Jordan

$$=\begin{pmatrix}1&0&0\\-1&-1&1\\2&1&0\end{pmatrix}\begin{pmatrix}J_1^k & \\ & J_1^k\end{pmatrix}\begin{pmatrix}1&0&0\\-2&0&1\\-1&1&1\end{pmatrix}=\begin{pmatrix}1&0&0\\-1&-1&1\\2&1&0\end{pmatrix}\begin{pmatrix}1&k&0\\0&1&0\\0&0&2^k\end{pmatrix}\begin{pmatrix}1&0&0\\-2&0&1\\-1&1&1\end{pmatrix}$$

$$=\begin{pmatrix}-2k+1&0&k\\2k+1-2^k&2^k&-k-1+2^k\\-4k&0&2k+1\end{pmatrix}$$

为r_i阶Jordan块$J_i=\begin{bmatrix}\lambda_i&1&0&0\\0&\lambda_i&\ddots&0\\0&0&\ddots&1\\0&0&\cdots&\lambda_i\end{bmatrix}_{r_i\times r_i}$

有$J_i^k=\begin{bmatrix}\lambda_i^k&C_k^1\lambda_i^{k-1}&\cdots&C_k^{r_i-1}\lambda_i^{k-r_i+1}\\0&\lambda_i^k&\ddots&C_k^{r_i-2}\lambda_i^{k-r_i+2}\\0&0&\ddots&\vdots\\0&0&\cdots&\lambda_i^k\end{bmatrix}_{r_i\times r_i}$

其中$C_k^i=\dfrac{k!}{i!(k-i)!}$

已知$A=\begin{pmatrix}-1&0&1\\1&2&0\\-4&0&3\end{pmatrix}$，求$e^A$，用Jordan标准形求

（1）首先求相似变换矩阵$P=\begin{pmatrix}1&0&0\\-1&-1&1\\2&1&0\end{pmatrix}$

则$J=P^{-1}AP=\begin{pmatrix}1&1&\\&1&\\&&2\end{pmatrix}$，其中$J_1=\begin{bmatrix}1&1\\0&1\end{bmatrix}$,$J_2=[2]$

那么，$e^A=P\begin{pmatrix}f(J_1)&&&\\&f(J_2)&&\\&&\ddots&\\&&&f(J_s)\end{pmatrix}P^{-1}=P\begin{pmatrix}e^{J_1}&0\\0&e^{J_2}\end{pmatrix}P^{-1}$

那么，$e^{J_1}=f(J_1t)|_{t=1}=\begin{bmatrix}f(\lambda)&\frac{f'(\lambda)}{1!}\\0&f(\lambda)\end{bmatrix}\Bigg|_{\lambda=1}=\begin{bmatrix}e&e\\0&e\end{bmatrix}$, $e^{J_2}=[e^2]$

$$e^A=f(At)|_{t=1}=P\begin{pmatrix}f(J_1t)&&\\&\ddots&\\&&f(J_st)\end{pmatrix}P^{-1}=P\begin{bmatrix}e&e&0\\0&e&0\\0&0&e^2\end{bmatrix}P^{-1}$$

$$=\begin{bmatrix}-e&0&e\\3e-e^2&e^2&-2e+e^2\\-4e&0&3e\end{bmatrix}$$

$A=\begin{bmatrix}-1&1&0\\-4&3&0\\1&0&2\end{bmatrix}$，计算$A^7-A^5-19A^4+28A^3+6A-4I$；$A^{-1}$；$A^{100}$.

$\varphi(\lambda)=\det(\lambda I-A)=\begin{vmatrix}\lambda+1&-1&0\\4&\lambda-3&0\\-1&0&\lambda-2\end{vmatrix}=\lambda^3-4\lambda^2+5\lambda-2$

$f(\lambda)=\lambda^7-\lambda^5-19\lambda^4+28\lambda^3+6\lambda-4$ $\Rightarrow$ $\boxed{\text{根据Cayley定理：}}$

$f(\lambda)=g(\lambda)\varphi(\lambda)+r(\lambda)$

$g(\lambda)=\lambda^4+4\lambda^3+10\lambda^2+3\lambda-2$

$r(\lambda)=-3\lambda^2+22\lambda-8$

$f(A)=g(A)\varphi(A)+r(A)$

$=r(A)$

$=-3A^2+22A-8I$

例 3.5 已知$\boldsymbol{A}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$，求$e^{\boldsymbol{A}t}$.

解 可求得$\det(\lambda\boldsymbol{I}-\boldsymbol{A})=\lambda^2+1$. 由 Hamilton-Cayley 定理知$\boldsymbol{A}^2+\boldsymbol{I}=\boldsymbol{O}$，从而$\boldsymbol{A}^2=-\boldsymbol{I},\boldsymbol{A}^3=-\boldsymbol{A},\boldsymbol{A}^4=\boldsymbol{I},\boldsymbol{A}^5=\boldsymbol{A},\cdots$，即

$$\boldsymbol{A}^{2k}=(-1)^k\boldsymbol{I},\quad \boldsymbol{A}^{2k+1}=(-1)^k\boldsymbol{A}\quad(k=1,2,\cdots).$$

故

$$e^{\boldsymbol{A}t}=\sum_{k=0}^{+\infty}\frac{1}{k!}\boldsymbol{A}^kt^k=\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\cdots\right)\boldsymbol{I}+\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\cdots\right)\boldsymbol{A}$$

$$=(\cos t)\boldsymbol{I}+(\sin t)\boldsymbol{A}=\begin{bmatrix}\cos t&\sin t\\-\sin t&\cos t\end{bmatrix}.$$

已知 4 阶方阵 $\boldsymbol{A}$ 的特征值为 $\pi,-\pi,0,0$，求 $\sin\boldsymbol{A},\cos\boldsymbol{A}$.

解 因为$\det(\lambda\boldsymbol{I}-\boldsymbol{A})=(\lambda-\pi)(\lambda+\pi)\lambda^2=\lambda^4-\pi^2\lambda^2$，所以$\boldsymbol{A}^4-\pi^2\boldsymbol{A}^2=\boldsymbol{O}$. 于是

$$\boldsymbol{A}^4=\pi^2\boldsymbol{A}^2,\quad \boldsymbol{A}^5=\pi^2\boldsymbol{A}^3,\quad \boldsymbol{A}^6=\pi^4\boldsymbol{A}^2,\quad \boldsymbol{A}^7=\pi^4\boldsymbol{A}^3,\cdots,$$

即

$$\boldsymbol{A}^{2k}=\pi^{2k-2}\boldsymbol{A}^2,\quad \boldsymbol{A}^{2k+1}=\pi^{2k-2}\boldsymbol{A}^3\quad(k=2,3,\cdots).$$

故

$$\sin\boldsymbol{A}=\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)!}\boldsymbol{A}^{2k+1}=\boldsymbol{A}-\frac{1}{3!}\boldsymbol{A}^3+\sum_{k=2}^{+\infty}\frac{(-1)^k}{(2k+1)!}\pi^{2k-2}\boldsymbol{A}^3$$

$$=\boldsymbol{A}-\frac{1}{3!}\boldsymbol{A}^3+\frac{1}{\pi^3}\boldsymbol{A}^3\left(\sum_{k=2}^{+\infty}\frac{(-1)^k}{(2k+1)!}\pi^{2k+1}\right)$$

$$=\boldsymbol{A}+\frac{\sin\pi-\pi}{\pi^3}\boldsymbol{A}^3=\boldsymbol{A}-\frac{1}{\pi^2}\boldsymbol{A}^3,$$

$$\cos\boldsymbol{A}=\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k)!}\boldsymbol{A}^{2k}=\boldsymbol{I}-\frac{1}{2!}\boldsymbol{A}^2+\sum_{k=2}^{+\infty}\frac{(-1)^k}{(2k)!}\pi^{2k-2}\boldsymbol{A}^2$$

$$=\boldsymbol{I}+\frac{\cos\pi-1}{\pi^2}\boldsymbol{A}^2=\boldsymbol{I}-\frac{2}{\pi^2}\boldsymbol{A}^2.$$

例 3.10 已知$\boldsymbol{A}=\begin{bmatrix}3&1&-1\\-2&0&2\\-1&-1&3\end{bmatrix}$，求$e^{\boldsymbol{A}t},\sin\boldsymbol{A}$.

解 例 1.9 已求得 $\boldsymbol{A}$ 的 Jordan 标准形为

$$\boldsymbol{J}=\begin{bmatrix}2&&\\&2&1\\&&2\end{bmatrix},$$

于是 $\boldsymbol{A}$ 的最小多项式为$m_{\boldsymbol{A}}(\lambda)=(\lambda-2)^2$. 设

$$r(\lambda)=b_1\lambda+b_0,$$

由

$$\begin{cases}r(2)=2b_1+b_0=e^{2t},\\r'(2)=b_1\qquad=te^{2t},\end{cases}$$

解得

$$\begin{cases}b_1=te^{2t},\\b_0=(1-2t)e^{2t}.\end{cases}$$

于是

$$e^{\boldsymbol{A}t}=b_1\boldsymbol{A}+b_0\boldsymbol{I}=e^{2t}\begin{bmatrix}1+t&t&-t\\-2t&1-2t&2t\\-t&-t&1+t\end{bmatrix}.$$

又由

$$\begin{cases}r(2)=2b_1+b_0=\sin2,\\r'(2)=b_1\qquad=\cos2,\end{cases}$$

解得

$$\begin{cases}b_1=\cos2,\\b_0=\sin2-2\cos2.\end{cases}$$

从而

$$\sin\boldsymbol{A}=b_1\boldsymbol{A}+b_0\boldsymbol{I}=\begin{bmatrix}\sin2+\cos2&\cos2&-\cos2\\-2\cos2&\sin2-2\cos2&2\cos2\\-\cos2&-\cos2&\sin2+\cos2\end{bmatrix}.$$

$$\begin{cases}\dfrac{dx_1(t)}{dt}=-x_1(t)+x_3(t)\\\dfrac{dx_2(t)}{dt}=x_1(t)+2x_2(t)\\\dfrac{dx_3(t)}{dt}=-4x_1(t)+3x_3(t)\\x_1(0)=1,x_2(0)=0,x_3(0)=1\end{cases}$$

令 $A=\begin{pmatrix}-1&0&1\\1&2&0\\-4&0&3\end{pmatrix}$ $x(t)=\begin{pmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{pmatrix}$ $\boldsymbol{c}=\begin{pmatrix}1\\0\\1\end{pmatrix}$

$\boldsymbol{x(t)=e^{A(t-t_0)}c}$

则 $\begin{cases}\dfrac{dx(t)}{dt}=Ax(t)\\x(0)=\boldsymbol{c}\end{cases}$ $e^{At}=\begin{pmatrix}e^t-2te^t&0&te^t\\-e^{2t}+e^t+2te^t&e^{2t}&e^{2t}-e^t-te^t\\-4te^t&0&e^t+2te^t\end{pmatrix}$

$$x(t)=e^{At}\boldsymbol{c}=\begin{pmatrix}e^t-te^t\\te^t\\e^t-2te^t\end{pmatrix}$$

$$\begin{cases}\dfrac{dx_1(t)}{dt}=-x_1(t)+x_3(t)+1\\\dfrac{dx_2(t)}{dt}=x_1(t)+2x_2(t)-1\\\dfrac{dx_3(t)}{dt}=-4x_1(t)+3x_3(t)+2\\x_1(0)=1,x_2(0)=0,x_3(0)=1\end{cases}$$

令 $A=\begin{pmatrix}-1&0&1\\1&2&0\\-4&0&3\end{pmatrix}$ $x(t)=\begin{pmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{pmatrix}$ $\boldsymbol{c}=\begin{pmatrix}1\\0\\1\end{pmatrix}$ $f(t)=\begin{pmatrix}1\\-1\\2\end{pmatrix}$

则 $\begin{cases}\dfrac{dx(t)}{dt}=Ax(t)+f(t)\\x(0)=\boldsymbol{c}\end{cases}$ $e^{At}=\begin{pmatrix}e^t-2te^t&0&te^t\\-e^{2t}+e^t+2te^t&e^{2t}&e^{2t}-e^t-te^t\\-4te^t&0&e^t+2te^t\end{pmatrix}$

$x(t)=e^{At}\boldsymbol{c}=\begin{pmatrix}e^t-te^t\\te^t\\e^t-2te^t\end{pmatrix}$

$$\boldsymbol{x(t)=e^{A(t-t_0)}c+e^{At}\int_{t_0}^{t}e^{-A\tau}f(\tau)d\tau}$$

$$\int_0^t e^{-A\tau}f(\tau)d\tau = \int_0^t \begin{pmatrix} e^{-\tau} \\ -e^{-\tau} \\ 2e^{-\tau} \end{pmatrix} d\tau = \begin{pmatrix} 1-e^{-t} \\ -1+e^{-t} \\ 2-2e^{-t} \end{pmatrix}$$

$$x(t) = e^{A(t-t_0)}c + e^{At}\int_{t_0}^t e^{-A\tau}f(\tau)d\tau = \begin{pmatrix} (2-t)e^t - 1 \\ (t-1)e^t + 1 \\ (3-2t)e^{-t} - 2 \end{pmatrix}$$

由Doolittle分解的紧凑计算格式得:

$r_{11} = a_{11} = 2, r_{12} = a_{12} = -1, r_{13} = a_{13} = 3$

$l_{21} = \frac{a_{21}}{r_{11}} = \frac{1}{2}, l_{31} = \frac{a_{31}}{r_{11}} = 1$

$r_{22} = a_{22} - l_{21}r_{12} = \frac{5}{2}, r_{23} = a_{23} - l_{21}r_{13} = -\frac{1}{2}$

$l_{32} = \frac{a_{32} - l_{31}r_{12}}{r_{22}} = 2$

$r_{33} = a_{33} - l_{31}r_{13} - l_{32}r_{23} = 1$

三角分解的例子:

Doolittle分解中的矩阵为$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}, R = \begin{bmatrix} 2 & -1 & 3 \\ 0 & \frac{5}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$，记$D_R = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{5}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

则A的Crout分解为

$$A = LR = (LD_R)(D_R^{-1}R) = \begin{bmatrix} 2 & 0 & 0 \\ 1 & \frac{5}{2} & 0 \\ 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

A的LDR分解为

$$A = LDR = LD_RR = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{5}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

例 4.2 已知矩阵 $\boldsymbol{A} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$，求 $\boldsymbol{A}$ 的 Cholesky 分解.

解 容易验证 $\boldsymbol{A}$ 是实对称正定矩阵. 由 Cholesky 分解的紧凑计算格式得

$$g_{11} = \sqrt{a_{11}} = \sqrt{5}, \quad g_{21} = \frac{a_{21}}{g_{11}} = -\frac{2}{\sqrt{5}}, \quad g_{31} = \frac{a_{31}}{g_{11}} = 0,$$

$$g_{22} = \sqrt{a_{22} - |g_{21}|^2} = \sqrt{\frac{11}{5}}, \quad g_{32} = \frac{1}{g_{22}}(a_{32} - g_{31}\overline{g}_{21}) = -\sqrt{\frac{5}{11}},$$

$$g_{33} = \sqrt{a_{33} - |g_{31}|^2 - |g_{32}|^2} = \sqrt{\frac{6}{11}},$$

故 $\boldsymbol{A}$ 的 Cholesky 分解为

$$\boldsymbol{A} = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ -\frac{2}{\sqrt{5}} & \sqrt{\frac{11}{5}} & 0 \\ 0 & -\sqrt{\frac{5}{11}} & \sqrt{\frac{6}{11}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & -\frac{2}{\sqrt{5}} & 0 \\ 0 & \sqrt{\frac{11}{5}} & -\sqrt{\frac{5}{11}} \\ 0 & 0 & \sqrt{\frac{6}{11}} \end{bmatrix}.$$

设 $\boldsymbol{a}_1 = (0,0,2)^{\mathrm{T}}, \boldsymbol{a}_2 = (3,4,1)^{\mathrm{T}}, \boldsymbol{a}_3 = (1,-2,2)^{\mathrm{T}}$，则 $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ 线性无关. 正交化得

$$\boldsymbol{p}_1 = \boldsymbol{a}_1 = (0,0,2)^{\mathrm{T}}, \quad \boldsymbol{p}_2 = \boldsymbol{a}_2 - \frac{1}{2}\boldsymbol{p}_1 = (3,4,0)^{\mathrm{T}},$$

$$\boldsymbol{p}_3 = \boldsymbol{a}_3 - \boldsymbol{p}_1 + \frac{1}{5}\boldsymbol{p}_2 = \left(\frac{8}{5}, -\frac{6}{5}, 0\right)^{\mathrm{T}}.$$

再单位化

$$\boldsymbol{q}_1 = \frac{1}{2}\boldsymbol{p}_1 = (0,0,1)^{\mathrm{T}}, \quad \boldsymbol{q}_2 = \frac{1}{5}\boldsymbol{p}_2 = \left(\frac{3}{5}, \frac{4}{5}, 0\right)^{\mathrm{T}},$$

$$\boldsymbol{q}_3 = \frac{1}{2}\boldsymbol{p}_3 = \left(\frac{4}{5}, -\frac{3}{5}, 0\right)^{\mathrm{T}},$$

于是

$$\boldsymbol{a}_1 = \boldsymbol{p}_1 = 2\boldsymbol{q}_1, \quad \boldsymbol{a}_2 = \frac{1}{2}\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{q}_1 + 5\boldsymbol{q}_2,$$

$$\boldsymbol{a}_3 = \boldsymbol{p}_1 - \frac{1}{5}\boldsymbol{p}_2 + \boldsymbol{p}_3 = 2\boldsymbol{q}_1 - \boldsymbol{q}_2 + 2\boldsymbol{q}_3.$$

故 $\boldsymbol{A}$ 的 QR 分解为

$$\boldsymbol{A} = \begin{bmatrix} 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & \frac{4}{5} & -\frac{3}{5} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 5 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

$$\|A\|_{m_1} = \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|$$

$$\|A\|_F = \left(\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2\right)^{\frac{1}{2}}$$

$$\|A\|_{m_\infty} = n \cdot \max_{1 \le i,j \le n} |a_{ij}|$$

$$\|x\|_1 = \sum_{i=1}^n |x_i|$$

$$\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$$

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}, \ 1 \le p \le +\infty$$

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|$$