MATH 2800-01: Mathematics Major Seminar

Assignment # 04 15 Oct 2025 David Haberkorn

Problem 1

Result: The equation $x^5 + 2x - 5 = 0$ has a unique real number solution between $x = 1$ and $x = 2$.

Proof: Let $f(x) = x^5 + 2x - 5$. Notice that $f(x)$ is continuous because it is a polynomial, meaning it is continuous in $\mathbb{R}$.
Assume, to the contary, that $f(x)$ has two real number solutions. This implies that there are values $c, d \in (1, 2), c \neq d$ where $f(c) = 0$ and $f(d) = 0$. In other words,

$$f(c) = c^5 + 2c - 5 = 0$$
$$f(d) = d^5 + 2d - 5 = 0$$

Through algebraic manipulation, we get

$$\begin{aligned}
c^5 + 2c - 5 &= d^5 + 2d - 5 \\
c^5 + 2c &= d^5 + 2d \\
c^5 - d^5 &= 2d - 2c \\
c^5 - d^5 &= 2(d - c) \\
(c - d)(c^4 + c^3d + c^2d^2 + cd^3 + d^4) &= 2(d - c) \\
(c - d)(c^4 + c^3d + c^2d^2 + cd^3 + d^4) &= -2(c - d) \\
(c - d)(c^4 + c^3d + c^2d^2 + cd^3 + d^4) + 2(c - d) &= 0 \\
(c - d)(c^4 + c^3d + c^2d^2 + cd^3 + d^4 + 2) &= 0
\end{aligned}$$

In order for the product of the two terms above to equal zero, at least one of $(c-d)$ or $(c^4+c^3d+c^2d^2+cd^3+d^4+2)$ must be equal to zero. For the first case, we get

$$\begin{aligned}
c - d &= 0 \\
c &= d.
\end{aligned}$$

For the second case, we get

$$c^4 + c^3d + c^2d^2 + cd^3 + d^4 + 2 = 0$$
$$c^4 + c^3d + c^2d^2 + cd^3 + d^4 = -2$$

Notice that since $c, d \in (1, 2)$, they are both positive numbers. Also notice that the sum of powers and products of positive numbers must also be a positive number. Since the sum totaling to -2 is not possible, the only real values are from before, where we found $c = d$.
However, this is a contradiction since we stated that c and d are unique. Therefore, the equation $x^5 + 2x - 5 = 0$ has only one unique real number solution between $x = 1$ and $x = 2$. $\square$

Problem 2

Result: For every positive integer $n \geq 2$, the equation $x^n + (x+1)^n = (x+2)^n$ is false.
Disproof (by counterexample): Let $x = 3$ and $n = 2$. Plugging these values into the above equation, we arrive at

$$\begin{aligned} x^n + (x+1)^n &= (x+2)^n \\ 3^2 + (3+1)^2 &= (3+2)^2 \\ 3^2 + 4^2 &= 5^2 \\ 9 + 16 &= 25 \\ 25 &= 25 \end{aligned}$$

Thus, the statement has been disproven. $\square$

Problem 3

Result: If $a \geq 2$ and b are integers, then $a \nmid b$ or $a \nmid b+1$.
Proof, by contradiction: Assume, to the contrary, that both $a \mid b$ and $a \mid b+1$, for some $a \geq 2, b, \in \mathbb{Z}$. This implies that there are two integers s and t such that

$$\begin{aligned} b &= as \\ b+1 &= at \end{aligned}$$

Through algebra, we get

$$\begin{aligned} b+1 &= at \\ b &= at+1 \\ as &= at+1 \\ 1 &= a(s-t) \end{aligned} \tag{1}$$

Notice that according to equation (1), and since $(s-t) \in \mathbb{Z}$, we get that a divides 1. This implies that $a = \pm 1$, which is a contradiction since we assumed that $a \geq 2$.
Therefore, the result must be true. $\square$

Problem 4

Result: $\sqrt{3}$ is irrational.

Proof, by contradiciton: Assume, to the contrary, that $\sqrt{3}$ is rational. By definition, $\sqrt{3}$ can be expressed as

$$\sqrt{3} = \frac{m}{n}$$

for some integers m, n where n is nonzero and the fraction is in its most simplified form.
Through algebra, we get

$$\begin{aligned} \sqrt{3} &= \frac{m}{n} \\ 3 &= \frac{m^2}{n^2} \\ 3n^2 &= m^2 \end{aligned} \tag{2}$$

The implication of equation (2) is that 3 divides m^2, or in other words, $3 \mid m^2$. From the lemma provided, we know that $3 \mid m^2$ if and only if $3 \mid m$. Therefore, we know that $m = 3p$ for some integer p.
Again, through algebra, we find

$$\begin{aligned} 3n^2 &= (3p)^2 \\ 3n^2 &= 9p^2 \\ n^2 &= 3p^2 \end{aligned}$$

Without loss of generality, we find that $n = 3q$ for some integer q.
However, we have arrived upon a contradiction, since we said that the original fraction $\frac{m}{n} = \frac{3p}{3q}$ is in its most simplified form. Therefore, the original statement must be true. $\square$

Problem 5

Result: There exist no positive integers m, n such that $m^2 - n^2 = 1$.

Proof, by contradiction: Assume, to the contrary, that there do exist two integers m, n such that $m^2 - n^2 = 1$. Through algebra, we get

$$3 \tag{3}$$

Problem 6

Result: For any integer n, $5|n^2$ if and only if $5|n$

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Proof: We must prove both implications, so we begin by proving that for any integer n, $5|n^2$ if $5|n$.
Left to Right: We will prove this implication using contrapositive.
Let $n \in \mathbb{Z}$, and assume that $5 \nmid n$. By definition, there does not exist an integer a such that $n = 5a$, or in other words, $n \neq 5a$.

$$\begin{aligned} n &\neq 5a \\ n^2 &\neq (5a)^2 \\ n^2 &\neq 25a^2 \\ n^2 &\neq 5(5a^2) \end{aligned}$$

Since n^2 cannot be written as the product of two integers, $5a^2$ and 5, $5 \nmid n^2$.
Right to left: We will prove this implication directly.
Let $n \in \mathbb{Z}$, and assume that $5|n$. By definition, there exists an integer a such that $n = 5a$. Therefore,

$$\begin{aligned} n =& 5a \\ n^2 =& (5a)^2 \\ n^2 =& 25a^2 \\ n^2 =& 5(5a^2) \end{aligned} \tag{4}$$

Since n^2 can be expressed as the product of two integers, 5 and $5a^2$, $5|n^2$ $\square$

Problem 7

Result: If $a, b \in \mathbb{R}$, then $ab \leq \sqrt{a^2}\sqrt{b^2}$.
Proof: Let $a, b \in \mathbb{R}$. Therefore,

$$\begin{aligned} ab &\leq \sqrt{a^2}\sqrt{b^2} \\ ab &\leq (a)(b) \\ ab &\leq ab \end{aligned} \tag{5}$$

The proof has been satisfied. It should be noted that for the cases where $a, b \leq 0$, we square the values before taking the square root, meaning the expression is still valid for all $a, b, \in \mathbb{R}$. $\square$

Problem 8

Result: Let $a, b \in \mathbb{R}$. If $a > 0$ and $b > 0$, then $\frac{a}{b} + \frac{b}{a} \geq 2$.
Proof: Let $a, b \in \mathbb{R}$. Therefore,

$$\begin{aligned} \frac{a}{b} + \frac{b}{a} &\geq 2 \\ a + \frac{b^2}{a} &\geq 2b \\ a^2 + b^2 &\geq 2ab \\ a^2 + b^2 - 2ab &\geq 0 \end{aligned}$$

Now recognize that this inequality is very similar to that outlined in the Law of Cosines, which is pictured in inequality (5).

$$a^2 + b^2 - 2ab\cos(\theta) = c^2 \tag{6}$$

Observe that when the angle, θ, is zero degrees, then the opposite side (length c) has a length of zero units. Also observe that $cos(0) = 1$. Therefore,

$$\begin{aligned} a^2 + b^2 - 2ab\cos(\theta) &\geq c^2 \\ a^2 + b^2 - 2ab\cos(0) &\geq 0 \\ a^2 + b^2 - 2ab &\geq 0 \\ a^2 + b^2 &\geq 2ab \end{aligned}$$

The proof has been completed. $\square$