MATH 239, Winter 2020 Assignment 1 Due: 11:59PM, 17 January 2020.

Problem 1.

(a) Let n be a positive integer. How many binary strings are there of length $2n+1$ that have more 1's than 0's? Prove your answer using a bijection. (You should clearly specify the bijection, as well as its domain, codomain and inverse, and show that the function indeed maps elements of the domain to elements of the codomain). You may represent a binary string of length k as $b_1 b_2 \cdots b_k$ where each $b_i \in \{0, 1\}$.

(b) Let n be a positive integer. How many binary strings of length $2n$ are there that have more 1's than 0's? Prove your answer.

Problem 2.

(a) Let $n \geq 0$ be an integer and let S_0 be the collection of ordered pairs of disjoint subsets of $\{1, \dots, n\}$; that is,

$$S_0 = \{(A, B) : A \subseteq \{1, \dots, n\}, B \subseteq \{1, \dots, n\}, A \cap B = \varnothing\}.$$

Show using a bijection that $|S_0| = 3^n$.

(b) Let S be the collection of all ordered pairs of subsets of $\{1, \dots, n\}$, not necessarily disjoint. That is,

$$S = \{(A, B) : A \subseteq \{1, \dots, n\}, B \subseteq \{1, \dots, n\}\}.$$

Show by computing the size of S in two different ways that

$$4^n = \sum_{k=0}^{n} 3^k \binom{n}{k}.$$

Problem 3. Let $n \geq 0$ and $k \geq 0$ be integers, and let S be the set of all k-tuples $(a_1, \dots, a_k)$ of nonnegative integers whose sum is n. (For example, if $k = 3$ and $n = 2$, then $S = \{(0,0,2), (0,2,0), (2,0,0), (0,1,1), (1,0,1), (1,1,0)\}$.) Prove using a bijection that $|S| = \binom{n+k-1}{k-1}$. (Hint: put S into bijection with the set of binary strings of length $n+k-1$ having exactly n zeroes and $k-1$ ones, thinking of the zeroes as small dots, and the 1's as vertical lines).