Exercise Sheet 3

Machine Learning 2, SS16

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Exercise 1

(a) For data-span constructed $w_x = X\alpha_x$ and $w_y = Y\alpha_y$ the primal problem is:

$$\max_{\alpha_x,\alpha_y} {\alpha_x}^\top X^\top C_{xy} Y \alpha_y$$
$$\text{s.t. } {\alpha_x}^\top X^\top C_{xx} X\alpha_x - 1 = 0 \qquad , \qquad {\alpha_y}^\top Y^\top C_{yy} Y\alpha_y - 1 = 0$$

Lagrangian (the factor 1/2 is introduced just for convenience):

$$\mathcal{L} = \alpha_x^\top X^\top C_{xy} Y\alpha_y - \frac{1}{2}\lambda_x(\alpha_x^\top X^\top C_{xx} X\alpha_x - 1) - \frac{1}{2}\lambda_y(\alpha_y^\top Y^\top C_{yy} Y\alpha_y - 1)$$

Partial derivatives:

$$\frac{\partial\mathcal{L}}{\partial\alpha_x^\top} = X^\top C_{xy} Y\alpha_y - \lambda_x X^\top C_{xx} X\alpha_x \stackrel{!}{=} 0 \qquad , \qquad \frac{\partial\mathcal{L}}{\partial\alpha_y^\top} = Y^\top C_{yx} X\alpha_x - \lambda_y Y^\top C_{yy} Y\alpha_y \stackrel{!}{=} 0$$

We now multiply with $\alpha_x^\top$, $\alpha_y^\top$

$$\alpha_x^\top X^\top C_{xy} Y\alpha_y = \lambda_x\alpha_x^\top X^\top C_{xx} X\alpha_x \qquad , \qquad \alpha_y^\top Y^\top C_{yx} X\alpha_x = \lambda_y\alpha_y^\top Y^\top C_{yy} Y\alpha_y$$
$$\Longrightarrow \alpha_x^\top X^\top C_{xy} Y\alpha_y = \lambda_x\alpha_x^\top X^\top C_{xx} X\alpha_x \qquad , \qquad \alpha_x^\top X^\top C_{xy} Y\alpha_y = \lambda_y\alpha_y^\top Y^\top C_{yy} Y\alpha_y$$

From the auto-cov constraints follows

$$\alpha_x^\top X^\top C_{xy} Y\alpha_y = \lambda_x \underbrace{\alpha_x^\top X^\top C_{xx} X\alpha_x}_{=1} = \lambda_y \underbrace{\alpha_y^\top Y^\top C_{yy} Y\alpha_y}_{=1} \quad \Longrightarrow \quad \lambda_x = \lambda_y$$

Now the derivatives can be rewritten as follows:

$$X^\top C_{xy} Y\alpha_y \stackrel{!}{=} \lambda_x X^\top C_{xx} X\alpha_x \qquad , \qquad Y^\top C_{yx} X\alpha_x \stackrel{!}{=} \lambda_x Y^\top C_{yy} Y\alpha_y$$

The same in blockmatrix form:

$$\begin{bmatrix} 0 & X^\top C_{xy} Y \\ Y^\top C_{yx} X & 0 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix} \stackrel{!}{=} \lambda_x \begin{bmatrix} X^\top C_{xx} X & 0 \\ 0 & Y^\top C_{yy} Y \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix}$$
$$\Longrightarrow \begin{bmatrix} X^\top C_{xx} X & 0 \\ 0 & Y^\top C_{yy} Y \end{bmatrix}^{-1} \begin{bmatrix} 0 & X^\top C_{xy} Y \\ Y^\top C_{yx} X & 0 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix} \stackrel{!}{=} \lambda_x I \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix}$$

(a) $X^\top C_{xx} X$ and $Y^\top C_{xx} Y$ are positive semi-definite. At least after regularizing one of these blocks, the diagonal block matrix becomes positive definite / invertible, which leads to a non-trivial solution.
(b) In comparison, the same conditions occur when viewing the Jacobian of $\mathcal{L}$. The Jacobian has to be negative definite. Since the Jacobian is symmetric, this is true iff the determinants of the principle minors alternate. We already know that the first principle minor $X^\top C_{xx} X$ or $Y^\top C_{yy} Y$ can only be positive. Then, the second principle minor should be negative, which means $-AA - BB - AB - BA < 0$. This is true as this is the quadratic form $-(A-B)^\top(A-B) < 0$. Ultimately, positive $X^\top C_{xx} X$ or $Y^\top C_{yy} Y$ is solely necessary for a solution likewise.
(c) By finding the solutions ${\alpha_x}^*$ and ${\alpha_y}^*$, the dual variable λ_x is identified as this is an eigenvalue problem. Each eigenvalue λ_x corresponds to an eigenvector $[\alpha_x, \alpha_y]^\top$. Therefore, the lagrangian does not depend on λ_x which means $\forall\lambda_x.\mathcal{L}({\alpha_x}^*, {\alpha_y}^*, \lambda_x) = \mathcal{L}({\alpha_x}^*, {\alpha_y}^*)$. For the dual problem we therefore find $\min_{\lambda_x}\max_{\alpha_x,\alpha_y}\mathcal{L}(\alpha_x, \alpha_y, \lambda_x) = \min_{\lambda_x}\mathcal{L}({\alpha_x}^*, {\alpha_y}^*, \lambda_x) = \mathcal{L}({\alpha_x}^*, {\alpha_y}^*)$