

# Exercise Sheet 10

Machine Learning 2, SS16

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## Exercise 1a

For arbitrary  $n$  let  $\vec{x}_1, \dots, \vec{x}_n \in \mathcal{X}$  be arbitrary samples and  $\vec{v} \in \mathbb{R}^n$  be an arbitrary vector.

$$\begin{aligned} \forall_{l=1}^L \cdot \sum_{i,j=1}^n v_i v_j k_l(x_i, x_j) \geq 0 &\implies \sum_{l=1}^L \sum_{i,j=1}^n v_i v_j k_l(x_i, x_j) \geq 0 \implies \sum_{i,j=1}^n v_i v_j \sum_{l=1}^L k_l(x_i, x_j) \geq 0 \\ &\stackrel{\vec{\beta} \geq 0}{\implies} \sum_{i,j=1}^n v_i v_j \sum_{l=1}^L \beta_l k_l(x_i, x_j) \geq 0 \stackrel{\text{Def. k}}{\implies} \sum_{i,j=1}^n v_i v_j k(x_i, x_j) \geq 0 \end{aligned}$$

## Exercise 1b

Let  $x, x' \in \mathcal{X}$  be two arbitrary samples.

$$\begin{aligned} \sum_{l=1}^L \beta_l k_l(x, x') &= \sum_{l=1}^L \beta_l \phi_l(x)^\top \phi_l(x') = \sum_{l=1}^L (\sqrt{\beta_l} \phi_l(x)^\top) (\sqrt{\beta_l} \phi_l(x')) \\ &= \underbrace{\begin{bmatrix} \sqrt{\beta_1} \phi_1(x)^\top & \dots & \sqrt{\beta_L} \phi_L(x)^\top \end{bmatrix}}_{\phi(x)^\top} \underbrace{\begin{bmatrix} \sqrt{\beta_1} \phi_1(x') \\ \dots \\ \sqrt{\beta_L} \phi_L(x') \end{bmatrix}}_{\phi(x')} \end{aligned}$$

$\phi(x)^\top$  and  $\phi(x')$  are block partitioned matrices, i.e. the  $\phi_l$ 's are simply concatenated together in one very long vector. So, the result is:

$$\phi(x) = \begin{bmatrix} \sqrt{\beta_1} \phi_1(x) \\ \dots \\ \sqrt{\beta_L} \phi_L(x) \end{bmatrix}$$

## Exercise 2a

We also encode the classes via the canonical base vectors  $\mathbf{e}_y$  for all  $y \in \{1, \dots, C\}$ . For arbitrary  $n$  let  $\vec{x}_1, \dots, \vec{x}_n \in \mathcal{X}$  be arbitrary samples and  $\vec{v} \in \mathbb{R}^n$  be an arbitrary vector.

$$\begin{aligned}
 \sum_{i,j=1}^n v_i v_j k(x_i, x_j) \geq 0 &\implies \left( \sum_{i,j=1}^n v_i v_j k(x_i, x_j) \right) \left( \sum_{c=1}^C (\mathbf{e}_y)_c (\mathbf{e}_{y'})_c \right) \geq 0 \\
 &\implies \left( \sum_{c=1}^C \sum_{i,j=1}^n v_i v_j k(x_i, x_j) \right) (\mathbf{e}_y)_c (\mathbf{e}_{y'})_c \geq 0 \\
 &\implies \left( \sum_{c=1}^C \sum_{i,j=1}^n v_i v_j k(x_i, x_j) \right) \mathbb{1}_{[y=y']} \geq 0
 \end{aligned}$$

## Exercise 2b

We also encode the classes via the canonical base vectors  $\mathbf{e}_y$  for all  $y \in \{1, \dots, C\}$ .

$$\begin{aligned}
 k(x, x') \mathbb{1}_{[y=y']} &= \phi(x)^\top \phi(x') \mathbf{e}_y^\top \mathbf{e}_{y'} = \sum_{i=1}^h \phi_i(x) \phi_i(x') \sum_{c=1}^C (\mathbf{e}_y)_c (\mathbf{e}_{y'})_c = \sum_{i=1}^h \sum_{c=1}^C \phi_i(x) \phi_i(x') (\mathbf{e}_y)_c (\mathbf{e}_{y'})_c \\
 &= \sum_{c=1}^C \sum_{i=1}^h \phi_i(x) (\mathbf{e}_y)_c \phi_i(x') (\mathbf{e}_{y'})_c = \underbrace{((\phi(x) \times \mathbf{e}_y) \vec{\mathbb{1}}_C)^\top}_{\phi_{\text{struct}}(x,y)} \underbrace{(\phi(x') \times \mathbf{e}_{y'}) \vec{\mathbb{1}}_C}_{\phi_{\text{struct}}(x',y')}
 \end{aligned}$$