Exercise Sheet 1

Machine Learning 2, SS16

April 28, 2016

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Exercise 2

(i) We have to prove $w^\top Cw \overset{?}{=} \epsilon \overset{(3)}{=} |x - \sum_j w_j \eta_j|^2$.

$$\begin{aligned}
&w^\top Cw \\
&= w^\top(\mathbb{1}_k x^\top - \eta)(\mathbb{1}_k x^\top - \eta)^\top w \\
&= w^\top(\mathbb{1}_k x^\top - \eta)(x\mathbb{1}_k^\top - \eta^\top)w \\
&= w^\top \mathbb{1}_k x^\top x \mathbb{1}_k^\top w - w^\top \mathbb{1}_k x^\top \eta^\top w - w^\top \eta x \mathbb{1}_k^\top w + w^\top \eta\eta^\top w \\
&= (w^\top \mathbb{1}_k x^\top)(x\mathbb{1}_k^\top w) - 2(w^\top \mathbb{1}_k x^\top)(\eta^\top w) + (w^\top \eta)(\eta^\top w) \\
&= |(w^\top \mathbb{1}_k x^\top) - (w^\top \eta)|^2 \\
&= |w^\top(\mathbb{1}_k x^\top - \eta)|^2 \\
&= |\sum_j w_j(x - \eta_j)|^2
\end{aligned}$$

Since $\sum_i w_i = 1$ we find $\sum_i w_i x = x$, which leads to the desired result. $\square$

We now perform Lagrange optimization:

$$\Lambda(w, \lambda) = w^\top Cw - \lambda(w^\top \mathbb{1}_k - 1)$$
$$\frac{\partial \Lambda}{\partial w} = 2Cw - \lambda \mathbb{1}_k \overset{!}{=} 0 \quad \Longrightarrow \quad 2Cw = \lambda \mathbb{1}_k \quad \Longrightarrow \quad w = \frac{\lambda}{2} C^{-1} \mathbb{1}_k$$
$$\frac{\partial \Lambda}{\partial \lambda} = w^\top \mathbb{1}_k - 1 \overset{!}{=} 0 \quad \Longrightarrow \quad w^\top \mathbb{1}_k = 1$$

We now replace w in the constraint $w^\top \mathbb{1}_k = 1$. Since $C = C^\top$ we find

$$\frac{\lambda}{2} = \frac{1}{\mathbb{1}_k^\top C^{-1} \mathbb{1}_k}$$

(ii) Replacing $\lambda/2$ in the deduced definition of w leads to the desired result. $\square$

(iii) Multiplying by C from the left in the deduced definition of w leads to $2Cw = \lambda \mathbb{1}_k$. W.l.o.g. we focus on the case $\lambda = 2$, which leads to the desired result. $\square$