Problem 1

Result: The equation $x^5 + 2x - 5 = 0$ has a unique real number solution between $x = 1$ and $x = 2$.

Proof: Let $f(x) = x^5 + 2x - 5$. Notice that $f(x)$ is continuous because it is a polynomial, meaning it is continuous in $\mathbb{R}$.
Assume, to the contary, that $f(x)$ has two real number solutions. This implies that there are values $c, d \in (1, 2), c \neq d$ where $f(c) = 0$ and $f(d) = 0$. In other words,

$$f(c) = c^5 + 2c - 5 = 0$$
$$f(d) = d^5 + 2d - 5 = 0$$

Through algebraic manipulation, we get

$$\begin{aligned} c^5 + 2c - 5 &= d^5 + 2d - 5 \\ c^5 + 2c &= d^5 + 2d \\ c^5 - d^5 &= 2d - 2c \\ c^5 - d^5 &= 2(d - c) \\ (c-d)(c^4 + c^3d + c^2d^2 + cd^3 + d^4) &= 2(d - c) \\ (c-d)(c^4 + c^3d + c^2d^2 + cd^3 + d^4) &= -2(c - d) \\ (c-d)(c^4 + c^3d + c^2d^2 + cd^3 + d^4) + 2(c-d) &= 0 \\ (c-d)(c^4 + c^3d + c^2d^2 + cd^3 + d^4 + 2) &= 0 \end{aligned}$$

In order for the product of the two terms above to equal zero, at least one of $(c-d)$ or $(c^4+c^3d+c^2d^2+cd^3+d^4+2)$ must be equal to zero. For the first case, we get

$$\begin{aligned} c - d &= 0 \\ c &= d. \end{aligned}$$

For the second case, we get

$$\begin{aligned} c^4 + c^3d + c^2d^2 + cd^3 + d^4 + 2 &= 0 \\ c^4 + c^3d + c^2d^2 + cd^3 + d^4 &= -2 \end{aligned}$$

Notice that since $c, d \in (1, 2)$, they are both positive numbers. Also notice that the sum of powers and products of positive numbers must also be a positive number. Since the sum totaling to -2 is not possible, the only real values are from before, where we found $c = d$.
However, this is a contradiction since we stated that c and d are unique. Therefore, the equation $x^5 + 2x - 5 = 0$ has only one unique real number solution between $x = 1$ and $x = 2$. $\square$

Problem 2

Result: For every positive integer $n \geq 2$, the equation $x^n + (x+1)^n = (x+2)^n$ is false.
Disproof (by counterexample): Let $x = 3$ and $n = 2$. Plugging these values into the anove equation, we arrive at

$$\begin{aligned} x^n + (x+1)^n &= (x+2)^n \\ 3^2 + (3+1)^2 &= (3+2)^2 \\ 3^2 + 4^2 &= 5^2 \\ 9 + 16 &= 25 \\ 25 &= 25 \end{aligned}$$

Thus, the statement has been disproven. $\square$

Problem 3

Result: If a and b are two distinct real numbers, then either $\frac{a+b}{2} > a$ or $\frac{a+b}{2} > b$.

Proof Strategy: We will use constrapositive. We will work to show that if $\frac{a+b}{2} \leq a$ and $\frac{a+b}{2} \leq b$, then a and b are not distinct, meaning $a = b$. We will work through two cases, then combine the two cases into one result. ♦

Proof, by contrapositive: There are two cases we must prove. Let case 1 be $\frac{a+b}{2} \leq a$, and let case 2 be $\frac{a+b}{2} \leq b$, for any two distinct real numbers a and b.

Case 1: Let $\frac{a+b}{2} \leq a$, for $a, b \in \mathbb{R}$. Therefore,

$$\begin{aligned} \frac{a+b}{2} &\leq a \\ a+b &\leq 2a \\ b &\leq a \qquad (1) \end{aligned}$$

Inequality (1) is our first result.

Case 2: Let $\frac{a+b}{2} \leq b$, for $a, b \in \mathbb{R}$. Therefore,

$$\begin{aligned} \frac{a+b}{2} &\leq b \\ a+b &\leq 2b \\ a &\leq b \qquad (2) \end{aligned}$$

Inequality (2) is our second result. *Combining cases:* Combining results (1) and (2), we get the dual inequalities

$$b \leq a \qquad a \leq b$$

For both to be true, then a must equal b, which is the result we have been attempting to prove. □

Problem 4

Result: If xy and $x+y$ are even and $x, y \in \mathbb{Z}$, then both x and y are even.

Proof Strategy: We will use constrapositive. We will work to show that if x or y is odd, then either xy or $x+y$ is odd. ♦

Proof, by contrapositive: Let $x, y \in \mathbb{Z}$, and let x or y be odd. By definition, there exists integers m and n such that $x = 2m+1$ and $y = 2n+1$. Without loss of generality, we assume that x is odd.
We work to show that xy or $x+y$ is odd when x is odd. y can be even or odd, so we will prove two cases.
Case 1: Let y be odd. By definition, $y = 2k+1$ for some $k \in \mathbb{Z}$. Therefore,

$$xy = (2m+1)(2k+1) = 4mk + 2m + 2k + 1 = 2(2mk + m + k) + 1$$

By definition, since $2mk + m + k \in \mathbb{Z}$, xy is odd.
Case 2: Let y be even. By definition, $y = 2l$ for some $l \in \mathbb{Z}$. Therefore,

$$x + y = (2m+1) + (2l) = 2m + 2l + 1 = 2(m+l) + 1$$

By definition, since $m + l \in \mathbb{Z}$, $x+y$ is odd.
Both cases have been proven. □

Problem 5

Result: For any integer x, $3x+1$ is even if and only if $5x-2$ is odd.

Proof Strategy: We will use two cases, one where x is even, and one where x is odd. We will then prove the biconditional for both cases. ⧫

Proof: Let x be an integer. We will prove the biconditional with two cases.
Case 1: Let x be an even integer. By definition, $x = 2a$ for some integer a. Therefore,

$$3x + 1 = 3(2a) + 1 = 6a + 1 = 2(3a) + 1$$
$$5x - 2 = 5(2a) - 2 = 10a - 2 = 2(5a)$$

By definition, since $3a, 5a \in \mathbb{Z}$, then $3x+1$ is odd and $5x-2$ is even.
Since neither $3x+1$ is even, nor $5x-2$ is odd when x is even, we can mark this case irrelevant.
Case 2: Let x be an odd integer. By definition, $x = 2b+1$ for some integer b. Therefore,

$$3x + 1 = 3(2b+1) + 1 = 6b + 3 + 1 = 6b + 4 = 2(3b+2)$$
$$5x - 2 = 5(2b+1) - 2 = 10b + 5 - 2 = 10b + 2 + 1 = 2(5b+1) + 1$$

Since $3b+2, 5b+1 \in \mathbb{Z}$, $3x+1$ is even and $5x-2$ is odd.
Since both implications of the biconditional are met when x is odd, and neither implication is met when x is even, the biconditional is satisfied. $\square$.

Problem 6

Result: For any integer n, $5|n^2$ if and only if $5|n$

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Proof: We must prove both implications, so we begin by proving that for any integer n, $5|n^2$ if $5|n$.
Left to Right: We will prove this implication using contrapositive.
Let $n \in \mathbb{Z}$, and assume that $5 \nmid n$. By definition, there does not exist an integer a such that $n = 5a$, or in other words, $n \neq 5a$.

$$\begin{aligned} n &\neq 5a \\ n^2 &\neq (5a)^2 \\ n^2 &\neq 25a^2 \\ n^2 &\neq 5(5a^2) \end{aligned}$$

Since n^2 cannot be written as the product of two integers, $5a^2$ and 5, $5 \nmid n^2$.
Right to left: We will prove this implication directly.
Let $n \in \mathbb{Z}$, and assume that $5|n$. By definition, there exists an integer a such that $n = 5a$. Therefore,

$$\begin{aligned} n =&5a \\ n^2 =&(5a)^2 \\ n^2 =&25a^2 \\ n^2 =&5(5a^2) \end{aligned} \tag{3}$$

Since n^2 can be expressed as the product of two integers, 5 and $5a^2$, $5|n^2$ $\square$

Problem 7

Result: If $a, b \in \mathbb{R}$, then $ab \leq \sqrt{a^2}\sqrt{b^2}$.
Proof: Let $a, b \in \mathbb{R}$. Therefore,

$$ab \leq \sqrt{a^2}\sqrt{b^2}$$
$$ab \leq (a)(b)$$
$$ab \leq ab \tag{4}$$

The proof has been satisfied. It should be noted that for the cases where $a, b \leq 0$, we square the values before taking the square root, meaning the expression is still valid for all $a, b, \in \mathbb{R}$. $\square$

Problem 8

Result: Let $a, b \in \mathbb{R}$. If $a > 0$ and $b > 0$, then $\frac{a}{b} + \frac{b}{a} \geq 2$.
Proof: Let $a, b \in \mathbb{R}$. Therefore,

$$\begin{aligned} \frac{a}{b} + \frac{b}{a} &\geq 2 \\ a + \frac{b^2}{a} &\geq 2b \\ a^2 + b^2 &\geq 2ab \\ a^2 + b^2 - 2ab &\geq 0 \end{aligned}$$

Now recognize that this inequality is very similar to that outlined in the Law of Cosines, which is pictured in inequality (5).

$$a^2 + b^2 - 2ab\cos(\theta) = c^2 \tag{5}$$

Observe that when the angle, θ, is zero degrees, then the opposite side (length c) has a length of zero units. Also observe that $cos(0) = 1$. Therefore,

$$\begin{aligned} a^2 + b^2 - 2ab\cos(\theta) &\geq c^2 \\ a^2 + b^2 - 2ab\cos(0) &\geq 0 \\ a^2 + b^2 - 2ab &\geq 0 \\ a^2 + b^2 &\geq 2ab \end{aligned}$$

The proof has been completed. $\square$