Assignment # 08 10 Dec 2025 David Haberkorn

Problem 1

Result: Suppose that $\lim_{x\to a} f(x) = L$, where $L > 0$. Then $\lim_{x\to a} \sqrt{f(x)} = \sqrt{L}$.
Proof: Let $\epsilon > 0$. Since $\lim_{x\to a} f(x) = L$, there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x)-L| < \epsilon\sqrt{L}$. Now suppose $0 < |x-a| < \delta$. We have

$$|\sqrt{f(x)} - \sqrt{L}| = \frac{|f(x) - L|}{|\sqrt{f(x)} + \sqrt{L}|} \leq \frac{|f(x) - L|}{\sqrt{L}} < \frac{\delta}{\sqrt{L}} = \frac{\sqrt{L}\epsilon}{\sqrt{L}} = \epsilon$$

Thus, $\lim_{x\to a} \sqrt{f(x)} = \sqrt{L}$. $\square$

Problem 2

Result: The function $Q : \mathbb{R} - \{-1\}$ is defined as

$$Q(x) = \begin{cases} \dfrac{x^2 - 3x + 2}{x^2 - 1}, & x \in \mathbb{R} \setminus \{-1, 1\}, \\ -\dfrac{1}{2}, & x = 1. \end{cases}$$

$Q(x)$ is continuous at $x = 1$.

Proof: In order to show continuity at $x = 1$, we must show that

1. $\lim_{x \to 1} Q(x)$ exists,
2. $\lim_{x \to 1} Q(x) = Q(1)$,
3. $Q(1)$ is defined.

From the definition of $Q(x)$ above, we have that $Q(1) = -\frac{1}{2}$, which satisfies the third criteria above.
To show the first and second criteria, we employ an $\epsilon - \delta$ proof.
Let $\epsilon > 0$, and choose $\delta = \min 1, \frac{2}{3}\epsilon$. Now suppose $|x - 1| < 1$.

$$\begin{aligned} |x - 1| &< 1 \\ -1 < x - 1 &< 1 \\ 1 < x + 1 &< 3 \end{aligned}$$

This implies that the quantity $|x + 1| = x + 1$. Now suppose $0 < |x - 1| < \delta$. We have

$$\left| \frac{x^2 - 3x + 2}{x^2 - 1} - \frac{-1}{2} \right| = \left| \frac{x - 2}{x + 1} + \frac{1}{2} \right| = \left| \frac{3(x - 1)}{2(x + 1)} \right| = \frac{3\,|x - 1|}{2(x + 1)} \leq \frac{3\,|x - 1|}{2} < \frac{3\delta}{2} \leq \frac{3}{2} \cdot \frac{2}{3}\epsilon = \epsilon$$

Therefore, the limit exists.
Evaluating $Q(1)$, we have

$$\frac{x^2 - 3x + 2}{x^2 - 1} = \frac{x - 2}{x + 1} \to \frac{(1) - 2}{(1) + 1} = -\frac{1}{2}$$

Since the limit both exists and is equal to $Q(1)$, we have satisfied all three necessary criteria. Therefore, the function $Q(x)$ is continuous at $x = 1$. $\square$

Problem 3

Result: Suppose two functions, one the bounded function $g : \mathbb{R} \to \mathbb{R}$, meaning there exists a positive real number B such that $|g(x)| < B$ for each $x \in \mathbb{R}$, and the other $f : \mathbb{R} \to \mathbb{R}$ where $a \in \mathbb{R}$ such that $\lim_{x\to a} f(x) = 0$. Given these two functions, then $\lim_{x\to a} f(x)g(x) = 0$.
Proof: Let $\epsilon > 0$, and choose $\delta = \frac{\epsilon}{B}$. Suppose $0 < |x - a| < \delta$. We have

$$|f(x)g(x) - 0| = |f(x)g(x)| = |f(x)|\,|g(x)| < |f(x)|\,B < \delta B = B \cdot \frac{\epsilon}{B} = \epsilon$$

Therefore, $\lim_{x\to a} f(x)g(x) = 0$.
Problem: Use the above result to determine $\lim_{x\to 0} x^2 \sin(\frac{1}{x})$.
Solution: Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ and let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \sin(\frac{1}{x})$. Notice that when evaluating $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 = 0$. Also notice that for all $s \in \mathbb{R}$, $-1 < \sin(s) < 1$ Since $s = \frac{1}{t}$ is a real number for all $t \in \mathbb{R} - \{0\}$, the function $g(x) = \sin(\frac{1}{x})$ is a bounded function.
From above, when evaluating the limit of a function that evaluates to 0, and multiplying it by a bounded function, the result is the same limit as the original function. In this example, we have a function (namely f) whose limit as x approaches 0 evaluates to 0. When multiplying it by the bounded function g, we have the same limit as x approaches 0. Therefore,

$$\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0.$$

$\square$

Problem 4

Result: Let $f : [1, \infty) \to [0, \infty)$ be defined by $f(x) = \sqrt{x-1}$. f is both continuous and differentiable at $x = 10$.
Proof: Let $f : [1, \infty) \to [0, \infty)$ be defined by $f(x) = \sqrt{x-1}$. We will start by showing that f is continuous at $x = 10$.
Evaluating $\lim_{x \to 10} \sqrt{x-1}$, we have

$$\lim_{x \to 10} \sqrt{x-1} = \sqrt{10-1} = 3$$

Let $\epsilon > 0$, and choose $\delta = 3\epsilon$. Now suppose $0 < |x - 10| < \delta$. We have

$$\left|\sqrt{x-1} - 3\right| = \left|\sqrt{x-1} - 3\right| \cdot \frac{\left|\sqrt{x-1}+3\right|}{\left|\sqrt{x-1}+3\right|} = \left|\frac{x-10}{\sqrt{x-1}+3}\right|$$

Since $\sqrt{x-1}$ is always positive (from our domain), we have

$$\left|\frac{x-10}{\sqrt{x-1}+3}\right| \leq \left|\frac{x-10}{3}\right| < \frac{\delta}{3} = \frac{3\epsilon}{3} = \epsilon$$

Because f is defined at $x = 10$, and the limit as x approaches 10 both exists and is equal to $f(10)$, we have that f is continuous.
We now work to show that f is differentiable at $x = 10$. Consider the definition of the derivative, where

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. From here, we have

$$f'(10) = \lim_{x \to 10} \frac{f(x) - f(10)}{x - 10} = \lim_{x \to 10} \frac{\sqrt{x-1} - 3}{x - 10} = \lim_{x \to 10} \frac{\sqrt{x-1} - 3}{x - 10} \cdot \frac{\sqrt{x-1}+3}{\sqrt{x-1}+3} = \lim_{x \to 10} \frac{(x-10)}{(x-10)(\sqrt{x-1}+3)}$$

Because we are evaluating the limit as x approaches 10, rather than $x = 10$, we can say that $x \neq 10$ and simplify the $(x - 10)$ values from the numerator and denominator.

$$\lim_{x \to 10} \frac{(x-10)}{(x-10)(\sqrt{x-1}+3)} = \lim_{x \to 10} \frac{1}{\sqrt{x-1}+3}$$

Now, because the function is continuous, we can evaluate the limit as follows.

$$\lim_{x \to 10} \frac{1}{\sqrt{x-1}+3} = \frac{1}{\sqrt{(10)-1}+3} = \frac{1}{6}$$

Since the limit exists, f is differentiable at $x = 10$. $\square$