Derivation of attack and release times

Extra notes to go with lecture notes DAFX 5 AE 6 slide 44-47

Step response

$$g(t)=1-e^{-t/t}$$
$$t \to kT_s$$
$$g(k)=1-z_0^k, z_0=e^{-T_s/t}=e^{-1/d}$$
$$g(k)=0, 1-z_0, 1-z_0^2, \ldots$$
$$G(z)=\sum_{k=0}^{\infty} g(k)z^{-k}$$
$$=\sum_{k=1}^{\infty} z^{-k}-\sum_{k=1}^{\infty} z_0{}^k z^{-k}$$
$$\frac{1}{1-z^{-1}}-\frac{1}{1-z_0z^{-1}}$$
$$\frac{z}{z-1}-\frac{1}{1-z_0z^{-1}}$$
$$\frac{(1-z_0z^{-1})z-(z-1)}{(1-z_0z^{-1})(z-1)}$$
$$=\frac{1-z_0}{(1-z_0z^{-1})(z-1)}$$

impulse response

$$H(z)=\frac{z-1}{z}G(z)$$
$$=\frac{z-1}{z}\frac{1-z_0}{(1-z_0z^{-1})(z-1)}$$
$$=\frac{1-z_0}{z(1-z_0z^{-1})}=\frac{(1-z_0)z^{-1}}{(1-z_0z^{-1})}=\frac{Y(z)}{X(z)}$$
$$Y(z)-z_0z^{-1}Y(z)=(1-z_0)z^{-1}X(z)$$
$$y[n]=z_0y[n-1]+(1-z_0)x[n-1]$$

ELEC 350 notes section 5.8.2

De-emphasis with digital filter

This filter can be implemented digitally with a single pole IIR filter with difference equation

$$y[n]-(1-\alpha)y[n-1]=\alpha x[n] \text{ or}$$
$$y_n=(1-\alpha)y_{n-1}+\alpha x_n$$

with transfer function derived as follows:

$$Y(z)-z^{-1}(1-\alpha)Y(z)=\alpha X(z)$$
$$H(z)=\frac{Y(z)}{X(z)}=\frac{\alpha}{1-(1-\alpha)z^{-1}}=\frac{\alpha z}{z-(1-\alpha)}$$

This filter has a single pole at $1-\alpha$ and a single zero at 0.

The impulse response may be found by inverse z-transform or iteration to find

$$h[n]=\alpha(1-\alpha)^n u[n]$$

This filter is implemented in GNURadio GRC as a single pole IIR filter block with parameter α, see
http://gnuradio.org/doc/sphinx-3.7.0/filter/filter_blk.html#gnuradio.filter.single_pole_iir_filter_cc

The Matlab representation is $y=\text{filter}(b,a,x)$ for a general filter, where we define the vectors $a=[1\ a_1\ \ a_2],\ b=[1\ b_1\ \ b_2]$, see
www.mathworks.com/help/matlab/ref/filter.html

for the filter with difference equation

$$y[n]+a_1y[n-1]+a_2y[n-2]=b_0x[n]+b_1x[n-1]+b_2x[n-2]$$

and transfer function

$$H(z)=\frac{b_0+b_1z^{-1}+b_2z^{-2}}{1+a_1z^{-1}+a_2z^{-2}}$$

For the filter considered here $y[n]-(1-\alpha)y[n-1]=\alpha x[n]$

Thus $a=[a_1\ \ a_2]=[1\ \ -(1-\alpha)]$ and $b=[b_0]=[\alpha]$.

The filter characteristics (including frequency response, phase response, phase delay, group delay, pole-zero plot, impulse response) can be viewed using the Matlab command **fvtool(b,a)**, see www.mathworks.com/help/signal/ref/fvtool.html

For example, if $\alpha = 0.1$ then set the values in **fvtool** to be $b = 0.1$ and $a = [1 \quad -0.9]$

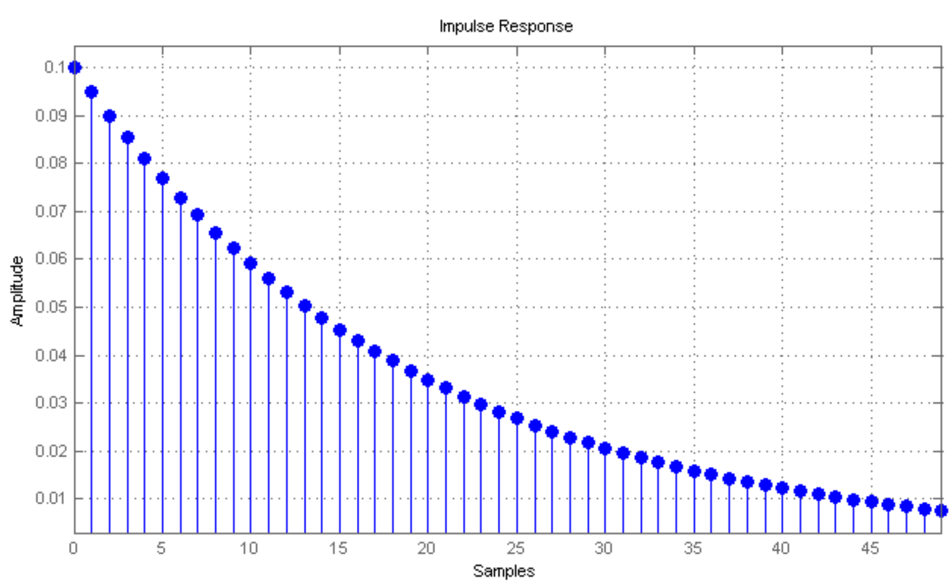
The impulse response of the digital filter is $h[n] = \alpha(1-\alpha)^n u[n]$ and thus is an exponential decay similar to that of the analog RC filter.

The signal decays to $1/e = 36.8$ % of its initial value in d samples such that

$$\frac{h[n=d]}{h[n=0]} = \frac{h_d}{h_0} = \frac{\alpha(1-\alpha)^d}{\alpha} = (1-\alpha)^d = e^{-1} \text{ or } 1-\alpha = e^{-1/d}$$

Given the sampling rate of the filter f_s and the desired time constant $t = RC$ we set the number of samples d needed for the filter output to decay to $1/e = 36.8$ % of its initial value during the time from $t = 0$ to $t = RC$ to be $d/f_s = RC$ or $d = RCf_s$

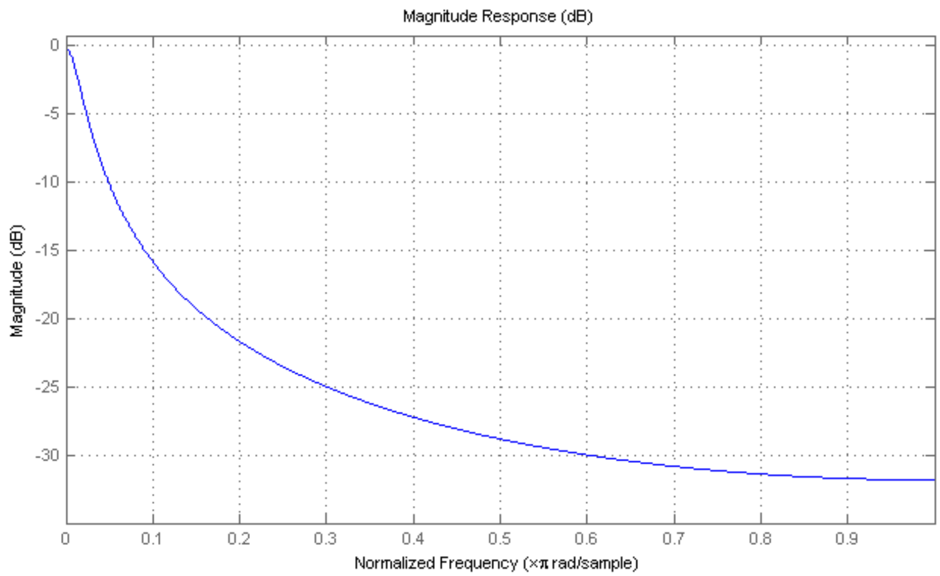
For example, if we assume a sampling rate of 250 KHz and time constant of 75 usec, we set $d = RCf_s = 75{\cdot}10^{-6}{\cdot}0.25{\cdot}10^{6} = 19$ samples.



The value of α is found via $1-\alpha = e^{-1/d}$. In this example $1-\alpha = e^{-1/19} = 0.9487$

An expression for the frequency response is obtained by choosing $z = e^{j2\pi f/f_s}$ around the unit circle in the z-plane, so that

$$H(f) = H(z)|_{z=e^{j2\pi f/f_s}} = \frac{\alpha}{1-(1-\alpha)z^{-1}}|_{z=e^{j2\pi f/f_s}} = \frac{\alpha}{1-(1-\alpha)e^{-j2\pi f/f_s}} = \frac{1}{\alpha^{-1}-(\alpha^{-1}-1)e^{-j2\pi f/f_s}}$$



The gain at $f = 0$ or $z = 1$ is $H(z=1) = \dfrac{\alpha}{1-(1-\alpha)} = \dfrac{\alpha}{\alpha} = 1$

The gain at $f = f_s/2$ or $z = e^{j\pi} = -1$ is $H(z=-1) = \dfrac{\alpha}{1+(1-\alpha)} = \dfrac{\alpha}{2+\alpha} = \dfrac{1}{1+2/\alpha}$

The gain is reduced by 3dB at a cutoff frequency f_C such that $1/d = 2\pi f_C / f_s$ or $(1-\alpha) = e^{-1/d} = e^{-2\pi f_C/f_s}$.

The gain at f_C such that $(1-\alpha) = e^{-2\pi f_C/f_s}$ is

$$H(f = f_C) = \frac{\alpha}{1-(1-\alpha)e^{-j2\pi f/f_s}} = \frac{1-e^{-2\pi f_C/f_s}}{1-e^{-2\pi f_C/f_s}e^{-j2\pi f_C/f_s}}$$

Here f_C / f_s is normalized to the sampling rate so that $0 \le f_C / f_s \le 0.5$

Recall for the analog filter the cutoff frequency $\dfrac{1}{2\pi RC} = \dfrac{1}{2\pi 75 \cdot 10^{-6}} = 2122$ Hz.

We want the gain for the digital filter at 2122 Hz to be 3 dB down.

In this case, assuming a sampling rate of 250 KHz, the normalized frequency is

$$\frac{f_C}{f_s} = \frac{2122}{250000} = 0.00848$$

Substituting these values into the 3dB cutoff frequency result $(1-\alpha) = e^{-2\pi f_C/f_s}$ we find

$e^{-2\pi f_C/f_s} = e^{-0.00848} = 0.9481 = (1-\alpha)$ consistent with the value of α found via $1-\alpha = e^{-1/d}$ above, and also consistent with the frequency (magnitude) response plot.

Thus a digital de-emphasis filter with time constant 75 usec, cutoff frequency 2122 Hz and sampling rate $f_s = 250000$ Hz may be built using a single pole IIR filter with parameter $\alpha = 1-0.948 = 0.052$