Bicycle Model

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1 Definitions

1.1 System States:

x_1 : X coordinate of the vehicle in global frame
x_2 : Y coordinate of the vehicle in global frame
x_3 : heading of the chassis in global frame(ψ)
x_4 : velocity of the center of gravity in body frame(longitudinal direction
x_5 : velocity of the center of gravity in body frame(lateral direction)
x_6 : angular velocity of the chassis in body frame

1.2 System Inputs:

u_1 : steering angle of the front Wheel(δ)
u_2 : acceleration of the vehicle(+/-(braking))

1.3 Vehicle Parameters:

$mass$: Vehicle Mass(Kg)
I_{zz} : moment of inertia about z axis

1.4 Time Varying Parameters

β : chassis Side Slip Angle
v_{cg} : velocity of the center of gravity in body frame
α : slip angle
F_z : Vertical Force acting on the tire
$cpcg_{dist-\gamma}$: distance between tire contact patch and vehicle cg($\gamma \in [f, r]$)
$cpcg_{angle-\gamma}$: angle between vehicle's longitudinal axis and line joining tire contact patch and vehicle cg ($\gamma \in [f, r]$)
CP_{vel} : contact patch velocity
LSR_f : Limited Slip Ratio Front
LSR_r : Limited Slip Ratio Rear
F_{yf} : front tire force in lateral direction
F_{xf} : front tire force in longitudinal direction
F_{yr} : rear tire force in lateral direction
F_{xr} : rear tire force in longitudinal direction
τ : torque at wheel

2 Non Linear Dynamics

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \\ \dot{x_5} \\ \dot{x_6} \end{bmatrix} = \begin{bmatrix} x_4\cos(x_3) - x_5\sin(x_3) \\ x_4\sin(x_3) + x_5\cos(x_3) \\ x_6 \\ \frac{1}{mass}(F_{xr} + F_{xf}\cos(u_1) - F_{yf}\sin(u_1) + mass * x_5 * x_6 + \frac{\tau}{radius}) \\ \frac{1}{mass}(F_{yr} + F_{xf}\sin(u_1) + F_{yf} * \cos(u_1) - mass * x_5 * x_6) \\ \frac{1}{I_{zz}}((F_{xf}\sin(u_1) + F_{yf}\cos(u_1))cpcg_{dist-f} - F_{yr}cpcg_{dist-r} \end{bmatrix}$$

3 Linearized Dynamics

Linearizing the dynamics around a reference trajectory $(\tilde{x}(t), \tilde{u}(t))$:-

- $\dot{x}_1 = x_4 \cos x_3 - x_5 \sin x_3$

$$\boxed{\begin{array}{ll} \dfrac{\delta\dot{x}_1}{\delta x_1} = 0 & \dfrac{\delta\dot{x}_1}{\delta x_4} = \cos\tilde{x}_3 \\ \dfrac{\delta\dot{x}_1}{\delta x_2} = 0 & \dfrac{\delta\dot{x}_1}{\delta x_5} = -\sin\tilde{x}_3 \\ \dfrac{\delta\dot{x}_1}{\delta x_3} = -\tilde{x}_4\sin\tilde{x}_3 - \tilde{x}_5\cos\tilde{x}_3 & \dfrac{\delta\dot{x}_1}{\delta x_6} = 0 \end{array}} \quad \rightarrow \quad \text{With respect to states } x(t)$$

$$\boxed{\dfrac{\delta\dot{x}_1}{\delta u_1} = 0 \qquad \dfrac{\delta\dot{x}_1}{\delta u_2} = 0} \quad \rightarrow \quad \text{With respect to states } u(t)$$

- $\dot{x}_2 = x_4 \sin x_3 + x_5 \cos x_3$

$$\boxed{\begin{array}{ll} \dfrac{\delta\dot{x}_2}{\delta x_1} = 0 & \dfrac{\delta\dot{x}_2}{\delta x_4} = \sin\tilde{x}_3 \\ \dfrac{\delta\dot{x}_2}{\delta x_2} = 0 & \dfrac{\delta\dot{x}_2}{\delta x_5} = \cos\tilde{x}_3 \\ \dfrac{\delta\dot{x}_2}{\delta x_3} = \tilde{x}_4\cos\tilde{x}_3 - \tilde{x}_5\sin\tilde{x}_3 & \dfrac{\delta\dot{x}_2}{\delta x_6} = 0 \end{array}} \quad \rightarrow \quad \text{With respect to states } x(t)$$

$$\boxed{\dfrac{\delta\dot{x}_2}{\delta u_1} = 0 \qquad \dfrac{\delta\dot{x}_2}{\delta u_2} = 0} \quad \rightarrow \quad \text{With respect to states } u(t)$$

- $\dot{x}_3 = x_6$

$$\boxed{\begin{array}{ll} \dfrac{\delta\dot{x}_3}{\delta x_1} = 0 & \dfrac{\delta\dot{x}_3}{\delta x_4} = 0 \\ \dfrac{\delta\dot{x}_3}{\delta x_2} = 0 & \dfrac{\delta\dot{x}_3}{\delta x_5} = 0 \\ \dfrac{\delta\dot{x}_3}{\delta x_3} = 0 & \dfrac{\delta\dot{x}_3}{\delta x_6} = 1 \end{array}} \quad \rightarrow \quad \text{With respect to states } x(t)$$

$$\boxed{\dfrac{\delta\dot{x}_3}{\delta u_1} = 0 \qquad \dfrac{\delta\dot{x}_3}{\delta u_2} = 0} \quad \rightarrow \quad \text{With respect to states } u(t)$$

- $\dot{x}_4 = \frac{1}{mass}\left(F_{xr} + F_{xf}\cos(u_1) - F_{yf}\sin(u_1) + mass \times x_5 \times x_6 + \frac{\tau}{radius}\right)$

$$\boxed{\begin{array}{ll} \dfrac{\delta \dot{x}_4}{\delta x_1} = 0 & \dfrac{\delta \dot{x}_4}{\delta x_4} = 0 \\ \dfrac{\delta \dot{x}_4}{\delta x_2} = 0 & \dfrac{\delta \dot{x}_4}{\delta x_5} = \tilde{x}_6 \\ \dfrac{\delta \dot{x}_4}{\delta x_3} = 0 & \dfrac{\delta \dot{x}_4}{\delta x_6} = \tilde{x}_5 \end{array}} \quad \rightarrow \quad \text{With respect to states } x(t)$$

$$\boxed{\frac{\delta \dot{x}_4}{\delta u_1} = \frac{1}{mass}\left(-F_{xf}\sin u_1 - F_{yf}\cos u_1\right) \qquad \frac{\delta \dot{x}_4}{\delta u_2} = 0} \quad \rightarrow \quad \text{With respect to states } u(t)$$

- $\dot{x}_5 = \frac{1}{mass}\left(F_{yr} + F_{xf}\sin(u_1) + F_{yf}\cos(u_1) - mass \times x_5 \times x_6\right)$

$$\boxed{\begin{array}{ll} \dfrac{\delta \dot{x}_5}{\delta x_1} = 0 & \dfrac{\delta \dot{x}_5}{\delta x_4} = 0 \\ \dfrac{\delta \dot{x}_5}{\delta x_2} = 0 & \dfrac{\delta \dot{x}_5}{\delta x_5} = -\tilde{x}_6 \\ \dfrac{\delta \dot{x}_5}{\delta x_3} = 0 & \dfrac{\delta \dot{x}_5}{\delta x_6} = -\tilde{x}_4 \end{array}} \quad \rightarrow \quad \text{With respect to states } x(t)$$

$$\boxed{\frac{\delta \dot{x}_5}{\delta u_1} = \frac{1}{mass}\left(F_{xf}\cos\tilde{u}_1 - F_{yf}\sin\tilde{u}_1\right) \qquad \frac{\delta \dot{x}_5}{\delta u_2} = 0} \quad \rightarrow \quad \text{With respect to states } u(t)$$

- $\dot{x}_6 = \frac{1}{I_{zz}}((F_{xf}\sin(u_1) + F_{yf}\cos(u_1))cpcg_{dist-f} - F_{yr}cpcg_{dist-r})$

$$\boxed{\begin{array}{ll} \dfrac{\delta \dot{x}_6}{\delta x_1} = 0 & \dfrac{\delta \dot{x}_6}{\delta x_4} = 0 \\ \dfrac{\delta \dot{x}_6}{\delta x_2} = 0 & \dfrac{\delta \dot{x}_6}{\delta x_5} = 0 \\ \dfrac{\delta \dot{x}_6}{\delta x_3} = 0 & \dfrac{\delta \dot{x}_6}{\delta x_6} = 0 \end{array}} \quad \rightarrow \quad \text{With respect to states } x(t)$$

$$\boxed{\frac{\delta \dot{x}_6}{\delta u_1} = \frac{1}{I_{zz}}\left(F_{xf}\cos\tilde{u}_1 - F_{yf}\sin\tilde{u}_1\right)cpcg_{dist-f} \qquad \frac{\delta \dot{x}_6}{\delta u_2} = 0} \quad \rightarrow \quad \text{With respect to states } u(t)$$

Final Linearized Equation:-

$$\boxed{\begin{array}{l} \delta\dot{x} = A\delta x + B\delta u \\ \delta\dot{x} = \begin{bmatrix} 0 & 0 & -\tilde{x}_4\sin\tilde{x}_3 - \tilde{x}_5\cos\tilde{x}_3 & \cos\tilde{x}_3 & -\sin\tilde{x}_3 & 0 \\ 0 & 0 & \tilde{x}_4\cos\tilde{x}_3 - \tilde{x}_5\sin\tilde{x}_3 & \sin\tilde{x}_3 & \cos\tilde{x}_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \tilde{x}_6 & \tilde{x}_5 \\ 0 & 0 & 0 & 0 & -\tilde{x}_6 & -\tilde{x}_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\delta x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{mass}\left(-F_{xf}\sin u_1 - F_{yf}\cos u_1\right) & 0 \\ \frac{1}{mass}\left(F_{xf}\cos\tilde{u}_1 - F_{yf}\sin\tilde{u}_1\right) & 0 \\ \frac{1}{I_{zz}}\left(F_{xf}\cos\tilde{u}_1 - F_{yf}\sin\tilde{u}_1\right)cpcg_{dist-f} & 0 \end{bmatrix}\delta u \end{array}}$$