MATH 239, Winter 2020 Assignment 2 Due: 11:59PM, 24 January 2020.

Problem 1.

(a) Let

$$F(x) = \sum_{n=0}^{\infty} n^2 x^n.$$

Use the negative binomial theorem to find integers c_1, c_2, c_3 such that

$$F(x) = c_1(1-x)^{-1} + c_2(1-x)^{-2} + c_3(1-x)^{-3}.$$

(b) Let $n \geq 0$. Compute the coefficient of x^n in $F(x)(1 + x + x^2 + \cdots) = F(x)(1-x)^{-1}$ and use part (a) to give a formula for

$$1 + 2^2 + 3^2 + \cdots + n^2.$$

Problem 2.

(a) Let a and b be positive integers with $b \geq a$. Prove algebraically that

$$\binom{a+b}{a} = \sum_{i=0}^{a} \binom{a}{i}\binom{b}{a-i}.$$

(Hint: Look at $(1+x)^{a+b}$ in two different ways.)

(b) Show that for each positive integer n,

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.$$

(c) Let p be a prime number. Show that $\binom{2p}{p} \equiv 2 \pmod{p}$.

Problem 3.

(a) Let $F(x) = x + x^2 + x^3 + \cdots$ and let $G(x) = x - x^2 + x^3 - x^4 \cdots$. Show that for $k \geq 1$ and $n \geq k$,

$$[x^n]F(x)^k = \binom{n-1}{k-1}$$

and if $n < k$ then $[x^n]F(x)^k = 0$.

(b) Show that $G(F(x)) = x$.