Mathematical Proofs
A Transition to Advanced Mathematics

Chapter 14
Proofs in Calculus

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Proofs in Calculus

The proofs that occur in calculus are considerably different than any of those we have seen thus far. The functions encountered in calculus are real-valued functions defined on sets of real numbers. That is, each function that we study in calculus is of the type

$$f : X \to \mathbf{R}, \text{ where } X \subseteq \mathbf{R}.$$

In the study of limits, we are often interested in such functions having the property that either

(1) $X = \mathbf{N}$ and increasing values in the domain $\mathbf{N}$ result in functional values approaching some real number L or

(2) the function is defined for all real numbers near some specified real number a and domain values approaching a result in functional values approaching some real number L.

We begin with (1), where $X = \mathbf{N}$.

Definition

A **sequence** (of real numbers) is a real-valued function defined on the set of natural numbers; that is, a **sequence** is a function

$$f : \mathbf{N} \to \mathbf{R}.$$

If $f(n) = a_n$ for each $n \in \mathbf{N}$, then

$$f = \{(1, a_1), (2, a_2), (3, a_3), \ldots\}.$$

Since only the numbers $a_1, a_2, a_3, \ldots$ are relevant in f, this sequence is often denoted only by $a_1, a_2, a_3, \ldots$ or by $\{a_n\}$.

Definition

The numbers a_1, a_2, a_3, etc. are called the **terms** of the sequence $\{a_n\}$, with a_1 being the first term, a_2 the second term, etc. Thus, a_n is the nth term of the sequence.

$\left\{\frac{1}{n}\right\}$ is the sequence $1, 1/2, 1/3, \ldots$;
$\left\{\frac{n}{2n+1}\right\}$ is the sequence $1/3, 2/5, 3/7, \cdots$.

In these two examples, the nth term of a sequence is given and, from this, we can easily find the first few terms and, in fact, any particular term. On the other hand, finding the nth term of a sequence whose first few terms are given can be challenging.

For example, the nth term of the sequence

$$\frac{1}{2},\ \frac{1}{4},\ \frac{1}{6},\ \ldots$$

is $1/2n$; the nth term of the sequence

$$1+\frac{1}{2},\ 1+\frac{1}{4},\ 1+\frac{1}{8},\ \cdots$$

is $1+1/2^n$; the nth term of the sequence

$$1,\ \frac{3}{5},\ \frac{1}{2},\ \frac{5}{11},\ \frac{3}{7},\ \frac{7}{17},\ \ldots$$

is $(n+1)/(3n-1)$; the nth term of the sequence

$$1,\ -1,\ 1,\ -1,\ 1,\ -1,\ \ldots$$

is $(-1)^{n+1}$; while the nth term of the sequence $1, 4, 9, 16, \cdots$ is n^2.

Limits of Sequences

For the sequence $\left\{\frac{1}{n}\right\}$, the larger the integer n, the closer $1/n$ is to 0; and for the sequence $\left\{\frac{n}{2n+1}\right\}$, the larger the integer n, the closer $n/(2n+1)$ is to $1/2$. On the other hand, for the sequence $\{n^2\}$, as the integer n become larger, n^2 becomes increasingly large and does not approach any real number.

For some sequences $\{a_n\}$, there is a real number L (or at least there appears to be a real number L) such that the larger the integer n becomes, the closer a_n is to L.

Definition

A sequence $\{a_n\}$ of real numbers is said to **converge** to the real number L if for every real number $\epsilon > 0$, there exists a positive integer N such that if n is an integer with $n > N$, then $|a_n - L| < \epsilon$.

The number ϵ is a measure of how close the terms a_n are required to be to the number L and N indicates a position in the sequence beyond which the required condition is satisfied.

Definition

If a sequence $\{a_n\}$ converges to L, then $\{a_n\}$ is a **convergent sequence** and L is referred to as the **limit** of $\{a_n\}$ and we write

$$\lim_{n\to\infty} a_n = L.$$

If a sequence does not converge, it is said to **diverge**. Consequently, if a sequence $\{a_n\}$ diverges, then there is *no* real number L such that $\lim_{n\to\infty} a_n = L$.

Limits of Sequences

For a real number x, the *ceiling* $\lceil x \rceil$ of x is the smallest integer greater than or equal to x.

$$\lceil 8/3 \rceil = 3, \left\lceil \sqrt{2} \right\rceil = 2, \lceil -1.6 \rceil = -1 \text{ and } \lceil 5 \rceil = 5.$$

Example 1

Result The sequence $\left\{ \frac{1}{n} \right\}$ converges to 0.

Proof We want to show $\lim_{n \to \infty} \frac{1}{n} = 0$.

Let $\varepsilon > 0$.

Choose $N = \lceil \frac{1}{\varepsilon} \rceil \geq \frac{1}{\varepsilon}$. For $n > N$, we have

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \frac{1}{N} \leq \varepsilon.$$

$\square$

Example 2

Result The sequence $\left\{3 + \dfrac{2}{n^2}\right\}$ converges to 3.

Proof We want to show $\lim_{n \to \infty} 3 + \frac{2}{n^2} = 3$.

Let $\varepsilon > 0$. Choose $N = \left\lceil \sqrt{2/\varepsilon} \right\rceil \geq \sqrt{2/\varepsilon}$.

For $n > N$, we have

$$\left| 3 + \frac{2}{n^2} - 3 \right| = \frac{2}{n^2} < \frac{2}{N^2} < \frac{2}{(2/\varepsilon)} = \varepsilon.$$

Therefore, $\lim_{n \to \infty} 3 + \frac{2}{n^2} = 3$.

$\square$

Example 3

Result The sequence $\left\{\dfrac{n}{2n+1}\right\}$ converges to $\dfrac{1}{2}$.

Proof Suppose $\varepsilon > 0$. Choose $N = \left\lceil \frac{1}{4\varepsilon} - \frac{1}{2} \right\rceil$ (so $N \geq \frac{1}{4\varepsilon} - \frac{1}{2}$)

For $n > N$, we have

$$\left| \frac{n}{2n+1} - \frac{1}{2} \right| = \left| \frac{2n-2n-1}{2(2n+1)} \right| = \left| \frac{-1}{4n+2} \right| = \frac{1}{4n+2}$$

$$< \frac{1}{4N+2} \leq \frac{1}{4\left(\frac{1}{4\varepsilon} - \frac{1}{2}\right) + 2} = \frac{1}{\frac{1}{\varepsilon} - 2 + 2} = \varepsilon.$$

Thus, $\displaystyle\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$.

□

Example 4

Result The sequence $\left\{(-1)^{n+1}\right\}$ is divergent.

Proof By contradiction, suppose it is.

Thus, $\forall \varepsilon > 0$, $\exists N > 0$ such that $|a_n - L| < \varepsilon$ for any $n > N$.

Pick $\varepsilon = 1$. Let $k > N$ such that k is odd.

This means

$$|(-1)^{k+1} - L| < 1 \Rightarrow |1 - L| < 1 \Rightarrow -1 < 1 - L < 1$$

$$\Rightarrow -2 < -L < 0 \Rightarrow 0 < L < 2.$$

In a similar fashion, let $k > N$ be an even integer.

This means

$$|(-1)^{k+1} - L| < 1 \Rightarrow |-1 - L| < 1 \Rightarrow |1 + L| < 1$$

$$\Rightarrow -1 < 1 + L < 1 \Rightarrow -2 < L < 0.$$

Both of these imply $0 < L < 0$. This is a contradiction.

Thus $\{(-1)^{n+1}\}_{n=1}^{\infty}$ must diverge.

A sequence $\{a_n\}$ may diverge because as n becomes larger, a_n becomes larger and eventually exceeds any given real number. If a sequence has this property, then $\{a_n\}$ is said to diverge to infinity.

Definition

More formally, a sequence $\{a_n\}$ **diverges to infinity**, written $\lim_{n\to\infty} a_n = \infty$, if for every positive number M, there exists a positive integer N such that if n is an integer such that $n > N$, then $a_n > M$.

The sequence $\{(-1)^{n+1}\}$, although divergent, does not diverge to infinity. However, the sequence $\{n^2 + \frac{1}{n}\}$ *does* diverge to infinity.

Example 5

Result $\lim_{n\to\infty} \left(n^2 + \dfrac{1}{n}\right) = \infty.$

Proof Suppose $M > 0$. Choose $N = \lceil \sqrt{M} \rceil \geq \sqrt{M}$.

Then for $n > N$,

$$n^2 + \frac{1}{n} > n^2 > N^2 \geq (\sqrt{M})^2 = M.$$

Thus $\{n^2 + \frac{1}{n}\}_{n=1}^{\infty}$ diverges.

□

Definition

Let f be a real-valued function defined on a set X of real numbers and let $a \in \mathbf{R}$ such that f is defined in some deleted neighborhood of a.

A number L is the **limit** of a function $f(x)$ as x approaches a, written $\lim_{x \to a} f(x) = L$, if for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that for every real number x with $0 < |x - a| < \delta$, it follows that $|f(x) - L| < \epsilon$.
This implies that if $0 < |x - a| < \delta$, then certainly $f(x)$ is defined.

If there exists a number L such that $\lim_{x \to a} f(x) = L$, then we say that the limit $\lim_{x \to a} f(x)$ exists and is equal to L; otherwise, this limit does not exist.

Thus, to show that $\lim_{x \to a} f(x) = L$, it is necessary to specify $\epsilon > 0$ first and then show the existence of a real number $\delta > 0$.

Ordinarily, the smaller the value of ϵ, the smaller the value of δ. However, we must be certain that the number δ selected satisfies the requirement regardless of how small (or large) ϵ may be.

Even though our choice of δ depends on ϵ, it should not depend on which real number x with $0 < |x - a| < \delta$ is being considered.

Example 6

Result $\lim_{x \to 4} (3x - 7) = 5.$

Proof. Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{3}$ Then for $0 < |x-4| < \delta$, we have

$$|3x-7-5| = |3(x-4)| = 3|x-4| < 3\delta = 3\,\frac{\varepsilon}{3} = \varepsilon.$$

Thus $\lim_{x \to 4} (3x-7) = 5.$

$\square$

Example 7

Result $\lim_{x\to\frac{3}{2}} \dfrac{4x^2-9}{2x-3} = 6.$

Proof. Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{2}$. Then for $0 < |x - 3/2| < \delta$, we have,

$$\left|\frac{4x^2-9}{2x-3} - 6\right| = \left|\frac{(2x-3)(2x+3)}{2x-3} - 6\right| = |2x+3-6|$$

$$= 2|x - 3/2| < 2\delta = 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\lim_{x\to 3/2} \dfrac{4x^2-9}{2x-3} = 6$.

$\square$

Example 8

Result $\lim_{x\to 3} x^2 = 9.$

Proof. Let $\varepsilon > 0$. Choose $\delta < -3+\sqrt{9+\varepsilon}$. Then for $0 < |x-3| < \delta$, we have

$-\delta < x-3 < \delta \Rightarrow -\delta+3 < x < \delta+3 \Rightarrow -\delta+6 < x+3 < \delta+6,$

so $|x+3| < \delta+6$. Thus

$$|x^2-9| = |x+3| \cdot |x-3| < (\delta+6)\delta < (3+\sqrt{9+\varepsilon})(-3+\sqrt{9+\varepsilon})$$
$$= -9+9+\varepsilon$$
$$= \varepsilon.$$

Thus $\lim_{x\to 3} x^2 = 9$.

□