## Problem 1

**Result:** The function $f : \mathbb{Z} \to \mathbb{Z}$, defined by $f(n) = 5n + 2$ is injective, but not surjective.
**Proof:** We first prove that the function is injective. Let the function $f$ be defined as above, and suppose that $f(x) = f(y)$ for some arbitrary integers $x$ and $y$. From this assumption, we have

$$\begin{aligned} f(x) &= f(y) \\ 5x + 2 &= 5y + 2 \\ 5x &= 5y \\ x &= y \end{aligned}$$

By the definition of injective, we have shown that the function $f$ is injective.
To show that $f$ is not surjective, we demonstrate a counterexample. We seek two integers $q, r$ such that $f(q) = r$ has no solution.
Consider $r = 1$. In this case, we have

$$\begin{aligned} f(q) &= 1 \\ 5n + 2 &= 1 \\ 5n &= -1 \\ n &= -\frac{1}{5} \end{aligned}$$

Because $-\frac{1}{5}$ is not an integer, we have shown that $f$ is not surjective. $\square$

## Problem 2

**Result:** The function $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{5\}$ defined by $f(x) = \frac{5x+1}{x-2}$ is bijective.
**Proof:** In order to show bijectivity, we must show that the function defined above is both injective and surjective. We begin by showing injectivity.
Let the function $f$ be defined as above, and suppose that $f(x) = f(y)$ for some arbitrary real numbers $x$ and $y$, excluding $x, y = 2$. From this assumption, we have

$$\begin{aligned} f(x) &= f(y) \\ \frac{5x+1}{x-2} &= \frac{5y+1}{y-2} \\ (5x+1)(y-2) &= (x-2)(5y+1) \\ 5xy - 10x + y - 2 &= 5xy + x - 10y - 2 \\ -10x + y &= -10y + x \\ -11x &= -11y \\ x &= y \end{aligned}$$

By the definition of injective, we have shown that the function $f$ is injective.
We now work to show surjectivity.
Let the function $f$ be defined as above, and suppose two real numbers $q, r$ such that $f(q) = r$. From this supposition, we have

$$\begin{aligned} f(q) &= r \\ \frac{5q+1}{q-2} &= r \\ 5q + 1 &= r(q-2) \\ 5q + 1 &= rq - 2r \\ 5q - rq &= -2r - 1 \\ (5-r)q &= -2r - 1 \\ q &= \frac{-2r-1}{5-r} \qquad (1) \end{aligned}$$

Recall from the definition of the function that the codomain is the set of real numbers, excluding $5$. Since the solution above holds for all real numbers excluding $5$, the function is surjective.
However, we need to check if the integer $q$ can be an element of the domain. We do this using contradiction. Suppose that $q = 2$, which is strictly not in the domain. From this, we have

$$\begin{aligned} 2 &= \frac{-2r-1}{5-r} \\ 10 - 2r &= -2r - 1 \\ 10 &= -1 \end{aligned}$$

which is clearly a contradiction. Thus, the equation found in (1) holds true.
Because the function is both injective and surjective, it is, by definition, bijective. $\square$

## Problem 3

**Result:** Let $A$ be a nonempty set and let $f : A \to A$ be a function. If $f \circ f = i_a$, then $f$ is bijective.

**Proof:** Let $A$ be a nonempty set and let $f : A \to A$ be a function, and suppose that $f \circ f = i_a$. In order to show that $f$ is bijective, we must show that $f$ is both one-to-one and onto. We start with one-to-one.

Suppose that there are some elements $x, y \in A$ such that $f(x) = f(y)$. We work to show that $x = y$.

$$f(x) = f(y)$$
$$f(f(x)) = f(f(y))$$

By the definition of the identity function, we have

$$f(f(x)) = f(f(y))$$
$$x = y$$

Thus, $f$ is one-to-one.

We now work to show that $f$ is also onto. Let $r$ be an arbitrary element in $A$, and let another arbitrary element $q \in A$ be such that $f(r) = q$. We know that $q$ exists because $f$ is defined by $f : A \to A$. Through algebra, we have

$$f(r) = q$$
$$f(f(r)) = f(q)$$
$$r = f(q)$$

Since we have shown that for an arbitrary element $r$ in the codomain $A$, there exists an element $q$ in the domain $A$ (specifically $q = f(r)$) such that $f(q) = r$, the function $f$ is onto.

Since $f$ is both one-to-one and onto, it is, by definition, bijective. $\square$

## Problem 4

**Result:** $4(k^2+k) < (2k+1)^2$ for every positive integer k.
**Proof:** Let $k$ be a positive integer. Then, by algebra, we have

$$\begin{aligned} 4(k^2+k) &< (2k+1)^2 \\ 4k^2+4k &< 4k^2+4k+1 \\ 4k &< 4k+1 \\ 0 &< 1 \end{aligned}$$

Thus, the statement has been proven. $\square$

**Result:** $\sum_{m=1}^{n} \frac{1}{\sqrt{m}} \leq 2\sqrt{n}-1$ for every positive integer $n$.
**Proof:** We use induction. Consider the case where $n=1$.

$$\sum_{m=1}^{1} \frac{1}{\sqrt{m}} = \frac{1}{\sqrt{1}} = 1$$
$$2\sqrt{n}-1 = 2\sqrt{1}-1 = 2-1 = 1$$

Since $1 \leq 1$, the base case holds.
Now, assume that the hypothesis holds for some positive integer $k \geq 1$. In other words,

$$\sum_{m=1}^{k} \frac{1}{\sqrt{m}} \leq 2\sqrt{k}-1$$

We must show that the hypothesis holds for $n=k+1$. In other words,

$$\sum_{m=1}^{k+1} \frac{1}{\sqrt{m}} \leq 2\sqrt{k+1}-1$$

We start with the left-hand side of the equation:

$$\sum_{m=1}^{k+1} \frac{1}{\sqrt{m}} = \left(\sum_{m=1}^{k} \frac{1}{\sqrt{m}}\right) + \frac{1}{\sqrt{k+1}} \leq (2\sqrt{k}-1) + \frac{1}{\sqrt{k+1}}$$

To complete the proof, we need to show that $(2\sqrt{k}-1) + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1}-1$. By algebra, we have

$$\begin{aligned} (2\sqrt{k}-1) + \frac{1}{\sqrt{k+1}} &\leq 2\sqrt{k+1}-1 \\ 2\sqrt{k} + \frac{1}{\sqrt{k+1}} &\leq 2\sqrt{k+1} \\ \frac{1}{\sqrt{k+1}} &\leq 2\sqrt{k+1} - 2\sqrt{k} \\ \frac{1}{\sqrt{k+1}} &\leq 2(\sqrt{k+1} - \sqrt{k}) \end{aligned}$$

We multiply the right-hand side by the conjugate $\frac{\sqrt{k+1}+\sqrt{k}}{\sqrt{k+1}+\sqrt{k}}$:

$$\begin{aligned} 2(\sqrt{k+1}-\sqrt{k}) &= 2 \cdot \frac{(\sqrt{k+1}-\sqrt{k})(\sqrt{k+1}+\sqrt{k})}{\sqrt{k+1}+\sqrt{k}} \\ &= 2 \cdot \frac{(k+1)-k}{\sqrt{k+1}+\sqrt{k}} \\ &= \frac{2}{\sqrt{k+1}+\sqrt{k}} \end{aligned}$$

Continuing, we have

$$\begin{aligned} \frac{1}{\sqrt{k+1}} &\leq \frac{2}{\sqrt{k+1}+\sqrt{k}} \\ \sqrt{k+1}+\sqrt{k} &\leq 2\sqrt{k+1} \\ \sqrt{k} &\leq \sqrt{k+1} \end{aligned}$$

Since $k$ is a positive integer, $k < k+1$, and thus $\sqrt{k} \leq \sqrt{k+1}$ is true. Therefore, the original inequality holds for $n = k+1$.

By the principle of mathematical induction, the statement $\sum_{m=1}^{n} \frac{1}{\sqrt{m}} \leq 2\sqrt{n} - 1$ holds for every positive integer $n$. □

## Problem 5

**Result:** A sequence $\{a_n\}$ is defined recursively by $a_1 = 1$, $a_2 = 2$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. The $n$th term is also defined as $a_n = 2^{n-1}$.

**Proof:** We proceed with strong induction. Consider the cases $n = 1$ and $n = 2$. For $n = 1$, $a_1 = 1$ and $2^{1-1} = 2^0 = 1$. The first base case holds. For $n = 2$, $a_2 = 2$ and $2^{2-1} = 2^1 = 2$. The second base case holds.
Now, assume for an arbitrary integer $k \geq 2$ that $a_i = 2^{i-1}$ for all integers $i$ with $1 \leq i \leq k$.
We show that the statement holds for $n = k + 1$, i.e., $a_{k+1} = 2^{(k+1)-1} = 2^k$.

$$\begin{aligned}
a_{k+1} &= a_k + 2a_{k-1} \\
&= 2^{k-1} + 2 \cdot 2^{(k-1)-1} \\
&= 2^{k-1} + 2^{1+(k-2)} \\
&= 2^{k-1} + 2^{k-1} \\
&= 2 \cdot 2^{k-1} \\
&= 2^{1+(k-1)} \\
&= 2^k
\end{aligned}$$

Thus, $a_{k+1} = 2^k$. By the principle of strong mathematical induction, the statement holds for all positive integers $n$. $\square$

## Problem 6

**Result:** For every integer $n \geq 28$, there exist nonnegative integers $x$ and $y$ such that

$$n = 5x + 8y.$$

**Proof:** We use the Strong Principle of Mathematical Induction.

First, we verify the statement for the integers $n = 28, 29, 30, 31$, and $32$:

$$\begin{aligned} 28 &= 5(0) + 8(3), \\ 29 &= 5(1) + 8(3), \\ 30 &= 5(6) + 8(0), \\ 31 &= 5(3) + 8(2), \\ 32 &= 5(4) + 8(2). \end{aligned}$$

Thus, the statement is true for $n = 28, 29, 30, 31$, and $32$.
Now assume that for some integer $k \geq 32$, the statement holds for all integers $m$ satisfying

$$28 \leq m \leq k.$$

We work to show that the statement holds for $n = k + 1$. Since $k \geq 32$, we have

$$(k + 1) - 5 = k - 4 \geq 28.$$

By the inductive hypothesis, the integer $k - 4$ can be written as

$$k - 4 = 5x + 8y$$

for some nonnegative integers $x$ and $y$. We add $5$ to both sides:

$$\begin{aligned} k + 1 &= (k - 4) + 5 \\ &= 5x + 8y + 5 \\ &= 5(x + 1) + 8y. \end{aligned}$$

Since $x + 1$ and $y$ are nonnegative integers, this expresses $k + 1$ in the required form.
Therefore, by the Strong Principle of Mathematical Induction, every integer $n \geq 28$ can be written as $5x + 8y$ for some nonnegative integers $x$ and $y$. $\square$