Formal Languages | Kurt Medley
Midterm

1. Give functions $f: N \to N$ that satisfy the following

a) f is total and one-to-one but not onto
$f(x) = x^2$ $f(1) = 1^2 = 1$, $f(2) = 2^2 = 4$, $f(3) = 3^2 = 9$, etc.
total 1-1 each element of X maps to a distinct element in the range.

b) f is total and onto but not one-to-one
$f(x) =$ if $(x == 0)$ return 1, else x
$f(0) = 1$, $f(1) = 1$, $f(2) = 2$

c) f is total, one-to-one, and onto but not the identity
$f(x) = s(x) - 1$ $f(0) = 1 - 1 = 0$, $f(1) = 2 - 1 = 1$, $f(2) = 3 - 1 = 2$, etc.

d) f is not total, but onto
$f(x)$: if $2 \bmod x = 0$, $x/2$

2. Show that the binary relation LT, less than, is not an equivalence relation

$LT = \{[i,j] \mid i < j \text{ and } i, j \in \mathbb{N}\}$

$i < i$ is irreflexive given $i < j$ and i for all $i \in \mathbb{N}$
$i < j$, implies $j < i$ is Not a symmetric relation as $1 < 2$ does not imply $2 < 1$
$i < j$ and $j < k$ implies $a < k$ for all $i, j, k \in \mathbb{N}$ implies

3. Give a recursive definition of the relation greater than on $\mathbb{N} \times \mathbb{N}$ using the successor operator s.

A recursive definition of $>$ $GT = \{[m,n] \mid m > n\}$
i) Basis $[s(z), z]$ is $\in GT$
ii) if $[m,n]$ is $\in GT$, then $[s(m), n] \in GT$ and $[s(m), s(n)] \in GT$
iii) $[m,n] \in GT$ only if it can be obtained from $[s(z), z]$ by a finite number of applications of the operations in the recursive step

4. Prove that $2 + 5 + 8 + \dots + (3n-1) = \frac{n(3n+1)}{2}$ for all $n > 0$

Basis step; $n = 1$
$3(1) - 1 = 2$
$\frac{1(3(1)+1)}{2} = 2$ ✓

I.S. Assume true for $n = k$,
show true for $n = k+1$
Assume: $2 + 5 + 8 + \dots + (3k-1) = \frac{k}{2}(3k+1)$
Show: $\underbrace{2 + 5 + 8 + \dots + (3k-1)} + (3(k+1)-1) = \frac{k+1}{2}(3(k+1)+1)$

$$\frac{k}{2}(3k+1) + (3(k+1)-1) \stackrel{?}{=} \frac{k+1}{2}(3(k+1)+1)$$
$$\frac{k}{2}(3k+1) + 3k + 2 \stackrel{?}{=} \frac{k+1}{2}(3k+4)$$
$$\frac{3k^2}{2} + \frac{k}{2} + 3k + 2 \stackrel{?}{=} \frac{k+1}{2}(3k+4)$$
$$\frac{3k^2}{2} + \frac{k}{2} + \frac{6k}{2} + 2 \stackrel{?}{=} \frac{3k(k+1)}{2} + \frac{4(k+1)}{2}$$
$$\frac{3k^2}{2} + \frac{7k}{2} + 2 \stackrel{?}{=} \frac{3k^2}{2} + \frac{3k}{2} + \frac{4k}{2} + \frac{4}{2}$$
$$\frac{3k^2}{2} + \frac{7k}{2} + 2 \stackrel{\checkmark}{=} \frac{3k^2}{2} + \frac{7k}{2} + 2$$

5. Give a recursive definition of the set of strings over $\{a, b\}$ that contain twice as many a's as b's

Let L be the strings over $\Sigma = \{a, b\}$
B.C. : $\lambda \in L$
Rec : If $u \in L$ and u can divided into xyzw for some arbitrary string where $xyzw \in \Sigma^*$ then $xayazbw \in L$, $xaybzaw \in L$ and $xbyazaw \in L$
closure. $u \in L$ if it can be obtained from λ and a finite # of apps. of Rec.
This configuration allows twice as many a's as b's in any arrangement.

6. Let L be the language over $\{a, b\}$ generated by the recursive definition
a) B.C. $\lambda \in L$
b) Rec. If $u \in L$ then $aaub \in L$
c) Closure A string w is in L only if it can be obtained from λ by a finite number of apps of the recursive step

i) $L_0 = \{\lambda\}$, $L_1 = \{aab\}$ $L_2 = \{aaaabb\}$ $L_3 = \{aaaaaabbb\}$
ii) $\{(aa)^n b^n \mid n \geq 0\}$ $n=0$ λ $n=1$ aab, $n=2$ $aaaabb$
iii) Prove by mathematical induction that for every string u in L, the number of a's in u is twice the number of b's in u. Let $n_a(u)$, $n_b(u)$ denote the number of a's and the number of b's, respectively.

1. $n_a(\lambda) = 0 = 2n_b(\lambda) = 0$
2. Assume for strings generated by the k^{th} recursive step that $n_a(u_k) = 2n_b(u_k)$
3. Prove for the k+1th step. One recursive rule
If $u \in L$, then $aaub \in L$, this rule adds 2 a's and 1 b to the string
so $n_a(u) + 2 = 2(n_b(u) + 1)$

7. For each of the following context free grammars, use set notation to define the language generated by the grammar.

a) $S \rightarrow aaSB \mid \lambda$ $\{(aa)^n b^m \mid n \geq 1, m \geq n\} \cup \{\lambda\}$ 1. $(aa)^n$ always precedes any number of b's
$B \rightarrow bB \mid b$

Sample iteration

aaSB
aa(aaSB)B
aaaaλBB
aaaaλbBbB
aaaab(bB)b(bB)
aaaabbbbbb

$(aa)^3 b^4$
aaaaaabbbb
$S \rightarrow aaSB \rightarrow aa(aaSB)B \rightarrow aaaa(aaSB)BB$
$\rightarrow aaaaaa\lambda BBB \rightarrow aaaaaa(bB)BB \rightarrow aaaaaabbbb$ ✓
aaaabbbbbb
$S \rightarrow aaSB \rightarrow aa(aaSB)B \rightarrow aaaa\lambda BB \rightarrow aaaa(bB)B$
$\rightarrow aaaab(bB)B \rightarrow aaaabb(bB)B \rightarrow aaaabbb(bB)B$
$\rightarrow aaaabbbbbb$ ✓

b) $S \rightarrow aSbb \mid A$ $\{a^n (bb)^n c^m \mid n \geq 0, m \geq 1\}$
$A \rightarrow cA \mid c$

Sample iterations:

a(aSbb)
aaAbb
aa(cA)bb
aac(cA)bb
aacccbb

$S \rightarrow A \rightarrow (cA) \rightarrow c(cA) \rightarrow cc(cA) \rightarrow cccc$
$S \rightarrow aSbb \rightarrow a(aSbb)bb \rightarrow aa(aSbb)bbbb \rightarrow aaaAbbbbbb$
$\rightarrow aaa(c)bbbbbb \rightarrow aaacbbbbbb$
Every string $u_i \in G$, must be derived using one application of A, thus terminating the recursive apps of S, and ending with at least 1 c.

c) $S \rightarrow abSdc \mid A$ $\quad \{(ab)^n S^n (dc)^n\} \rightarrow \{(ab)^n (cd)^m (ba)^m (dc)^n \mid n \geq 0, m \geq 0\}$ $L(G)$

$A \rightarrow cdAba \mid \lambda$ $\quad \lambda$

the string abdc can be generated 0 or more times with or w/o the string cdba; vise versa string cdba, λ allows 0 apps of either $n \mid m$.

Sample iterations

$S \rightarrow abSdc \rightarrow ab(abSdc)dc \rightarrow abab(abSdc)dcdc \rightarrow abababAdcdcdc$

$\rightarrow abababdcdcdcdc \mid ababab(cdAba)dcdcdc \rightarrow abababcdbadcdcdc$

$S \rightarrow A \rightarrow cdAba \rightarrow cd(cdAba)ba \rightarrow cdcdbaba$

$S \rightarrow abSdc \rightarrow abAdc \rightarrow ab(cdAba)dc \rightarrow abcdbadc$

$\Rightarrow$ $n=0, m=0$ λ, $n=1, m=0$ $abdc$, $n=2, m=0$ $ababdcdc$, $n=3, m=0$ $abababdcdcdc$

$n=1, m=1$ $abcdbadc$, $n=2, m=2$ $ababcdcdbabadcdc$

d) $S \rightarrow aSb \mid A$ $\quad L(G)\{a^n b^n c^m a^k (ab)^j b^k d^m \mid n \geq 0, m > 0, k \geq 0, j = 1\}$

$A \rightarrow cAd \mid cBd$

$B \rightarrow aBb \mid ab$

sample iterations

$S \rightarrow aSb \rightarrow a(aSb)b \rightarrow aa(aSb)bb \rightarrow aaa(B)bb \rightarrow aaa(aBb)bb \rightarrow aaaa(ab)bbb$

$S \rightarrow A \rightarrow cAd \rightarrow c(cAd)d \rightarrow cc(cBd)dd \rightarrow ccc(aBb)ddd \rightarrow ccca(ab)bddd$

$\rightarrow cccaabbddd$

$n=0, m=3, K=1, j=1 \rightarrow cccaabbddd$

e) $S \rightarrow aSB \mid aB$ $\quad \{a^n S B^n\} \rightarrow \{a^n b^m \mid 1 \leq n \leq m \leq 2n\}$

$B \rightarrow bb \mid b$ $\quad S \rightarrow abb \mid ab$ $\quad S \rightarrow aaSBB \rightarrow aaaSBBB \rightarrow aaaaSBBBB \rightarrow aaaaaBBBBB$

sample iterations

$S \rightarrow aSB \rightarrow a(aSB)B \rightarrow aa(aB)B \rightarrow aaabbbb$

$S \rightarrow aB \rightarrow a(bb) \rightarrow abb \mid ab.$

8. Give a regular grammar that generates the set of strings over $\{a,b\}$ in which the substring aa occurs exactly once.

$S \rightarrow abS \mid bS \mid A$ $\quad S \rightarrow abS \rightarrow ab(abS) \rightarrow ababbbabbab$

$A \rightarrow aa \mid B$ $\quad (ab \cup b)^* aa (ba \cup b)^*$

$B \rightarrow baB \mid bB \mid aa$

$S \rightarrow abS \rightarrow abB \rightarrow abbaB \rightarrow abbabbaa$

9. Let G be the grammar: Prove that $L(G) = \{a^n b^n \mid 0 \leq n < m\}$

$S \rightarrow aSb \mid B$ $\quad S \rightarrow a(aSb)b \rightarrow aa(B)bb$ $\quad n \geq 0 \quad m > n$

$B \rightarrow bB \mid b$ $\quad \rightarrow aa(bB)bb \rightarrow aabbbb$

$S \rightarrow B \rightarrow b$

Derivation $\quad$ rule $S \rightarrow aBb \rightarrow abb$

$S \Rightarrow a^n S^n b^m$ $\quad S \rightarrow aSb$ $\quad$ B.C. For $S \rightarrow B$

$\Rightarrow a^n B^m b^m$ $\quad S \rightarrow B$ $\quad n_b(u) > n_a(u)$

$\Rightarrow a^n b^m b^m$ $\quad B \rightarrow b$ $\quad$ Here my base case attempts to establish that for $n_a(u) = 0$ (which implies rule $S \rightarrow B$, generates 1 or more b's. This stands for $\{a^n b^n \mid 0 \leq n < m\}$

$\Rightarrow a^n b^m$ $\quad m > n$

$S \Rightarrow B^m$ $\quad S \rightarrow B$ $\quad$ Now assume n applications of $S \rightarrow aSb$ are added before falling to $S \rightarrow B$

$\Rightarrow (b)^m$ $\quad B \rightarrow b$

$m > n$ $\quad n_b(u) + 1 \overset{?}{>} n_a(u) + 1$

$\{a^n b^n \mid 0 \leq n < m\} \subseteq L(G)$ $\quad$ clearly $n_b(u) + 1 + (n_b(u))^m > n_a(u) + 1$ where $(n_b(u))^m$ represents m apps of $B \rightarrow$. Thus $L(G) \subseteq \{a^n b^n \mid 0 \leq n < m\}$

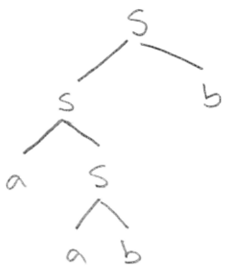
Let $n_a(u)$ and $n_b(u)$ denote the number of a's and b's in $L(G)$

10. Let G be the grammar
$S \to aS \mid Sb \mid ab$

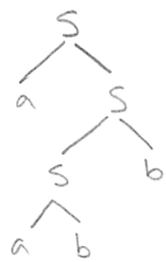
a) Give a regular expression for L(G)
$S \to a(Sb) \to a(aS)b \to aa(aS)b \to aaa(ab)b \to aaaabb$
$S \to Sb \to (Sb)b \to (aS)bb \to a(ab)bb \to aabbb$
✓ aabb $S \to aS \to a(Sb) \to a(ab)b \to aabb$
x abab $S \to aS \to a(Sb) \to aabb$
aaabbb $S \to aS \to a(aS) \to aa(Sb) \to aa(Sb)b \to aa(ab)bb \to aaabbb$
a^+b^+

b) Construct two leftmost derivations of the string aabb.
i $S \to Sb \to (aS)b \to a(ab)b \to aabb$.
ii $S \to aS \to a(Sb) \to aSb \to a(ab)b \to aabb$

i aabb



ii aabb



c) omitted