MATH 239, Winter 2020 Assignment 4 Due: 11:59PM, 7 February 2020.

Problem 1. Let S denote the collection of all binary strings that have length at least one, begin with 0 (when reading left to right), and are such that each occurrence of 0 in the string is followed by a block of either one or two consecutive 1's. (For example, 011011101 is not allowed, because the second zero is followed by three consecutive 1's.)

(a) Find a decomposition of S using only finite sets, disjoint unions, unambiguous concatenations, and unambiguous $*$ operations. (You do not have to use every available operations.)

(b) We declare the weight of a binary string to be its length and we let $\Phi_S(x)$ denote the generating function for S. Find a simplified rational expression for $\Phi_S(x)$.

Problem 2. Let $S = \{1, 10\}^*$ and let S_n denote the set of strings in S of length n.

(a) Compute $|S_n|$ for small n and guess a formula for $|S_n|$.

(b) Give a combinatorial proof of the identity $|S_n| = |S_{n-1}| + |S_{n-2}|$ for $n \geq 2$.

Problem 3. Let S be the set of all binary strings that have an even number of 1's.

(a) For $k \geq 0$, let S_k be the set of all binary strings that have exactly k occurrences of 1. Find a rational expression for $\Phi_{S_k}(x)$, where the weight of a string is its length.

(b) Give a formula for the number of strings in S_k of length n when $n \geq k$ and $k \geq 1$.

(c) Let S be the set of all binary strings with an even number of 1's. Find a rational expression for $\Phi_S(x)$.