Problem 1

Result: The function $f : \mathbb{Z} \to \mathbb{Z}$, defined by $f(n) = 5n + 2$ is injective, but not surjective.
Proof: We first prove that the function is injective. Let the function f be defined as above, and suppose that $f(x) = f(y)$ for some arbitrary integers x and y. From this assumption, we have

$$\begin{aligned} f(x) &= f(y) \\ 5x + 2 &= 5y + 2 \\ 5x &= 5y \\ x &= y \end{aligned}$$

By the definition of injective, we have shown that the function f is injective.
To show that f is not surjective, we demonstrate a counterexample. We seek two integers q, r such that $f(q) = r$ has no solution.
Consider $r = 1$. In this case, we have

$$\begin{aligned} f(q) &= 1 \\ 5n + 2 &= 1 \\ 5n &= -1 \\ n &= -\frac{1}{5} \end{aligned}$$

Because $-\frac{1}{5}$ is not an integer, we have shown that f is not surjective. $\square$

Problem 2

Result: The function $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{5\}$ defined by $f(x) = \frac{5x+1}{x-2}$ is bijective.
Proof: In order to show bijectivity, we must show that the function defined above is both injective and surjective. We begin by showing injectivity.
Let the function f be defined as above, and suppose that $f(x) = f(y)$ for some arbitrary real numbers x and y, excluding $x, y = 2$. From this assumption, we have

$$\begin{aligned}
f(x) &= f(y) \\
\frac{5x+1}{x-2} &= \frac{5y+1}{y-2} \\
(5x+1)(y-2) &= (x-2)(5y+1) \\
5xy - 10x + y - 2 &= 5xy + x - 10y - 2 \\
-10x + y &= -10y + x \\
-11x &= -11y \\
x &= y
\end{aligned}$$

By the definition of injective, we have shown that the function f is injective.
We now work to show surjectivity.
Let the function f be defined as above, and suppose two real numbers q, r such that $f(q) = r$. From this supposition, we have

$$\begin{aligned}
f(q) &= r \\
\frac{5q+1}{q-2} &= r \\
5q + 1 &= r(q-2) \\
5q + 1 &= rq - 2r \\
5q - rq &= -2r - 1 \\
(5-r)q &= -2r - 1 \\
q &= \frac{-2r-1}{5-r} \qquad (1)
\end{aligned}$$

Recall from the definition of the function that the codomain is the set of real numbers, excluding 5. Since the solution above holds for all real numbers excluding 5, the function is surjective.
However, we need to check if the integer q can be an element of the domain. We do this using contradiction. Suppose that $q = 2$, which is strictly not in the domain. From this, we have

$$\begin{aligned}
2 &= \frac{-2r-1}{5-r} \\
10 - 2r &= -2r - 1 \\
10 &= -1
\end{aligned}$$

which is clearly a contradiction. Thus, the equation found in (1) holds true.
Because the function is both injective and surjective, it is, by definition, bijective. $\square$

Problem 3

Result: Let A be a nonempty set and let $f : A \to A$ be a function. If $f \circ f = i_a$, then f is bijective.
Proof: Let A be a nonempty set and let $f : A \to A$ be a function, and suppose that $f \circ f = i_a$. In order to show that f is bijective, we must show that f is both one-to-one and onto. We start with one-to-one.
Suppose that there are some elements $x, y \in A$ such that $f(x) = f(y)$. We work to show that $x = y$.

$$\begin{aligned} f(x) &= f(y) \\ f(f(x)) &= f(f(y)) \end{aligned}$$

By the definition of the identity function, we have

$$\begin{aligned} f(f(x)) &= f(f(y)) \\ x &= y \end{aligned}$$

Thus, f is one-to-one.
We now work to show that f is also onto. Let r be an arbitrary element in A, and let another arbitrary element $q \in A$ be such that $f(r) = q$. We know that q exists because f is defined by $f : A \to A$. Through algebra, we have

$$\begin{aligned} f(r) &= q \\ f(f(r)) &= f(q) \\ r &= f(q) \end{aligned}$$

Since we have shown that for an arbitrary element r in the codomain A, there exists an element q in the domain A (specifically $q = f(r)$) such that $f(q) = r$, the function f is onto.
Since f is both one-to-one and onto, it is, by definition, bijective. $\square$

Problem 4

Result: Let the composition $g \circ f : (0,1) \to \mathbb{R}$ of two functions f and g be given by $(g \circ f)(x) = \frac{4x-1}{2\sqrt{x-x^2}}$ where $f : (0,1) \to (-1,1)$ is defined by $f(x) = 2x - 1$ for $x \in (0,1)$. The function g is given by $g(y) = \frac{2y-1}{2\sqrt{1-y^2}}$.
Proof: Let the function f and the composition $g \circ f$ be defined as above. We work to find the function g.
From the definitions above, we have

$$\begin{aligned} f(x) &= 2x - 1 \\ y &= 2x - 1 \\ y + 1 &= 2x \\ \frac{y+1}{2} &= x \end{aligned} \tag{1}$$

Expression (1) above is the inverse of $f(x)$, or in other words, $f^{-1}(x)$.
Plugging the inverse function into the composition $(g \circ f)(x)$, we will be able to 'undo' the function f to be left with g.

$$\begin{aligned} (g \circ f)(x) &= \frac{4x-1}{2\sqrt{x-x^2}} \\ g(y) &= \frac{4(\frac{y+1}{2}) - 1}{2\sqrt{(\frac{y+1}{2}) - (\frac{y+1}{2})^2}} \\ g(y) &= \frac{2y+1}{2\sqrt{1-y^2}} \end{aligned}$$

Thus, we have our function $g(y) = \frac{2y+1}{2\sqrt{1-y^2}}$. $\square$

Problem 5

Result: Let $A = \mathbb{R} - \{1\}$ and define $f : A \to A$ by $f(x) = \frac{x}{x-1}$ for all $x \in A$. When this is the case, then f is bijective, and $f^{-1} = f = \frac{x}{x-1}$.

Proof: Let the function f be defined as above. In order to show bijectivity, we must show injectivity and surjectivity. We begin with injectivity.

Suppose there are some elements $x, y \in A$ such that $f(x) = f(y)$. We work to show that $x = y$.

$$
\begin{aligned}
f(x) &= f(y) \\
\frac{x}{x-1} &= \frac{y}{y-1} \\
x(y-1) &= y(x-1) \\
xy - x &= xy - y \\
-x &= -y \\
x &= y
\end{aligned}
$$

Thus, f is injective.

To show surjectivity, let q, r be arbitrary elements in A such that $f(q) = r$. We work to show that there exists such a q.

$$
\begin{aligned}
f(q) &= r \\
\frac{q}{q-1} &= r \\
q &= (q-1)r \\
q &= qr - r \\
0 &= qr - q - r \\
r &= qr - q \\
r &= q(r-1) \\
\frac{r}{r-1} &= q \\
q &= \frac{r}{r-1}
\end{aligned}
$$

Recalling that the domain A is defined by all real numbers, excluding $\{1\}$, we see that there does in fact exist such a q.

Since f is both injective and surjective, we have that it is, in fact, bijective.

In order to find the inverse of f, we use algebra. Let $f(x) = y$ and let $x \in A$.

$$
\begin{aligned}
f(x) &= \frac{x}{x-1} \\
y &= \frac{x}{x-1}
\end{aligned}
$$

Without loss of generality (from the proof of surjectivity), we have

$$\begin{aligned} x &= \frac{y}{y-1} \\ f^{-1}(x) &= \frac{y}{y-1} \end{aligned}$$

$\square$

Problem 6

Result: The sequence $\{\frac{n+2}{2n+3}\}$ is convergent to $\frac{1}{2}$.

Proof: We want to show that $\lim_{n\to\infty}(\frac{n+2}{2n+3}) = \frac{1}{2}$.
Let $\epsilon > 0$. By algebra, we have

$$\begin{aligned}
\left|\frac{n+2}{2n+3} - \frac{1}{2}\right| &< \epsilon \\
\left|\frac{2n+4}{4n+6} - \frac{2n+3}{4n+6}\right| &< \epsilon \\
\left|\frac{1}{4n+6}\right| &< \epsilon \\
\frac{1}{4n+6} &< \epsilon \\
4n+6 &> \frac{1}{\epsilon} \\
n &> \frac{1}{4\epsilon} - \frac{3}{2}
\end{aligned}$$

We now choose $N = \lceil \frac{1}{4\epsilon} - \frac{3}{2} \rceil \geq (\frac{1}{4\epsilon} - \frac{3}{2})$. For $n > N$, we have

$$\left|\frac{n+2}{2n+3} - \frac{1}{2}\right| = \frac{1}{4n+6} < \frac{1}{4N+6} \leq \frac{1}{4(\frac{1}{4\epsilon} - \frac{3}{2}) + 6} = \frac{1}{\frac{4}{4\epsilon} - \frac{12}{2} + 6} = \frac{1}{(\frac{1}{\epsilon})} = \epsilon$$

Therefore, we have that the sequence $\{\frac{n+2}{2n+3}\}$ is convergent to $\frac{1}{2}$. $\square$

Problem 7

Result: $\lim_{x\to 3}(\frac{3x+1}{4x+3}) = \frac{2}{3}$.

Proof: Let $\epsilon > 0$. Choose $\delta = \epsilon$. Then, for $0 < |x-3| < \delta$, we have

$$\left|\frac{3x+1}{4x+3} - \frac{2}{3}\right| = \left|\frac{x-3}{12x+9}\right| < |x-3| < \delta = \epsilon$$

Thus, $\lim_{x\to 3}(\frac{3x+1}{4x+3}) = \frac{2}{3}$. $\square$