x-axis. To ensure pure rolling one must enforce the (rolling) constraint

$$v = r\dot{\theta}, \tag{2.22}$$

where r is the disc's radius. Projecting the velocity vector onto the x and y axes gives

$$\dot{x} = r\dot{\theta}\cos\psi \tag{2.23}$$

$$\dot{y} = r\dot{\theta}\sin\psi. \tag{2.24}$$

One can establish that these constraints are nonholonomic, by showing that the disc can be rolled from an arbitrary initial configuration $x_i, y_i, \psi_i, \theta_i$ to an arbitrary final configuration $x_f, y_f, \psi_f, \theta_f$ as follows:

(1) Place the disc at $x_i, y_i, \psi_i, \theta_i$;
(2) roll the disc from $x_i, y_i, \psi_i, \theta_i$ to x_f, y_f along any path of length $r(\theta_f - \theta_i + 2k\pi)$;
(3) rotate the disc around the vertical axis to ψ_f.

This confirms that the two constraints do not impose a constraint on the generalized coordinates.

Nonholonomy can also be established using (2.19). Since the problem has four generalized coordinates, there are six conditions associated with each of (2.23) and (2.24). Relevant to the present argument are these:

$$\frac{\partial\beta_1}{\partial\psi} = 0 \tag{2.25}$$

$$-\frac{\partial\beta_1}{\partial\psi} r\cos\psi + \beta_1 r\sin\psi = 0 \tag{2.26}$$

$$\frac{\partial\beta_2}{\partial\psi} = 0 \tag{2.27}$$

$$-\frac{\partial\beta_2}{\partial\psi} r\sin\psi - \beta_2 r\cos\psi = 0, \tag{2.28}$$

in which β_1 and β_2 are the integrating factors related to (2.23) and (2.24) respectively. It follows that $\beta_1 r\sin\psi = 0$ and $\beta_2 r\cos\psi = 0$, which means that $\beta_1 \equiv 0$ and $\beta_2 \equiv 0$, indicating that (2.23) and (2.24) are nonholonomic. In the special case that $\psi = k\pi$ (k integer), β_1 is arbitrary indicating that (2.23) is holonomic and can be integrated to give $x = r\theta +$ a constant. In this case the disc is rolling parallel to the x-axis and $\dot{y} = 0$; (2.24) plays no further part. A similar argument can be developed using (2.24) when $\psi = \pi/2 + k\pi$. In each of these special cases only two variables are involved and differential relationships between two variables are always integrable [44].

2.2.6 Kinetic energy

Leibniz (1646–1716) noticed that in many mechanical systems the quantity $\sum_i m_i \boldsymbol{v}_i^T \boldsymbol{v}_i$ was conserved. He called this the *vis viva* of the system. This quantity (apart from a factor of two), is what we now call kinetic energy, which for rigid bodies generally takes the form

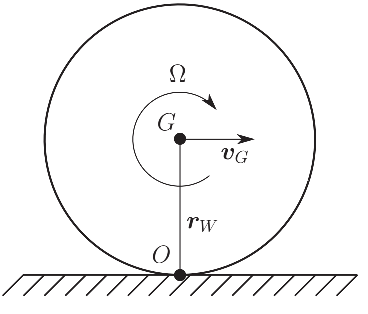
$$T = \frac{m}{2}\boldsymbol{v}_G^T\boldsymbol{v}_G + \frac{1}{2}\boldsymbol{\omega}^T J_G\boldsymbol{\omega}, \tag{2.29}$$

where m is the body's mass, $\boldsymbol{v}_G$ the velocity of its mass centre, $\boldsymbol{\omega}$ is the body's angular velocity, and J_G is the body's inertia tensor with respect to its mass centre (see Section 2.2.2). The first term accounts for the body's translation energy, while the second accounts for its rotation energy. Alternatively, the kinetic energy can be written as

$$T = \frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{J}_O\boldsymbol{\omega}, \tag{2.30}$$

in which the inertia J_O is computed with respect to a fixed point O in the body, which is instantaneously absolutely stationary.

Example 2.4 Consider the non-slipping rolling wheel in the figure below:



The wheel's kinetic energy is given by

$$T = \frac{m}{2}\boldsymbol{v}_G^T\boldsymbol{v}_G + \frac{1}{2}\boldsymbol{\Omega}^T J_G\boldsymbol{\Omega},$$

where m is the wheel's mass and J_G is its spin moment of inertia. If $\boldsymbol{r}_W$ is a vector pointing from O to G, then for non-slip rolling

$$\boldsymbol{v}_G = \boldsymbol{\Omega} \times \boldsymbol{r}_W$$

and

$$\begin{aligned} T &= \frac{m}{2}(\boldsymbol{\Omega} \times \boldsymbol{r}_W)^T(\boldsymbol{\Omega} \times \boldsymbol{r}_W) + \frac{1}{2}\boldsymbol{\Omega}^T J_G\boldsymbol{\Omega} \\ &= \frac{1}{2}\boldsymbol{\Omega}^T \left(m\boldsymbol{r}_W^2 + J_G\right)\boldsymbol{\Omega} \\ &= \frac{1}{2}\boldsymbol{\Omega}^T J_O\boldsymbol{\Omega} \end{aligned}$$

by the parallel axis theorem. Point O is fixed in the wheel and momentarily absolutely stationary.

2.2.7 Potential energy

Consider a mechanical system that is subject to a force $\boldsymbol{F} = [F_x, F_y, F_z]^T$ and an infinitesimal displacement $d\boldsymbol{s} = [dx, dy, dz]^T$ of the force's point of application. The work done is

$$dW = \boldsymbol{F} \cdot d\boldsymbol{s} = F_x dx + F_y dy + F_z dz, \tag{2.31}$$

which can be rewritten in terms of the generalized coordinates that describe the system configuration as

$$dW = \sum_{i=1}^{n} Q_i dq_i, \tag{2.32}$$

where the Q_i s are the components of the *generalized force* $\boldsymbol{Q}$.

Of central importance in analytical mechanics are forces that are derivable from a scalar *work function* $V(q_1, q_2, \cdots, q_n, t)$,[2] for which

$$Q_i = -\frac{\partial V}{\partial q_i}. \tag{2.33}$$

In the case that V is not explicitly time dependent, the related forces are *conservative*, which implies that the work done depends on the initial and final states only, and not on the path taken.[3] In this case the work function can be interpreted as a *potential energy function* $V = V(\boldsymbol{q})$. The work done by the force between an initial state $\boldsymbol{q}_i$ and a final states $\boldsymbol{q}_f$ is

$$W = V(\boldsymbol{q}_i) - V(\boldsymbol{q}_f). \tag{2.34}$$

Typical potential energy expressions are those related to the energy stored in a spring with stiffness k and deflection l

$$V = k\frac{l^2}{2} \Rightarrow Q_k = -kl \tag{2.35}$$

and to gravitational acceleration g of a body with mass M and centre of mass at $\boldsymbol{r}_G$

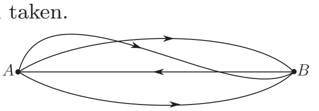
$$V = -M\boldsymbol{r}_G \cdot \boldsymbol{g} \Rightarrow Q_g = Mg; \tag{2.36}$$

the potential is negative when the vertical axis is pointing downwards.

---

[2] In the most general case the work function may be of the form $V = V(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ [44].

[3] Consider the closed contour C in the figure below, and suppose that the work done going from A to B is independent of the path taken.



This means that $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$, since one could choose the forward path from A to B to be the return path taken in reverse. It then follows from Stoke's theorem that $\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 0$ for any surface bounded by C. Since this is true for any contour C containing A and B it follows that $\nabla \times \mathbf{F} = 0$; conservative force fields are irrotational. We therefore conclude that $\mathbf{F} = \nabla V$ for some scalar potential function V.

Forces derivable from work functions are called *monogenic*, while forces such as friction, that are not derivable from work functions, are called *polygenic* [44].

2.2.8 Lagrange's equations

Lagrange's equations are now introduced. We will assume temporarily that the system is not subject to constraints and that all the external forces are monogenic. Hamilton's principle, or the *principle of stationary action*, states that the motion of an arbitrary mechanical system evolves in such a way that a certain definite integral, called the *action integral*, is stationary for all possible variations of the system configuration. The action integral $\mathcal{S}$ is given by

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}(q_1, \cdots q_n, \dot{q}_1, \cdots \dot{q}_n, t)dt \qquad \mathcal{L} = T - V, \tag{2.37}$$

where $\mathcal{L}$ is the Lagrangian, T is the kinetic energy, and V is the potential energy of the system. If $\boldsymbol{q}$ is the vector of generalized coordinates, (2.37) can be written more compactly as

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)dt. \tag{2.38}$$

The principle of stationary action requires that the first variation in the action integral

$$\delta\mathcal{S} = \delta \int_{t_1}^{t_2} \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)dt \tag{2.39}$$

vanishes. Standard variational arguments [44] show that condition (2.39) can only be satisfied if the system Lagrangian satisfies the n Euler–Lagrange equations [43]

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = 0. \tag{2.40}$$

If holonomic constraints are present, the generalized coordinates $q_1, \cdots, q_n$ are not independent, but are restricted by constraints of the form

$$f_j(\boldsymbol{q}, t) = 0, \qquad\qquad (j = 1, \cdots, m), \tag{2.41}$$

which can be written in vector form as $\boldsymbol{f}(\boldsymbol{q}, t) = 0$. These equations allow m of the q_i s to be expressed in terms of the remaining $n - m$ variables; the problem is thus reduced to one having $n - m$ degrees of freedom. We can proceed in two ways. Either we eliminate $n - m$ generalized coordinates 'up front', thus transforming the constrained problem in n variables into an unconstrained problem in $n - m$ variables, or we append the constraint conditions to the integrand of the action integral using *Lagrange multipliers*. In the Lagrange multiplier method the integrand in (2.37) is modified to

$$\hat{\mathcal{L}} = \mathcal{L} - \boldsymbol{\lambda}^T \boldsymbol{f}, \tag{2.42}$$

in which $\boldsymbol{\lambda}$ and $\boldsymbol{f}$ are vectors of Lagrange multipliers and constraints respectively. Following the introduction of the Lagrange multipliers, (2.40) can be expressed as

$$\frac{d}{dt}\left(\frac{\partial \hat{\mathcal{L}}}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial \hat{\mathcal{L}}}{\partial \boldsymbol{q}} = 0, \tag{2.43}$$

which highlights the fact that holonomic constraints can be treated within a variational framework. If one considers the constraints to be part of the potential energy function

$$\hat{V} = V + \boldsymbol{\lambda}^T \boldsymbol{f}, \tag{2.44}$$

$\hat{V}$ will generate the conservative forces, as well as the forces of constraint. Equation (2.43) can also be expressed as

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = \boldsymbol{k}, \tag{2.45}$$

in which the generalized reaction force $\boldsymbol{k}$ is given by

$$\boldsymbol{k} = -A(\boldsymbol{q})^T \boldsymbol{\lambda}; \tag{2.46}$$

$\boldsymbol{k}$ is an n-vector, $\boldsymbol{\lambda}$ is an m-vector and $A(\boldsymbol{q})$ is an $m \times n$ matrix, whose entries are given by

$$A_{ji}(\boldsymbol{q}) = \left\{\frac{\partial f_j}{\partial q_i}\right\}.$$

The Lagrange multipliers provide the monogenic forces of reaction that maintain the kinematic constraints.[4]

Nonholonomic systems (coined by Hertz in 1894) are typified by velocity-dependent constraints that are not derivable from position constraints. As a result of the velocity constraints, nonholonomic systems are not variational, but the basic mechanics are still governed by Newton's second law. There are some fascinating differences between nonholonomic systems and classical Lagrangian systems. First, the momenta conjugate to cyclic coordinates (Section 2.3.2) may not be conserved as they would be in a holonomic system [41]; further details follow. Second, even in the absence of external forces and dissipation, nonholonomic systems may possess asymptotically stable relative equilibria despite being (energy) conservative. This property will be illustrated in the context of Čhaplygin's Sleigh (Section 2.8), the rolling disc (Section 2.10), and the Whipple bicycle (Chapter 5).

Since nonholonomic constraints place no restrictions on the generalized coordinates, initial values on the q_i s can be assigned arbitrarily. The velocities, however, are constrained by

$$F_{j1}\dot{q}_1 + \cdots + F_{jn}\dot{q}_n + F_{jt} = 0; \qquad (j = 1, \cdots, m), \tag{2.47}$$

in which the F_{ij} s are given functions of the q_i s. In this case we can assign arbitrarily n initial coordinates and $n - m$ initial velocities; (2.47) serves to eliminate the other

[4] Note that no $\partial \boldsymbol{f}/\partial t$ term appears in (2.46); virtual displacements only consider displacements of the coordinates. Under these conditions the forces of constraint remain perpendicular to the virtual displacements (even if the constraint is time-varying). In the case of time-varying constraints, however, the real work done by the constraint force may be non-zero.

m initial velocities. In the case of nonholonomic constraints the forces of constraint are not drivable from a work function, but are given by

$$\boldsymbol{k} = -A(\boldsymbol{q})^T \boldsymbol{\lambda} \tag{2.48}$$

where the matrix $A(q)$ is given by

$$A_{ji}(\boldsymbol{q}) = \{F_{ji}\} \qquad A(\boldsymbol{q})\dot{\boldsymbol{q}} = -F_t,$$

which is similar in form to (2.46). Once more these forces act on the system so as to enforce the given nonholonomic constraints; (2.48) does not include the terms F_{jt} even when the constraints are rheonomic.

Finally, one can characterize external polygenic forces (e.g. aerodynamic forces and friction) in terms of their virtual work [44]. Let that work be

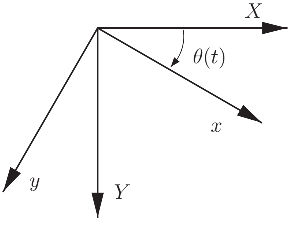
$$\delta W_j = Q_{j1}\delta q_1 + \cdots + Q_{jn}\delta q_n \qquad (j = 1, \cdots, m), \tag{2.49}$$

where Q_{ji} are the components of the generalized forces and δq_i are the components of the virtual displacement. These (polygenic) forces again produce a right-hand-side term in the Lagrange equations

$$k_i = \sum_{j=1}^{m} Q_{ji} \qquad (i = 1, \cdots, n). \tag{2.50}$$

We conclude this subsection with a number of examples that either illustrate key points of the theory, or else uncover issues we have not discussed in our brief overview of classical mechanics.

Example 2.5 Consider a particle whose position is (x, y) in a rotating coordinate system x–y, which rotates relative to an inertial frame X–Y through some time-varying angle $\theta(t)$.



The coordinates of the particle in the inertial frame are given by

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{2.51}$$

The particle's velocity comes from differentiating (2.51) with respect to time

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \left\{ \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} - \dot{\theta} \begin{bmatrix} y \\ -x \end{bmatrix} \right\}. \tag{2.52}$$

The particle's acceleration is thus

$$\begin{bmatrix} \ddot{X} \\ \ddot{Y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \left\{ \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} - \dot{\theta}^2 \begin{bmatrix} x \\ y \end{bmatrix} - \ddot{\theta} \begin{bmatrix} y \\ -x \end{bmatrix} - 2\dot{\theta} \begin{bmatrix} \dot{y} \\ -\dot{x} \end{bmatrix} \right\}. \tag{2.53}$$

The first terms on the right-hand side of (2.53) are rotating-frame acceleration terms, the second terms are the *centrifugal acceleration* terms, the third terms are the *Euler acceleration* terms that derive from the angular acceleration of the moving frame (relative to the inertial frame), while the last terms are the *Coriolis acceleration* terms. These accelerations (forces) are sometimes referred to as *fictitious*, *apparent*, or *non-inertial* accelerations (forces), because they result from the use of non-inertial frames of reference.

Example 2.6 Consider the system of interconnected masses illustrated in Figure 2.2.

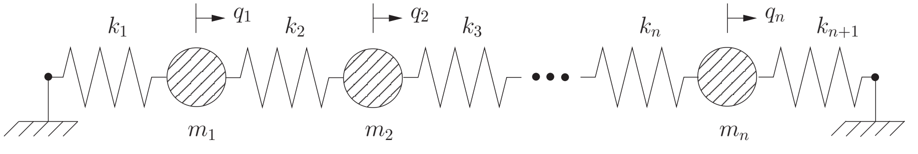


Figure 2.2: A string of interconnected springs and masses with colinear translational freedoms. The masses have mass m_i, the springs have stiffness k_i, and the displacement of the masses from their rest positions is denoted q_i.

A direct application of Newton's second law gives

$$M\ddot{\boldsymbol{q}} + K\boldsymbol{q} = 0 \tag{2.54}$$

in which the mass matrix is

$$M = \begin{bmatrix} m_1 & 0 & \cdots & 0 & 0 \\ 0 & m_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & m_n \end{bmatrix} \tag{2.55}$$

and the stiffness matrix is

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & \ddots & \vdots \\ 0 & \vdots & \vdots & \vdots & 0 \\ \vdots & \ddots & k_{n-1} & k_{n-1} + k_n & -k_n \\ 0 & \cdots & 0 & -k_n & k_n + k_{n+1} \end{bmatrix}. \tag{2.56}$$

By introducing the vector of intermediate variables $\boldsymbol{p}$ we can write these equations as

$$\begin{bmatrix} 0 & I \\ M & 0 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -K \end{bmatrix} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{bmatrix}, \tag{2.57}$$

or equivalently

$$\begin{bmatrix} \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} 0 & -M^{-1}K \\ I & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{bmatrix}. \tag{2.58}$$

If a matrix H satisfies $JH = (JH)^T$, in which

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

then H is called Hamiltonian [55]. The eigenvalues of Hamiltonian matrices have well-known symmetry properties—if λ_i is an eigenvalue of H, then so is $-\lambda_i$. If the entries of H are real, and if λ_i is an eigenvalue, then so is $\bar{\lambda}_i$ (the bar denotes complex conjugate). Since the mass and stiffness matrices are symmetric, it is easy to show that the $2n \times 2n$ matrix in (2.58) is Hamiltonian. The $2n$ eigenvalues of H are given by $\pm i\omega_i$, where the ω_i s are the system's n modal frequencies.

The kinetic energy of each mass is $\frac{1}{2}m_i\dot{q}_i^2$, and so the system's total kinetic energy is given by the quadratic form $T(\dot{\boldsymbol{q}}) = \frac{1}{2}(\dot{\boldsymbol{q}}^T M\dot{\boldsymbol{q}})$. From this it follows that the inertial force term in (2.54) is given by the vector $\frac{d}{dt}(\partial T/\partial\dot{\boldsymbol{q}}) = M\ddot{\boldsymbol{q}}$. It is also clear that the strain energy in the central springs is given by $\frac{1}{2}k_i(q_i - q_{i-1})^2$, while that in the two end springs is given by $\frac{1}{2}k_1q_1^2$ and $\frac{1}{2}k_{n+1}q_n^2$; we make no distinction between the energy associated with compression and that associated with spring extensions. This means that the total strain (potential) energy is given by the quadratic form $V(\boldsymbol{q}) = \frac{1}{2}(\boldsymbol{q}^T K\boldsymbol{q})$ and the stiffness force term in (2.54) is given by the vector $(\partial V/\partial\boldsymbol{q}) = K\boldsymbol{q}$.

Example 2.7 When some coordinate systems are used the mass matrix becomes a function of the displacement coordinates. To see how this comes about we consider briefly the spring pendulum in Figure 2.3. Suppose the origin of an inertial polar coordinate system is located at the pendulum pivot. In this coordinate system the mass centre of the pendulum bob is located at

$$l = (r_0 + r)e^{i\theta}. \tag{2.59}$$

The complex quantity in (2.59) can be thought of as a radial vector that points away from the pendulum pivot at an angle of θ radians to the vertical; see Figure 2.3. The acceleration of the bob's mass centre can be found by differentiating the position vector twice

$$\ddot{l} = (\ddot{r} - (r_0 + r)\dot{\theta}^2)e^{i\theta} + i(2\dot{\theta}\dot{r} + (r_0 + r)\ddot{\theta})e^{i\theta}; \tag{2.60}$$

the second term in (2.60) represents a radial vector at an angle of $\theta + \frac{\pi}{2}$ to the vertical. Summing forces in the radial direction gives

$$m\ddot{r} = mg\cos\theta - kr + m(r_0 + r)\dot{\theta}^2 \tag{2.61}$$

in which k is the spring stiffness. Summing forces in the tangential direction gives

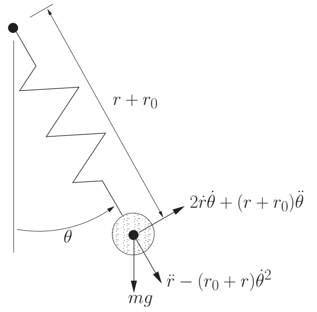


Figure 2.3: Spring pendulum system. The pendulum angle is θ, the pendulum's relaxed length is r_0, the pendulum's extended length is $r_0 + r$, and the bob mass is m.

$$(r_0 + r)\ddot{\theta} + 2\dot{\theta}\dot{r} = -g \sin\theta. \tag{2.62}$$

Equations (2.61) and (2.62) can be combined as

$$\begin{bmatrix} m & 0 \\ 0 & r_0 + r \end{bmatrix} \begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} mg\cos\theta - kr + m(r_0 + r)\dot{\theta}^2 \\ -2\dot{\theta}\dot{r} - g\sin\theta \end{bmatrix} \tag{2.63}$$

which is of the general form

$$M(\boldsymbol{q})\ddot{\boldsymbol{q}} = F(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

in which the mass matrix is position dependent. It is clear that the swing mode is coupled to the heave mode through a Coriolis acceleration term, while the heave mode is coupled to the swing mode via both gravitational and centripetal acceleration terms.[5]

2.2.9 Lagrange's equations in quasi-velocities

The Lagrange equations, as discussed in Section 2.2.8, are not always an efficient way of deriving a compact set of equations of motion. It is difficult, for example, to study the motion of tops and gyroscopes using the Euler–Lagrange equations in generalized coordinates and generalized velocities. As an alternative, Euler showed that expressing the equations of motion of moving bodies in terms of angular velocity components, relative to body-fixed axes, was effective in producing compact descriptions of the system dynamics. These angular velocity components are examples of *quasi-velocities* that may be used instead of conventional generalized velocities. Another example of the application of quasi-velocities is in vehicle dynamics, where velocities in vehicle-fixed axes gives a compact description of the vehicle's motion.

[5] For those already familiar with Lagrange mechanics, it should be easy to check that the pendulum system's kinetic energy is given by $T(\dot{\boldsymbol{q}}) = \frac{m}{2}(\dot{r}^2 + (r_0 + r)^2\dot{\theta}^2)$; that the potential energy is given by $V(\boldsymbol{q}) = \frac{1}{2}kr^2 - mg(r_0 + r)\cos\theta$; and that the Lagrangian is given by $\mathcal{L} = T - V$. Direct substitution of this Lagrangian into (2.40) gives (2.63). Is there anything you can say about $T(\dot{\boldsymbol{q}}) + V(\boldsymbol{q})$?

In the classical Lagrangian formalism nonholonomic constraints are appended to Lagrange's equations using Lagrange multipliers; see (2.45). The subsequent elimination of these multipliers using (2.47) can be a cumbersome process. In the approach to be described here, a judicious choice of quasi-velocities allows for the inclusion of nonholonomic constraints without invoking Lagrange multipliers. Procedurally, the unconstrained equations are derived first, with the constraints then enforced by setting the associated quasi-velocities to zero. The equations of motion relating to the nonholonomic constraints can then be used to find the forces of constraint; an example of the application of this process will be given in Section 2.8.

In Section 2.2.8 it was shown that the Lagrange's equations of motion take the form

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = \boldsymbol{k} \tag{2.64}$$

where $\mathcal{L}$ is given by (2.37), $\boldsymbol{q}$ is the vector of generalized variables, $\dot{\boldsymbol{q}}$ is the vector of generalized velocities, and $\boldsymbol{k}$ is the vector of generalized forces.

As explained above, it is often convenient to replace $\dot{\boldsymbol{q}}$ with the vector of quasi-velocities $\boldsymbol{v} = [v_1, \cdots, v_n]^T$. Following [56], we assume that the quasi-velocities are a linear function of the the generalized velocities as follows:

$$v_i = \alpha_{1i}\dot{q}_1 + \alpha_{2i}\dot{q}_2 + \cdots + \alpha_{ni}\dot{q}_n = \sum_{j=1}^{n} \alpha_{ji}\dot{q}_j, \tag{2.65}$$

in which the α_{ji} s are functions of the generalized coordinates $\boldsymbol{q}$. The corresponding set of differentials ds_i are given by

$$ds_i = \sum_{j=1}^{n} \alpha_{ji} dq_j, \qquad i = 1, 2, \cdots, n. \tag{2.66}$$

It follows from Schwarz theorem that (2.66) cannot be integrated to obtain the s_i s unless $\frac{\partial \alpha_{ji}}{\partial q_k} = \frac{\partial \alpha_{ki}}{\partial q_j}$. This is why the v_i s are called *quasi-velocities* [56–58]. In matrix form (2.65) becomes

$$\boldsymbol{v} = \alpha^T \dot{\boldsymbol{q}}; \tag{2.67}$$

the reason for introducing the transpose will soon become apparent. We will assume that α is non-singular and so $\dot{\boldsymbol{q}} = \alpha^{-T}\boldsymbol{v}$; quasi-velocities v_i corresponding to nonholonomic constraints will necessarily be zero.

Our aim is now to rewrite (2.64) in terms of $\boldsymbol{q}$ and $\boldsymbol{v}$ instead of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$. Starting with the first term of (2.64), in which $\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \bar{\mathcal{L}}(\boldsymbol{q}, \boldsymbol{v})$, there holds

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \sum_{j=1}^{n} \frac{\partial \bar{\mathcal{L}}}{\partial v_j}\frac{\partial v_j}{\partial \dot{q}_i} = \sum_{j=1}^{n} \alpha_{ij}\frac{\partial \bar{\mathcal{L}}}{\partial v_j}. \tag{2.68}$$

Using matrix notation, (2.68) can be used to show that

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right) = \frac{d}{dt}\left(\alpha\frac{\partial \bar{\mathcal{L}}}{\partial \boldsymbol{v}}\right) = \alpha\frac{d}{dt}\left(\frac{\partial \bar{\mathcal{L}}}{\partial \boldsymbol{v}}\right) + \dot{\alpha}\frac{\partial \bar{\mathcal{L}}}{\partial \boldsymbol{v}}, \tag{2.69}$$

where

$$\dot{\alpha}_{ij} = \sum_{k=1}^{n} \frac{\partial \alpha_{ij}}{\partial q_k} \dot{q}_k = \dot{\boldsymbol{q}}^T \frac{\partial \alpha_{ij}}{\partial \boldsymbol{q}} = \boldsymbol{v}^T \alpha^{-1} \frac{\partial \alpha_{ij}}{\partial \boldsymbol{q}}. \tag{2.70}$$

The second term of (2.64) can be rewritten as

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\partial \bar{\mathcal{L}}}{\partial q_i} + \left(\frac{\partial \boldsymbol{v}}{\partial q_i}\right)^T \frac{\partial \bar{\mathcal{L}}}{\partial \boldsymbol{v}} = \frac{\partial \bar{\mathcal{L}}}{\partial q_i} + \dot{\boldsymbol{q}}^T \frac{\partial \alpha}{\partial q_i} \frac{\partial \bar{\mathcal{L}}}{\partial \boldsymbol{v}} = \frac{\partial \bar{\mathcal{L}}}{\partial q_i} + \boldsymbol{v}^T \alpha^{-1} \frac{\partial \alpha}{\partial q_i} \frac{\partial \bar{\mathcal{L}}}{\partial \boldsymbol{v}}. \tag{2.71}$$

The Lagrange equations of motion in terms of the quasi-coordinates $\boldsymbol{q}$ and $\boldsymbol{v}$ can now be obtained from (2.69)–(2.71) after premultiplication by α^{-1}:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}(\boldsymbol{q}, \boldsymbol{v})}{\partial \boldsymbol{v}}\right) + \alpha^{-1} \gamma \frac{\partial \mathcal{L}(\boldsymbol{q}, \boldsymbol{v})}{\partial \boldsymbol{v}} - \alpha^{-1} \frac{\partial \mathcal{L}(\boldsymbol{q}, \boldsymbol{v})}{\partial \boldsymbol{q}} = \alpha^{-1} \boldsymbol{k}, \tag{2.72}$$

where the bar in $\bar{\mathcal{L}}$ has been dropped, and where the $n \times n$ matrix γ is given by

$$\gamma = \begin{bmatrix} \ddots & \vdots & \iddots \\ \cdots & \boldsymbol{v}^T \alpha^{-1} \frac{\partial \alpha_{ij}}{\partial \boldsymbol{q}} & \cdots \\ \iddots & \vdots & \ddots \end{bmatrix} - \begin{bmatrix} \vdots \\ \boldsymbol{v}^T \alpha^{-1} \frac{\partial \alpha}{\partial q_i} \\ \vdots \end{bmatrix}. \tag{2.73}$$

It is worth emphasizing that each term in the first part of (2.73) is a scalar quantity, while those in the second part are $1 \times n$ row vectors. The right-hand-side term $\alpha^{-1}\boldsymbol{k}$ in (2.72) is the projection of the generalized forces onto the quasi-velocity system.

Example 2.8 We will derive the Lagrange equations in quasi-velocities for a body moving in a horizontal x–y plane. The body has translational velocities u, v along an axis system attached to its mass centre, and yaw ω. The absolute position of the body is given by x, y while the orientation is given by the angle ψ.
The quasi-velocity vector $\boldsymbol{v} = [u, v, \omega]^T$ replaces $\dot{\boldsymbol{q}} = [\dot{x}, \dot{y}, \dot{\psi}]^T$, with the corresponding Lagrangian given by $\mathcal{L} = T(\boldsymbol{v})$, where T is the kinetic energy. The relationship between $\boldsymbol{v}$ and $\dot{\boldsymbol{q}}$ is obtained by trigonometry as

$$\alpha^T = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.74}$$

which is introduced into (2.73) to give

$$\alpha^{-T}\gamma = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ -v & u & 0 \end{bmatrix}. \tag{2.75}$$

The resulting equations of motion are

$$\frac{d}{dt}\frac{\partial T}{\partial u} - \omega \frac{\partial T}{\partial v} = k_u \tag{2.76}$$

$$\frac{d}{dt}\frac{\partial T}{\partial v} + \omega \frac{\partial T}{\partial u} = k_v \tag{2.77}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \omega} - v \frac{\partial T}{\partial u} + u \frac{\partial T}{\partial v} = k_\omega \tag{2.78}$$

which corresponds to those reported in, for example, [8]. In Section 4.2 the same equations will be derived with the alternative Newton–Euler approach, in the context of modelling a car moving on a flat road.

Example 2.9 We will now derive Lagrange's equations in quasi-velocities for a body moving in space. The body has translational velocities u, v, w and rotational velocities $\omega_x, \omega_y, \omega_z$ along a body-fixed axis system attached to its mass centre. The absolute position of the body is given by x, y, z, while the orientation is given by three angles θ, μ, ϕ.
The quasi-velocity vector $\boldsymbol{v} = [u, v, w, \omega_x, \omega_y, \omega_z]^T$ is used in place of $\dot{\boldsymbol{q}} = [\dot{x}, \dot{y}, \dot{z}, \dot{\theta}, \dot{\mu}, \dot{\phi}]$. The Lagrangian is $\mathcal{L}(\boldsymbol{v}, \boldsymbol{q}) = T(\boldsymbol{v}) - V(\boldsymbol{q})$, where T is the kinetic energy and V is the potential energy. The relationship between $\boldsymbol{v}$ and $\dot{\boldsymbol{q}}$ is given by[6]

$$\alpha^T = \begin{bmatrix} c_\theta c_\mu & s_\theta c_\mu & -s_\mu & 0 & 0 & 0 \\ -s_\theta c_\phi + c_\theta s_\mu s_\phi & c_\theta c_\phi + s_\theta s_\mu s_\phi & c_\mu s_\phi & 0 & 0 & 0 \\ s_\theta s_\phi + c_\theta s_\mu c_\phi & -c_\theta s_\phi + s_\theta s_\mu c_\phi & c_\mu c_\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & -s_\mu & 0 & 1 \\ 0 & 0 & 0 & c_\mu s_\phi & c_\phi & 0 \\ 0 & 0 & 0 & c_\mu c_\phi & -s_\phi & 0 \end{bmatrix}$$

which is introduced into (2.73) to give

$$\alpha^{-T}\gamma = \begin{bmatrix} 0 & -\omega_z & \omega_y & 0 & 0 & 0 \\ \omega_z & 0 & -\omega_x & 0 & 0 & 0 \\ -\omega_y & \omega_x & 0 & 0 & 0 & 0 \\ 0 & -w & v & 0 & -\omega_z & \omega_y \\ w & 0 & -u & \omega_z & 0 & -\omega_x \\ -v & u & 0 & -\omega_y & \omega_x & 0 \end{bmatrix}. \tag{2.79}$$

The resulting equations of motion are

[6] A general method for deriving the relationships between quantities defined in different axis systems will be explained in Section 2.5. In this example the orientation is obtained according to the yaw–pitch–roll (θ–μ–ϕ) convention. The 3×3 diagonal blocks in (2.79) are skew-symmetric matrix representations of $\boldsymbol{\omega} \times \cdot$, where $\boldsymbol{\omega}$ is the angular velocity of the moving body with respect to a stationary axis system.

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial u} - \omega_z\frac{\partial\mathcal{L}}{\partial v} + \omega_y\frac{\partial\mathcal{L}}{\partial w} + \sin\mu\frac{\partial\mathcal{L}}{\partial z} = k_u \tag{2.80}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial v} + \omega_z\frac{\partial\mathcal{L}}{\partial u} - \omega_x\frac{\partial\mathcal{L}}{\partial w} - \cos\mu\sin\phi\frac{\partial\mathcal{L}}{\partial z} = k_v \tag{2.81}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial w} - \omega_y\frac{\partial\mathcal{L}}{\partial u} + \omega_x\frac{\partial T}{\partial v} - \cos\mu\cos\phi\frac{\partial\mathcal{L}}{\partial z} = k_w \tag{2.82}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial \omega_x} - w\frac{\partial\mathcal{L}}{\partial v} + v\frac{\partial\mathcal{L}}{\partial w} - \omega_z\frac{\partial\mathcal{L}}{\partial \omega_y} + \omega_y\frac{\partial\mathcal{L}}{\partial \omega_z} = k_{\omega_x} \tag{2.83}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial \omega_y} + w\frac{\partial\mathcal{L}}{\partial u} - u\frac{\partial\mathcal{L}}{\partial w} + \omega_z\frac{\partial\mathcal{L}}{\partial \omega_x} - \omega_x\frac{\partial\mathcal{L}}{\partial \omega_z} = k_{\omega_y} \tag{2.84}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial \omega_z} - v\frac{\partial\mathcal{L}}{\partial u} + u\frac{\partial\mathcal{L}}{\partial v} - \omega_y\frac{\partial\mathcal{L}}{\partial \omega_x} + \omega_x\frac{\partial\mathcal{L}}{\partial \omega_y} = k_{\omega_z}\,. \tag{2.85}$$

In Section 7.5 the same equations will be derived using a Newton–Euler formulation for a car moving on a three-dimensional road.

2.3 Conservation laws

The dynamic properties of a mechanical system are described in terms of the temporal evolution of its generalized coordinates from a given initial condition. While the generalized coordinates are almost always time-varying, there are certain quantities that remain constant during the motion and depend only on the initial conditions. These quantities are called *integrals of the motion* [43], some of which are of great importance and derive from the homogeneity (translation invariance) and isotropy (rotation invariance) of space and time. These constants of the motion are referred to as *conserved quantities*.[7]

2.3.1 Energy

We consider first the conservation law that derives from the *homogeneity of time*. If all the impressed forces can be derived from a potential, and the resulting system Lagrangian is time-shift invariant (scleronomic), there holds

$$\frac{d\mathcal{L}}{dt} = \frac{\partial\mathcal{L}}{\partial\boldsymbol{q}}\cdot\dot{\boldsymbol{q}} + \frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\cdot\ddot{\boldsymbol{q}}; \tag{2.86}$$

in which the $\cdot$ denotes a dot product. There is no $\frac{\partial\mathcal{L}}{\partial t}$ term, since $\mathcal{L}$ is not an explicit function of time (by assumption). In the case of unconstrained systems, the $\frac{\partial\mathcal{L}}{\partial\boldsymbol{q}}$ term can be replaced by $\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}})$ using (2.40) so that

[7] In some important work E. Noether (1918) showed that every symmetry of the action of a physical system has a corresponding conservation law. By 'symmetry' we mean any transformation of the generalized coordinates, the associated generalized velocities, and time that leaves the value of the Lagrangian unaffected.

$$\begin{aligned}\frac{d\mathcal{L}}{dt} &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right)\cdot\dot{\boldsymbol{q}} + \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\cdot\ddot{\boldsymbol{q}} \\ &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\cdot\dot{\boldsymbol{q}}\right)\end{aligned}$$

or

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\cdot\dot{\boldsymbol{q}} - \mathcal{L}\right) = 0.$$

The quantity

$$E = \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\cdot\dot{\boldsymbol{q}} - \mathcal{L} \tag{2.87}$$

is therefore constant as a result of temporal homogeneity. In the case that time-invariant holonomic constraints are included, (2.87) still holds with $\mathcal{L}$ replaced by $\hat{\mathcal{L}}$ given by (2.42).

Homogeneity in time results in energy conservation in the case of (scleronomic) non-holonomic systems too. To see this observe that the $\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}$ term in (2.86) can be replaced by $\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}) + A(\boldsymbol{q})^T\boldsymbol{\lambda}$ using (2.45) and (2.46). Since the nonholonomic constraints are satisfied during the motion, $A(\boldsymbol{q})^T\boldsymbol{\lambda}\cdot\dot{\boldsymbol{q}} = \boldsymbol{\lambda}^T A(\boldsymbol{q})\dot{\boldsymbol{q}} = 0$, and the arguments relating to holonomic systems hold good.

In sum, the requirements for the conservation of energy are that the impressed forces be derivable from a potential function and that the Lagrangian and constraints are not explicitly dependent on time.

If the Lagrangian of a mechanical system is of the form $\mathcal{L} = T(\boldsymbol{q},\dot{\boldsymbol{q}}) - V(\boldsymbol{q})$, in which T is quadratic in the velocities, there holds

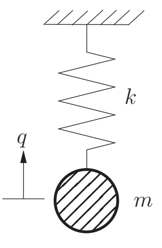
$$\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\cdot\dot{\boldsymbol{q}} = \frac{\partial T}{\partial \dot{\boldsymbol{q}}}\cdot\dot{\boldsymbol{q}} = 2T$$

and (2.87) can be rewritten as

$$E = T(\boldsymbol{q},\dot{\boldsymbol{q}}) + V(\boldsymbol{q}), \tag{2.88}$$

which is the sum of the system's kinetic and potential energy; time homogeneity leads to the *law of conservation of energy.*

Example 2.10 Consider the one-dimensional harmonic oscillator shown here:



Ignoring gravity, the system's kinetic and potential energies are given by

$$T = \frac{1}{2}m\dot{q}^2 \qquad \text{and} \qquad V = \frac{1}{2}kq^2$$

respectively. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}m\dot{q}^2 - \left(\frac{1}{2}kq^2\right).$$

Solving (2.40) gives

$$\ddot{q} + \omega^2 q = 0,$$

where

$$\omega = \sqrt{\frac{k}{m}}.$$

It is easy to establish that $q(t) = A\sin\omega t$ and that $E = T + V = A^2k/2$. In this case the Lagrangian is time-shift invariant, and system's total internal energy is constant and a function only of the initial conditions.

2.3.2 Linear momentum

Another conservation law derives from homogeneity, that is, translation invariance. In this case we derive the system properties that remain unchanged under an infinitesimal translational generalized displacement of the entire system. Let us consider, therefore, a small (constant) displacement $\delta\boldsymbol{q}$ in the generalized coordinates, with the generalized velocities unchanged. That is

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\boldsymbol{q}}\delta\boldsymbol{q}. \tag{2.89}$$

Since $\delta\mathcal{L} = 0$ by assumption, and $\delta\boldsymbol{q}$ is arbitrary, there holds

$$\frac{\partial\mathcal{L}}{\partial\boldsymbol{q}} = 0. \tag{2.90}$$

In the case of an unconstrained system (2.40) gives

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\right) = 0$$

and so the generalized momenta

$$\boldsymbol{p} = \frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}} \tag{2.91}$$

are constant (conserved). In the case of holonomic constraints, the same arguments hold with $\mathcal{L}$ replaced by $\hat{\mathcal{L}}$ in (2.42). In the case of nonholonomic systems, (2.45) and (2.46) give

$$\boldsymbol{p} = -\int_0^t A(\boldsymbol{q})^T\boldsymbol{\lambda}d\tau, \tag{2.92}$$

which may be time-varying.

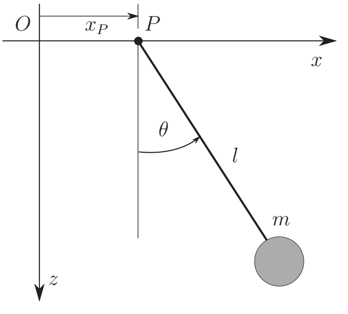
Generalized coordinates that do not appear in the Lagrangian, while their derivatives do, are called *cyclic*, *ignorable*, or *kinosthenic* variables [41, 44]. These variables satisfy (2.90) and so the related momenta are conserved if the Lagrange equations have no right-hand-side terms. Typical examples are the position and yaw of a vehicle on a flat road and the spin angle of a homogeneous wheel.

Example 2.11 Consider a particle of mass m in a horizontal Cartesian inertial reference frame. In this case the Lagrangian is

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2).$$

Since the Lagrangian is not explicitly time dependent, energy is conserved. It also follows from (2.40) that $m\dot{x}$ and $m\dot{y}$ are constant. This is an elementary illustration of the conservation of linear momentum, which derives from a Lagrangian that is not a function of position.

Example 2.12 Consider in the following figure a simple planar pendulum suspended from pivot point P, which is free to move along the x-axis.



The pendulum rod is massless and has length l; the bob mass is m. The pendulum's angle of rotation is θ. The system's kinetic energy is computed using (2.29)

$$T = \frac{m}{2}\left(\dot{x}_P^2 + 2l\dot{x}_P\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right), \tag{2.93}$$

and the system's potential energy is given by (2.36)

$$V = -mgl\cos\theta. \tag{2.94}$$

The Lagrangian is

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}_P^2 + 2l\dot{x}_P\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right) + mgl\cos\theta. \tag{2.95}$$

Since the Lagrangian is not an explicit function of time, the system is conservative.

The equations of motion are derived using (2.40). The θ-related equation is

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0, \tag{2.96}$$

which gives

$$l\ddot{\theta} = -\ddot{x}_P\cos\theta - g\sin\theta, \tag{2.97}$$

which is the familiar pendulum equation when x_P is fixed (anywhere along the x-axis).

Since the Lagrangian is neither an explicit function of time nor a function of x_P, the system is conservative with the linear momentum

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_P} = m(\dot{x}_P + l\dot{\theta}\cos\theta) \tag{2.98}$$

conserved.

Now suppose that x_P is subject to the more general constraint

$$f(x_P, t) = x_P - u(t) = 0. \tag{2.99}$$

The augmented Lagrangian (2.42) is given by

$$\hat{\mathcal{L}} = \frac{m}{2}\left(\dot{x}_P^2 + 2l\dot{x}_P\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right) + mgl\cos\theta + \lambda(x_P - u(t)), \tag{2.100}$$

and the swing equation becomes

$$l\ddot{\theta} = -\left(\ddot{u}(t)\cos\theta + g\sin\theta\right). \tag{2.101}$$

The linear displacement equation is

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_P}\right) - \frac{\partial \mathcal{L}}{\partial x_P} = -A(x_P)\lambda, \tag{2.102}$$

where

$$A(x_P) = \frac{\partial f(x_P, t)}{\partial x_P} = 1. \tag{2.103}$$

Therefore

$$\lambda = m\left(l\dot{\theta}^2\sin\theta - \ddot{u}(t) - l\ddot{\theta}\cos\theta\right) \tag{2.104}$$

which gives the force of constraint

$$k_{x_P} = -A(x_P)\lambda = m\left(\ddot{u}(t) + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta\right). \tag{2.105}$$

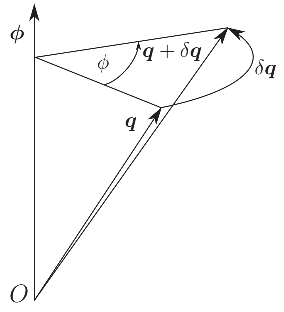
There are three cases to consider. If $u = x_P = 0$ (2.97) becomes

$$l\ddot{\theta} = -g\sin\theta \tag{2.106}$$

and the system is conservative. In the case that $\dot{u} = \dot{x}_P$ is constant, one can replace the original coordinate system with one in which x_P is stationary. This new axis system is *inertial*, (2.97) applies, and the system is again conservative. If $\ddot{u} = \ddot{x}_P(t) \neq 0$, the swing equation becomes (2.101), (real) work is done by the constraint force and the system is no longer conservative.

2.3.3 Angular momentum

Spatial isotropy is related to the mechanical property of systems that remain invariant under spatial rotations. In this case we consider next an infinitesimal rotation of the entire system and then seek the condition under which the Lagrangian remains unchanged. As shown in the diagram, the vector $\delta\boldsymbol{\phi}$ has a prescribed axis of rotation and magnitude ϕ.



The change in the generalized coordinates $\delta\boldsymbol{q}$ resulting from the infinitesimal but fixed rotation $\boldsymbol{\phi}$ is given by

$$\delta\boldsymbol{q} = \delta\boldsymbol{\phi} \times \boldsymbol{q},$$

while the change in the generalized velocities is given by

$$\delta\dot{\boldsymbol{q}} = \delta\boldsymbol{\phi} \times \dot{\boldsymbol{q}}.$$

If these expressions are substituted into the condition for the variation in the Lagrangian

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\boldsymbol{q}} \cdot \delta\boldsymbol{q} + \frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}} \cdot \delta\dot{\boldsymbol{q}} \tag{2.107}$$

to remain unchanged, we obtain

$$\begin{aligned} 0 &= \frac{\partial\mathcal{L}}{\partial\boldsymbol{q}} \cdot \delta\boldsymbol{\phi} \times \boldsymbol{q} + \frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}} \cdot \delta\boldsymbol{\phi} \times \dot{\boldsymbol{q}} \\ &= \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\right) \cdot \delta\boldsymbol{\phi} \times \boldsymbol{q} + \frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}} \cdot \delta\boldsymbol{\phi} \times \dot{\boldsymbol{q}} \end{aligned}$$

in the case of a holonomic system. Using a standard triple product identity one obtains

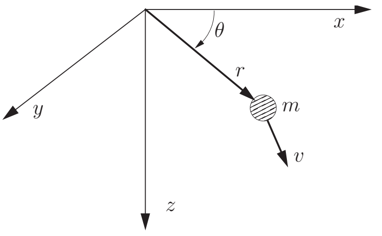
$$\begin{aligned} 0 &= \delta\boldsymbol{\phi}\left(\boldsymbol{q} \times \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\right) + \dot{\boldsymbol{q}} \times \frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\right) \\ &= \delta\boldsymbol{\phi}\left(\frac{d}{dt}\left(\boldsymbol{q} \times \boldsymbol{p}\right)\right), \end{aligned}$$

in which $\boldsymbol{p} = \frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}$ is the generalized momentum. Since $\delta\boldsymbol{\phi}$ is arbitrary, it follows that $(d/dt)\,(\boldsymbol{q} \times \boldsymbol{p}) = 0$ and so

$$\boldsymbol{H} = \boldsymbol{q} \times \boldsymbol{p} \tag{2.108}$$

is constant; $\boldsymbol{H}$ is the *angular momentum*, or the *moment of momentum*.

Example 2.13 Consider once more a free particle of mass m in a polar inertial reference frame. As shown in the following sketch,



the particle is free to move in the x–y plane, with the z-axis orthogonal to the horizontal plane. The Lagrangian is again the kinetic energy of the particle and is given by

$$\mathcal{L} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$$

in this case. The Euler–Lagrange equations in r and θ are given by

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0 \tag{2.109}$$

and

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0 \tag{2.110}$$

respectively. Equation (2.109) gives

$$\ddot{r} = r\dot{\theta}^2 \tag{2.111}$$

that is the centrifugal acceleration usually associated with circular motion. Equation (2.110) shows that

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta},$$

is constant, which is the magnitude of the angular momentum. If one recognizes the vectorial implications of this result, we see that the angular momentum (as a vector), $\boldsymbol{H}$, is given by (Section 2.2.2)

$$\boldsymbol{H} = r\hat{\boldsymbol{e}}_r \times m(\dot{r}\hat{\boldsymbol{e}}_r + r\dot{\theta}\hat{\boldsymbol{e}}_\theta),$$

in which $\hat{\boldsymbol{e}}_r$ is a unit vector in the radial direction in the x–y plane, while $\hat{\boldsymbol{e}}_\theta$ is also in the x–y plane and orthogonal to $\hat{\boldsymbol{e}}_r$. By simplifying the cross product, one sees that the angular momentum is in the direction of $\hat{\boldsymbol{e}}_z$ along the z-axis:

$$\boldsymbol{H} = mr^2\dot{\theta}\hat{\boldsymbol{e}}_z.$$

Solving (2.110) gives

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r}. \tag{2.112}$$

It is easy to show that (2.111) and (2.112) have solutions

$$r\cos\theta = at + b \tag{2.113}$$

$$r\sin\theta = ct + d \tag{2.114}$$

with the constants a, b, c, and d determined by the initial conditions.[8] As one would expect, this solution represents rectilinear motion in polar coordinates.

2.4 Hamilton's equations

In the Lagrangian framework the state of the system is described in terms of the n generalized coordinates q_i, their derivatives $\dot{q}_i$, and the time t. In an alternative framework, due to *Hamilton*, the system state is described in terms of the generalized coordinates q_i, the generalized momenta p_i, and t. Hamilton's canonical equations replace the original n (second-order) Lagrangian differential equations with $2n$ first-order equations. As we will now show Lagranges equations are converted into the Hamiltonian formalism using the *Legendre Transformation.* The discovery of the canonical equations led to a new era in theoretical mechanics.

2.4.1 Legendre transform

The Legendre transformation connects two alternative descriptions of the same physics through functions of related ('conjugate') variables. Suppose a function of n variables is given

$$F = F(u_1, \cdots, u_n) \tag{2.115}$$

and that we introduce a new set of variables by means of the transformation

$$\boldsymbol{v} = \frac{\partial F}{\partial \boldsymbol{u}}, \tag{2.116}$$

where $\boldsymbol{u} = [u_1, \cdots, u_n]^T$. If

$$det\left[\frac{\partial^2 F}{\partial u_i \partial u_j}\right] \neq 0 \tag{2.117}$$

the v_i s are independent and (2.116) can be solved for the u_i s in terms of the v_i s. In mechanics, the regularity condition is satisfied if $\mathcal{L}$ is given as the kinetic energy T with a positive definite generalized mass matrix minus the potential energy V. We will not consider degenerate systems in which the u_i s are not independent.

A new function G can now be defined as follows

$$G = \boldsymbol{u}^T \boldsymbol{v} - F. \tag{2.118}$$

By solving (2.116) for the u_i s, we can express G in terms of the v_i s alone

$$G = G(v_1, \cdots, v_n). \tag{2.119}$$

Taking variations in G, one obtains

[8] Differentiate (2.113) and (2.114) twice, and eliminate $\ddot{r}$ and $\ddot{\theta}$ using (2.111) and (2.112).

$$\delta G = \sum_{i=1}^{n} \frac{\partial G}{\partial v_i} \delta v_i \tag{2.120}$$

$$= \sum_{i=1}^{n} \left(u_i \delta v_i + \left(v_i - \frac{\partial F}{\partial u_i} \right) \delta u_i \right) \tag{2.121}$$

$$= \sum_{i=1}^{n} u_i \delta v_i. \tag{2.122}$$

using (2.116). This means that

$$\frac{\partial G}{\partial v_i} \delta v_i = \sum_{i=1}^{n} u_i \delta v_i,$$

and so

$$\boldsymbol{u} = \frac{\partial G}{\partial \boldsymbol{v}}, \tag{2.123}$$

which establishes the duality of the Legendre transform. The new variables are the partial derivatives of the old functions with respect to the old variables and the old variables are the partial derivatives of the new functions with respect to the new variables. In sum:

Old System	*New System*
variables: $u_1, \cdots, u_n$;	variables: $v_1, \cdots, v_n$;
Function $F = F(u_1, \cdots, u_n)$;	Function $G = G(v_1, \cdots, v_n)$;

System Transformations

$\boldsymbol{v} = \dfrac{\partial F}{\partial \boldsymbol{u}}$;	$\boldsymbol{u} = \dfrac{\partial G}{\partial \boldsymbol{v}}$;
$G = \boldsymbol{u}^T \boldsymbol{v} - F$;	$F = \boldsymbol{u}^T \boldsymbol{v} - G$;
$G = G(v_1, \cdots, v_n)$.	$F = F(u_1, \cdots, u_n)$.

In the case that F is a function of two independent sets of variables

$$F(w_1, \cdots, w_m; u_1, \cdots, u_m), \tag{2.124}$$

where the w_i s play no part in the transformation. A new function G is again defined as

$$G = \boldsymbol{u}^T \boldsymbol{v} - F, \tag{2.125}$$

in which the w_i s are passive variables. If we return to (2.121), and compute the complete variation of G by allowing the u_i s and w_i s to vary arbitrarily, we obtain

$$\frac{\partial F}{\partial \boldsymbol{w}} = -\frac{\partial G}{\partial \boldsymbol{w}} \tag{2.126}$$

and (2.123) using (2.116) as before.

2.4.2 Canonical equations

As shown in (2.37), the Lagrangian function $\mathcal{L}$ is a function of n position coordinates q_i, n velocities $\dot{q}_i$, and possibly the time t. We can now apply Legendre's transformation to $\mathcal{L}$, with the $\dot{q}_i$ the n active variables, and the q_i s and t the $n+1$ passive variables. In this scheme

$$\boldsymbol{p} = \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \tag{2.127}$$

are the n *generalized momenta*. The new function is

$$\mathcal{H} = \boldsymbol{p}^T \dot{\boldsymbol{q}} - \mathcal{L}, \tag{2.128}$$

which we recognize from (2.87) as the total system energy. Solving (2.127) for the $\dot{q}_i$ s, and substituting them into (2.128), gives

$$\mathcal{H} = \mathcal{H}(q_1, \cdots, q_n; p_1, \cdots, p_n, t). \tag{2.129}$$

The basic features of the original and transformed systems are:

Old System	*New System*
Function: Lagrangian function $\mathcal{L}$;	Function: Hamiltonian function $\mathcal{H}$;
Variables: generalized velocities $\dot{q}_i$;	Variables: generalized momenta p_i;

The passive variables are the generalized positions and time. The two systems are therefore described by

$$\boldsymbol{p} = \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}; \qquad \dot{\boldsymbol{q}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}};$$

$$\mathcal{H} = \boldsymbol{p}^T \dot{\boldsymbol{q}} - \mathcal{L}; \qquad \mathcal{L} = \boldsymbol{p}^T \dot{\boldsymbol{q}} - \mathcal{H};$$

$$\mathcal{H} = \mathcal{H}(q_1, \cdots, q_n; p_1, \cdots, p_n; t). \qquad \mathcal{L} = \mathcal{L}(q_1, \cdots, q_n; \dot{q}_1, \cdots, \dot{q}_n; t).$$

The passive variables transform according to (2.126)

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}} \tag{2.130}$$

$$\frac{\partial \mathcal{L}}{\partial t} = -\frac{\partial \mathcal{H}}{\partial t}. \tag{2.131}$$

If we substitute (2.40) into (2.130), we obtain

$$\dot{\boldsymbol{p}} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}}. \tag{2.132}$$

The Lagrangian equations of motion have thus been replaced by a new set of equations called the *canonical equations of Hamilton:*

$$\dot{\boldsymbol{q}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} \qquad\qquad \dot{\boldsymbol{p}} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}}. \tag{2.133}$$

These equations are equivalent to the original Lagrangian equations; they are just in a new form.

In Section 2.2.8 we showed that constrained mechanical systems can be described by Euler–Lagrange equations with extra terms corresponding to the constraint forces. These equations take the form

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = -A^T(\boldsymbol{q})\boldsymbol{\lambda}, \tag{2.134}$$

in which

$$A(\boldsymbol{q})\dot{\boldsymbol{q}} = -F_t \tag{2.135}$$

describes the constraints. In the case of constrained problems, we can substitute (2.134) into (2.130) to obtain

$$\dot{\boldsymbol{p}} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}} - A^T(\boldsymbol{q})\boldsymbol{\lambda}. \tag{2.136}$$

Further details relating to the incorporation of nonholonomic constraints into mechanical system modelling in a Hamiltonian framework can be found in [59].

Example 2.14 Consider once more the one-dimensional harmonic oscillator of Example 2.10. The Hamiltonian is given by (2.128)

$$\mathcal{H}(p,\, q) = \frac{1}{2m}p^2 + \frac{k}{2}q^2,$$

where $p = m\dot{q}$. Therefore

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m} \qquad\qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = -kq.$$

These two Hamiltonian equations are clearly equivalent to the second-order Lagrange equation obtained previously.

2.4.3 Poisson brackets

Werner Heisenberg formulated a theory of quantum mechanics, known as matrix mechanics, in which an object called the *commutator* $[A,\, B] = AB - BA$ plays a central role; A and B are matrices. Matrices A and B are said to commute under multiplication if $AB - BA = 0$. It is self-evident that the commutator is *antisymmetric*:

$$[A,\, B] = -[B,\, A].$$

Linearity is easily checked:

$$\begin{aligned}[\alpha A + \beta B,\, C] &= (\alpha A + \beta B)C - C(\alpha A + \beta B)\\ &= \alpha(AC - CA) + \beta(BC - CB)\\ &= \alpha[A,\, C] + \beta[B,\, C].\end{aligned}$$

The commutator also satisfies a *product rule*:

$$\begin{aligned}[AB,\, C] &= ABC - CAB\\ &= ABC - CAB - ACB + ACB\\ &= A[B,\, C] + [A,\, C]B.\end{aligned}$$

Finally, the commutator satisfies the *Jacobi identity*:

$$\begin{aligned}0 &= A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B\\ &\quad + C(AB - BA) - (AB - BA)C\\ &= [A,\, [B,\, C]] + [B,\, [C,\, A]] + [C,\, [A,\, B]].\end{aligned}$$

A function theoretic equivalent of the commutator is another bracket, called the *Poisson bracket*, which has an analogous definition. In the context of mechanics, if we suppose that $F(\boldsymbol{q}, \boldsymbol{p})$ and $G(\boldsymbol{q}, \boldsymbol{p})$ are functions of the generalized position and momentum, then the Poisson bracket is defined as

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}. \tag{2.137}$$

The Poisson bracket also satisfies the *antisymmetry*, *lineariy*, *product rule*, and *Jacobi identity* properties:

$$\begin{aligned}\{F, G\} &= -\{G, F\};\\ \{\alpha F + \beta G, H\} &= \alpha\{F, H\} + \beta\{G, H\};\\ \{FG, H\} &= F\{G, H\} + \{F, H\}G;\\ 0 &= \{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\}.\end{aligned} \tag{2.138}$$

Functions that commute with the Hamiltonian are of great importance in mechanics. For any function $F(\boldsymbol{q}, \boldsymbol{p}, t)$, where $\boldsymbol{p}$ and $\boldsymbol{q}$ are governed by $\mathcal{H}$, there holds

$$\begin{aligned}\frac{dF}{dt} &= \frac{\partial F}{\partial \boldsymbol{q}} \cdot \dot{\boldsymbol{q}} + \frac{\partial F}{\partial \boldsymbol{p}} \cdot \dot{\boldsymbol{p}} + \frac{\partial F}{\partial t}\\ &= \{F, \mathcal{H}\} + \frac{\partial F}{\partial t}.\end{aligned} \tag{2.139}$$

Functions of the dynamical variables that remain constant during the motion of the system are the *integrals of the motion*. It follows from (2.139) that F is an integral of the motion if

$$\{F, \mathcal{H}\} + \frac{\partial F}{\partial t} = 0.$$

If the integral of the motion is not explicitly dependent on time, then

$$\{F, \mathcal{H}\} = 0.$$

If $\mathcal{H}$ is not explicitly time dependent,

$$\{\mathcal{H}, \mathcal{H}\} = 0$$

shows that $\frac{d\mathcal{H}}{dt} = 0$ and therefore that the total system energy is conserved.

It is immediate from (2.137) that the canonical equations (2.133) can be written as

$$\{p_j, \mathcal{H}\} = -\frac{\partial \mathcal{H}}{\partial q_j} \quad \text{and} \quad \{q_j, \mathcal{H}\} = \frac{\partial \mathcal{H}}{\partial p_j}. \tag{2.140}$$

The first equation shows that if p_j commutes with $\mathcal{H}$, $\mathcal{H}$ is independent of q_j, which is therefore a conserved quantity. This establishes an important relationship between ignorable coordinates and conserved quantities in the language of Poisson brackets.

An important property of the Poisson bracket is Poisson's theorem, which says that if F and G commute with $\mathcal{H}$, then so does $\{F, G\}$. This is easy to show if F and G do not depend explicitly on time using the Jacobi identity:

$$0 = \{F, \{G, \mathcal{H}\}\} + \{G, \{\mathcal{H}, F\}\} + \{\mathcal{H}, \{F, G\}\},$$

then $\{F, \mathcal{H}\} = 0$ and $\{G, \mathcal{H}\} = 0$ implies that $\{\mathcal{H}, \{F, G\}\} = 0$. As shown in [43] the extension to the case that F and G are explicitly time dependent is only slightly more complicated to prove.

Poisson's theorem, in principle, allows one to find a third integral of the motion given two others. Not all the integrals of the motion computed in this way will be 'new'; some may be old integrals, or constants that may be zero. If a body is in free motion in space, then p_x, p_y, and p_z are all constant. Yet knowing that $\dot{p}_x = 0$ and $\dot{p}_y = 0$ does not allow one to deduce that $\dot{p}_z = 0$, since $\{p_x, p_y\} = 0$ provides no new information.

Example 2.15 The *Kepler problem* is one of the fundamental problems in classical mechanics and is as ubiquitous as the harmonic oscillator in the mechanics literature.[9] The Kepler problem is also a simple and informative illustration of the use of Poisson brackets. Goldstein [41] dedicates a whole chapter to two-body central-force problems and so a far more detailed treatment of this problem can be found there. In essence, the Kepler problem studies the dynamics of a unit point mass in an inverse-square law central force field:

$$\ddot{\boldsymbol{r}} + \frac{\kappa \boldsymbol{r}}{r^3} = 0, \tag{2.141}$$

where $\boldsymbol{r}$ is a position (radius) vector, κ is a constant, and $r = |\boldsymbol{r}|$. The central force motion of two bodies about their mass centre can always be reduced to an equivalent one-body problem [41].

[9] The Kepler laws of planetary motion have been of interest ever since the appearance of Newton's *Principia*. In 1911 Rutherford proposed a planetary model for the hydrogen atom, whereby an electron rotates on a planetary orbit about a charged atomic core. This model is unstable (and therefore wrong), because the rotating electron would radiate away its energy until the atom collapsed.

The first thing to note is that the motion is planar. If we cross (2.141) with $\boldsymbol{r}$ we obtain

$$\boldsymbol{r} \times \ddot{\boldsymbol{r}} = 0$$

and so the angular momentum vector

$$\boldsymbol{H} = \boldsymbol{r} \times \dot{\boldsymbol{r}}, \tag{2.142}$$

is constant. Since $\boldsymbol{H}$ is perpendicular to both $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$, the constancy of $\boldsymbol{H}$ implies that the orbit is confined to a plane.

We can therefore describe the problem in terms of cylindrical (rather than spherical) coordinates with Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\dot{r}^2 + (r\dot{\theta})^2\right) + \frac{\kappa}{r},$$

and Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{\kappa}{r},$$

where

$$p_r = \dot{r} \quad \text{and} \quad p_\theta = r^2\dot{\theta}.$$

Unit vectors in the $\boldsymbol{r}$, $\boldsymbol{\theta}$, and $\boldsymbol{z}$ directions are denoted $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{z}}$ respectively. Since these vectors are orthogonal, the cross product of any two provides the third; $\hat{\boldsymbol{z}}$ is chosen perpendicular to the motion.

Using (2.140) one can see that

$$\dot{p}_\theta = \{p_\theta, \mathcal{H}\} = 0,$$

which shows that p_θ is a conserved quantity. Since $\mathcal{H}$ is not explicitly time-varying, the system energy is also conserved.

The motion-specifying equation (2.141) can be written as

$$0 = \ddot{\boldsymbol{r}} + \frac{\kappa \boldsymbol{r}}{r^3} = \ddot{\boldsymbol{r}} + \frac{\kappa}{H}\dot{\theta}\hat{\boldsymbol{r}} = \frac{d}{dt}\left(\dot{\boldsymbol{r}} - \frac{\kappa}{H}\hat{\boldsymbol{\theta}}\right),$$

since $\dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\hat{\boldsymbol{r}}$. The *Hamiltonian vector*

$$\boldsymbol{L} = \dot{\boldsymbol{r}} - \frac{\kappa}{H}\hat{\boldsymbol{\theta}}$$

is thus also a conserved quantity. Since the cross product of two conserved quantities is also conserved,

$$\boldsymbol{K} = \boldsymbol{L} \times \boldsymbol{H} = \dot{\boldsymbol{r}} \times \boldsymbol{H} - \kappa\hat{\boldsymbol{r}}$$

which is the *Laplace–Runge–Lenz* vector, is also conserved [41]. The vectors $\boldsymbol{H}$, $\boldsymbol{L}$, and $\boldsymbol{K}$ are mutually orthogonal, with $\boldsymbol{H}$ normal to the orbital plane.

Carrying out the requisite differentiations gives [41, 60]:

$$
\begin{aligned}
\{\boldsymbol{L}, \mathcal{H}\} &= 0 \\
\{\boldsymbol{K}, \mathcal{H}\} &= 0 \\
\{H_i, H_j\} &= \epsilon_{ijk} H_k \\
\{H_i, K_j\} &= \epsilon_{ijk} K_k \\
\{K_i, K_j\} &= -2\mathcal{H}\epsilon_{ijk} H_k,
\end{aligned}
$$

where ϵ_{ijk} is the Levi–Civita symbol. These identities highlight the conserved quantities in the Kepler problem in terms of Poisson brackets and Poisson's theorem.

2.5 Frames, velocity, and acceleration

In vehicle dynamics it is common to use at least two orthonormal coordinate systems to describe the motion: a reference (inertial) coordinate system and a moving coordinate system attached to the vehicle. The most general displacement of a rigid body, and thus also of an orthonormal reference frame, is a translation plus a rotation; this is *Chasles' theorem*.[10] The orientation of one frame relative to another can be described by a maximum of three successive rotations about the coordinate axes (*Euler's first theorem*), or by a single rotation about a specific axis (*Euler's second theorem*). Therefore the relative motion between two reference frames in a three-dimensional space is usually described by three translations and three rotations. We begin this section by deriving the vectorial relationships relating position, velocity, and acceleration in an inertial frame and in a moving frame. We then introduce the matrix notation used to deal with the modelling of a generic three-dimensional motion and the related position, velocity, and acceleration relationships.

A moving reference frame is characterized by the position of its origin $\boldsymbol{r}_{OC}$ and its angular velocity $\boldsymbol{\omega}$ with respect to the inertial frame.

The position $\boldsymbol{r}_{OP}$ of a point P in the inertial frame can be described by

$$\boldsymbol{r}_{OP} = \boldsymbol{r}_{OC} + \boldsymbol{r}_{CP}, \tag{2.143}$$

where $\boldsymbol{r}_{OC}$ is the position of the origin of the moving frame and $\boldsymbol{r}_{CP}$ is the position of P in the moving frame.

The velocity $\boldsymbol{v}_P$ of the point P can be obtained by differentiating (2.143) with respect to time

$$\boldsymbol{v}_P = \frac{d\boldsymbol{r}_{OP}}{dt} = \dot{\boldsymbol{r}}_{OC} + \boldsymbol{\omega} \times \boldsymbol{r}_{CP} + \dot{\boldsymbol{r}}_{CP}, \tag{2.144}$$

where $\dot{\boldsymbol{r}}_{OC}$ is the velocity of the origin of the moving frame, $\boldsymbol{\omega} \times \boldsymbol{r}_{CP}$ is the transferred velocity, and $\dot{\boldsymbol{r}}_{CP}$ is the velocity of P relative to the moving frame; the 'dot' denotes

[10] There is a stronger form of the theorem stating that the most general displacement is a translation plus a rotation along a given axis, the so-called *screw axis*.

component-wise differentiation with respect to time. In the case of a moving frame of reference, the complete time derivation has a 'dot term' and a $\boldsymbol{\omega}\times$ angular velocity term.[11]

The acceleration $\boldsymbol{a}_{OP}$ of the point P is obtained by differentiating (2.144) with respect to time

$$\boldsymbol{a}_P = \frac{d^2\boldsymbol{r}_{OP}}{dt^2} = \ddot{\boldsymbol{r}}_{OC} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{CP} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{CP}) + 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}}_{CP} + \ddot{\boldsymbol{r}}_{CP}, \tag{2.145}$$

where $\ddot{\boldsymbol{r}}_{OC}$ is the acceleration of the origin of the moving frame, $\dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{CP}$ is the Euler acceleration term, $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{CP})$ is the centrifugal term, $2\boldsymbol{\omega} \times \dot{\boldsymbol{r}}_{CP}$ is the Coriolis term, and $\ddot{\boldsymbol{r}}_{CP}$ is the acceleration of P relative to the moving frame.

It goes without saying that the vector quantities in (2.143), (2.144), and (2.145) must be expressed in the same reference frame prior to any vector addition calculations. The vector $\boldsymbol{a}_P$ represents the absolute acceleration of P, which may be expressed in the coordinates of the moving reference frame if so desired.

In the case that P is stationary in the moving frame, the velocity and acceleration expressions simplify to

$$\boldsymbol{v}_P = \dot{\boldsymbol{r}}_{OC} + \boldsymbol{\omega} \times \boldsymbol{r}_{CP} \tag{2.146}$$

$$\boldsymbol{a}_P = \ddot{\boldsymbol{r}}_{OC} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{CP} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{CP}). \tag{2.147}$$

It is often convenient to use transformation matrices to change the coordinates of a point P expressed in frame $\mathcal{C}$ (for example the moving frame attached to the vehicle) $\boldsymbol{r}^{\mathcal{C}}_{CP}$ to a different frame $\mathcal{O}$ (for example the inertial ground frame) $\boldsymbol{r}^{\mathcal{O}}_{OP}$. The superscript is used to indicate the coordinate system in which the coordinates are expressed. Equation (2.143) can be written in the following matrix form

$$\begin{bmatrix} \boldsymbol{r}^{\mathcal{O}}_{OP} \\ 1 \end{bmatrix} = \mathcal{T}_{OC} \begin{bmatrix} \boldsymbol{r}^{\mathcal{C}}_{CP} \\ 1 \end{bmatrix} \tag{2.148}$$

where

$$\mathcal{T}_{OC} = \begin{bmatrix} \mathcal{R}_{OC} & \boldsymbol{r}^{\mathcal{O}}_{OC} \\ 0 & 1 \end{bmatrix} \tag{2.149}$$

is the 4×4 transformation matrix, which consists of a 3×3 rotation matrix $\mathcal{R}_{OC}$ and a 3×1 translation vector $\boldsymbol{r}^{\mathcal{O}}_{OC}$. The columns of $\mathcal{R}_{OC}$ are the components of the unit vectors of the frame $\mathcal{C}$ axes expressed in the coordinates of the frame $\mathcal{O}$, while the $\boldsymbol{r}^{\mathcal{O}}_{OC}$ is the position of the origin of the frame $\mathcal{C}$ expressed in the coordinates of frame $\mathcal{O}$. General rotation matrices satisfy the conditions $\mathcal{R}\mathcal{R}^T = I$ with the added constraint

[11] The relationship between the time derivative in an inertial frame $\mathcal{O}$ and the time derivative in a moving frame $\mathcal{C}$ moving with angular velocity $\boldsymbol{\omega}$ is (Chapter IV in [42])

$$\left.\frac{d\,\cdot}{dt}\right|_{\mathcal{O}} = \left.\frac{d\,\cdot}{dt}\right|_{\mathcal{C}} + \boldsymbol{\omega} \times \,\cdot\,.$$

$\det(\mathcal{R}) = 1$; rotation matrices form a subset of the orthogonal matrices [41]. In some references one sees three-dimensional rotations referred to as members of SO(3), which is the (three-dimensional) special orthogonal group [53].

The inverse of the transformation matrix $\mathcal{T}_{OC}$ is

$$\mathcal{T}_{OC}^{-1} = \begin{bmatrix} \mathcal{R}_{OC}^T & -\mathcal{R}_{OC}^T \boldsymbol{r}_{OC}^{\mathcal{O}} \\ 0 & 1 \end{bmatrix}. \tag{2.150}$$

The coordinates of P expressed in frame $\mathcal{C}$ can be obtained from those given in frame $\mathcal{O}$ by inversion of (2.148)

$$\begin{bmatrix} \boldsymbol{r}_{CP}^{\mathcal{C}} \\ 1 \end{bmatrix} = \mathcal{T}_{OC}^{-1} \begin{bmatrix} \boldsymbol{r}_{OP}^{\mathcal{O}} \\ 1 \end{bmatrix}. \tag{2.151}$$

The transformations defined above for points can be applied to vectors by replacing the '1' on the right-hand side of (2.148) with a '0', which suppresses the translational component of the transformation:

$$\begin{bmatrix} \boldsymbol{r}_{CP}^{\mathcal{O}} \\ 0 \end{bmatrix} = \mathcal{T}_{OC} \begin{bmatrix} \boldsymbol{r}_{CP}^{\mathcal{C}} \\ 0 \end{bmatrix}. \tag{2.152}$$

The vectors on both sides of (2.152) have subscript 'CP', while in the case of points $\boldsymbol{r}_{OP}$ and $\boldsymbol{r}_{CP}$ are used; see (2.148).

While the translation component of the transformation is straightforward, the rotational element component requires further discussion. Every rotation matrix can be expressed in terms of an axis-of-rotation unit-vector $\boldsymbol{n}$ and a rotation angle φ [41,53]. For any rotation matrix $\mathcal{R}$ there exists a unit vector $\boldsymbol{n}$ and an angle φ such that $\mathcal{R} = \mathcal{R}(\boldsymbol{n}, \varphi)$, where

$$\mathcal{R}(\boldsymbol{n}, \varphi) = I + (1 - \cos(\varphi))S^2(\boldsymbol{n}) + \sin(\varphi)S(\boldsymbol{n}) \tag{2.153}$$

is *Rodrigues' rotation formula*, and $S(\boldsymbol{n})$ is a skew-symmetric matrix

$$S(\boldsymbol{n}) = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}, \tag{2.154}$$

in which n_1, n_2, and n_3 are the x-, y-, and z-axis components of $\boldsymbol{n}$.

The matrix $\mathcal{R}_{123}$ related to the orientation of a body in space can be defined by three successive rotations about given axes performed in a specific sequence.[12] The combined rotation is given by

$$\mathcal{R}_{123} = \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 \,. \tag{2.155}$$

In principle, there are $3^3 = 27$ possible combinations of the three basic rotations, but only 12 of them can be used to represent arbitrary three-dimensional rotations.

[12] The order does not matter for 'small' rotations.

These 12 combinations avoid degenerate consecutive rotations around the same axis (such as x–x–y) that would reduce the degrees of freedom that can be represented. The most widespread sequences are z–x–z (usually referred to as Euler 'x-convention'), z–y–z (Euler 'y-convention', popular in quantum and nuclear mechanics), x–y–z (usually referred to as Tait–Bryan), z–y–x (heading–attitude–bank, popular in aerospace) and z–x–y (yaw–roll–pitch, popular in vehicle dynamics).

The rotated frame x–y–z may be imagined to be initially aligned with X–Y–Z, before undergoing the three elemental rotations. Its successive orientations may be described as:

1. X–Y–Z (initial configuration),
2. X'–Y'–Z' (first rotation),
3. X''–Y''–Z'' (second rotation),
4. x–y–z (final configuration).

Suppose that the sequence z–x–y is chosen

$$\mathcal{R}_1 = \mathcal{R}(\boldsymbol{e}_z, \psi) = \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.156}$$

$$\mathcal{R}_2 = \mathcal{R}(\boldsymbol{e}_x, \phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix} \tag{2.157}$$

$$\mathcal{R}_3 = \mathcal{R}(\boldsymbol{e}_y, \theta) = \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}, \tag{2.158}$$

in which $\mathcal{R}(\,\cdot\,, \cdot\,)$ is defined in (2.153), with $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ unit vectors in the x-, y-, and z-axis directions, respectively, in the appropriate reference frame. We have used the shorthand $s_{(\,\cdot\,)}$ and $c_{(\,\cdot\,)}$ to denote the sine and cosine of angle $(\,\cdot\,)$. The resulting yaw–roll–pitch rotation matrix is

$$\mathcal{R} = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3 = \begin{bmatrix} -s_\psi s_\phi s_\theta + c_\psi c_\theta & -s_\psi c_\phi & s_\psi s_\phi c_\theta + c_\psi s_\theta \\ c_\psi s_\phi s_\theta + s_\psi c_\theta & c_\psi c_\phi & -c_\psi s_\phi c_\theta + s_\psi s_\theta \\ -c_\phi s_\theta & s_\phi & c_\phi c_\theta \end{bmatrix} \tag{2.159}$$

The three-rotations matrix in (2.159) can be alternatively expressed in terms of the single-rotation matrix $\mathcal{R}(\boldsymbol{n}, \varphi)$ in (2.153). The axis of rotation $\boldsymbol{n}$ is the eigenvector corresponding to the unity eigenvalue of (2.159)

$$\mathcal{R}\boldsymbol{n} = \boldsymbol{n}, \tag{2.160}$$

since the rotation axis remains the same in the initial and final reference frames. It follows from (2.153) that the angle φ can be obtained from

$$\cos\varphi = \frac{Tr(\mathcal{R}) - 1}{2} \qquad \sin\varphi = \frac{\mathcal{R}(kj) - \mathcal{R}(jk)}{2n_i} \qquad (i, j, k = 1, 2, 3 \text{ cyclic}), \tag{2.161}$$

where $Tr(\mathcal{R})$ is the trace of $\mathcal{R}$, n_i is the i-th entry of $\boldsymbol{n}$, and $\mathcal{R}(kj)$ is the kj-th entry of $\mathcal{R}$.

Our next goal is to link rotation matrices to the vector cross product defined by

$$\mathbf{a} \times \boldsymbol{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

in which $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors. Since the cross product (by $\mathbf{a}$) is a linear operation, it has a matrix representation $S(\mathbf{a})$ and so

$$\mathbf{a} \times \mathbf{b} = S(\mathbf{a})\mathbf{b}, \tag{2.162}$$

where $S(\mathbf{a})$ is the skew symmetric matrix defined in (2.154). It is important to note that rotations commute with the cross product. That is $\mathcal{R}(\mathbf{a} \times \mathbf{b}) = (\mathcal{R}\mathbf{a}) \times (\mathcal{R}\mathbf{b})$.[13] It follows that $\mathcal{R}(\mathbf{a} \times \mathbf{b}) = \mathcal{R}S(\mathbf{a})(\mathcal{R}^T\mathcal{R})\mathbf{b}$ and so

$$\mathcal{R}S(\boldsymbol{a})\mathcal{R}^T = S(\mathcal{R}\boldsymbol{a}). \tag{2.163}$$

We will use the properties of cross products to derive a number of standard formulae relating to motions in rotating reference frames. Since $I = \mathcal{R}\mathcal{R}^T$, $d(\mathcal{R}\mathcal{R}^T)/dt = \dot{\mathcal{R}}\mathcal{R}^T + (\dot{\mathcal{R}}\mathcal{R}^T)^T = 0$, and we see that $\dot{\mathcal{R}}\mathcal{R}^T = -(\dot{\mathcal{R}}\mathcal{R}^T)^T$, which shows that $\dot{\mathcal{R}}\mathcal{R}^T$ is skew-symmetric. Parallel arguments using $d(\mathcal{R}^T\mathcal{R})/dt = 0$ show that $\mathcal{R}^T\dot{\mathcal{R}}$ is skew-symmetric too.

We can now derive expressions for the velocities and accelerations using matrix notation. The absolute velocity of a point P, expressed in the frame $\mathcal{O}$, is obtained by differentiating (2.148) with respect to time

$$\boldsymbol{v}_P^{\mathcal{O}} = \dot{\mathcal{T}}_{OC} \begin{bmatrix} \boldsymbol{r}_{CP}^{\mathcal{C}} \\ 1 \end{bmatrix} + \mathcal{T}_{OC} \begin{bmatrix} \dot{\boldsymbol{r}}_{CP}^{\mathcal{C}} \\ 0 \end{bmatrix} \tag{2.164}$$

$$= \dot{\boldsymbol{r}}_{OC}^{\mathcal{O}} + \dot{\mathcal{R}}_{OC}\boldsymbol{r}_{CP}^{\mathcal{C}} + \mathcal{R}_{OC}\dot{\boldsymbol{r}}_{CP}^{\mathcal{C}}; \tag{2.165}$$

the first term is the translational velocity of the origin of the moving frame, the second is the transferred velocity, and the third is the relative velocity. The second term in (2.165) can be rewritten as $(\dot{\mathcal{R}}_{OC}\mathcal{R}_{OC}^T)\boldsymbol{r}_{CP}^{\mathcal{O}}$ and the third as $\dot{\boldsymbol{r}}_{CP}^{\mathcal{O}}$. Comparison with (2.144) shows that the skew-symmetric matrix for the angular velocity of the frame $\mathcal{C}$ expressed in the frame $\mathcal{O}$ is

$$S(\boldsymbol{\omega}_{OC}^{\mathcal{O}}) = \dot{\mathcal{R}}_{OC}\mathcal{R}_{OC}^T. \tag{2.166}$$

It is often convenient to express the velocity of the moving frame in its own coordinate system. Premultiplied by the inverse transformation matrix, (2.164) and (2.165) become

$$\boldsymbol{v}_P^{\mathcal{C}} = (\mathcal{T}_{OC}^{-1}\dot{\mathcal{T}}_{OC}) \begin{bmatrix} \boldsymbol{r}_{CP}^{\mathcal{C}} \\ 1 \end{bmatrix} + \begin{bmatrix} \dot{\boldsymbol{r}}_{CP}^{\mathcal{C}} \\ 0 \end{bmatrix} \tag{2.167}$$

$$= \mathcal{R}_{OC}^T\dot{\boldsymbol{r}}_{OC}^{\mathcal{O}} + (\mathcal{R}_{OC}^T\dot{\mathcal{R}}_{OC})\boldsymbol{r}_{CP}^{\mathcal{C}} + \dot{\boldsymbol{r}}_{CP}^{\mathcal{C}}\,. \tag{2.168}$$

[13] This can be established as follows: if $\mathbf{a} \times \mathbf{b} = \mathbf{c}$, then $\mathcal{R}(\mathbf{a} \times \mathbf{b}) = \mathcal{R}\mathbf{c}$. If $\mathbf{a}' = \mathcal{R}\mathbf{a}$, $\mathbf{b}' = \mathcal{R}\mathbf{b}$, and $\mathbf{c}' = \mathcal{R}\mathbf{c}$, then $\mathbf{a}' \times \mathbf{b}' = \mathbf{c}'$, since the relative orientations and lengths of these vectors have not changed. Consequently $(\mathcal{R}\mathbf{a}) \times (\mathcal{R}\mathbf{b}) = \mathcal{R}\mathbf{c} = \mathcal{R}(\mathbf{a} \times \mathbf{b})$ as required.

Comparison with (2.144) shows that the angular velocity of frame $\mathcal{C}$ expressed in its own coordinate system is given by

$$S(\boldsymbol{\omega}^{\mathcal{C}}_{OC}) = \mathcal{R}^T_{OC}\dot{\mathcal{R}}_{OC}. \tag{2.169}$$

The acceleration vector can be computed by differentiating the velocity in (2.164) with respect to time:

$$\begin{aligned}
\boldsymbol{a}^{\mathcal{O}}_P &= \frac{d}{dt}\left(\dot{\mathcal{T}}_{OC}\mathcal{T}^{-1}_{OC}\begin{bmatrix}\boldsymbol{r}^{\mathcal{O}}_{OP}\\1\end{bmatrix} + \mathcal{T}_{OC}\begin{bmatrix}\dot{\boldsymbol{r}}^{\mathcal{C}}_{CP}\\0\end{bmatrix}\right)\\
&= \frac{d}{dt}\left(\dot{\mathcal{T}}_{OC}\mathcal{T}^{-1}_{OC}\right)\begin{bmatrix}\boldsymbol{r}^{\mathcal{O}}_{OP}\\1\end{bmatrix} + \dot{\mathcal{T}}_{OC}\mathcal{T}^{-1}_{OC}\left(\dot{\mathcal{T}}_{OC}\begin{bmatrix}\boldsymbol{r}^{\mathcal{C}}_{CP}\\1\end{bmatrix}\right.\\
&\quad \left.+ \mathcal{T}_{OC}\begin{bmatrix}\dot{\boldsymbol{r}}^{\mathcal{C}}_{CP}\\0\end{bmatrix}\right) + \dot{\mathcal{T}}_{OC}\begin{bmatrix}\dot{\boldsymbol{r}}^{\mathcal{C}}_{CP}\\0\end{bmatrix} + \mathcal{T}_{OC}\begin{bmatrix}\ddot{\boldsymbol{r}}^{\mathcal{C}}_{CP}\\0\end{bmatrix}\\
&= \frac{d}{dt}\left(\dot{\mathcal{T}}_{OC}\mathcal{T}^{-1}_{OC}\right)\begin{bmatrix}\boldsymbol{r}^{\mathcal{O}}_{OP}\\1\end{bmatrix} + (\dot{\mathcal{T}}_{OC}\mathcal{T}^{-1}_{OC})(\dot{\mathcal{T}}_{OC}\mathcal{T}^{-1}_{OC})\begin{bmatrix}\boldsymbol{r}^{\mathcal{O}}_{OP}\\1\end{bmatrix}\\
&\quad +2(\dot{\mathcal{T}}_{OC}\mathcal{T}^{-1}_{OC})\begin{bmatrix}\dot{\boldsymbol{r}}^{\mathcal{O}}_{CP}\\0\end{bmatrix} + \mathcal{T}_{OC}\begin{bmatrix}\ddot{\boldsymbol{r}}^{\mathcal{C}}_{CP}\\0\end{bmatrix} \qquad (2.170)\\
&= \ddot{r}^{\mathcal{O}}_{OC} + \frac{d}{dt}(\dot{\mathcal{R}}_{OC}\mathcal{R}^T_{OC})\boldsymbol{r}^{\mathcal{O}}_{CP} + (\dot{\mathcal{R}}_{OC}\mathcal{R}^T_{OC})(\dot{\mathcal{R}}_{OC}\mathcal{R}^T_{OC})\boldsymbol{r}^{\mathcal{O}}_{CP}\\
&\quad +2(\dot{\mathcal{R}}_{OC}\mathcal{R}^T_{OC})\dot{\boldsymbol{r}}^{\mathcal{O}}_{CP} + \mathcal{R}_{OC}\ddot{\boldsymbol{r}}^{\mathcal{C}}_{CP}\,. \qquad (2.171)
\end{aligned}$$

Again, comparison with (2.145) allows one to recognize the acceleration of the origin of the frame, the Euler acceleration, the centrifugal acceleration, the Coriolis acceleration, and the relative acceleration. The absolute acceleration in (2.170) and (2.171) can be expressed in the moving frame by premultiplication by $\mathcal{T}^{-1}$ and $\mathcal{R}^T$ respectively.

We conclude the section by noting that the summation rule holds for the angular velocity. Assuming a combination of the three rotation $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$, there holds $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{\omega}_3$. If we use the three rotations in (2.156)–(2.158) we find that the angular velocities of the frames in their own coordinate systems are

$$S(\boldsymbol{\omega}^1_1) = \mathcal{R}^T_1\dot{\mathcal{R}}_1 = \begin{bmatrix}0 & -\dot{\psi} & 0\\ \dot{\psi} & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \to \boldsymbol{\omega}^1_1 = \begin{bmatrix}0\\0\\\dot{\psi}\end{bmatrix} \tag{2.172}$$

$$S(\boldsymbol{\omega}^2_2) = \mathcal{R}^T_2\dot{\mathcal{R}}_2 = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -\dot{\phi}\\ 0 & \dot{\phi} & 0\end{bmatrix} \to \boldsymbol{\omega}^2_2 = \begin{bmatrix}\dot{\phi}\\0\\0\end{bmatrix} \tag{2.173}$$

$$S(\boldsymbol{\omega}^3_3) = \mathcal{R}^T_3\dot{\mathcal{R}}_3 = \begin{bmatrix}0 & 0 & \dot{\theta}\\ 0 & 0 & 0\\ -\dot{\theta} & 0 & 0\end{bmatrix} \to \boldsymbol{\omega}^3_3 = \begin{bmatrix}0\\\dot{\theta}\\0\end{bmatrix} \tag{2.174}$$

and the resulting angular velocity is

$$\boldsymbol{\omega}_{123}^3 = \boldsymbol{\omega}_1^3 + \boldsymbol{\omega}_2^3 + \boldsymbol{\omega}_3^3 = \begin{bmatrix} -s_\theta\dot{\psi}\, c_\phi + c_\theta\dot{\theta} \\ s_\phi\dot{\psi} + \dot{\theta} \\ c_\theta\dot{\psi}\, c_\phi + s_\theta\dot{\theta} \end{bmatrix} \tag{2.175}$$

which is expressed in the coordinates of the final frame.

2.6 Equilibria, stability, and linearization

We will make extensive use of linear models, because linear equations are easy to solve and their stability properties are readily assessed using eigenvalues. Also, in many nonlinear problems in mechanics, linearized models provide a satisfactory approximate solution. In cases when linear models are not enough, a study of the local dynamics is often a useful initial step. When studying linear behaviour one examines small perturbations around an equilibrium (trim) condition. If

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t)),$$

then $\boldsymbol{x}_0$ is a trim state if $f(\boldsymbol{x}_0) = 0$. In the case that $T = \frac{1}{2}\boldsymbol{p}^T M(\boldsymbol{q})^{-1}\boldsymbol{p}$, $V = V(\boldsymbol{q})$, and $\mathcal{H} = T + V$, and there are no cyclic coordinates, we have:

Theorem 2.1 *The point $\boldsymbol{q} = \boldsymbol{q}_0$, $\boldsymbol{p} = \boldsymbol{p}_0$ is an equilibrium position if and only if $\boldsymbol{p}_0 = 0$ and $\boldsymbol{q}_0$ is a stationary point of the potential energy:*

$$\left.\frac{\partial V}{\partial \boldsymbol{q}}\right|_{\boldsymbol{q}_0} = 0. \tag{2.176}$$

Proof Under the assumptions relating to the energy functions, Hamilton's equations take the form:

$$\begin{bmatrix} \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}} \\ \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} \end{bmatrix} = \begin{bmatrix} -\frac{\partial T}{\partial \boldsymbol{q}} - \frac{\partial V}{\partial \boldsymbol{q}} \\ M^{-1}(\boldsymbol{q})\boldsymbol{p} \end{bmatrix}. \tag{2.177}$$

Suppose the system is in equilibrium. The second row of (2.177) shows that $\dot{\boldsymbol{q}} = 0$ implies that $\boldsymbol{p}_0 = 0$, and the first row establishes that $\dot{\boldsymbol{p}} = 0$ implies (2.176), since $\frac{\partial T}{\partial \boldsymbol{q}} = -\frac{1}{2}\boldsymbol{p}^T M^{-1}\frac{\partial M}{\partial \boldsymbol{q}} M^{-1}\boldsymbol{p}$ vanishes when $\boldsymbol{p}_0 = 0$. If (2.176) is satisfied and $\boldsymbol{p}_0 = 0$, the first row of (2.177) establishes that $\dot{\boldsymbol{p}} = 0$, while the second row shows that $\dot{\boldsymbol{q}} = 0$ and so the system is in equilibrium. □

The next result shows that trim conditions corresponding to minima in the potential energy function are stable in the sense of Lyapunov [45].

Theorem 2.2 *If the point $\boldsymbol{q}_0$ is a local minimum in the potential energy function, the equilibrium $\boldsymbol{q} = \boldsymbol{q}_0$ is stable in the sense of Lyapunov.*

Proof Suppose $V(\boldsymbol{q}_0) = h$. For sufficiently small $\epsilon > 0$, the solution set $\boldsymbol{q} : V(\boldsymbol{q}) \leq h + \epsilon$, containing $\boldsymbol{q}_0$, will be in a small neighbourhood of $\boldsymbol{q}_0$. Furthermore, the corresponding region of the state space $\{\boldsymbol{p}, \boldsymbol{q} : E(\boldsymbol{q}, \boldsymbol{p}) \leq h + \epsilon\}$ will be in a small neighbourhood of $\boldsymbol{p} = 0$ and $\boldsymbol{q}_0$; see Figure 2.4. Since the energy remains constant throughout the flow, $E(\boldsymbol{q}_0, \boldsymbol{p}_0) \leq h + \epsilon$, implies that $E(\boldsymbol{q}, \boldsymbol{p}) \leq h + \epsilon$. Thus for any initial condition $(\boldsymbol{q}_0, \boldsymbol{p}_0)$ close to $(\boldsymbol{q}_0, 0)$, the corresponding flow $(\boldsymbol{q}(t), \boldsymbol{p}(t))$ also remains close to $(\boldsymbol{q}_0, 0)$. □

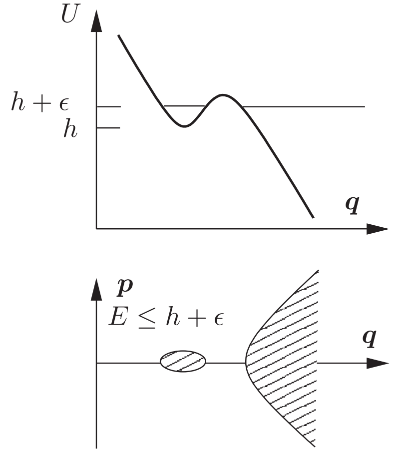


Figure 2.4: Stable equilibrium associated with $V(\boldsymbol{q}_0) = h$; the shaded areas correspond to $E(\boldsymbol{p}, \boldsymbol{q}) \leq h + \epsilon$.

The final result of this section shows that one can derive the linear equations of motion by considering only the quadratic parts of the energy functions. We can assume without loss of generality that the coordinate system has been chosen so that the equilibrium position is $\boldsymbol{q}_0 = 0$.

Theorem 2.3 *In order to linearize the Lagrangian system*

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}$$

in the neighbourhood of $\boldsymbol{q} = 0$, *one may replace the kinetic energy by* $T_2 = \frac{1}{2}\dot{\boldsymbol{q}}^T M(0)\dot{\boldsymbol{q}}$ *and the potential energy by its quadratic part*

$$V_2 = \frac{1}{2}\boldsymbol{q}^T \left(\left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\boldsymbol{q}=0} \right) \boldsymbol{q}.$$

Proof Since the inertial force terms are determined by partial derivatives with respect to the $\dot{q}_i$ s, the linear inertial forces are determined by the quadratic terms in T, which are precisely those in T_2. The imposed forces derive from partial derivatives with respect to the q_i s, and the linear imposed force terms come from the quadratic terms in V, which are those in V_2. □

2.7 Time-reversal symmetry and dissipation

Time reversibility is a topic that lies on the border of physics and philosophy and has caused more than its fair share of controversy. In one oft-quoted reference [61], the idea that '*all known laws of physics are invariant under time reversal*' is studied; alternative discussions on time reversibility can be found in the excellent texts [62,63]. In the case of classical mechanics it can be argued that Newton's first and third laws make no reference to the direction of time and would therefore have identical forms in a time-reversed universe. In the case of Newton's second law one might consider a particle of mass m at position $\boldsymbol{r}$ in a force field of the form $\boldsymbol{F}(\boldsymbol{r})$ and so

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F}(\boldsymbol{r}). \tag{2.178}$$

This equation can be solved subject to boundary conditions on $\boldsymbol{r}(t_0)$ and $\dot{\boldsymbol{r}}(t_0)$. If $m\ddot{\boldsymbol{\gamma}}(t) = \boldsymbol{F}(\boldsymbol{\gamma}(t))$, then $\boldsymbol{\gamma}(t)$ is a solution to (2.178). If t is replaced by $\tau = -t$ one obtains

$$m\ddot{\boldsymbol{\gamma}}(\tau) = \boldsymbol{F}(\boldsymbol{\gamma}(\tau)) \tag{2.179}$$

and so $\boldsymbol{\gamma}(\tau)$ is a possible trajectory if $\boldsymbol{\gamma}(t)$ is; recall $\frac{d^2}{dt^2} = \frac{d^2}{d\tau^2}$. Under the given assumptions on the force, Newton's second law is *time reversal invariant*. The time reversal leaves $\boldsymbol{r}$ unchanged, but reverses the sign of the velocities $d\boldsymbol{r}/dt = -(d\boldsymbol{r}/d\tau)$. The same arguments show that the system has time-reversal symmetry even with a more general force of the form $\boldsymbol{F}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t)$ that satisfies

$$\boldsymbol{F}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) = \boldsymbol{F}(\boldsymbol{r}, -\dot{\boldsymbol{r}}, -t) \tag{2.180}$$

under the mapping $t \to -t$, $\boldsymbol{r} \to \boldsymbol{r}$, $\dot{\boldsymbol{r}} \to -\dot{\boldsymbol{r}}$. These arguments can be developed in a similar way in a Lagrangian framework [64].

Example 2.16 Consider an ideal lossless simple pendulum described by

$$\ddot{\theta} l + g\theta = 0,$$

which has solution

$$\begin{aligned}\theta(t) &= \frac{\dot{\theta}_0}{\omega}\sin(\omega t) + \theta_0\cos(\omega t)\\ \dot{\theta}(t) &= \dot{\theta}_0\cos(\omega t) - \theta_0\omega\sin(\omega t),\end{aligned}$$

where $\omega = \sqrt{g/l}$, and θ_0 and $\dot{\theta}_0$ are initial conditions on the pendulum's position and velocity respectively. If we now reverse the sign of t and $\dot{\theta}_0$, there holds

$$\begin{aligned}\theta(-t) &= -\frac{\dot{\theta}_0}{\omega}\sin(-\omega t) + \theta_0\cos(-\omega t) = \theta(t)\\ \dot{\theta}(-t) &= -\dot{\theta}_0\cos(-\omega t) - \theta_0\omega\sin(-\omega t) = -\dot{\theta}(t).\end{aligned}$$

This means that if we were to film a few cycles of the pendulum's motion it would be impossible to tell if one was watching the film being played forwards or backwards. The backward time motion satisfies the same laws of physics as the forward motion and in this case (2.180) holds. If we now consider a more realistic pendulum with losses, it will be obvious if the observed motion is running forwards or backwards, because the forward motion will show decay, while the reversed observation will show growth. Linear damping forces take the form $-c\dot{\theta}$, for c constant, which do not satisfy (2.180).

The previous example suggests an association between conservative systems and time-reversibility. The next two examples show that conservative systems are not necessarily time reversible, nor are time-reversible systems necessarily conservative! The first is adapted from [65] and shows that conservative systems need not be time reversible. The second example shows that time-reversible systems need not be conservative [66].

Example 2.17 Consider a unit mass with one translational degree of freedom in a well-like potential field described by $V(x) = (x^4 - x^6)/2$. The kinetic energy of the unit mass is $T(\dot{x}) = \dot{x}^2/2$. The initial position of the mass is at $x = +1\,\mathrm{m}$; see Figure 2.5. A direct application of Lagrange's equation shows that the mass's equation of motion is $\ddot{x} = 3x^5 - 2x^3$. Solving this equation for $x(0) = 1$ and $\dot{x}(0) = 0$ gives $x(t) = (1-t^2)^{-1/2}$ and so $\dot{x}(t) = t(1 - t^2)^{-3/2}$. This means that the kinetic energy of the mass, as a function of time, is given by $T(t) = t^2(1 - t^2)^{-3}/2$, while $T(t) + V(t) = 0$. It is now clear that the mass's position, velocity, and kinetic energy all approach infinity in the limit as $t \to 1$. At the same time the kinetic energy approaches infinity, the potential energy approaches minus infinity, thereby keeping the total system energy constant (and in fact zero). For times greater than unity the system comprises a force field alone, since the mass has left the universe being considered. This system cannot be

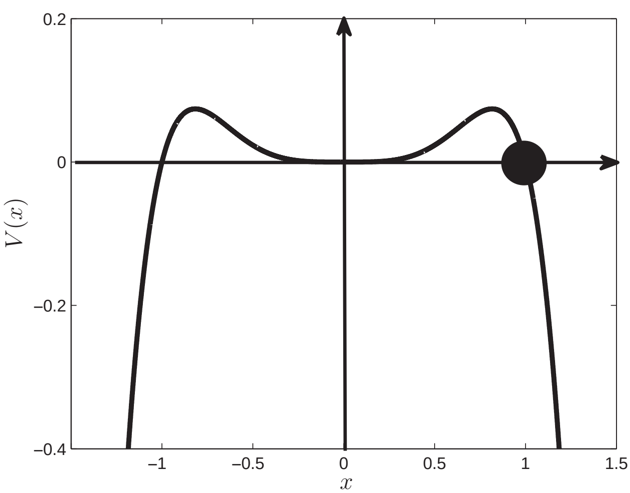


Figure 2.5: Unit mass in a potential force field.

time reversed despite being conservative. As a result of the singularity at $t = 1$, we cannot deduce the system's reversibility from the equations of motion alone and time reversibility may only be a 'local' property.

Example 2.18 If $x(t)$ and $y(t)$ are solutions to the equations

$$\begin{aligned} \dot{x} &= -2\cos x - \cos y \qquad & x(t_0) &= x_0 \\ \dot{y} &= -2\cos y - \cos x \qquad & y(t_0) &= y_0, \end{aligned} \tag{2.181}$$

then so are $-x(-t)$ and $-y(-t)$. This means that this system is time reversible. The fixed points satisfy $\dot{x} = \dot{y} = 0$. Solving the corresponding algebraic equations gives $\cos x^* = 0$ and $\cos y^* = 0$, or $(x^*, y^*) = (\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$. Linearization around (x^*, y^*) gives

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 2\sin x^* & \sin y^* \\ \sin x^* & 2\sin y^* \end{bmatrix},$$

which has eigenvalues (1, 3) at $(\frac{\pi}{2}, \frac{\pi}{2})$, $\pm\sqrt{3}$ at $(\pm\frac{\pi}{2}, \mp\frac{\pi}{2})$, and $(-1, -3)$ at $(-\frac{\pi}{2}, -\frac{\pi}{2})$. As shown in the phase portrait in Figure 2.6, the fixed points are respectively a repeller, two saddle points, and an attractor. A conservative system must be free of attractors, because the system energy cannot be both time-invariant and non-constant throughout the basin of attraction.

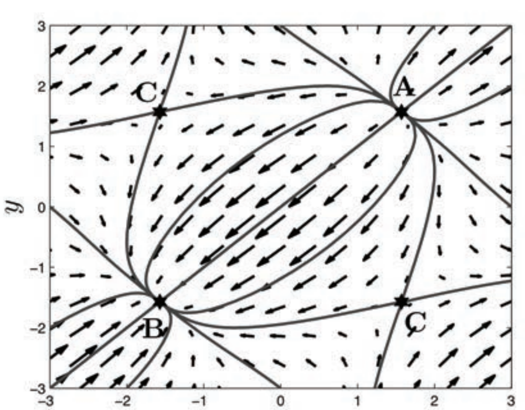


Figure 2.6: Phase portrait of the system given in (2.181). Point A is a repeller, point B an attractor, and the points C are saddles.

Engineers are always making use of time-reversibility-destroying effects such as friction, damping, hysteresis, and Ohmic losses, even though physicists might take the view that these loss-creating influences are 'non-physical' contrivances that are not part of the fundamental make-up of nature. It might be suggested that one has to consider the more basic properties of nature's micro-structure, where forces are conservative and relate to such things as atomic interactions. The damped-pendulum example poses a deep theoretical problem that is encapsulated in Loschmidt's objection [61], also known as the irreversibility paradox, which asks: 'how can a macroscopic system be irreversible (like the damped pendulum), while its microscopic constituents are conservative?' This puts the time-reversal symmetry of (almost) all known low-level fundamental physical processes at odds with any attempt to infer from them the second law of thermodynamics that describes the behaviour of macroscopic systems. In an attempt to explain this paradox, one could argues that the motion of the damped pendulum is slowed by the transference of kinetic energy from the pendulum to the surrounding medium atoms in the form of heat [61]. Since the laws of physics governing atomic interactions are reversible, each collision must be reversed, causing a cooperative transfer of energy back to the bob, which will then be consequently accelerated returning it to its original position (if one waits long enough). The *Poincaré recurrence theorem* states that conservative systems will, after a sufficiently long but finite time (the *Poincaré recurrence time*), return to a state 'close' to the initial state.

During modelling exercises one must be mindful of the difference between the true properties of nature and the characteristics of the model being used to describe it. As we have shown, if one assumes that air resistance can be described by a velocity-dependent damping force (linear damping), the *predicted* motion of the pendulum will be irreversible. If one believes Newton's laws, there is nothing wrong with the mechanics, but the model being used to describe nature may be brought into question. Classical mechanics (as a theory) is neutral on the subject of time reversibility and is equally compatible with forces that produce reversible behaviours, and phenomenological or empirical forces such as friction, which cause irreversible motions.

Bearing Example 2.16 in mind, one might postulate that conservative (position-dependent) forces result in reversible motions, while time- and velocity-dependent forces such as friction do not. The next two examples show that the relationship between forces that result in reversible motions and ones that do not is more subtle than this. Indeed, conservative systems may 'appear' dissipative if one does not look too far into the future (much less than the Poincaré recurrence time). The first example is taken from statistical mechanics, while the second is taken from engineering.

Example 2.19 In physics there are many systems that are describable by equations of motion of the form

$$m\ddot{x} + c\dot{x} + \frac{\partial V(x)}{\partial x} = F \tag{2.182}$$

in which $c\dot{x}$ is a damping term and F is a driving force. A simple example is the *Brownian motion* of colloidal particles floating in a liquid medium. In this case m is the mass of the particle, c is a damping coefficient, $V(x)$ is the potential acting on the particle, and F is a stochastic force, which causes the particles to undergo irregular 'jiggle' motions. This force is describable in terms of its mean and variance:

$$\begin{aligned} \mathcal{E}\{F(t)\} &= 0, \\ \mathcal{E}\{F(t)F(t')\} &= 2mk_B Tc\delta(t-t'), \end{aligned} \tag{2.183}$$

where $\mathcal{E}\{\,\cdot\,\}$ represents the expected value, or statistical average, over the ensemble of realizations of the force $F(t)$ [67]. Boltzmann's constant is given by k_B, and T is the

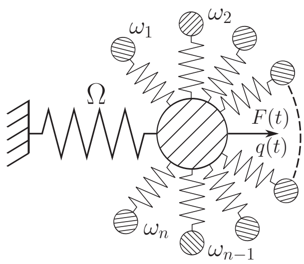


Figure 2.7: Harmonic oscillator in a thermal bath.

temperature. The appearance of the damping constant c in the variance means that the damping force and the random force are related. This relationship between the deterministic damping and the random forces is the subject of the *fluctuation–dissipation theorem*, which states that the random force must have a power spectrum determined by the damping [67]. In sum, (2.182) is a second-order stochastic differential equation driven by white noise and is known in the physics literature as the *Langevin equation*.

In classical studies of Brownian motion, the so-called *Caldeira–Leggett system-plus-reservoir model* is used [68–70], which is illustrated in Figure 2.7. Since this system is conservative, it follows from the Poincaré recurrence theorem that this system cannot display damping effects, because this would preclude a return to the system's initial state.

The Lagrangian associated with this system is

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 + \sum_{i=1}^{n} \frac{m_i}{2}\dot{q}_i^2 - \frac{1}{2}\Omega^2 q^2 - \sum_{i=1}^{n} \frac{k_i}{2}\left(q_i - q\right)^2, \tag{2.184}$$

in which the particle mass is assumed to be unity. The primary oscillator frequency is Ω, the heat bath spring stiffnesses are $k_i = \gamma^2$, the heat bath masses are $m_i = \gamma^2\left(\frac{n}{i\omega_c}\right)^2$, with the equispaced bath frequencies given by $\omega_i = i\omega_c/n$, and ω_c is some cut-off frequency. Direct calculation using Lagrange's equations (2.40) gives

$$\ddot{q} = -\Omega^2 q + \sum_{i=1}^{n} k_i\left(q_i - q\right) \tag{2.185}$$

$$\ddot{q}_i = -\omega_i^2\left(q_i - q\right). \tag{2.186}$$

The Laplace transform of (2.185) is

$$q(s)\left(s^2 + \Omega^2 + \sum_{i=1}^{n} k_i\right) = \sum_{i=1}^{n} k_i q_i(s) + sq(0^+) + \dot{q}(0^+) \tag{2.187}$$

while the Laplace transform of (2.186) is

$$q_i(s) = \frac{\omega_i^2 q(s)}{s^2 + \omega_i^2} + \frac{sq_i(0^+) + \dot{q}_i(0^+)}{s^2 + \omega_i^2}. \tag{2.188}$$

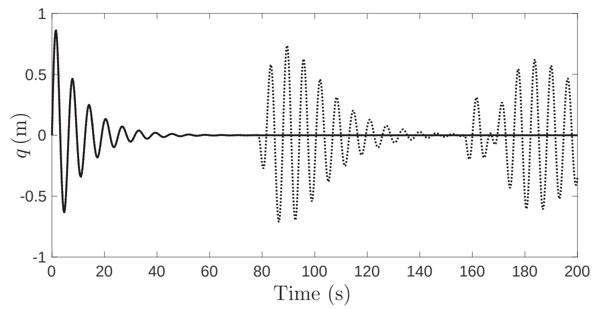


Figure 2.8: Impulse response of a harmonic oscillator in a thermal bath; $\Omega = 1$, $\gamma = 0.1$, $n = 250$ and $\omega_c = 20$. The dotted curve is the response of the full model (2.189), while the solid curve is the response of the second-order damped model given in (2.190).

Substituting (2.188) into (2.187), assuming that $q(0^+) = 0$ and $\dot{q}(0^+) = 0$, gives

$$q(s)\left(s^2+\Omega^2+s\frac{n\pi\gamma^2}{2\omega_c}\left(\frac{2\omega_c}{\pi n}\sum_{i=1}^{n}\frac{s}{s^2+(\frac{i\omega_c}{n})^2}\right)\right)=\overbrace{\sum_{i=1}^{n}\left(\frac{sq_i(0^+)+\dot{q}_i(0^+)}{s^2+(\frac{i\omega_c}{n})^2}\right)}^{F(s)}. \tag{2.189}$$

The right-hand side of (2.189) can be interpreted as random forcing that comes from the initial positions and momenta of the bath masses. Since

$$\lim_{n\to\infty}\left(\frac{2\omega_c}{\pi n}\sum_{i=1}^{n}\frac{s}{s^2+(\frac{i\omega_c}{n})^2}\right)=1,$$

when n is large, one might expect (2.189) to behave like

$$\ddot{q}+\frac{n\pi\gamma^2}{2\omega_c}\dot{q}+\Omega^2 q=F(t), \tag{2.190}$$

which is in the form of the Langevin equation (2.182), which predicts damping.

In problems of this type, high-order conservative descriptions are often replaced by simple low-order models with dissipation. Figure 2.8 shows the impulse responses of the full-order system (2.189) and the second-order system (2.190). It is clear that on some finite horizon, which may be large (but smaller than the Poincaré recurrence time), the high-order conservative model is almost indistinguishable from a low-order dissipative description.

Example 2.20 We now return to Example 2.6, and consider again the one-dimensional mass–spring transmission line illustrated in Figure 2.9. The input of this system is the force F, while the output is the velocity $\dot{q}$ of the left-hand mass m. A force balance on the first mass gives

$$m\ddot{q}=F+k(q_1-q). \tag{2.191}$$

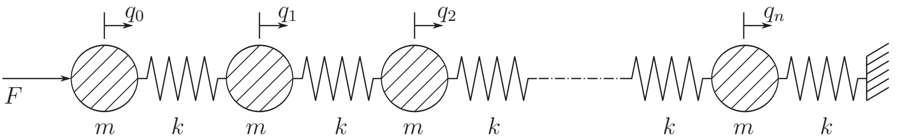


Figure 2.9: Transmission line lumped approximation.

Forces balances on the interior masses yield

$$m\ddot{q}_i = k(q_{i-1} - 2q_i + q_{i+1}) \qquad i = 1, \cdots, n-1, \tag{2.192}$$

while a force balance on the last mass is given by

$$m\ddot{q}_n = k(q_{n-1} - 2q_n). \tag{2.193}$$

The step response of this system is given in Figure 2.10 and it can be seen that the mechanical admittance[14] is $F/\dot{q} = \pm 80$, which corresponds to pure (positive and negative) damping, with the sign switching every 50 s. As one would expect, there is

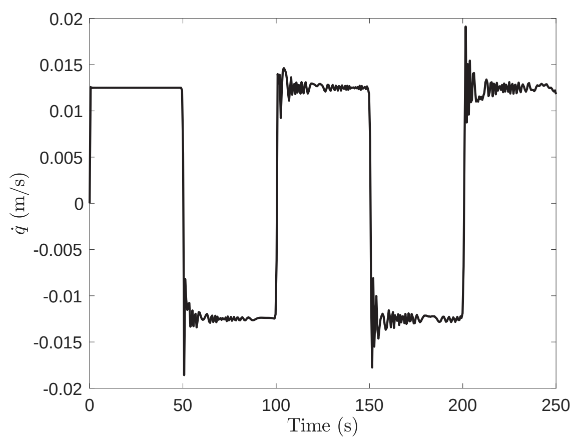


Figure 2.10: Step response $\dot{q}$ of the lumped transmission line model as a function of time; $k = 1600$, $m = 4$ and $n = 500$.

a travelling wave in the lumped-mass approximation of the transmission system. This is best understood by considering the limiting case when $n \to \infty$.

Consider the transmission element illustrated in Figure 2.11, which is at a distance x along the line. It is immediate from that figure that the propagated force is given

[14] By exploiting the mechanical-electrical analogy, where current $\leftrightarrow$ force and voltage $\leftrightarrow$ velocity, it follows that the admittance is given by either current/voltage or force/velocity [71].

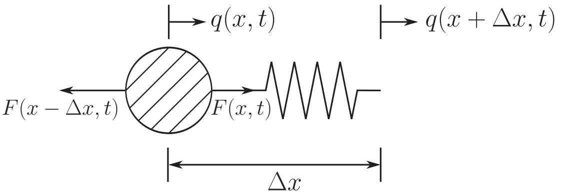


Figure 2.11: Small element of a continuous transmission line.

by

$$\begin{aligned} F(x,t) &= k(q(x+\Delta x,t) - q(x,t)) \\ &= K\frac{q(x+\Delta x,t) - q(x,t)}{\Delta x}, \end{aligned} \tag{2.194}$$

in which $K = k\Delta x$.[15] Taking limits in (2.194) and differentiating with respect to time gives

$$\frac{\partial F(x,t)}{\partial t} = K\frac{\partial \dot{q}(x,t)}{\partial x}. \tag{2.195}$$

A force balance on the mass element gives

$$\Delta x \rho \frac{\partial \dot{q}(x,t)}{\partial t} = F(x+\Delta x,t) - F(x,t)$$

in which $m = \rho\Delta x$, where ρ is the lengthwise mass density. Taking limits gives

$$\rho\frac{\partial \dot{q}(x,t)}{\partial t} = \frac{\partial F(x,t)}{\partial x}. \tag{2.196}$$

If we now differentiate (2.195) with respect to time, and (2.196) with respect to position, and eliminate the mixed partial derivatives, we obtain

$$\frac{\partial^2 F(x,t)}{\partial t^2} = \frac{K}{\rho}\frac{\partial^2 F(x,t)}{\partial x^2}, \tag{2.197}$$

which is the well-known wave equation in which $v = \sqrt{\frac{K}{\rho}}$ is the wave-propagation velocity. In the same way one can also obtain

$$\frac{\partial^2 \dot{q}(x,t)}{\partial t^2} = \frac{K}{\rho}\frac{\partial^2 \dot{q}(x,t)}{\partial x^2}; \tag{2.198}$$

the force and velocity waves satisfy the same wave equation.

[15] In the limit as the spring length goes to zero $k \to \infty$. For that reason we introduce K, which is well defined in this limit.

There is an important relationship between the force and velocity waves. Using (2.195) we obtain

$$\begin{aligned}\frac{\partial F(x,t)}{\partial t} &= K\frac{\partial \dot{q}(x,t)}{\partial t}\frac{\partial t}{\partial x}\\ &= \pm\frac{K}{v}\frac{\partial \dot{q}(x,t)}{\partial t}\\ &= \pm\sqrt{K\rho}\frac{\partial \dot{q}(x,t)}{\partial t}.\end{aligned}$$

It now follows by integration that

$$F(x,t) = D_O\dot{q}(x,t), \tag{2.199}$$

in which $D_O = \pm\sqrt{K\rho}$ is the characteristic damping of the transmission line. In the case of the forward wave D_O is positive, while D_O is negative for the backward wave.

It can be shown by direct calculation that

$$f(x,t) = F_+\overrightarrow{f}(t-\frac{x}{v}) + F_-\overleftarrow{f}(t+\frac{x}{v}) \tag{2.200}$$

is a solution to (2.197) (and by analogy to (2.198)) in which v is the wave-propagation velocity. The first term in (2.200) represents a forward-travelling wave with amplitude F_+, while the second term corresponds to a backward-propagating wave with amplitude F_-. Since the wave equation is linear, these terms can be considered separately. Suppose a pulse at $t = 0$ is given by $\overrightarrow{f}(x,0)$; see Figure 2.12. At a later moment in time, Δt say, the difference $\Delta t - x/v$ will have the same value as at $t = 0$ if we consider position $x + \Delta tv$. This means that the pulse $\overrightarrow{f}(x,0)$ will move unaltered in shape to position $x+\Delta tv$. Since Δt is arbitrary, the pulse will move continuously and unaltered in shape to the right with velocity v. Parallel arguments show that the second term in (2.200) represents a wave propagating in the negative x-direction.

This means that the transmission line, which is made up of only lossless components, is time reversible (reverse the sign of t and v in (2.200)), and yet the driving-point admittance $F(x,t)/\dot{q}(x,t) = \pm D_O$ is a damper. In our example the propagation velocity is $v = \sqrt{K/\rho} = \sqrt{1600/4} = 20$ and the length is 500. This means that the round-trip time is 50 as shown. It is now clear that, in principle, the line's driving-point admittance can be 'designed' to look like a (positive) damper for arbitrarily long periods.

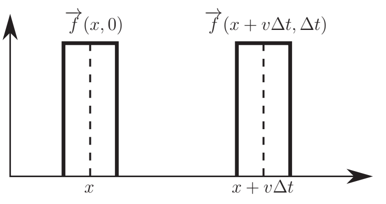


Figure 2.12: A wave moving along a lossless transmission line, with velocity v, that is unaltered in shape.

2.8 Čhaplygin's sleigh

The most widely studied class of mechanical systems involves conservative forces constrained by holonomic constraints. Systems of this type cannot have exponential stability in any of their configuration variables. It is therefore paradoxical that the most widely studied of all mechanical systems cannot have the most fundamental of all engineering requirements—stability! In contrast, the rolling disc (Section 2.10), or axisymmetric tops spinning on a level surface, are conservative nonholonomic systems that can have exponential stability in some of their configuration variables. In order to examine these issues in more detail we study here a well-known example that demonstrates that exponential stability in conservative systems can be a direct consequence of nonholonomy [72].

Čhaplygin's sleigh is a nonholonomic system comprising a rigid body that is free to slide over a frictionless horizontal plane; see Figure 2.13. The rigid body has mass m with centre of mass G and polar moment of inertia J. There is a skate C at a distance l from G. The skate and the body-fixed unit vector $\hat{\boldsymbol{e}}_x$ are aligned with CG. The body-fixed unit vector $\hat{\boldsymbol{e}}_y$ lies on the plane and is normal to $\hat{\boldsymbol{e}}_x$. The unit vector $\hat{\boldsymbol{e}}_z = \hat{\boldsymbol{e}}_x \times \hat{\boldsymbol{e}}_y$ is normal to the plane. The unit vector $\hat{\boldsymbol{e}}_x$ has yaw angle ψ relative to the positive inertial x-axis. The skate constraint ensures that the velocity of point C on the sleigh is $\boldsymbol{v}_C = u\hat{\boldsymbol{e}}_x$ and that the reactive force on the sleigh is thus $F_r\hat{\boldsymbol{e}}_y$. The absolute position of G is $\boldsymbol{r}_G = \boldsymbol{r}_C + l\hat{\boldsymbol{e}}_x$. Since $\dot{\hat{\boldsymbol{e}}}_x = \omega\hat{\boldsymbol{e}}_y$, where $\omega = \dot{\psi}$ is the sleigh's angular velocity, and $\dot{\hat{e}}_y = -\omega\hat{\boldsymbol{e}}_x$, the velocity and acceleration of the mass centre are given by

$$\boldsymbol{v}_G = u\hat{\boldsymbol{e}}_x + \omega l\hat{\boldsymbol{e}}_y \quad \text{and} \quad \boldsymbol{a}_G = \dot{u}\hat{\boldsymbol{e}}_x + u\omega\hat{\boldsymbol{e}}_y + \dot{\omega}l\hat{\boldsymbol{e}}_y - \omega^2 l\hat{\boldsymbol{e}}_x \tag{2.201}$$

respectively.

We can now derive the equations of motion by taking moments about G using (2.1) and (2.11):

$$\boldsymbol{F} = m\boldsymbol{a}_G \quad \text{and} \quad (\boldsymbol{r}_C - \boldsymbol{r}_G) \times \boldsymbol{F} = J\dot{\omega}\hat{\boldsymbol{e}}_z. \tag{2.202}$$

If we 'dot' the first of these equations with $\hat{\boldsymbol{e}}_x$, while making use of (2.201), we get

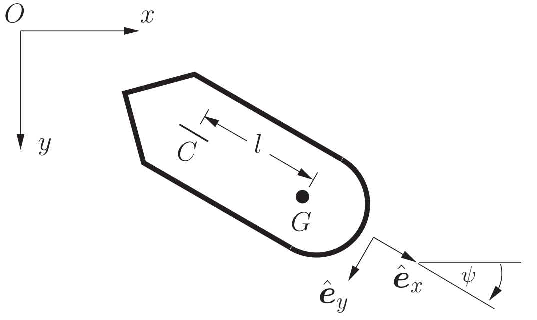


Figure 2.13: Čhaplygin's sleigh is a rigid body that is constrained to move on a frictionless plane. The origin of the body-fixed axis system is at C.

$$\dot{u} = \omega^2 l. \tag{2.203}$$

Since the sleigh is constrained in the $\hat{\boldsymbol{e}}_y$ direction, the reaction force must be in this direction, and it cannot do work. If we 'dot' the first equation in (2.202) with $\hat{\boldsymbol{e}}_y$, we get the reaction force as

$$\boldsymbol{F}_r = m(u\omega + l\dot{\omega})\hat{\boldsymbol{e}}_y. \tag{2.204}$$

If we substitute $\boldsymbol{r}_G - \boldsymbol{r}_C = l\hat{\boldsymbol{e}}_x$ in the second equation in (2.202) one obtains

$$\dot{\omega} = -\frac{lF_r}{J} \tag{2.205}$$

$$= -\frac{ml}{J + ml^2}u\omega. \tag{2.206}$$

The dynamics of the sleigh are fully determined by the solution of (2.203) and (2.206), which are quadratically coupled first-order nonlinear differential equations. The sleigh's initial speed $u_0 = u(0)$ and its initial angular velocity $\omega_0 = \omega(0)$ are assumed specified. When solving these equations we will specify the initial conditions in terms of the sleigh's initial speed u_0, and its total system energy. In order to keep track of the yaw angle and mass centre of the sleigh (in inertial coordinates) we need to solve

$$\begin{bmatrix} \dot{x}_G \\ \dot{y}_G \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} c_\psi & s_\psi & 0 \\ -s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ \omega l \\ \omega \end{bmatrix} \tag{2.207}$$

in which the initial conditions $x_G(0)$, $y_G(0)$, and $\psi(0)$ are assumed specified. This means that the sleigh's dynamics are described by five first-order differential equations.

The sleigh's equations of motion can also be obtained using Lagrange's equation. To do this we will assume that the velocity component of the mass centre in the (inertial) x-direction is given by $\dot{x}_G$, while that in the y-direction is given by $\dot{y}_G$. The system Lagrangian (and the system internal energy in this case) is given by

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}_G^2 + \dot{y}_G^2\right) + \frac{J}{2}\omega^2; \tag{2.208}$$

note that all the position variables are cyclic. Referring to Figure 2.13, we see that

$$0 = \omega l + \dot{x}_G \sin\psi - \dot{y}_G \cos\psi \tag{2.209}$$

ensures that the sleigh is absolutely stationary in the $\hat{\boldsymbol{e}}_y$ direction; this is a nonholonomic constraint that must be appended to (2.208). Using (2.45) and (2.48) gives

$$m\ddot{x}_G = -\lambda \sin\psi \tag{2.210}$$

$$m\ddot{y}_G = \lambda \cos\psi \tag{2.211}$$

$$J\dot{\omega} = -\lambda l. \tag{2.212}$$

In order to reconcile (2.203), (2.204), and (2.206) with (2.210), (2.211), and (2.212), we observe that (2.211)$\cos\psi$ − (2.210)$\sin\psi$ gives

$$m\left(\ddot{y}_G \cos\psi \;-\; \ddot{x}_G \sin\psi\right) = \lambda. \tag{2.213}$$

The expression $\ddot{y}_G \cos\psi \;-\; \ddot{x}_G \sin\psi$ is the acceleration of G in the $\hat{\boldsymbol{e}}_y$ direction; as shown in (2.201), this is just $(\dot{\omega}l \;+\; u\omega)\hat{\boldsymbol{e}}_y$. Hence

$$m(u\omega + \dot{\omega}l) = \lambda = F_r$$

by (2.204); (2.205) and (2.212) are thus equivalent. Next, (2.210)$\cos\psi +$ (2.211)$\sin\psi$ gives

$$0 = \ddot{x}_G \cos\psi + \ddot{y}_G \sin\psi. \tag{2.214}$$

We now use (2.214) and $\dot{\boldsymbol{v}}_G = (\dot{u} - l\omega^2)\hat{\boldsymbol{e}}_x$, which comes from differentiating $\boldsymbol{r}_G = \boldsymbol{r}_C + l\hat{\boldsymbol{e}}_x$ twice with respect to time, to establish that

$$\begin{aligned} 0 &= \ddot{x}_G \cos\psi + \ddot{y}_G \sin\psi \\ &= \dot{u} - l\omega^2, \end{aligned}$$

which is (2.203). It is clear that (2.205) and (2.212) are the same.

Finally, we derive the equations of motion of the sleigh using the Lagrangian approach in quasi-velocities; see Section 2.2.9. Initially we will neglect the nonholonomic constraint associated with the skate. Accordingly, the unconstrained velocity of the skate is $\boldsymbol{v}_C \;=\; u\hat{\boldsymbol{e}}_x + v\hat{\boldsymbol{e}}_y$, with the unconstrained kinetic energy given by $T = \frac{m}{2}(u^2 + (v + \omega l)^2) + \frac{J}{2}\omega^2$. The vector of quasi-velocities $[u, v, \omega]^T$ is used in place of $\dot{\boldsymbol{q}} = [\dot{x}, \dot{y}, \dot{\psi}]^T$; u is the longitudinal velocity of the skate, v is its lateral velocity, and ω its yaw rate. The relationship between the quasi-velocities and the generalized velocities is given by (2.65), which takes the form of (2.74). In anticipation of the introduction of the nonholonomic constraint, the generalized force is $\boldsymbol{k} = [0, F_r, 0]^T$. The unconstrained equations of motion are obtained from (2.76)–(2.78), and are given by

$$m(\dot{u} - \omega^2 l - \omega v) = 0 \tag{2.215}$$

$$m(u\omega + l\dot{\omega} + \dot{v}) = F_r \tag{2.216}$$

$$(J + ml^2)\dot{\omega} + m(lu\omega + uv - v\dot{u} + l\dot{v}) = 0. \tag{2.217}$$

In order to enforce the nonholonomic constraint we set $v = \dot{v} = 0$ in (2.215)–(2.217), in which case (2.215) becomes (2.203), and (2.217) becomes (2.206). Under the assumed velocity constraint, F_r in (2.216) is the constraint force and corresponds to (2.204).

We will now focus on the dynamics of the sleigh. In order to solve analytically (2.203) and (2.206) we begin by eliminating ω. From (2.203) we see that $\dot{\omega} = \ddot{u}/(2l\omega)$, which can be substituted into (2.206) to obtain

$$\ddot{u} = -\frac{2ml}{J + ml^2}(u\dot{u}) = -\frac{2ml}{J + ml^2}\left(\frac{1}{2}\frac{d}{dt}(u^2)\right). \tag{2.218}$$

Integrating (2.218) gives the Riccati equation

$$\dot{u} = -\frac{ml}{J + ml^2}u^2 + \frac{2lE}{J + ml^2}, \tag{2.219}$$

in which the second term is a constant of integration. If we now define $U = \sqrt{\frac{2E}{m}}$ and $r = (l\sqrt{2mE})/(J + ml^2)$, it is straightforward to verify that the Riccati equation (2.219) has solution[16]

$$u = U\frac{u_0 + U\tanh(rt)}{U + u_0\tanh(rt)}, \tag{2.220}$$

which is a function of U and the initial speed u_0. A closed-form expression for the yaw rate is obtained by substituting (2.220) into (2.203) and (2.219). This gives

$$\omega = \sqrt{\frac{2E(U^2 - u_0^2)}{J + ml^2}}\frac{1}{U\cosh(rt) + u_0\sinh(rt)}. \tag{2.221}$$

Substituting (2.203) into (2.219) gives

$$\left(\frac{J}{m} + l^2\right)\omega^2 + u^2 = U^2, \tag{2.222}$$

which is the equation of an ellipse that characterize the motion (flow) of the system. Referring to (2.208), we see that the system's total internal energy is given by

$$E = \frac{1}{2}\left(mu^2 + (ml^2 + J)\omega^2\right), \tag{2.223}$$

since $|\boldsymbol{v}_G|^2 = u^2 + l^2\omega^2$, and

$$U^2 = \frac{2E}{m} = u^2 + \left(\frac{J}{m} + l^2\right)\omega^2.$$

In the case that $\omega = 0$, $u = U$ and all the system energy is translational; it therefore follows that $|u_0| \leq U$. This means that $|u_0\tanh(\beta t)| \leq U$, establishing that (2.219) has no finite escape times. Also, $\lim_{t\to\infty} u(t) = U$ for all non-equilibrium initial conditions and the flow is attracted to the positive u-axis along constant energy trajectories as shown in Figure 2.14.

By exploiting the relationship between the speed, the energy, and the yaw rate given in (2.223), we can express ω in terms of the simpler expression

$$\omega = \frac{U\omega_0}{U\cosh(rt) + u_0\sinh(rt)}, \tag{2.224}$$

which admits both positive and negative values of ω_0 facilitating trajectories around the right- and left-hand sides of Figure 2.14. Equation (2.224) can be integrated to give the yaw angle

$$\psi = \frac{2U\omega_0}{r\sqrt{U^2 - u_0^2}}\left(\tan^{-1}\left\{\frac{U\tanh(\frac{rt}{2}) + u_0}{\sqrt{U^2 - u_0^2}}\right\} - \tan^{-1}\left\{\frac{u_0}{\sqrt{U^2 - u_0^2}}\right\}\right) + \psi_0, \tag{2.225}$$

with ψ_0 the initial yaw angle.

[16] The Riccati equation $\dot{u} = au^2 + b$ can be solved using the substitution $u = -\dot{w}/(aw)$, which results in the linear second-order equation $\ddot{w} + abw = 0$.

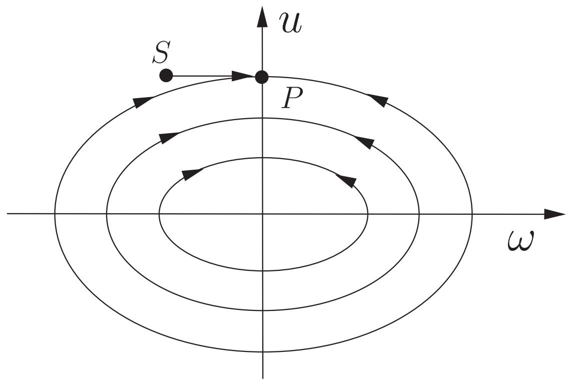


Figure 2.14: Flows of the Čhaplygin sleigh take the form of constant-energy ellipses. The only stable equilibrium points are along the positive u-axis; P is one such point. The point S is an exemplar initial condition for the linearized model, which then flows to P.

The sleigh's reaction force can be found by substituting (2.220) and (2.221) into (2.204), while making use of (2.206). This gives

$$\boldsymbol{F}_r = 2F_r^{max}\sqrt{1-\frac{u_0}{U}}\left(\frac{(\frac{u_0}{U}\cosh(rt)+\sinh(rt))}{(\frac{u_0}{U}\sinh(rt)+\cosh(rt))^2}\right) \tag{2.226}$$

with

$$|\boldsymbol{F}_r| \leq F_r^{max} = \frac{E(1-k^2)}{lk^3}, \tag{2.227}$$

where $k^2 = 1 + \frac{J}{ml^2}$. This result appears to have been found for the first time by Carathéodory [73].

If we linearize equations (2.203) and (2.206) around the equilibrium point $u = u_* \geq 0$ and $\omega = 0$, we see that these equations become decoupled, with solutions $\dot{u} = 0$ and $\omega = \delta\omega_0 e^{(-mlu^*t)/(J+ml^2)}$, in which $\delta\omega_0$ is an arbitrary small perturbation in ω. Thus if the system is perturbed to point S, say, in Figure 2.14, it will decay back to P along a horizontal constant-velocity trajectory. It is important to observe that since the exponential decay in ω is not accompanied by a change in the speed of the sleigh, energy is necessarily dissipated. Equation (2.223) becomes

$$E = \frac{1}{2}\left(mu_*^2 + (ml^2+J)\delta\omega_0^2 e^{(-2mlu_*t)/(J+ml^2)}\right),$$

which represents a dissipative motion since $\dot{E} < 0$ for any $u_* > 0$. The energy conserving properties of the nonlinear system come from the quadratic coupling terms in (2.203) and (2.206), which are destroyed by linearization.

Example 2.21 We consider the case $l = 0.1\,\text{m}$, $m = 1.0\,\text{kg}$, $J = 0.1\,\text{kg}\,\text{m}^2$, $u_0 = 1.99\,\text{m/s}$, and $E = 2\,\text{J}$ (which results in $\omega(0) = 0.6\,\text{rad/s}$ and $U = 2\,m/s$), with $x_G(0) = 0$, $y_G(0) = 0$, and $\psi(0) = 0$. As shown in Figure 2.15 (a), the sleigh's mass

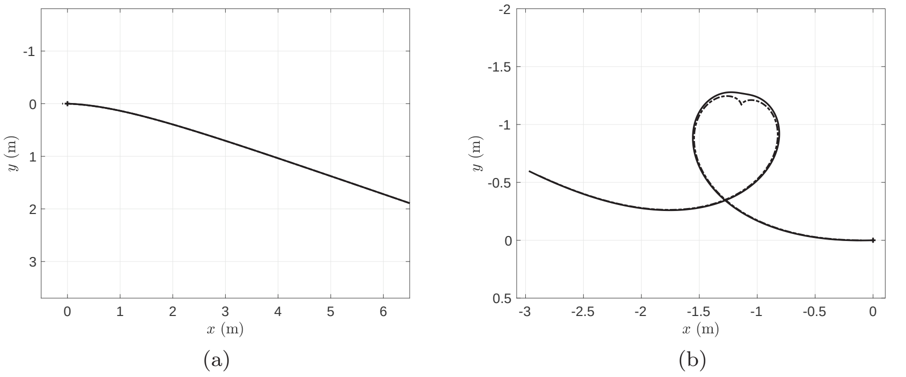


Figure 2.15: Trajectory of the mass centre and skate of the Čhaplygin sleigh: (a) positive initial velocity ($u_0 = 1.99\,\mathrm{m/s}$); (b) negative initial velocity ($u_0 = -1.99\,\mathrm{m/s}$). The mass centre is shown as the solid curve, while the skate position is the dot-dash curve; the system was simulated for 3.5 s.

centre moves to the right along the positive x-axis. Since $u_0 < U$, some of the initial stored energy is in the form of rotational kinetic energy. It follows from (2.220) that this energy is soon transferred into translational kinetic energy causing the sleigh to complete its turn, and then move away in a straight line.

In the case that $u_0 = -1.99\,\mathrm{m/s}$ ($\omega(0) = 0.6\,\mathrm{rad/s}$), when the skate starts out in front of the mass centre, the motion is more complex and consequently more interesting. First, it follows from (2.220), that $\lim_{t\to\infty} u(t) = 2\,\mathrm{m/s}$. As shown in Figure 2.16 (a), the forward speed starts at $u_0 = -1.99\,\mathrm{m/s}$, passes through zero at 1.65 s, and then approaches $2\,\mathrm{m/s}$. Since the system energy remains constant, energy is transferred from translational kinetic energy into rotational kinetic energy and then back again. During the period of high angular velocity, the sleigh 'flips around' so that the skate follows the mass centre in a stable configuration. The ground-plane trajectory is shown in Figure 2.15 (b). In this case the sleigh's mass centre moves off to the left along the negative x-axis following the skate. It then turns gently to the left and starts to slow down. At a relatively low speed it enters into the neighbourhood of the cusp, 'flips', and then moves away with the skate following the mass centre. The sleigh then speeds up and moves away along a rectilinear path. Figure 2.17 shows the reaction force at the sleigh's skate and F_r^{max} during the manoeuvre. The small negative initial reaction force cause an increase in the sleigh's angular velocity causing it to turn to the left; see Figure 2.15 (a). At approximately 1.2 s the reaction force reaches its minimum and begins to increase. When the reaction force changes sign (at approximately 1.7 s), the angular velocity is at its peak, the translational speed is zero, and the sleigh is in mid 'flip'. Once the sleigh has 'flipped', it continues to turn left despite a change in sign in the reaction force, but this comes from the fact that the body-fixed vector $\hat{\boldsymbol{e}}_y$ has flipped too. The reaction force then decays away as the sleigh transitions into its rectilinear terminal motion.

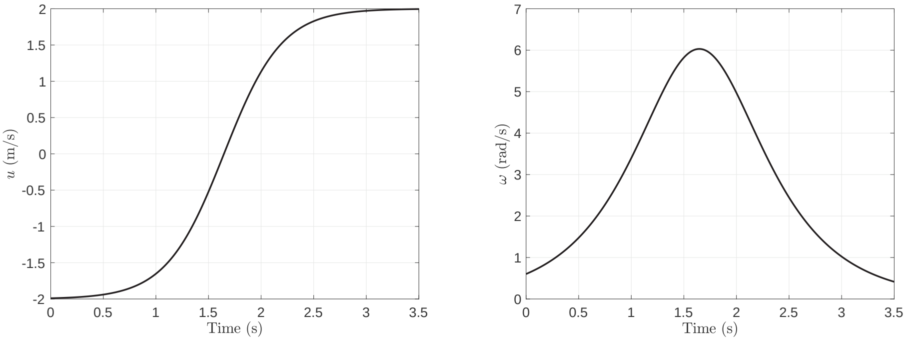


Figure 2.16: Speed (a) and yaw rate (b) of the Čhaplygin sleigh.

2.8.1 T-symmetry of the Čhaplygin sleigh

As we will now demonstrate, it is possible to establish the time reversibility of the Čhaplygin sleigh by studying the solutions of the describing equations. To begin, (2.220) and the odd symmetry of the hyperbolic tangent function established that $u(-t) = -u(t)$ when u_0 is replaced by $-u_0$. In Equation (2.224) the symmetry of the hyperbolic cosine function and the odd symmetry of the hyperbolic sine function show that $\omega(-t) = -\omega(t)$ when u_0 is replaced by $-u_0$, and ω_0 is replaced by $-\omega_0$. In the same way (2.225) shows that $\psi(t) = \psi(-t)$ when $u_0 \to -u_0$, $\omega_0 \to -\omega_0$, and ψ_0 remains unchanged. It now follows from (2.207) that $\dot{x}_G(t) = -\dot{x}_G(-t)$ and $\dot{y}_G(t) = -\dot{y}_G(-t)$. When these equations are integrated with respect to time, we find that $x_G(t) = x_G(-t)$ and $y_G(t) = y_G(-t)$, which proves that the sleigh follows the same trajectory under time reversals. Finally, (2.204) can be written as

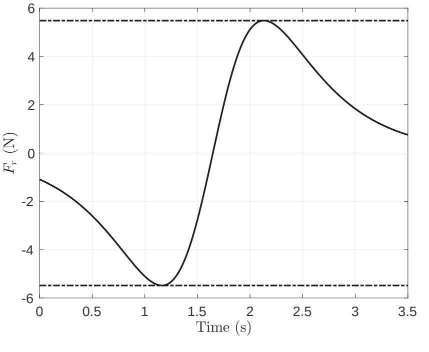


Figure 2.17: Sleigh reaction force. The dot-dash lines are given by $\pm F_r^{max}$ in equation (2.227).

$$\boldsymbol{F}_r = \frac{mJu\omega}{J+ml^2}\hat{\boldsymbol{e}}_y \tag{2.228}$$

using (2.206), which shows that the skate reaction force is invariant under time reversals satisfying (2.180).

2.9 Dynamics of a rolling ball

Rolling contacts are a key feature of all road-going vehicles. While modern modelling techniques involve the concept of 'slip' (Chapter 3), rolling contacts were modelled classically using nonholonomic constraints, which is a reasonable approximation for tyres rolling at very low speeds. In order to introduce this aspect of vehicle dynamics we will study the motion of a ball rolling on a rough surface, which is a well-known class of nonholonomic systems. When solving this problem one thinks of a point of contact P, which is a material point on the surface of the ball, that is in instantaneous contact with a corresponding material point P′ on the underlying rough surface. In the case of an ideal non-slipping contact one assumes that the points P and P′ are relatively stationary, that is, they have zero relative velocity. This modelling assumption imposes a kinematic constraint that forces the velocities of P and P′ to be equal in inertial space. In addition, the vertical components of the positions of P and P′ must remain equal so that the ball remains in contact with the surface below it. In the vernacular of nonholonomic systems the rolling contact is characterized by a holonomic constraint in the direction normal to the plane, and two nonholonomic constraints in the plane of the underlying rough surface. Rolling-ball problems are simplified by the fact that homogeneous axisymmetric spheres have uniform inertia properties characterized by the fact that any axis passing through the sphere's mass centre is a principal axis with a common moment of inertia. There is therefore no need for the use of Euler angles, or other similar device, to characterize the ball's orientation in space.

We begin with a study of a ball rolling down an inclined surface, which is an entry-level problem that is usually solved using an elementary two-dimensional analysis. The solution to this problem is followed by the analysis of a ball rolling on the surface of an inclined turntable. The analyses presented in this section follow for the most part the excellent treatment given to these problems in [46].

2.9.1 Ball on an incline

The study of the dynamics of a ball rolling down an inclined surface is essentially a planar problem, which we have nonetheless chosen to solve using three-dimensional vector mechanics. At first sight this might seem unnecessarily complicated, but this approach will soon pay dividends.

In this section we study the dynamics of a ball rolling down a non-rotating inclined turntable; see Figure 2.18 where $\Omega = 0$. In the next section we will consider the effect of $\Omega \neq 0$. The homogeneous and axisymmetric ball is assumed free to roll on the inclined surface without slipping. The ball has radius a, mass m, and moment of inertia μma^2 (for any axis passing through the ball's mass centre). In the case of a uniform sphere $\mu = \frac{2}{5}$, while in the case of a spherical 'thin' shell $\mu = \frac{2}{3}$.

The origin of an inertial coordinate system is located at the central point O on the surface of the inclined turntable. A unit vector $\mathbf{k}$ points down the turntable's spindle.

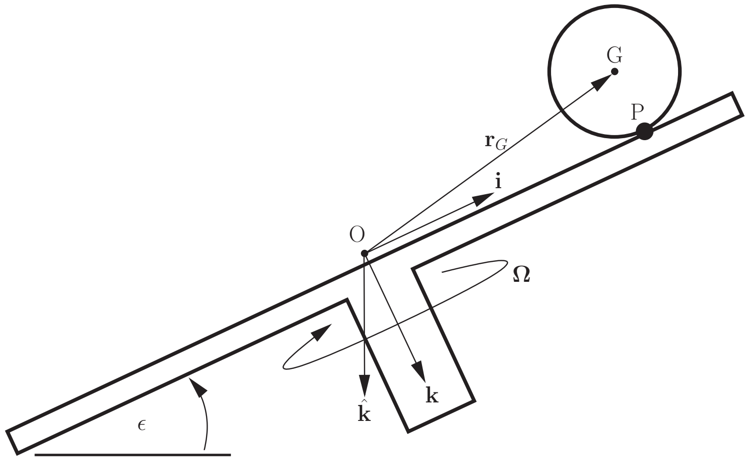


Figure 2.18: Rolling ball on an inclined turntable. The diagram shows a ball rolling on the surface of a turntable that is inclined at an angle ϵ to the horizontal. The turntable rotates at angular velocity $\boldsymbol{\Omega}$. The ball's contact point is located at P, while the ball's mass centre is at G. The unit vectors $\mathbf{i}$ and $\mathbf{k}$ form part of an orthogonal coordinate system that is completed by $\mathbf{j}$, which points out of the page. The unit vector $\hat{\mathbf{k}}$ points in the direction of gravity.

The unit vector $\mathbf{i}$, which lies in the page, points away from O in a radial 'uphill' direction. This right-handed orthogonal axis system is completed with the unit vector $\mathbf{j} = \mathbf{k} \times \mathbf{i}$ that points out of the page. The unit vector $\hat{\mathbf{k}}$ points away from O in the direction of gravity. The ball's mass centre is at G and is located by a vector $\mathbf{r}_G$. The radius vector $a\mathbf{k}$ points from G to P. The force of reaction at the turntable contact point is $\mathbf{F}_r$, while the gravitational force is $mg\hat{\mathbf{k}}$.

We will derive the equations of motion using Newton's laws as given in (2.1) and (2.11). Summing forces on the ball gives

$$m\ddot{\mathbf{r}}_G = \mathbf{F}_r + mg\hat{\mathbf{k}}. \tag{2.229}$$

If the ball's angular velocity is $\boldsymbol{\omega}$, then balancing moments at the ball's mass centre gives

$$\mu m a^2 \dot{\boldsymbol{\omega}} = a\mathbf{k} \times \mathbf{F}_r. \tag{2.230}$$

Eliminating the reaction force $\mathbf{F}_r$ from (2.229) and (2.230) yields

$$\mu m a^2 \dot{\boldsymbol{\omega}} = am\mathbf{k} \times (\ddot{\mathbf{r}}_G - g\hat{\mathbf{k}}). \tag{2.231}$$

Vector multiplication by $\mathbf{k}$ gives:

$$\mu a \dot{\boldsymbol{\omega}} \times \mathbf{k} = \Big(\mathbf{k} \times (\ddot{\mathbf{r}}_G - g\hat{\mathbf{k}})\Big) \times \mathbf{k}. \tag{2.232}$$

Since the centre of mass of the ball must move in a plane parallel to the incline, $\ddot{\mathbf{r}}_G \perp \mathbf{k}$, and it follows that $(\mathbf{k} \times \ddot{\mathbf{r}}_G) \times \mathbf{k} = \ddot{\mathbf{r}}_G$. It is also evident that $(\mathbf{k} \times \hat{\mathbf{k}}) \times \mathbf{k} = -\sin(\epsilon)\mathbf{i}$. Therefore (2.232) becomes

$$\mu a\dot{\boldsymbol{\omega}} \times \mathbf{k} = \ddot{\mathbf{r}}_G + g\sin(\epsilon)\mathbf{i}. \tag{2.233}$$

Under the 'no-slip' assumption, the contact point P that is regarded as a material point on the surface of the ball must be absolutely stationary. This means that its velocity (2.144) and acceleration (2.145) are

$$\dot{\mathbf{r}}_G + \boldsymbol{\omega} \times a\mathbf{k} = 0, \tag{2.234}$$
$$\ddot{\mathbf{r}}_G + \dot{\boldsymbol{\omega}} \times a\mathbf{k} = 0. \tag{2.235}$$

It therefore follows from (2.233) and (2.235) that

$$(\mu + 1)\ddot{\mathbf{r}}_G = -g\sin(\epsilon)\mathbf{i}. \tag{2.236}$$

As anticipated, the dynamics of the ball are governed by the three equations in (2.236) that are independent of position variables. Integrating this equation gives

$$\dot{\mathbf{r}}_G = -\frac{g}{1+\mu}\sin(\epsilon)t\mathbf{i} + \dot{\mathbf{r}}_G(0) \tag{2.237}$$

in which $\dot{\mathbf{r}}_G(0)$ is a vector-valued constant of integration that represents the initial velocity of the ball's mass centre, and t is the time.

In order to understand the spinning motion of the ball we return to (2.234), which we multiply vectorially by $\mathbf{k}$ to obtain

$$\begin{aligned} 0 &= \mathbf{k} \times (\dot{\mathbf{r}}_G + a\boldsymbol{\omega} \times \mathbf{k}) \\ &= \mathbf{k} \times \dot{\mathbf{r}}_G + a(\boldsymbol{\omega} - \mathbf{k}(\boldsymbol{\omega} \cdot \mathbf{k})) \\ &= \mathbf{k} \times \dot{\mathbf{r}}_G + a(\boldsymbol{\omega} - \omega_k\mathbf{k}), \end{aligned} \tag{2.238}$$

where $\omega_k = \boldsymbol{\omega} \cdot \mathbf{k}$ is the polar component of the angular velocity. In the second line we employ the useful vector identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$. Scalar multiplication of (2.231) by $\mathbf{k}$ gives $\dot{\boldsymbol{\omega}} \cdot \mathbf{k} = \dot{\omega}_k = 0$ and so ω_k is constant. We therefore have from (2.238)

$$\boldsymbol{\omega} = \omega_k\mathbf{k} + \frac{1}{a}\dot{\mathbf{r}}_G \times \mathbf{k}, \tag{2.239}$$

or equivalently substituting (2.237) into (2.239)

$$\boldsymbol{\omega} = \omega_k\mathbf{k} + \frac{g\sin(\epsilon)t}{a(1+\mu)}\mathbf{j} + \frac{1}{a}\dot{\mathbf{r}}_G(0) \times \mathbf{k}. \tag{2.240}$$

The first term is the constant polar spin term, the second term corresponds to the angular acceleration produced by the in-plane component of the reaction force $\mathbf{F}_r$, while the final term is due to the ball's initial translational velocity.

As an alternative, the balance of moments can be computed with respect to the material contact point P, which is stationary. In this case the angular momentum (2.8) is $\boldsymbol{H}_P = J_P\boldsymbol{\omega} = (\mu ma^2 + ma^2)\boldsymbol{\omega}$, where the parallel axis theorem (2.13) has been used to computed J_P, and the equation of motion (2.11) is $ma^2(\mu+1)\dot{\boldsymbol{\omega}} = -a\boldsymbol{k} \times mg\hat{\boldsymbol{k}}$ which is identical to (2.230) when (2.235) is introduced.

2.9.2 Ball on an inclined turntable

A more advanced problem that illustrates the analysis of systems containing non-holonomic constraints and rotating bodies is the ball and turntable illustrated in Figure 2.18. This examples builds upon the results obtained in the previous section by adding the effect of a constant turntable angular velocity $\mathbf{\Omega}$.

We start from the balance of moments (2.233), which remains good even when the turntable rotation Ω is included. The effect of rotation enters the problem through the rolling constraint. Since the ball rolls without slipping, it follows that any material contact point P on the surface of the ball must have the same instantaneous velocity as the corresponding material contact point P$'$ on the surface of the turntable. Unlike the $\Omega = 0$ case, the point P$'$ on the turntable's surface has a non-zero velocity $\mathbf{\Omega}\times(\mathbf{r}_G+a\mathbf{k})$, in which $\mathbf{r}_G + a\mathbf{k}$ is a vector pointing from O to P. It thus follows that

$$\dot{\mathbf{r}}_G + a\boldsymbol{\omega} \times \mathbf{k} - \mathbf{\Omega} \times (\mathbf{r}_G + a\mathbf{k}) = 0. \tag{2.241}$$

Since $\mathbf{\Omega}$ is parallel with $\mathbf{k}$, the last term in (2.241) is zero. Differenting this equation gives

$$\ddot{\mathbf{r}}_G + a\dot{\boldsymbol{\omega}} \times \mathbf{k} = \mathbf{\Omega} \times \dot{\mathbf{r}}_G. \tag{2.242}$$

Substituting (2.242) into (2.233) yields:

$$\ddot{\mathbf{r}}_G + g\sin(\epsilon)\mathbf{i} = \mu\left(\mathbf{\Omega} \times \dot{\mathbf{r}}_G - \ddot{\mathbf{r}}_G\right), \tag{2.243}$$

or what is the same

$$\ddot{\mathbf{r}}_G = \frac{\mu}{1+\mu}\mathbf{\Omega} \times \dot{\mathbf{r}}_G - \frac{g}{1+\mu}\sin(\epsilon)\mathbf{i}. \tag{2.244}$$

Again, the dynamics of the ball are governed by the three equations in (2.244) and no position variables appear. Integrating (2.244) (with respect to time t) gives:

$$\dot{\mathbf{r}}_G = \frac{\mu}{1+\mu}\mathbf{\Omega} \times (\mathbf{r}_G - \mathbf{c}_0) - \frac{g}{1+\mu}\sin(\epsilon)t\mathbf{i} \tag{2.245}$$

in which $\mathbf{c}_0$ is a vector-valued constant of integration. The first term describes circular motion about an initial centre of rotation $\mathbf{c}_0$, with angular speed $\frac{\mu|\mathbf{\Omega}|}{1+\mu}$ in the plane of the turntable. The second term is an inclination-related drift term. If $\epsilon = 0$, the ball will move in a circle with centre $\mathbf{c}_0$ and period

$$T = \frac{2\pi(1+\mu)}{\mu|\mathbf{\Omega}|}. \tag{2.246}$$

The solution of (2.245) is thus the superposition of an orbital motion and a drift term.

If the initial position $\mathbf{r}_G(0)$ is known, it follows from (2.245) that the initial velocity $\dot{\mathbf{r}}_G(0)$ and the initial centre of rotation $\mathbf{c}_0$ are related. It is immediate from (2.245) that

$$\dot{\mathbf{r}}_G(0) = \frac{\mu}{1+\mu}\mathbf{\Omega} \times (\mathbf{r}_G(0) - \mathbf{c}_0).$$

Vector multiplication by $\mathbf{k}$ gives

$$\mathbf{c}_0 = \mathbf{r}_G(0) - \left(\frac{1+\mu}{\mu|\mathbf{\Omega}|}\right) \dot{\mathbf{r}}_G(0) \times \mathbf{k}. \tag{2.247}$$

This equation shows that for a given initial position $\boldsymbol{r}_G(0)$, $\dot{\mathbf{r}}_G(0)$ and $\mathbf{c}_0$ are uniquely related.

Example 2.22 Figure 2.19 shows the trajectory of a ball rolling without slipping on a horizontal turntable that is rotating at $|\mathbf{\Omega}| = 1\,\text{rad/s}$. The radius of the ball is $a = 2.5\,\text{cm}$ and it is assumed to have a spherical shell ($\mu = \frac{2}{3}$). It follows from (2.246) that the period of rotation is $T = 5\pi\,\text{s}$. The ball's initial contact position is $\boldsymbol{i}$ with its initial mass centre velocity $\dot{\boldsymbol{r}}_G(0) = \boldsymbol{j}$. These initial conditions imply that the centre of rotation is at $\boldsymbol{c}_0 = -1.5\boldsymbol{i} - a\boldsymbol{k}$, and that a circular orbital radius of 2.5 m results.

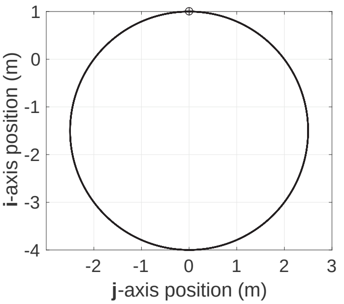


Figure 2.19: Trajectory of the mass centre of a hollow ball rolling on a level turntable. The simulation is run for $25\pi\,\text{s}$, which represents precisely 5 orbits of period $5\pi\,\text{s}$.

If $\mathbf{\Omega} = \mathbf{0}$, (2.245) shows that the ball will simply roll 'down the hill' according to (2.237). If both terms are present, a drifting circular motion results. At first sight it would appear that (2.245) suggests a cyclic motion that drifts 'downhill'. This is not true however, because the gravitational term interacts with the time-varying radius $\mathbf{r}_G$ to produce a 'sideways' epicycloidal motion. One can see this by rewriting (2.245) as

$$\dot{\mathbf{r}}_G = \frac{\mu}{1+\mu}\mathbf{\Omega} \times (\mathbf{r}_G - \mathbf{c}_0 + \frac{g\sin(\epsilon)t}{\mu|\mathbf{\Omega}|}\mathbf{j}). \tag{2.248}$$

In the case $|\mathbf{\Omega}| = 0$, this becomes, by substitution of (2.247)

$$\dot{\mathbf{r}}_G = -\frac{g}{1+\mu}\sin(\epsilon)t\mathbf{i} + \dot{\mathbf{r}}_G(0), \tag{2.249}$$

which is the same as (2.237); as expected we recover the stationary-incline solution. Equation (2.248) represents epicyclic motion about a centre of rotation $\mathbf{c}_0 - \frac{g\sin(\epsilon)t}{\mu|\mathbf{\Omega}|}\mathbf{j}$ that drifts in the negative $\mathbf{j}$ direction.

Example 2.23 Figure 2.20 shows the uniform sideways drifting of the ball's mass centre predicted by (2.248), when the turntable is inclined at 0.005 rad. The other parameters are the same as those in the previous example.

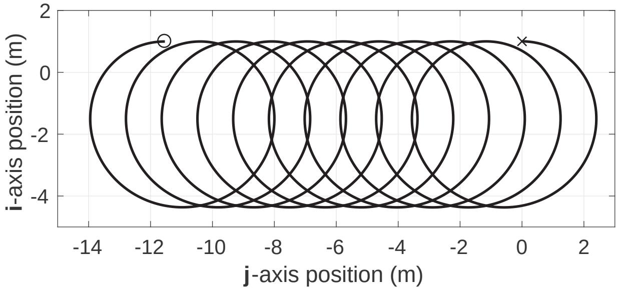


Figure 2.20: Trajectory of the mass centre of a hollow ball rolling on an inclined turntable. The turntable is rotating at 1 rad/s and is inclined at 0.005 rad. The initial position of the ball's mass centre is $\boldsymbol{r}_G(0) = \boldsymbol{i} - a\boldsymbol{k}$, while its initial velocity is $\dot{\boldsymbol{r}}_G(0) = \boldsymbol{j}$. The simulation is run for 50π s.

We conclude this section with a brief analysis of the ball's rotational behaviour. It follows from (2.229) and (2.230) that

$$\mu a\dot{\boldsymbol{\omega}} = \mathbf{k} \times \ddot{\mathbf{r}}_G + g\sin(\epsilon)\mathbf{j}. \tag{2.250}$$

Scalar multiplication of this equation by $\mathbf{k}$ gives $\dot{\boldsymbol{\omega}} \cdot \mathbf{k} = 0$ and so the polar spin of the ball ω_k is again an arbitrary constant. Vector multiplication of the rolling constraint equation (2.241) by $\mathbf{k}$ gives

$$\boldsymbol{\omega} = \mathbf{k}\omega_k - \frac{|\boldsymbol{\Omega}|}{a}(\mathbf{r}_G + a\mathbf{k}) - \frac{1}{a}\mathbf{k} \times \dot{\mathbf{r}}_G. \tag{2.251}$$

In order to determine the ball's angular velocity in terms of its position alone, we substitute (2.245) into (2.251) to obtain

$$\boldsymbol{\omega} = \mathbf{k}\omega_k - \boldsymbol{\Omega} - \frac{|\boldsymbol{\Omega}|}{a(1+\mu)}\mathbf{r}_G - \frac{\mu|\boldsymbol{\Omega}|}{a(1+\mu)}\mathbf{c}_0 + \frac{g\sin(\epsilon)t}{a(1+\mu)}\mathbf{j}. \tag{2.252}$$

The ball's initial angular velocity can be expressed in terms of its initial position and centre of rotation by

$$\boldsymbol{\omega}(0) = \mathbf{k}\omega_k - \boldsymbol{\Omega} - \frac{|\boldsymbol{\Omega}|}{a(1+\mu)}\mathbf{r}_G(0) - \frac{\mu|\boldsymbol{\Omega}|}{a(1+\mu)}\mathbf{c}_0. \tag{2.253}$$

Force-generating contact. Road vehicle tyres cannot realistically be represented by nonholonomic rolling constraints. Instead, they are best modelled as force producers, with the force proportional to some form of slip quantity. To see how this might come about we begin by representing the constraint equation (2.241) with an equivalent reaction or friction force. Substituting (2.244) into (2.229) yields

$$\boldsymbol{F}_r = \frac{m\mu}{1+\mu}\left(\boldsymbol{\Omega} \times \dot{\boldsymbol{r}}_G + g\sin(\epsilon)\mathbf{i}\right) - mg\cos(\epsilon)\boldsymbol{k}. \tag{2.254}$$

This friction force is clearly dynamically equivalent to the no-slipping constraint (2.241) and will therefore produce identical motions.

To obtain a first insight into tyre force-generating mechanisms, we define the slip vector

$$\boldsymbol{s} = \frac{(\dot{\mathbf{r}}_G + a\boldsymbol{\omega} \times \mathbf{k} - \boldsymbol{\Omega} \times \mathbf{r}_G)}{|\dot{\mathbf{r}}_G - \boldsymbol{\Omega} \times \mathbf{r}_G|}, \tag{2.255}$$

which is defined whenever $|\dot{\mathbf{r}}_G - \boldsymbol{\Omega} \times \mathbf{r}_G| \neq 0$. The numerator in this expression represents the departure, or slippage, from perfect rolling, while the denominator represents the speed of the ball's mass centre relative to the moving surface below it. We now suppose that whenever $\boldsymbol{s} \neq 0$ a reaction force is developed that acts to negate the slip. One possibility is simply to set

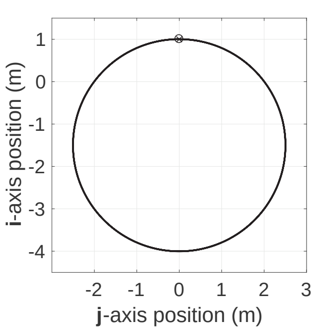
$$\boldsymbol{F}_r = -C_s\boldsymbol{s} - mg\cos(\epsilon)\boldsymbol{k} \tag{2.256}$$

in which C_s is a friction-related stiffness constant (longitudinal slip stiffness; see Chapter 3). Substituting (2.256) into (2.229) we see that

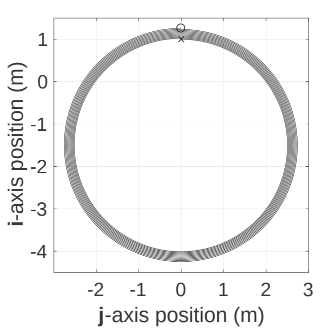
$$\boldsymbol{s} = -(m\ddot{\mathbf{r}}_G - mg\hat{\mathbf{k}})/C_s,$$

which shows that in the limit as $C_s \to \infty$ (2.241) is enforced.

Example 2.24 The simulation shown in Figure 2.21(a) shows that a friction force of the form (2.256), with $C_s = 200$, produces a motion that is almost identical to the non-slipping case given in Figure 2.19. If the friction coefficient is reduced to $C_s = 10$, the trajectory begins to deviate from that in Figure 2.19 as is shown in Figure 2.21(b). If the turntable is inclined at 0.005 rad, with $C_s = 10$, one obtains Figure 2.22, which should be compared with Figure 2.20 in order to appreciate the influence of contact slippage. The parameters and initial conditions are the same as those in the previous examples, with $\boldsymbol{\omega}(0) = \mathbf{0}$ and $m = 0.05$ kg.



(a)



(b)

Figure 2.21: Trajectory of the mass centre of a hollow slipping ball rolling on a level turntable. The horizontal turntable is rotating at 1 rad/s and the slipping stiffness is $C_s = 200$ (a) and $C_s = 10$ (b). The simulation is run for 25π s, which represents 5 orbits of period 5π s.

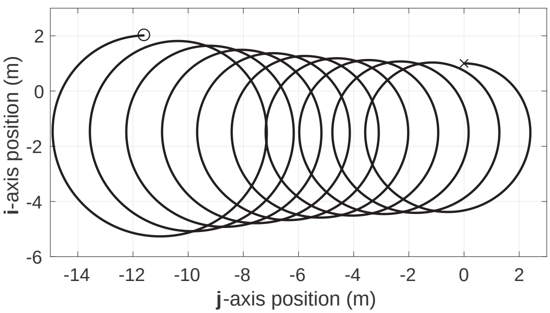


Figure 2.22: Trajectory of the mass centre of a hollow slipping ball rolling on an inclined turntable with $C_s = 10$. The turntable is rotating at 1 rad/s and is inclined at 0.005 rad. The simulation is run for 50π s.

2.10 Rolling disc

The rolling disc is a prototypical nonholonomic system comprising a thin, uniform disc that rolls without slipping on a horizontal surface. This famous problem has been studied by many authors including for example Goldstein [41], Neĭmark and Fufaev [48], O'Reilly [74], Pars [75], Routh [76], Synge and Griffith [40], and Bloch and his colleagues [77, 78]. This example is preparatory in the context of a study of road vehicles for several reasons:

1. It is relatively simple, but involves a rigid body with large rotations in three dimensions. This type of rigid-body behaviour occurs in both bicycles and motorcycles;
2. It involves two nonholonomic constraints that are associated with pure non-slip rolling, which are typical in the early modelling of bicycle wheels;
3. In common with bicycles and motorcycles, it has speed-dependent stability properties.

The rolling-disc problem involves six degrees of freedom related to the position and orientation of the body. There are one holonomic and two nonholonomic constraints relating to the rolling contact that constrain the disc's motion. The holonomic constraint ensures that the disc and plane remain in contact, while the nonholonomic constraints ensure pure rolling. As we will show, the disc's translational position, its yaw angle, and its rotational position are all cyclic variables that do not enter the equations of motion; see Section 2.3.2.

2.10.1 Introduction

The rotation of the disc will be represented by three Euler angles ψ, ϕ, and θ, which denote respectively the yaw, the roll, and the spin of the disc as it rolls across a smooth surface. We will take the reference configuration to be one in which the disc lies flat on a horizontal plane Π; see Figure 2.23. If the mass of the disc is m, with its radius R, then the inertia of the disc about its mass centre, and in its reference position, is:

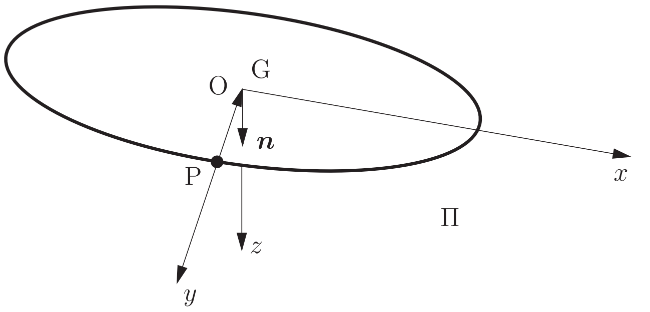


Figure 2.23: Disc in its reference position on a horizontal plane Π. The mass centre of the disc is at G, which in the reference position is coincident with the origin O of the inertial axis system Oxyz. The vector $\boldsymbol{n}$ is normal to the plane of the disc at G. A vector $\boldsymbol{r}_{PG}^{\text{ref}}$ points from the point P on the edge of the disc towards G.

$$J_G^{ref} = \begin{bmatrix} \frac{mR^2}{4} & 0 & 0 \\ 0 & \frac{mR^2}{4} & 0 \\ 0 & 0 & \frac{mR^2}{2} \end{bmatrix}. \tag{2.257}$$

Next, we rotate the disc through an angle ψ about $\boldsymbol{e}_z$, which is a unit vector in the positive z-axis direction. This is followed by a rotation about the now-rotated $\boldsymbol{e}_x$ vector by an angle ϕ, which points in the positive x-axis direction. Lastly, we rotate the disc through an angle θ about the now-twice-rotated e_z vector. To see how these calculations work out in detail we refer to the *Rodrigues formula* (2.153). Defining $\mathcal{R}_1 = \mathcal{R}(\boldsymbol{e}_z, \psi)$, $\bar{\mathcal{R}}_2 = \mathcal{R}(\mathcal{R}_1\boldsymbol{e}_x, \phi)$, and $\bar{\mathcal{R}}_3 = \mathcal{R}(\bar{\mathcal{R}}_2\mathcal{R}_1\boldsymbol{e}_z, \theta)$, the final rotation matrix is given by $\mathcal{R}_I = \bar{\mathcal{R}}_3\bar{\mathcal{R}}_2\mathcal{R}_1$, where

$$\mathcal{R}_I = \begin{bmatrix} c_\theta c_\psi - s_\theta c_\phi s_\psi & -c_\theta s_\psi c_\phi - c_\psi s_\theta & s_\phi s_\psi \\ s_\theta c_\phi c_\psi + c_\theta s_\psi & c_\psi c_\theta c_\phi - s_\psi s_\theta & -s_\phi c_\psi \\ s_\theta s_\phi & c_\theta s_\phi & c_\phi \end{bmatrix}. \tag{2.258}$$

Observe that this is the same as $\mathcal{R}_I = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3$, where $\mathcal{R}_2 = \mathcal{R}(\boldsymbol{e}_x, \phi)$ and $\mathcal{R}_3 = \mathcal{R}(\boldsymbol{e}_z, \theta)$, and that each of the Euler angles is time-varying.

In order to find the disc's (mass-centred) moment of inertia in an arbitrary position, expressed in the inertial coordinate system, we use:

$$J_G = \mathcal{R}_I J_G^{ref} \mathcal{R}_I^T = \frac{mR^2}{4} \begin{bmatrix} s_\psi^2 s_\phi^2 + 1 & -c_\psi s_\psi s_\phi^2 & s_\phi s_\psi c_\phi \\ -c_\psi s_\psi s_\phi^2 & 1 + c_\psi^2 s_\phi^2 & -s_\phi c_\psi c_\phi \\ s_\phi s_\psi c_\phi & -s_\phi c_\psi c_\phi & 1 + c_\phi^2 \end{bmatrix}. \tag{2.259}$$

The disc is seen in its general position in Figure 2.24.

2.10.2 Rolling constraints

If we were to consider the motion of the disc in free space, neglecting temporarily any interactions with the plane Π in Figure 2.23, it would have six degrees of freedom.

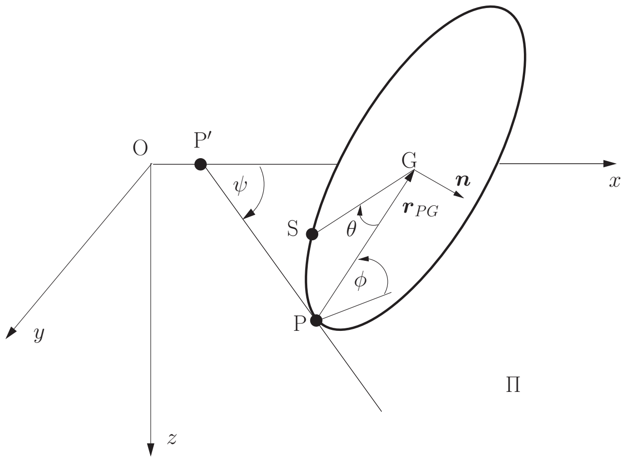


Figure 2.24: Disc rolling on a horizontal plane Π. The yaw angle is denoted by ψ, the roll angle by ϕ, and the spin angle by θ. The mass centre of the disc is at G, while the disc contacts the plane Π at P; the material point S is coincident with the (unspun) ground contact point P once every revolution of the disc. The vector $\boldsymbol{n}$ is normal to the plane of the disc at G. The axis system Oxyz is inertial.

These freedoms would be the three translations $x_G(t)$, $y_G(t)$, and $z_G(t)$ of the disc's mass centre, and the three Euler angles $\psi(t)$, $\phi(t)$, and $\theta(t)$. When non-slip rolling (on the plane Π) is considered, these freedoms are constrained. In order to find the constraint equations we begin by considering the vector $\boldsymbol{r}_{PG}^{\mathrm{ref}}$, which points from the point P on the periphery of the disc to the mass centre G in Figure 2.23. If this vector is allowed to yaw and then roll we obtain:

$$\boldsymbol{r}_{PG} = \bar{\mathcal{R}}_2 \mathcal{R}_1 \begin{bmatrix} 0 \\ -R \\ 0 \end{bmatrix} = -R \begin{bmatrix} -s_\psi c_\phi \\ c_\psi c_\phi \\ s_\phi \end{bmatrix}, \tag{2.260}$$

which is a vector pointing from the centre of the disc to the point of contact with the plane Π. Note that this vector is *not* fixed to the disc (body-fixed), but is 'unspun'; it has not been subjected to the θ rotation $\bar{\mathcal{R}}_3$. The vector $\mathbf{r}_{PG}$ can be seen in Figure 2.24.

This vector can also be found by first considering the vector

$$\mathbf{r} = \boldsymbol{n} \times \boldsymbol{e}_z,$$

which lies along the line of intersection $P'P$ between the plane of the disc and the ground plane Π. The vector $\mathbf{r}_{PG}$ can then be found using the alternative expression

$$\mathbf{r}_{PG} = -R \frac{\mathbf{r}}{\|\mathbf{r}\|} \times \boldsymbol{n}, \tag{2.261}$$

which is the more common approach to this calculation.

The velocity of the disc's mass centre can be found using:

$$\mathbf{v}_G = \mathbf{v}_P + \boldsymbol{\omega} \times \mathbf{r}_{PG}. \tag{2.262}$$

The disc's angular velocity $\boldsymbol{\omega}$ is

$$\boldsymbol{\omega} = \begin{bmatrix} \dot{\theta} s_\phi s_\psi + \dot{\phi} c_\psi \\ \dot{\phi} s_\psi - \dot{\theta} s_\phi c_\psi \\ \dot{\psi} + \dot{\theta} c_\phi \end{bmatrix}, \tag{2.263}$$

which can be computed using (2.166).

Since the point P, when regarded as a material point on the periphery of the disc in instantaneous contact with the ground, must be stationary, we substitute $\mathbf{v}_P = 0$ into (2.262) to obtain:

$$\begin{aligned} \mathbf{v}_G &= \boldsymbol{\omega} \times \mathbf{r}_{PG} \\ &= R \begin{bmatrix} \dot{\psi} c_\psi c_\phi - \dot{\phi} s_\psi s_\phi + \dot{\theta} c_\psi \\ \dot{\psi} s_\psi c_\phi + \dot{\phi} c_\psi s_\phi + \dot{\theta} s_\psi \\ -\dot{\phi} c_\phi \end{bmatrix}. \end{aligned} \tag{2.264}$$

The third entry in $\mathbf{v}_G$ can be integrated to give the holonomic constraint

$$z_G(t) = -R s_\phi. \tag{2.265}$$

The remaining two entries give the nonholonomic constraints:

$$\dot{x}_G(t) = R(\dot{\psi} c_\psi c_\phi - \dot{\phi} s_\psi s_\phi + \dot{\theta} c_\psi) \tag{2.266}$$

$$\dot{y}_G(t) = R(\dot{\psi} s_\psi c_\phi + \dot{\phi} c_\psi s_\phi + \dot{\theta} s_\psi). \tag{2.267}$$

2.10.3 Angular momentum balance

The disc's six degrees of freedom are the three translations $x_G(t)$, $y_G(t)$, and $z_G(t)$ of the disc's mass centre, and the three Euler angles $\psi(t)$, $\phi(t)$, and $\theta(t)$. The force acting at the disc's contact point P is $\mathbf{F}_P$; the direction of this force is unknown (at this stage). A linear momentum balance gives:

$$\mathbf{F}_P = m\mathbf{a}_G - mg\boldsymbol{e}_z; \tag{2.268}$$

note that the positive $\boldsymbol{e}_z$ direction points 'downwards'. An angular momentum balance about the mass centre G gives:

$$\begin{aligned} 0 &= \mathbf{r}_{PG} \times \boldsymbol{F}_P + J_G\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times J_G\boldsymbol{\omega} \\ &= m\mathbf{r}_{PG} \times (\mathbf{a}_G - g\boldsymbol{e}_z) + J_G\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times J_G\boldsymbol{\omega} \\ &= m\mathbf{r}_{PG} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_{PG} + \boldsymbol{\omega} \times \dot{\mathbf{r}}_{PG} - g\boldsymbol{e}_z) + J_G\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times J_G\boldsymbol{\omega}, \end{aligned} \tag{2.269}$$

where (2.262) was used to eliminate the disc's three translational freedoms.[17]

[17] In (2.269) we used the relationship

Direct calculation using (2.269) gives the disc's equations of motion as follows:

$$0 = \frac{4}{mR^2}\begin{bmatrix} -s_\psi c_\phi & c_\psi c_\phi & s_\phi \\ c_\psi & s_\psi & 0 \\ \frac{1}{2}s_\phi s_\psi & -\frac{1}{2}s_\phi c_\psi & \frac{1}{2}c_\phi \end{bmatrix}\begin{bmatrix} s_\phi\ddot{\psi} - 2\dot{\theta}\dot{\phi} \\ \frac{4c_\phi g}{R} + 6s_\phi\dot{\theta}\dot{\psi} + 5\ddot{\phi} + 5s_\phi c_\phi\dot{\psi}^2 \\ 3c_\phi\ddot{\psi} + 3\ddot{\theta} - 5s_\phi\dot{\phi}\dot{\psi} \end{bmatrix}. \tag{2.271}$$

Since

$$\det\left(\begin{bmatrix} -s_\psi c_\phi & c_\psi c_\phi & s_\phi \\ c_\psi & s_\psi & 0 \\ \frac{1}{2}s_\phi s_\psi & -\frac{1}{2}s_\phi c_\psi & \frac{1}{2}c_\phi \end{bmatrix}\right) = -\frac{1}{2} \tag{2.272}$$

the disc's motion is fully determined by the Euler angles, which must satisfy:

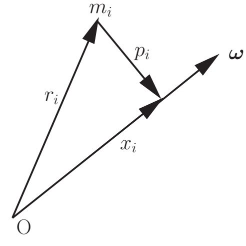
$$0 = s_\phi\ddot{\psi} - 2\dot{\theta}\dot{\phi} \tag{2.273}$$

$$0 = \frac{4c_\phi g}{R} + 6s_\phi\dot{\theta}\dot{\psi} + 5\ddot{\phi} + 5s_\phi c_\phi\dot{\psi}^2 \tag{2.274}$$

$$0 = 3c_\phi\ddot{\psi} + 3\ddot{\theta} - 5s_\phi\dot{\phi}\dot{\psi}. \tag{2.275}$$

$$\boldsymbol{M} = \frac{d\boldsymbol{H}}{dt} = \frac{\partial \boldsymbol{H}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{H} \tag{2.270}$$

in which $\boldsymbol{M}$ is the external moment and $\boldsymbol{H}$ is the rigid body's angular momentum. The apparent rate of change of $\boldsymbol{H}$ in a body-fixed moving frame of reference is $\frac{\partial \boldsymbol{H}}{\partial t} = J_G\dot{\boldsymbol{\omega}}$, since J_G is constant in this frame. The second term $\boldsymbol{\omega} \times \boldsymbol{H} = \boldsymbol{\omega} \times J_G\boldsymbol{\omega}$ comes from (2.147). A physical interpretation of the $\boldsymbol{\omega} \times J_G\boldsymbol{\omega}$ term may be of interest:



In the figure we consider a rigid body with instantaneous angular velocity vector $\boldsymbol{\omega}$. Let O be the body's mass centre and m_i the mass of a constituent mass element that is located at $\boldsymbol{r}_i$ relative to the mass centre. Let $\boldsymbol{p}_i$ represent a vector from m_i to $\boldsymbol{\omega}$ and perpendicular to it. Let $\boldsymbol{x}_i$ be the projection of $\boldsymbol{r}_i$ on to $\boldsymbol{\omega}$. Then:

$$\boldsymbol{p}_i = \boldsymbol{x}_i - \boldsymbol{r}_i = (\boldsymbol{r}_i \cdot \boldsymbol{\omega})\frac{\boldsymbol{\omega}}{\omega^2} - \boldsymbol{r}_i \qquad \text{where} \qquad \omega = |\boldsymbol{\omega}|$$

and so

$$\omega^2\boldsymbol{r}_i \times \boldsymbol{p}_i = (\boldsymbol{r}_i.\boldsymbol{\omega})(\boldsymbol{r}_i \times \boldsymbol{\omega}).$$

Now the angular momentum of the body is:

$$\boldsymbol{H} = \sum m_i\boldsymbol{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{r}_i) = \sum m_i(\boldsymbol{\omega}(\boldsymbol{r}_i \cdot \boldsymbol{r}_i) - \boldsymbol{r}_i(\boldsymbol{\omega} \cdot \boldsymbol{r}_i))$$

so that

$$\boldsymbol{\omega} \times \boldsymbol{H} = \sum m_i(\boldsymbol{\omega} \cdot \boldsymbol{r}_i)(\boldsymbol{r}_i \times \boldsymbol{\omega}) = \sum m_i\omega^2(\boldsymbol{r}_i \times \boldsymbol{p}_i).$$

Hence the term $\boldsymbol{\omega} \times \boldsymbol{H}$ is the moment about the body's mass centre of the centripetal forces acting on the constituent particles making up the body, as it spins around the instantaneous angular velocity vector $\boldsymbol{\omega}$.

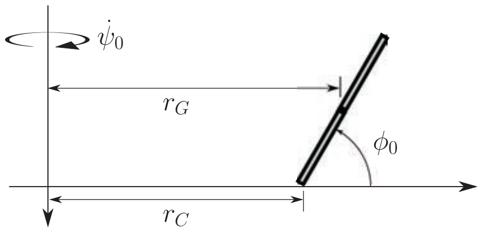


Figure 2.25: Rolling disc in steady motion.

2.10.4 Equilibrium solutions

The equations describing the disc's dynamics have a number of dynamic equilibria that we will consider briefly. To begin, we find the velocity of the disc's mass centre at a constant roll angle, and constant spin and yaw rates. It follows from (2.266) and (2.267) that for constant yaw and spin rates $\dot{\psi}_0$ and $\dot{\theta}_0$, respectively, and a constant roll angle ϕ_0, the velocity is given by

$$\boldsymbol{v}_G^0 = R(\dot{\psi}_0 c_{\phi 0} + \dot{\theta}_0) \begin{bmatrix} c_{\psi 0} \\ s_{\psi 0} \\ 0 \end{bmatrix}, \tag{2.276}$$

which has constant magnitude $|\boldsymbol{v}_G^0| = v_G^0 = R(\dot{\psi}_0 c_{\phi 0} + \dot{\theta}_0)$, and lies in the horizontal plane. It is clear from (2.276) and Figure 2.25, that under steady-state conditions, the disc's mass centre undergoes circular motion around a stationary vertical axis. The radius of rotation of the contact point is r_C, while that of the mass centre is r_G. Since $v_G^0 = \dot{\psi}_0 r_G$, there holds

$$r_G \dot{\psi}_0 = R(\dot{\psi}_0 c_{\phi_0} + \dot{\theta}_0), \tag{2.277}$$

with

$$r_C = r_G - R c_{\phi_0}. \tag{2.278}$$

Thus, for steady motion the 'spin' angular velocity $\dot{\theta}_0$ is related to the 'precession' angular velocity $\dot{\psi}_0$ by

$$\dot{\theta}_0 = \frac{r_C}{R} \dot{\psi}_0. \tag{2.279}$$

Eliminating $\dot{\theta}_0$ from (2.274) gives:

$$\dot{\psi}_0^2 = \frac{-4 g c_{\phi_0}}{s_{\phi_0}(6 r_C + 5 R c_{\phi_0})} \tag{2.280}$$

under steady-state conditions. In the case that $0 < \phi_0 < \pi/2$, the denominator in (2.280) is positive, since r_C is non-negative (by definition), while the numerator is negative. This leads to a non-physical imaginary precession angular velocity. We conclude that steady motion is not possible for this range of roll angles, which accords with the intuitive physical notion that the disc must 'lean' into circular motion.

In the case that $\pi/2 < \phi_0 < \pi$, the numerator in (2.280) is positive with the denominator also positive provided $r_C > -5Rc_{\phi_0}/6$, or what is the same

$$r_G > r_G^{min} = \frac{Rc_{\phi_0}}{6}. \tag{2.281}$$

In addition to the commonly observed $r_G > 0$ case, steady motion is still possible with small negative values of r_G. Once $\dot{\psi}_0$ has been found using (2.280), $\dot{\theta}_0$ follows from (2.279).

In the case that $r_G = 0$, $r_C = -Rc_{\phi_0}$, and $\dot{\theta}_0 = -\dot{\psi}_0 c_{\phi_0}$, there hold

$$\dot{\psi}_0 = \sqrt{\frac{4g}{Rs_{\phi_0}}} \tag{2.282}$$

and

$$\dot{\theta}_0 = c_{\phi_0}\sqrt{\frac{4g}{Rs_{\phi_0}}}. \tag{2.283}$$

Equations (2.282) and (2.283) correspond to the whirling motion associated with a spinning disc that has its centre of mass stationary. This is the motion associated with the toy known as Euler's disc [79].

In the case that $\phi_0 = \pi/2$, a solution is possible if $\dot{\psi}_0 = 0$ and $\dot{\theta}_0 > 0$ is arbitrary. This corresponds to the disc rolling in a straight line; if this motion is to be stable $\dot{\theta}_0$ must be 'large enough'. Alternatively, one could have $r_G = r_C = 0$, $\dot{\theta}_0 = 0$ and $\dot{\psi}_0$ arbitrary. This corresponds to the disc spinning on its side in a fixed vertical position (with arbitrary yaw velocity).

The reaction force given in (2.268) is easily evaluated under steady-state conditions by differentiating (2.276) with respect to time. This gives

$$\dot{\boldsymbol{v}}_G^0 = R\dot{\psi}_0(\dot{\psi}_0 c_{\phi 0} + \dot{\theta}_0)\begin{bmatrix} -s_{\psi_0} \\ c_{\psi_0} \\ 0 \end{bmatrix}, \tag{2.284}$$

which has magnitude

$$|\dot{\boldsymbol{v}}_G^0| = |\boldsymbol{a}_G^0| = a_G^0 = \dot{\psi}_0 v_G^0,$$

and points towards the centre of rotation. This means that the ground-contact force is

$$\boldsymbol{F}_P = m a_G^0 = m(v_G^0)^2/r_G,$$

which points towards the centre of rotation too.

Example 2.25 Consider a disc of radius $R = 0.025$ m under the rolling conditions given in Table 2.1. In the first case (2.281) has been violated and so steady motion is not possible. The second case corresponds to the common case of a disc rolling in a circle. In the third case the disc's mass centre is on the vertical axis of rotation as shown in Figure 2.25 and a 'whirling' motion results. When $r_G = 0$, (2.280) simplifies to (2.282) and it is clear that as $\phi_0 \to \pi$, the precession angular velocity $\dot{\psi}_0$ increases without bound. The fourth case illustrates this phenomenon.

Table 2.1 Rolling conditions

Case	ϕ_0	r_C	r_G	r_G^{min}	$\dot{\psi}_0$	$\dot{\theta}_0$
1	$3\pi/4$	0	-0.0177	-0.0029	—	—
2	$3\pi/4$	0.5	0.482	-0.0029	3.671	73.422
3	$3\pi/4$	0.0177	0.0	-0.0029	47.11	33.31
4	$179\pi/180$	0.025	0.0	-0.0042	299.89	299.85

2.10.5 Lagrange's equation

As is well known, it is also possible to derive the equations of motion using energy-based procedures. The disc's kinetic energy is:

$$T = \frac{m}{2}(\dot{x}_G^2(t) + \dot{y}_G^2(t) + \dot{z}_G^2(t)) + \frac{1}{2}\boldsymbol{\omega}_I^T I_{cm} \boldsymbol{\omega}_I, \tag{2.285}$$

while its potential energy is

$$V = -mgz_G(t). \tag{2.286}$$

Combining these, we obtain the Lagranian

$$\mathcal{L} = T - V. \tag{2.287}$$

We can immediately reduce the number of degrees of freedom (generalized coordinates) from six to five by eliminating $z_G(t)$ using the holonomic constraint (2.265); it is also possible to incorporate this holonomic constraint in its differential form. In order to incorporate the holonomic constraint (2.265), and the nonholonomic conditions (2.266) and (2.267) into the problem formulation, we will append these constraints to the Lagrangian using Lagrange multipliers. The equations of motion then follow from

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \left(\frac{\partial \mathcal{L}}{\partial q_j}\right) = -\sum_{k=1}^{3} \lambda_k a_{kj} \qquad \text{for} \quad j = 1, 2, \cdots 6 \tag{2.288}$$

in which (2.266) and (2.267) (and optionally) (2.265) have been written as

$$0 = \sum_{j=1}^{6} a_{ij}\dot{q}_j + a_i \qquad \text{for} \quad i = 1, \cdots, 3. \tag{2.289}$$

The (nine) equations in (2.288) and (2.289) give nine unknown quantities; six second derivatives and three Lagrange multipliers. The Lagrange multipliers specify the force of constraint $\mathbf{F}_P$. By direct calculation one obtains:

$$\lambda_1 = m\frac{d^2}{dt^2}x_G(t) \tag{2.290}$$

$$\lambda_2 = m\frac{d^2}{dt^2}y_G(t) \tag{2.291}$$

$$\lambda_3 = m\frac{d^2}{dt^2}z_G(t) - mg, \tag{2.292}$$

which we recognize as the three components of the contact-point reaction force. Eliminating the Lagrange multipliers and the translational freedoms again gives:

$$
\begin{aligned}
0 &= s_\phi \ddot{\psi} - 2\dot{\theta}\dot{\phi} \\
0 &= \frac{4c_\phi g}{R} + 6s_\phi \dot{\theta}\dot{\psi} + 5\ddot{\phi} + 5s_\phi c_\phi \dot{\psi}^2 \\
0 &= 3c_\phi \ddot{\psi} + 3\ddot{\theta} - 5s_\phi \dot{\phi}\dot{\psi}.
\end{aligned}
$$

2.10.6 Rolling stability

The stability of the rolling disc has been investigated by a number of authors including for example Routh [76], Synge and Griffith [40], and Neĭmark and Fufaev [48]. If one rolls a compact disc over table top it will roll stably provided the initial speed is 'high enough'. In this case the disc will even jump over small objects and then continue on its way; try it with a compact disc! Once the speed starts to reduce, the disc will begin to wobble and then it will tilt to one side and fall over—the roll dynamics of the disc become unstable. As we will show later, this type of behaviour is found in bicycle dynamics too.

Suppose the disc is rolling along a circular trajectory at constant speed on a horizontal plane Π. Under these conditions we have

$$\dot{\psi} = \dot{\psi}_0, \quad \phi = \phi_0, \quad \text{and} \quad \dot{\theta} = \dot{\theta}_0$$

in which $(\cdot)_0$ denotes a constant steady value. Expressions for the equilibrium values of the yaw and roll rates are given in (2.280) and (2.279) respectively.

Equations (2.273), (2.274), and (2.275) can be linearized for small perturbations from the previously specified steady-state cornering trim condition to obtain

$$0 = s_{\phi_0}\ddot{\psi} - 2\dot{\theta}_0\dot{\phi} \tag{2.293}$$

$$
\begin{aligned}
0 = {} & 5\ddot{\phi} - \frac{4g}{R}s_{\phi_0}\phi + 6s_{\phi_0}\dot{\theta}_0\dot{\psi} + 6s_{\phi_0}\dot{\theta}\dot{\psi}_0 + 6c_{\phi_0}\dot{\theta}_0\dot{\psi}_0\phi \\
& + 5\dot{\psi}_0^2\phi(c_{\phi_0}^2 - s_{\phi_0}^2) + 10c_{\phi_0}s_{\phi_0}\dot{\psi}_0\dot{\psi}
\end{aligned} \tag{2.294}
$$

$$0 = 3c_{\phi_0}\ddot{\psi} + 3\ddot{\theta} - 5s_{\phi_0}\dot{\psi}_0\dot{\phi} \tag{2.295}$$

in which ψ, ϕ, and θ are small perturbations. Integrating (2.293) and (2.295) gives respectively

$$\dot{\psi} = (C_1 + 2\dot{\theta}_0\phi)/s_{\phi_0} \tag{2.296}$$

$$\dot{\theta} = \frac{5}{3}s_{\phi_0}\dot{\psi}_0\phi - c_{\phi_0}(C_1 + 2\dot{\theta}_0\phi)/s_{\phi_0} + C_2, \tag{2.297}$$

in which C_1 and C_2 are constants of integration. Substituting (2.296) and (2.297) into (2.294) gives the following second-order linear differential equation for the roll angle:

$$\ddot{\phi} + \Xi\phi = \Delta \tag{2.298}$$

in which

$$\Xi = \frac{12}{5}\dot{\theta}_0^2 + \frac{14}{5}c_{\phi_0}\dot{\theta}_0\dot{\psi}_0 + \dot{\psi}_0^2 - \frac{4g}{5R}s_{\phi_0} \tag{2.299}$$

with Δ a constant that is irrelevant for present purposes. In the case that $\Xi < 0$ divergent roll behaviour will result. If $\Xi > 0$, bounded, but undamped roll oscillations will occur. The case $\Xi = 0$ requires a more sophisticaed (nonlinear) analysis as was pointed out in [48]. If one forms the equation

$$M(s)\begin{bmatrix}\phi(s)\\ \psi(s)\\ \theta(s)\end{bmatrix} = 0,$$

by taking Laplace transforms of (2.293) to (2.295), then the corresponding sixth-order characteristic polynomial is given by $\det(M(s))$. In the case that $\Xi > 0$ the roots of $\det(M(s))$ are of the form $\{i\lambda, -i\lambda, 0, 0, 0, 0\}$ in which $\lambda > 0$ for some real λ. If $\Xi < 0$ the roots of $\det(M(s))$ are of the form $\{\lambda, -\lambda, 0, 0, 0, 0\}$ in which $\lambda > 0$ is real. If $\Xi = 0$ the roots of $\det(M(s))$ are all at the origin. In the special case of $\dot{\psi}_0 = 0$ and $\phi_0 = \pi/2$, we obtain

$$\Xi = \frac{12}{5}\dot{\theta}_0^2 - \frac{4g}{5R}. \tag{2.300}$$

The condition $\Xi > 0$, or what is the same, $\dot{\theta}_0 > \sqrt{g/(3R)}$, is the condition for straight-running stability given in Routh [76].