Problem 1

Result: The series given by

$$1 + 5 + 9 + \cdots + (4n - 3) = 2n^2 - n$$

is true for every positive integer n.

Proof: We will use induction. Let $n = 1$. We have

$$\begin{aligned} 1 &= 2(1)^2 - 1 \\ 1 &= 1, \end{aligned}$$

so the statement holds for the base case.

Now, suppose that for some integer $k \geq 1$, the formula holds for $n = k$. In other words, the statement

$$1 + 5 + 9 + \cdots + (4k - 3) = 2k^2 - k$$

is true. We now work to show that the following statement is also true.

$$\begin{aligned} 1 + 5 + 9 + \cdots + (4(k+1) - 3) &= 2(k+1)^2 - (k+1) \\ 1 + 5 + 9 + \cdots + (4k - 3) + (4(k+1) - 3) &= 2(k+1)^2 - (k+1) \\ 2k^2 - k + (4(k+1) - 3) &= 2(k+1)^2 - (k+1) \\ 2k^2 + 3k + 1 &= 2(k^2 + 2k + 1) - (k+1) \\ 2k^2 + 3k + 1 &= 2k^2 + 3k + 1 \end{aligned}$$

Thus, by induction, the initial result is true. $\square$

Problem 2

Result: Let k be a positive integer greater than or equal to 2. Given k, then

$$\frac{k}{k+1} \geq \frac{2}{3}.$$

Proof: Let $k \geq 2$ be an integer. By cross-multiplying the expression above, we have

$$\begin{aligned} \frac{k}{k+1} &\geq \frac{2}{3} \\ 3k &\geq 2(k+1) \\ 3k &\geq 2k+2 \\ k &\geq 2 \end{aligned}$$

since $k \geq 2$ from the statement above, the statement has been proven. $\square$

Result: The expression $4^n > n^3$ for every positive integer n.
Proof: We will use induction.
Consider the case where $n = 1$. Applying this, we have

$$4^1/geq1^3 \rightarrow 4/geq1,$$

which is true.
Now, assume that the statement holds for $n = k$, where $k \geq 2 \in \mathbb{Z}$. Therefore,

$$4^k > k^3.$$

We work to show that the statement similarly holds for $n = k + 1$. Applying this, we have

$$\begin{aligned} 4^{(k+1)} = 4 \cdot 4^k >& 4k^3 > (k+1)^3 \\ 4k^3 >& (k+1)^3 \\ 4k^3 >& k^3 + 3k^2 + 3k + 1 \\ 3k^3 >& 3k^2 + 3k + 1 \end{aligned}$$

Which is true for all $k \geq 2$. Therefore, by induction, the statement has been proven. $\square$

Problem 3

Result: $7 \mid (3^{4n+1} - 5^{2n-1})$ for every positive integer n.
Proof: We use induction. Consider the case where $n = 1$. We have

$$
\begin{aligned}
&7 \mid (3^{4(1)+1} - 5^{2(1)-1}) \\
&7 \mid (3^5 - 5^1) \\
&7 \mid (243 - 5) \\
&7 \mid 238
\end{aligned}
$$

which holds because $7 \cdot 34 = 238$.
Now, assume that the hypothesis holds for $n = k$ where $k \geq 2 \in \mathbb{Z}$. From this assumption, and some integer a, we have

$$
\begin{aligned}
7 \mid (3^{4k+1} - 5^{2k-1}) \\
(3^{4k+1} - 5^{2k-1}) &= 7a \\
3^{4k+1} &= 7a + 5^{2k-1}
\end{aligned}
$$

We now work to show that the same hypothesis holds for $n = k + 1$, or in other words,

$$7 \mid (3^{4(k+1)+1} - 5^{2(k+1)-1}).$$

By definition of division, and some integer b, we have

$$
\begin{aligned}
3^{4(k+1)+1} - 5^{2(k+1)-1} &= 3^{4k+5} - 5^{2k+1} \\
&= 81 \cdot 3^{4k+1} - 25 \cdot 5^{2k-1} \\
&= 81 \cdot (7a + 5^{2k-1}) - 25 \cdot 5^{2k-1} \\
&= 81 \cdot 7a + 81 \cdot 5^{2k-1} - 25 \cdot 5^{2k-1} \\
&= 7(81a) + 56 \cdot 5^{2k-1} \\
&= 7(81a + 8 \cdot 5^{2k-1})
\end{aligned}
$$

Thus, since $81a + 8 \cdot 5^{2k-1}$ is an integer, the desired result has been achieved. $\square$

Problem 4

Result: $4(k^2+k) < (2k+1)^2$ for every positive integer k.
Proof: Let k be a positive integer. Then, by algebra, we have

$$\begin{gathered} 4(k^2+k) < (2k+1)^2 \\ 4k^2+4k < 4k^2+4k+1 \\ 4k < 4k+1 \\ 0 < 1 \end{gathered}$$

Thus, the statement has been proven. $\square$

Result: $\sum_{m=1}^{n} \frac{1}{\sqrt{m}} \leq 2\sqrt{n}-1$ for every positive integer n.
Proof: We use induction. Consider the case where $n=1$.

$$\begin{gathered} \sum_{m=1}^{1} \frac{1}{\sqrt{m}} = \frac{1}{\sqrt{1}} = 1 \\ 2\sqrt{n}-1 = 2\sqrt{1}-1 = 2-1 = 1 \end{gathered}$$

Since $1 \leq 1$, the base case holds.
Now, assume that the hypothesis holds for some positive integer $k \geq 1$. In other words,

$$\sum_{m=1}^{k} \frac{1}{\sqrt{m}} \leq 2\sqrt{k}-1$$

We must show that the hypothesis holds for $n=k+1$. In other words,

$$\sum_{m=1}^{k+1} \frac{1}{\sqrt{m}} \leq 2\sqrt{k+1}-1$$

We start with the left-hand side of the equation:

$$\sum_{m=1}^{k+1} \frac{1}{\sqrt{m}} = \left(\sum_{m=1}^{k} \frac{1}{\sqrt{m}}\right) + \frac{1}{\sqrt{k+1}} \leq (2\sqrt{k}-1) + \frac{1}{\sqrt{k+1}}$$

To complete the proof, we need to show that $(2\sqrt{k}-1) + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1}-1$. By algebra, we have

$$\begin{aligned} (2\sqrt{k}-1) + \frac{1}{\sqrt{k+1}} &\leq 2\sqrt{k+1}-1 \\ 2\sqrt{k} + \frac{1}{\sqrt{k+1}} &\leq 2\sqrt{k+1} \\ \frac{1}{\sqrt{k+1}} &\leq 2\sqrt{k+1}-2\sqrt{k} \\ \frac{1}{\sqrt{k+1}} &\leq 2(\sqrt{k+1}-\sqrt{k}) \end{aligned}$$

We multiply the right-hand side by the conjugate $\frac{\sqrt{k+1}+\sqrt{k}}{\sqrt{k+1}+\sqrt{k}}$:

$$
\begin{aligned}
2(\sqrt{k+1}-\sqrt{k}) &= 2 \cdot \frac{(\sqrt{k+1}-\sqrt{k})(\sqrt{k+1}+\sqrt{k})}{\sqrt{k+1}+\sqrt{k}} \\
&= 2 \cdot \frac{(k+1)-k}{\sqrt{k+1}+\sqrt{k}} \\
&= \frac{2}{\sqrt{k+1}+\sqrt{k}}
\end{aligned}
$$

Continuing, we have

$$
\begin{aligned}
\frac{1}{\sqrt{k+1}} &\leq \frac{2}{\sqrt{k+1}+\sqrt{k}} \\
\sqrt{k+1}+\sqrt{k} &\leq 2\sqrt{k+1} \\
\sqrt{k} &\leq \sqrt{k+1}
\end{aligned}
$$

Since k is a positive integer, $k < k+1$, and thus $\sqrt{k} \leq \sqrt{k+1}$ is true. Therefore, the original inequality holds for $n = k+1$.

By the principle of mathematical induction, the statement $\sum_{m=1}^{n} \frac{1}{\sqrt{m}} \leq 2\sqrt{n} - 1$ holds for every positive integer n. $\square$

Problem 5

Result: A sequence $\{a_n\}$ is defined recursively by $a_1 = 1$, $a_2 = 2$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. The nth term is also defined as $a_n = 2^{n-1}$.

Proof: We proceed with strong induction. Consider the cases $n = 1$ and $n = 2$. For $n = 1$, $a_1 = 1$ and $2^{1-1} = 2^0 = 1$. The first base case holds. For $n = 2$, $a_2 = 2$ and $2^{2-1} = 2^1 = 2$. The second base case holds.
Now, assume for an arbitrary integer $k \geq 2$ that $a_i = 2^{i-1}$ for all integers i with $1 \leq i \leq k$.
We show that the statement holds for $n = k + 1$, i.e., $a_{k+1} = 2^{(k+1)-1} = 2^k$.

$$\begin{aligned}
a_{k+1} &= a_k + 2a_{k-1} \\
&= 2^{k-1} + 2 \cdot 2^{(k-1)-1} \\
&= 2^{k-1} + 2^{1+(k-2)} \\
&= 2^{k-1} + 2^{k-1} \\
&= 2 \cdot 2^{k-1} \\
&= 2^{1+(k-1)} \\
&= 2^k
\end{aligned}$$

Thus, $a_{k+1} = 2^k$. By the principle of strong mathematical induction, the statement holds for all positive integers n. $\square$

Problem 6

Result: For every integer $n \geq 28$, there exist nonnegative integers x and y such that

$$n = 5x + 8y.$$

Proof: We use the Strong Principle of Mathematical Induction.

First, we verify the statement for the integers $n = 28, 29, 30, 31$, and 32:

$$\begin{aligned} 28 &= 5(0) + 8(3), \\ 29 &= 5(1) + 8(3), \\ 30 &= 5(6) + 8(0), \\ 31 &= 5(3) + 8(2), \\ 32 &= 5(4) + 8(2). \end{aligned}$$

Thus, the statement is true for $n = 28, 29, 30, 31$, and 32.
Now assume that for some integer $k \geq 32$, the statement holds for all integers m satisfying

$$28 \leq m \leq k.$$

We work to show that the statement holds for $n = k + 1$. Since $k \geq 32$, we have

$$(k + 1) - 5 = k - 4 \geq 28.$$

By the inductive hypothesis, the integer $k - 4$ can be written as

$$k - 4 = 5x + 8y$$

for some nonnegative integers x and y. We add 5 to both sides:

$$\begin{aligned} k + 1 &= (k - 4) + 5 \\ &= 5x + 8y + 5 \\ &= 5(x + 1) + 8y. \end{aligned}$$

Since $x + 1$ and y are nonnegative integers, this expresses $k + 1$ in the required form.
Therefore, by the Strong Principle of Mathematical Induction, every integer $n \geq 28$ can be written as $5x + 8y$ for some nonnegative integers x and y. $\square$