# Exercise Sheet 1

## Machine Learning 2, SS16

**April 30, 2016**

**Sascha Huk, 321249; Viktor Jeney, 348969; Mario Tambos, 380599; Jan Tinapp, 380549**

### Exercise 1

**(i)**

Given the following problem:

$$\min_w E(w) = \sum_i \left| \vec{x}_i - \sum_j w_{ij}\vec{x}_j \right|^2$$
$$s.t. \sum_j w_{ij} = 1$$

we are trying to prove that the multiplication of each vector $\vec{x}_i$ by a constant scalar $\alpha \in \mathbb{R}^+ \setminus \{0\}$ does not alter the problem's solution.

$$\min_w \sum_i \left| \alpha\vec{x}_i - \sum_j w_{ij}\alpha\vec{x}_j \right|^2$$
$$s.t. \sum_j w_{ij} = 1$$
$$\equiv$$
$$\min_w \sum_i \left| \alpha\vec{x}_i - \alpha \sum_j w_{ij}\vec{x}_j \right|^2$$
$$s.t. \sum_j w_{ij} = 1$$
$$\equiv$$
$$\min_w \sum_i \alpha^2 \left| \vec{x}_i - \sum_j w_{ij}\vec{x}_j \right|^2$$
$$s.t. \sum_j w_{ij} = 1$$
$$\equiv$$
$$\min_w \alpha^2 \sum_i \left| \vec{x}_i - \sum_j w_{ij}\vec{x}_j \right|^2$$
$$s.t. \sum_j w_{ij} = 1$$

Since the multiplication by $\alpha^2 \in \mathbb{R}^+ \setminus \{0\}$ doesn't change the minima with respect to $w$, the minimizaiton problem remains the same

**(ii)**

Given the following problem:

$$\min_w E(w) = \sum_i \left| \vec{x}_i - \sum_j w_{ij} \vec{x}_j \right|^2$$
$$s.t. \sum_j w_{ij} = 1$$

we are trying to prove that the addition of constant vector $\vec{v} \in \mathbb{R}^D$ to each vector $\vec{x}_i$ does not alter the problem's solution.

$$\min_w \sum_i \left| (x_i + \vec{v}) - \sum_j w_{ij}(x_j + \vec{v}) \right|^2$$
$$s.t. \sum_j w_{ij} = 1$$
$$\equiv$$
$$\min_w \sum_i \left| x_i + \vec{v} - \left( \sum_j w_{ij} x_j + w_{ij} \vec{v} \right) \right|^2$$
$$s.t. \sum_j w_{ij} = 1$$
$$\equiv$$
$$\min_w \sum_i \left| x_i + \vec{v} - \left( \sum_j w_{ij} x_j \right) - \left( \sum_j w_{ij} \vec{v} \right) \right|^2$$
$$s.t. \sum_j w_{ij} = 1$$
$$\overset{\text{minimization constraint}}{\equiv}$$
$$\min_w \sum_i \left| x_i + \vec{v} - \left( \sum_j w_{ij} x_j \right) - \vec{v} \right|^2$$
$$s.t. \sum_j w_{ij} = 1$$
$$\equiv$$
$$\min_w \sum_i \left| x_i - \sum_j w_{ij} x_j \right|^2$$
$$s.t. \sum_j w_{ij} = 1$$

**(iii)**

Given the following problem:

$$\min_w E(w) = \sum_i \left| \vec{x}_i - \sum_j w_{ij} \vec{x}_j \right|^2$$
$$s.t. \sum_j w_{ij} = 1$$

we are trying to prove that the multiplication of each vector $\vec{x}_i$ by a constant, orthogonal $D \times D$ matrix $U$ does not alter the problem's solution.

$$\min_w \sum_i \left| U\vec{x}_i - \sum_j w_{ij} U\vec{x}_j \right|^2$$
$$s.t. \sum_j w_{ij} = 1$$
$$\equiv$$
$$\min_w \sum_i \left| U\vec{x}_i - \sum_j U w_{ij} \vec{x}_j \right|^2$$
$$s.t. \sum_j w_{ij} = 1$$
$$\equiv$$
$$\min_w \sum_i \left| U\vec{x}_i - U \sum_j w_{ij} \vec{x}_j \right|^2$$
$$s.t. \sum_j w_{ij} = 1$$
$$\equiv$$
$$\min_w \sum_i \left| U \left( \vec{x}_i - \sum_j w_{ij} \vec{x}_j \right) \right|^2$$
$$s.t. \sum_j w_{ij} = 1$$

Since for all orthogonal matrices $U \in \mathbb{R}^{D \times D}$ and vectors $\vec{x} \in \mathbb{R}^D$ we have that $\left|U\vec{x}\right| = \left|\vec{x}\right|$

$$\equiv$$
$$\min_w \sum_i \left| \vec{x}_i - \sum_j w_{ij} \vec{x}_j \right|^2$$
$$s.t. \sum_j w_{ij} = 1$$

## Exercise 2

**(i)**

We have to prove $w^\top C w \stackrel{?}{=} \epsilon \stackrel{(3)}{=} |x - \sum_j w_j \eta_j|^2$.

$$
\begin{gathered}
w^\top C w \\
= w^\top (\mathbb{1}_k x^\top - \eta)(\mathbb{1}_k x^\top - \eta)^\top w \\
= w^\top (\mathbb{1}_k x^\top - \eta)(x \mathbb{1}_k^\top - \eta^\top) w \\
= w^\top \mathbb{1}_k x^\top x \mathbb{1}_k^\top w - w^\top \mathbb{1}_k x^\top \eta^\top w - w^\top \eta x \mathbb{1}_k^\top w + w^\top \eta \eta^\top w \\
= (w^\top \mathbb{1}_k x^\top)(x \mathbb{1}_k^\top w) - 2(w^\top \mathbb{1}_k x^\top)(\eta^\top w) + (w^\top \eta)(\eta^\top w) \\
= |(w^\top \mathbb{1}_k x^\top) - (w^\top \eta)|^2 \\
= |w^\top (\mathbb{1}_k x^\top - \eta)|^2 \\
= |\sum_j w_j (x - \eta_j)|^2
\end{gathered}
$$

Since $\sum_i w_i = 1$ we find $\sum_i w_i x = x$, which leads to the desired result. $\square$

We now perform Lagrange optimization:

$$
\begin{gathered}
\Lambda(w, \lambda) = w^\top C w - \lambda(w^\top \mathbb{1}_k - 1) \\
\frac{\partial \Lambda}{\partial w} = 2Cw - \lambda \mathbb{1}_k \stackrel{!}{=} 0 \quad \Longrightarrow \quad 2Cw = \lambda \mathbb{1}_k \quad \Longrightarrow \quad w = \frac{\lambda}{2} C^{-1} \mathbb{1}_k \\
\frac{\partial \Lambda}{\partial \lambda} = w^\top \mathbb{1}_k - 1 \stackrel{!}{=} 0 \quad \Longrightarrow \quad w^\top \mathbb{1}_k = 1
\end{gathered}
$$

We now replace w in the constraint $w^\top \mathbb{1}_k = 1$. Since $C = C^\top$ we find

$$
\frac{\lambda}{2} = \frac{1}{\mathbb{1}_k^\top C^{-1} \mathbb{1}_k}
$$

**(ii)**

Replacing $\lambda/2$ in the deduced definition of $w$ leads to the desired result. The candidate w is indeed a minimum since $\frac{\partial^2 \Lambda}{\partial w^2} = 2C$ (invertible covariance matrices have positive definite quadratic forms). $\square$