LINEARIZING DYNAMICS

General idea:- given a reference trajectory

$$(\tilde{x}(t), \tilde{u}(t))$$

reference state ref input.

Non-linear Dynamics:

$$\dot{x}(t) = f(x(t), u(t))$$

$$x(t) = \tilde{x}(t) + \Delta x(t)$$

deviation from ref.

$$u(t) = \tilde{u}(t) + \Delta u(t)$$

such that,

$$\dot{\tilde{x}}(t) = f(\tilde{x}(t), \tilde{u}(t))$$

linearizing about $(\tilde{x}(t), \tilde{u}(t))$

$$f(x(t), u(t)) = f(\tilde{x}(t) + \Delta x(t), \tilde{u}(t) + \Delta u(t))$$

Using 1^{st} order approximation:-

$$f(x(t), u(t)) = f(\tilde{x}(t), \tilde{u}(t)) + \nabla f\Big|_{x=\tilde{x}} \Delta x(t) + \nabla f\Big|_{u=\tilde{u}} \Delta u(t)$$

$$\Delta \dot{x}(t) = A\,\Delta x(t) + B\,\Delta u(t)$$

where $A = \nabla f \big|_{x=\tilde{x}}$ → partial derivative wrt states

$B = \nabla f \big|_{u=\tilde{u}}$ → partial derivatives wrt inputs

states :-

$$\begin{bmatrix} X \\ Y \\ \psi \\ V_x \\ V_y \\ \dot{\psi} \end{bmatrix}$$

inputs :-

$$\begin{bmatrix} \delta \\ D \end{bmatrix}$$

δ → steering angle

D → Driving command (± 1)

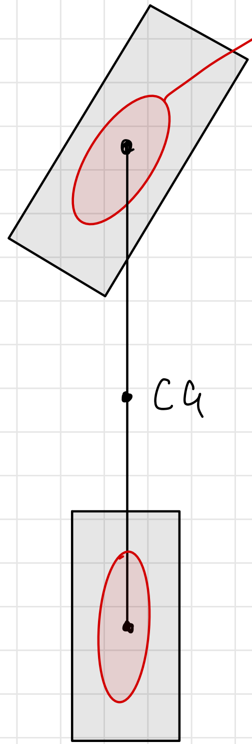
$$\begin{bmatrix} \text{throttle} \rightarrow +1 \\ \text{Braking} \rightarrow -1 \end{bmatrix}$$

$$
\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{\psi} \\ \dot{v}_x \\ \dot{v}_y \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} v_x \cos\psi - v_y \sin\psi \\ v_x \sin\psi + v_y \cos\psi \\ \dot{\psi} \\ \left[F_D + \left(F_{xr} + F_{xf}\cos\delta\right) - F_{yf}\sin\delta + m\cdot v_y\cdot\dot{\psi}\right] \frac{1}{m} \\ \left[F_{xf}\cdot\sin(\delta) + \left(F_{yr} + F_{yf}\cos(\delta)\right) - m\cdot v_x\cdot\dot{\psi}\right] \frac{1}{m} \\ \left[\left(F_{xf}\sin\delta + F_{yf}\cos\delta\right) d_{cp\text{-}cg\text{-}f} - F_{yr}\cdot d_{cp\text{-}cg\text{-}r}\right] \frac{1}{I_{zz}} \end{bmatrix}
$$

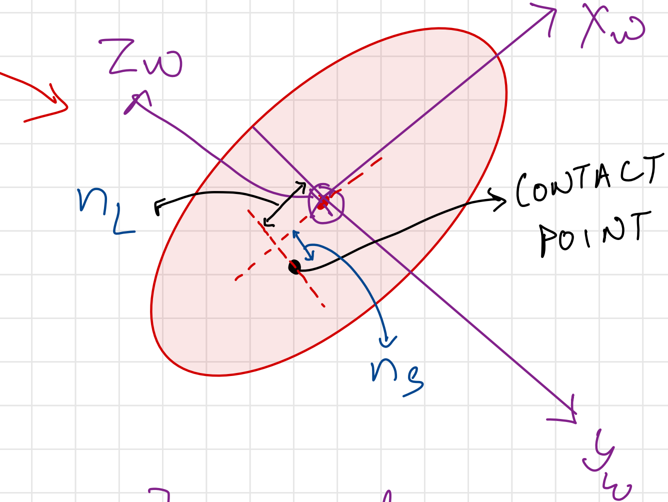
$F_{x\alpha}$, $F_{y\alpha}$, where $\alpha = \{f, r\}$ are tire forces

front and rear

$v_x, v_y \rightarrow$ velocity in vehicle frame.



→ tire contact patch :



$[x_w, y_w, z_w]$ → wheel coordinate system.

$[n_s, n_l]$ → casters

$d_{cp\text{-}cg}$ → distance between contact point and CG.

Linearizing around nominal/ref traj. $(\tilde{x}(t), \tilde{u}(t))$

$$\dot{X} = V_x \cos\psi - V_y \sin\psi$$

$$\left.\frac{\partial \dot{X}}{\partial X}\right|_{x=\tilde{x}} = 0$$

$$\left.\frac{\partial \dot{X}}{\partial Y}\right|_{x=\tilde{x}} = 0$$

$$\left.\frac{\partial X}{\partial \psi}\right|_{x=\tilde{x}} = -\tilde{V}_x \sin\tilde{\psi} - \tilde{V}_y \cos\tilde{\psi}$$

$$\left.\frac{\partial X}{\partial V_x}\right|_{x=\tilde{x}} = \cos\tilde{\varphi}$$

$$\left.\frac{\partial X}{\partial V_y}\right|_{x=\tilde{x}} = -\sin\tilde{\varphi}$$

$$\left.\frac{\partial X}{\partial \dot{\varphi}}\right|_{x=\tilde{x}} = 0$$

$$\frac{\partial \dot{X}}{\partial \delta} = \frac{\partial \dot{X}}{\partial D} = 0$$

$$\dot{\psi} = V_x \sin\psi + V_y \cos\psi$$

$$\left.\frac{\partial \dot{\psi}}{\partial X}\right|_{x=\tilde{x}} = 0$$

$$\left.\frac{\partial \dot{\psi}}{\partial V_x}\right|_{x=\tilde{x}} = \sin\tilde{\psi}$$

$$\left.\frac{\partial \dot{\psi}}{\partial Y}\right|_{x=\tilde{x}} = 0$$

$$\frac{\partial \dot{\psi}}{\partial V_y} = \cos\tilde{\psi}$$

$$\left.\frac{\partial \dot{\psi}}{\partial \psi}\right|_{x=\tilde{x}} = \tilde{V}_x \cos\tilde{\psi} - \tilde{V}_y \sin\tilde{\psi}$$

$$\frac{\partial \dot{\psi}}{\partial \dot{\psi}} = 0$$

$$\frac{\partial \dot{\psi}}{\partial S} = \frac{\partial \dot{\psi}}{\partial D} = 0$$

$$\dot{v}_x = \left(F_D + \left(F_{xr} + F_{xf}\cos\delta\right) - F_{yf}\sin\delta + m v_y \dot{\varphi}\right) \frac{1}{m}$$

$$\frac{\partial \dot{v}_x}{\partial X} = \frac{\partial \dot{v}_x}{\partial Y} = \frac{\partial \dot{v}_x}{\partial \varphi} = \frac{\partial \dot{v}_x}{\partial v_x} = 0$$

* $F_D \rightarrow$ will be a linear func^n of D

$F_D = C \cdot D$

$$\frac{\partial \dot{v}_x}{\partial v_y} = \tilde{\dot{\varphi}}$$

$$\frac{\partial \dot{v}_x}{\partial \delta} = -F_{xf}\sin\tilde{\delta} - F_{yf}\cos\tilde{\delta}$$

$$\frac{\partial \dot{v}_x}{\partial \dot{\varphi}} = \tilde{v}_y$$

$$\frac{\partial \dot{v}_x}{\partial D} = \pm F_D$$

{ $\pm$ depending throttle/Brake }

$$\dot{v}_y = \left(\left(F_{yr} + F_{yf} \cdot \cos(\delta) \right) + F_{xf} \sin\delta - m v_x \dot{\varphi} \right) \cdot 1/m$$

$$\frac{\partial \dot{v}_y}{\partial X} = \frac{\partial \dot{v}_y}{\partial Y} = \frac{\partial \dot{v}_y}{\partial \varphi} = \frac{\partial \dot{v}_y}{\partial v_y} = 0$$

$$\frac{\partial \dot{v}_y}{\partial v_x} = -\tilde{\dot{\varphi}} \qquad \frac{\partial \dot{v}_y}{\partial \dot{\varphi}} = -\tilde{v}_x$$

$$\frac{\partial \dot{v}_y}{\partial \delta} = -F_{yf} \sin\tilde{\delta} + F_{xf} \cos\tilde{\delta}$$

$$\frac{\partial \dot{v}_y}{\partial D} = 0$$

$$\ddot{\psi} = \left(\left(F_{xf} \sin \delta + F_{yf} \cdot \cos \delta \right) d_f - F_{yr} d_r \right) \frac{1}{I_{zz}}$$

$$\frac{\partial \ddot{\psi}}{\partial X} = \frac{\partial \ddot{\psi}}{\partial Y} = \frac{\partial \ddot{\psi}}{\partial \psi} = \frac{\partial \ddot{\psi}}{\partial v_x} = \frac{\partial \ddot{\psi}}{\partial v_y} = \frac{\partial \ddot{\psi}}{\partial \dot{\psi}} = 0$$

$$\frac{\partial \ddot{\psi}}{\partial \delta} = F_{xf} \cos \tilde{\delta} - F_{yf} \sin \tilde{\delta}$$