40. Prove $1 + 2^n < 3^n$ for all $n > 2$

BC $1 + 2^2 < 3^2$ $\quad$ $1 + 2^3 < 3^3$

$1 + 4 < 9$ ✓ $\quad$ $1 + 8 < 27$

$5 < 9$ $\quad$ $9 < 27$ ✓

IH Assume $1 + 2^k < 3^k$ $\quad k > 2$

PV $1 + 2^{(k+1)} \overset{?}{<} 3^{(k+1)}$

$\underbrace{1 + 2^k \cdot 2}_{} \overset{?}{<} 3^k \cdot 3$

$1 + 2^2 \cdot 2 < 3^2 \cdot 3$

$9 < 27$

$1 + 2^k < 3^k$

$(1 + 2^k) \cdot 2 - 1$

$2 + 2^{k+1}$

$1 + 2^k \cdot 2 < 3^k \cdot 3$

47. Prove that a strictly binary tree with n leaves contains $2n-1$ nodes

✓ $n = 1$ $\quad$ $2(1)-1$

$n = 2$ $\quad$ $2(2) - 1 = 3$ ✓

$n_1 + n_2$ leaves

$2(n_1 + n_2) - 1$ nodes

T1 $\quad$ n_1 leaves $\quad$ $2n_1 - 1$ nodes

T2 $\quad$ n_2 leaves $\quad$ $2n_2 - 1$ leaves

k + 1

T1 $\quad$ T2

n_1 leaves $\quad$ n_2 leaves

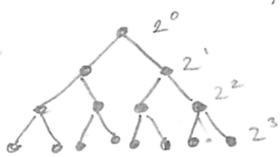
$2n_1 - 1$ nodes $\quad$ $2n_2 - 1$ nodes

$n_1 = 2$ $\quad$ $2(2)-1=3$

$n_2 = 2$ $\quad$ $2(2)-1=3$

PV. Any tree of depth D @ most n children, # leaves @ most n^d

<u>Lemma</u> PV a full tree has more leaves than a partial tree of the same depth



B.C. empty tree $d = 0$

no leaves $\quad n^0 = 1$

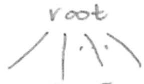
$0 < 1$ ✓

I.H. Assume the property holds for all trees $\leq k \geq 1$ depth

PV. n trees of depth $\leq k$ $\quad T_1, \ldots, T_n$

then make a new tree of depth $k+1$

root



T_1 $\quad T_2 \; T_3 \; T_n$

$n(n^k) = n^{k+1}$

$\leq n^k \ldots \leq n^k$

14. Let $X_1, \ldots, X_n$ be a partition of a set X. Define an equivalence relation $\equiv$ on X whose equivalence classes are precisely the sets $X_1, \ldots, X_n$

Let X be $\mathbb{N} \to \mathbb{N}$ where $[X_1, .., X_n] = [\text{odd } \mathbb{N}]$

$f(n)\, 2n+1 \quad 0+1 \quad 1$
$f(1)\, 2(1)+1 \quad 2+1 \quad 3$
$f(2)\, 2(2)+1 \quad 4+1 \quad 5$
$f(3)\, 2(3)+1 \quad 6+1 \quad 7$

$[1]_{\equiv odd} = [1, 3, 5, 7 \ldots n]_{\equiv odd}$

20. Prove that there are an uncountable number of total functions from $\mathbb{N}$ to $\{0,1\}$
$\mathbb{N} \to \{0,1\}$ $\quad$ $card(\{0,1\}) = 2$

let the set of functions $f: \mathbb{N} \to \{0,1\}$ be $F = \{f_1, f_2, f_3, \ldots, f_n\}$, Assume Countable

	1	2	3	4	5	6	7	8	9 ..
f_1	1	1	1	1	1	1	1	1	1
f_2	0	1	0	1	0	1	0	1	0
f_3	1	0	0	1	0	0	1	0	0
⋮									

using cantor's diagnolization argument, changing the n^{th} element of the n^{th} sequence, we can produce a new combination of 0's and 1's that isn't in the list. Thus $\mathbb{N} \to \{0,1\}$ is uncountable

30. Give a recursive definition of the relation GT on $\mathbb{N} \times \mathbb{N}$ using successor $s(n)$
$GT = \{[m,n] \mid m > n \text{ and } m, n \in \mathbb{N}\}$
i BC: $[1,0] \in GT$
ii Recursive Step: If $[m,n] \in GT$ then $[s(m), n] \in GT$ and $[s(m), s(n)] \in GT$
iii Closure: $[m,n] \in GT$ only if it can be obtained from $[1,0]$ by a finite # of applications

33. Give a recursive definition of the operation of multiplication of $\mathbb{N}$ using the operations s and additions

$\;\;\;\;\; n \;\; m$
mul 3 4 "Sum 3, 4 times"
mul 3 + 3 + 3 + 3
n

i B.C. "Multiplicative identity" If $n \mid m = 0$, then n mul $m = 0$ and m mul $n = 0$
ii $\quad n$ mul $1 = n$
Rec. $\quad n$ mul $m = (n$ mul $(m-1)) + n$

2 mul 2 $= (2$ mul $(1)) + 2$
$\quad = 2 + 2$
2 mul 3 $= (2$ mul $(2)) + 2$
$\quad = 2 + 2 + 2$