Problem 1

Result: Suppose that $\lim_{x\to a} f(x) = L$, where $L > 0$. Then $\lim_{x\to a} \sqrt{f(x)} = \sqrt{L}$.
Proof: Let $\epsilon > 0$. Since $\lim_{x\to a} f(x) = L$, there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x)-L| < \epsilon\sqrt{L}$. Now suppose $0 < |x - a| < \delta$. We have

$$|\sqrt{f(x)} - \sqrt{L}| = \frac{|f(x) - L|}{|\sqrt{f(x)} + \sqrt{L}|} \leq \frac{|f(x) - L|}{\sqrt{L}} < \frac{\delta}{\sqrt{L}} = \frac{\sqrt{L}\epsilon}{\sqrt{L}} = \epsilon$$

Thus, $\lim_{x\to a} \sqrt{f(x)} = \sqrt{L}$. $\square$

Problem 2

Result: The function $Q : \mathbb{R} - \{-1\}$ is defined as

$$Q(x) = \begin{cases} \dfrac{x^2 - 3x + 2}{x^2 - 1}, & x \in \mathbb{R} \setminus \{-1, 1\}, \\ -\dfrac{1}{2}, & x = 1. \end{cases}$$

$Q(x)$ is continuous at $x = 1$.

Proof: In order to show continuity at $x = 1$, we must show that

1. $\lim_{x\to 1} Q(x)$ exists,
2. $\lim_{x\to 1} Q(x) = Q(1)$,
3. $Q(1)$ is defined.

From the definition of $Q(x)$ above, we have that $Q(1) = -\frac{1}{2}$, which satisfies the third criteria above.
To show the first and second criteria, we employ an $\epsilon - \delta$ proof.
Let $\epsilon > 0$, and choose $\delta = \min 1, \frac{2}{3}\epsilon$. Now suppose $|x - 1| < 1$.

$$\begin{aligned} |x - 1| &< 1 \\ -1 < x - 1 &< 1 \\ 1 < x + 1 &< 3 \end{aligned}$$

This implies that the quantity $|x + 1| = x + 1$. Now suppose $0 < |x - 1| < \delta$. We have

$$\left|\frac{x^2 - 3x + 2}{x^2 - 1} - \frac{-1}{2}\right| = \left|\frac{x - 2}{x + 1} + \frac{1}{2}\right| = \left|\frac{3(x - 1)}{2(x + 1)}\right| = \frac{3\,|x - 1|}{2(x + 1)} \leq \frac{3\,|x - 1|}{2} < \frac{3\delta}{2} \leq \frac{3}{2} \cdot \frac{2}{3}\epsilon = \epsilon$$

Therefore, the limit exists.
Evaluating $Q(1)$, we have

$$\frac{x^2 - 3x + 2}{x^2 - 1} = \frac{x - 2}{x + 1} \to \frac{(1) - 2}{(1) + 1} = -\frac{1}{2}$$

Since the limit both exists and is equal to $Q(1)$, we have satisfied all three necessary criteria. Therefore, the function $Q(x)$ is continuous at $x = 1$. $\square$

Problem 3

Result: Let A be a nonempty set and let $f : A \to A$ be a function. If $f \circ f = i_a$, then f is bijective.
Proof: Let A be a nonempty set and let $f : A \to A$ be a function, and suppose that $f \circ f = i_a$. In order to show that f is bijective, we must show that f is both one-to-one and onto. We start with one-to-one.
Suppose that there are some elements $x, y \in A$ such that $f(x) = f(y)$. We work to show that $x = y$.

$$f(x) = f(y)$$
$$f(f(x)) = f(f(y))$$

By the definition of the identity function, we have

$$f(f(x)) = f(f(y))$$
$$x = y$$

Thus, f is one-to-one.
We now work to show that f is also onto. Let r be an arbitrary element in A, and let another arbitrary element $q \in A$ be such that $f(r) = q$. We know that q exists because f is defined by $f : A \to A$. Through algebra, we have

$$f(r) = q$$
$$f(f(r)) = f(q)$$
$$r = f(q)$$

Since we have shown that for an arbitrary element r in the codomain A, there exists an element q in the domain A (specifically $q = f(r)$) such that $f(q) = r$, the function f is onto.
Since f is both one-to-one and onto, it is, by definition, bijective. $\square$

Problem 4

Result: Let the composition $g \circ f : (0,1) \to \mathbb{R}$ of two functions f and g be given by $(g \circ f)(x) = \frac{4x-1}{2\sqrt{x-x^2}}$ where $f : (0,1) \to (-1,1)$ is defined by $f(x) = 2x - 1$ for $x \in (0,1)$. The function g is given by $g(y) = \frac{2y-1}{2\sqrt{1-y^2}}$.
Proof: Let the function f and the composition $g \circ f$ be defined as above. We work to find the function g.
From the definitions above, we have

$$\begin{aligned} f(x) &= 2x - 1 \\ y &= 2x - 1 \\ y + 1 &= 2x \\ \frac{y+1}{2} &= x \qquad (1) \end{aligned}$$

Expression (1) above is the inverse of $f(x)$, or in other words, $f^{-1}(x)$.
Plugging the inverse function into the composition $(g \circ f)(x)$, we will be able to 'undo' the function f to be left with g.

$$\begin{aligned} (g \circ f)(x) &= \frac{4x-1}{2\sqrt{x-x^2}} \\ g(y) &= \frac{4(\frac{y+1}{2}) - 1}{2\sqrt{(\frac{y+1}{2}) - (\frac{y+1}{2})^2}} \\ g(y) &= \frac{2y+1}{\sqrt{1-y^2}} \end{aligned}$$

Thus, we have our function $g(y) = \frac{2y+1}{2\sqrt{1-y^2}}$. □

Problem 5

Result: Let $A = \mathbb{R} - \{1\}$ and define $f : A \to A$ by $f(x) = \frac{x}{x-1}$ for all $x \in A$. When this is the case, then f is bijective, and $f^{-1} = f = \frac{x}{x-1}$.
Proof: Let the function f be defined as above. In order to show bijectivity, we must show injectivity and surjectivity. We begin with injectivity.
Suppose there are some elements $x, y \in A$ such that $f(x) = f(y)$. We work to show that $x = y$.

$$\begin{aligned} f(x) &= f(y) \\ \frac{x}{x-1} &= \frac{y}{y-1} \\ x(y-1) &= y(x-1) \\ xy - x &= xy - y \\ -x &= -y \\ x &= y \end{aligned}$$

Thus, f is injective.
To show surjectivity, let q, r be arbitrary elements in A such that $f(q) = r$. We work to show that there exists such a q.

$$\begin{aligned} f(q) &= r \\ \frac{q}{q-1} &= r \\ q &= (q-1)r \\ q &= qr - r \\ 0 &= qr - q - r \\ r &= qr - q \\ r &= q(r-1) \\ \frac{r}{r-1} &= q \\ q &= \frac{r}{r-1} \end{aligned}$$

Recalling that the domain A is defined by all real numbers, excluding $\{1\}$, we see that there does in fact exist such a q.
Since f is both injective and surjective, we have that it is, in fact, bijective.

In order to find the inverse of f, we use algebra. Let $f(x) = y$ and let $x \in A$.

$$\begin{aligned} f(x) &= \frac{x}{x-1} \\ y &= \frac{x}{x-1} \end{aligned}$$

Without loss of generality (from the proof of surjectivity), we have

$$\begin{aligned} x &= \frac{y}{y-1} \\ f^{-1}(x) &= \frac{y}{y-1} \end{aligned}$$

$\square$

Problem 6

Result: The sequence $\{\frac{n+2}{2n+3}\}$ is convergent to $\frac{1}{2}$.

Proof: We want to show that $\lim_{n\to\infty}(\frac{n+2}{2n+3}) = \frac{1}{2}$.
Let $\epsilon > 0$. By algebra, we have

$$\begin{aligned}
\left|\frac{n+2}{2n+3} - \frac{1}{2}\right| &< \epsilon \\
\left|\frac{2n+4}{4n+6} - \frac{2n+3}{4n+6}\right| &< \epsilon \\
\left|\frac{1}{4n+6}\right| &< \epsilon \\
\frac{1}{4n+6} &< \epsilon \\
4n+6 &> \frac{1}{\epsilon} \\
n &> \frac{1}{4\epsilon} - \frac{3}{2}
\end{aligned}$$

We now choose $N = \lceil \frac{1}{4\epsilon} - \frac{3}{2} \rceil \geq (\frac{1}{4\epsilon} - \frac{3}{2})$. For $n > N$, we have

$$\left|\frac{n+2}{2n+3} - \frac{1}{2}\right| = \frac{1}{4n+6} < \frac{1}{4N+6} \leq \frac{1}{4(\frac{1}{4\epsilon} - \frac{3}{2}) + 6} = \frac{1}{\frac{4}{4\epsilon} - \frac{12}{2} + 6} = \frac{1}{(\frac{1}{\epsilon})} = \epsilon$$

Therefore, we have that the sequence $\{\frac{n+2}{2n+3}\}$ is convergent to $\frac{1}{2}$. $\square$

Problem 7

Result: $\lim_{x\to 3}(\frac{3x+1}{4x+3}) = \frac{2}{3}$.

Proof: Let $\epsilon > 0$. Choose $\delta = \epsilon$. Then, for $0 < |x-3| < \delta$, we have

$$\left|\frac{3x+1}{4x+3} - \frac{2}{3}\right| = \left|\frac{x-3}{12x+9}\right| < |x-3| < \delta = \epsilon$$

However, we must justify our claim that $\left|\frac{x-3}{12x+9}\right| < |x-3|$. For this to be the case, we have that $|12x+9| > 1$. Notice that as the limit approaches 3, we have that $|12x+9| = |12(3)+9| = 45 > 1$. This works for all values of x around 3, therefore justifying the claim above.
Thus, $\lim_{x\to 3}(\frac{3x+1}{4x+3}) = \frac{2}{3}$. $\square$