Assignment # 02 23 Sept 2025 David Haberkorn

Problem 1

From the sets

- $A = \{n \in \mathbb{Z} : |n| < 2\}$
- $B = \{n \in \mathbb{Z} : n^3 = n\}$
- $C = \{n \in \mathbb{Z} : n^2 \leq n\}$
- $D = \{n \in \mathbb{Z} : n^2 \leq 1\}$ and
- $E = \{-1, 0, 1\}$

the sets that are equivalent are $A = B = D = E$.
A, B, D and E all describe the integer set $\{-1, 0, 1\}$, while C describes the integer set $\{0, 1\}$.

Problem 2

(a)

Statement: If $\{1\} \in \mathcal{P}(A)$, then $1 \in A$ but $\{1\} \notin A$.

Validity: The statement is false. The set A could contain both the elements 1 and $\{1\}$.

(b)

Statement: If A, B and C are sets such that $A \subset \mathcal{P}(B) \subset C$ and $|A| = 2$, then $|C|$ can be 5 but $|C|$ cannot be 4.

Validity: The statement is false. $|C|$ can be 5 when $A = \{\{1\}, \{2\}\}$, $B = \{1, 2\}$, $\mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, and $C = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, 10\}$. $|C|$ can be 4 when $A = \{\{1\}, \{2\}\}$, $B = \{1, 2\}$, $\mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, and $C = B$, which is the same as $B \subset C$.

(c)

Statement: If a set B has one more element than set A, then $\mathcal{P}(B)$ has at least two more elements than $\mathcal{P}(A)$.

Validity: The statement is false. Using the equation $|\mathcal{P}(S)| = 2^n$ where S is a set and $n = |S|$, we find that when $|A| = 1$ and $|B| = 0$, then $|\mathcal{P}(A)| = 2^{(1)} = 2$ and $|\mathcal{P}(B)| = 2^{(0)} = 1$, which disproves the statement.

Problem 3

Solution: To begin, assume each set is empty.

To satisfy requirement (a), we must add 1 to A and B. We cannot add 1 to C. **Requirement (a) has been satisfied.**

To satisfy requirement (b), we must add 2 to A and C. We cannot add 2 to B. **Requirement (b) has been satisfied.**

To satisfy requirement (c), 3 must be in A. In addition, 3 must be in either B or C, not both.

Currently, by necessity, $A = \{1, 2, 3\}$, $B = \{1\}$, and $C = \{2\}$. To satisfy $|A| = |B| = |C|$, we need to add elements to B and C.

The cardinality of each set must be equal to at least 3, since A must contain 1, 2 and 3.

To satisfy (d), we add 4 to sets B and C. **Requirement (d) has been satisfied.**

3 must be added to either B or C to satify (c), resulting in two cases.

Case 1: 3 is added to B: we now have 3 elements in A and B, and only 2 in C. The only element able to be added to C is 5, as adding any other element would contradict one of the previous requirements. However, when adding 5 to C, we find that requirement (f) is now contradicted. Therefore we must try the other case.

Case 2: 3 is added to C: We now have 3 elements in A and C, with only 2 in B. The only element able to be added to B is 5 (satisfying requirement (e)), as adding a different element would contradict one of the previous requirements. Checking for requirement (f), we now find that the sum of the elements of $B = 10$ and the sum of the elements of $C = 9$, fulfilling requirement (f).

Requirements (c), (e), and (f) have been satisfied, and the cardinality of each set is equal to 3.

Therefore, $A = \{1, 2, 3\}, B = \{1, 4, 5\}$, and $C = \{2, 3, 4\}$.

Problem 4

If $A = \{a, b, c, d, e, f, g\}$, then $S_1 = \{\{a, c, e, g\}, \{b, f\}, \{d\}\}$ and $S_3 = \{A\}$ are partitions of A.

- S_2 does not include the element g.
- S_4 includes the empty set, and a partition must be made of non-empty sets.
- S_5 includes the element b twice. In order to be a partition, the intersection of any two elements of the partition set must be equal to the empty set.

Problem 5

Let $A = \{1, 2, \cdots, 12\}$. A partition that satisfies the requirements listed below could be

$$S = \{\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}, \{9,10,11,12\}\}.$$

This satisfies the following requirements:

- $|S| = 5$
- $\left| \bigcup_{X \in T} X \right| = 10$ where $T \subset S$ and $T = \{\{3,4\}, \{5,6\}, \{7,8\}, \{9,10,11,12\}\}$.
- None of the subsets within S have a cardinality of 3.

Problem 6

For a real number r, let $A_r = \{r, r+1\}$. Let $S = \{x \in \mathbb{R} : x^2 + 2x - 1 = 0\}$.

(a)

When $s, t \in S$ and $s < t$, then

$$\begin{aligned} B =& A_s \times A_t \\ =& \{-1-\sqrt{2}, -\sqrt{2}\} \times \{1-\sqrt{2}, 2-\sqrt{2}\} \\ =& \{(-1-\sqrt{2}, 1-\sqrt{2}), (-1-\sqrt{2}, 2-\sqrt{2}), (-\sqrt{2}, 1-\sqrt{2}), (-\sqrt{2}, 2-\sqrt{2})\} \end{aligned}$$

(b)

Let $C = \{(a,b) \in B : ab\}$. The sum of the elements of C is:

$$\begin{aligned} \sum_{x \in C} x &= \sum_{(a,b)\in B} a \cdot b \\ &= (-1-\sqrt{2}) \cdot (1-\sqrt{2}) + (-1-\sqrt{2}) \cdot (2-\sqrt{2}) + \\ &\quad (-\sqrt{2}) \cdot (1-\sqrt{2}) + (-\sqrt{2}) \cdot (2-\sqrt{2}) \\ &= (1) + (-\sqrt{2}) + (-\sqrt{2}+2) + (-2\sqrt{2}+2) \\ &= 5 - 4\sqrt{2} \end{aligned}$$

Problem 7

Let $A = \{x \in \mathbb{R} : |x-1| \leq 2\}$, $B = \{x \in \mathbb{R} : |x| \geq 1\}$, and $C = \{x \in \mathbb{R} : |x+2| \leq 3\}$.

(a)

$$
\begin{aligned}
A &= \{x \in \mathbb{R} : |x-1| \leq 2\} \\
|x-1| &\leq 2 \\
-2 &\leq x-1 \leq 2 \\
-1 &\leq x \leq 3 \\
\boxed{A = [-1,3]} \\
B &= \{x \in \mathbb{R} : |x| \geq 1\} \\
x &< -1 \text{ or } x > 1 \\
\boxed{B = (-\infty,-1] \cup [1,\infty)} \\
C &= \{x \in \mathbb{R} : |x+2| \leq 3\} \\
|x+2| &\leq 3 \\
-3 &\leq x+2 \leq 3 \\
-5 &\leq x \leq 1 \\
\boxed{C = [-5,1]}
\end{aligned}
$$

(b)

- $A \cup B = [-1,3] \cup ((-\infty,-1] \cup [1,\infty)) = \mathbb{R}$
- $A \cap B = [-1,3] \cap ((-\infty,-1] \cup [1,\infty)) = -1 \cup [1,3]$
- $B \cap C = ((-\infty,-1] \cup [1,\infty)) \cap [-5,1] = [-5,-1] \cup 1$
- $B - C = ((-\infty,-1] \cup [1,\infty)) - [-5,1] = (-\infty,-5] \cup [1,\infty)$