EL9343
Data Structure and Algorithm

Lecture 13: NP-Completeness

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Last Lecture

- Single-Source Shortest Paths
  - Nonnegative edge weights: Dijkstra’s algorithm
  - Unweighted graphs: BFS
  - General case: Bellman-Ford algorithm
  - DAG: Topological sort + one pass Bellman-Ford

- All Pairs Shortest Paths
  - Nonnegative edge weights: |V| times of Dijkstra’s algorithm
  - Unweighted graphs: |V| times of BFS
  - General case: Floyd-Warshall algorithm

NP Complete Problems

- The course so far: techniques for designing efficient algorithms, e.g., divide-and-conquer, dynamic programming,greedy algorithms.
- What happens if you can't find an efficient algorithm?
  - Is it your "fault" or the problem's?
- Showing that a problem has an efficient algorithm is, relatively, easy. "All" that is needed is to demonstrate an algorithm.
- Proving that no efficient algorithm exists for a particular problem is difficult. How can we prove the nonexistence of something?
- We will now learn about NP Complete Problems, which provide us with a way to approach this question.

NP-Complete Problems

This is a very large class of thousands of practical problems for which

- It is not known if the problems have “efficient” solutions
- It is known that if any one of the NP-Complete Problems has an efficient solution then all of the NP-Complete Problems have efficient solutions
- Researchers have spent innumerable man-years trying to find efficient solutions to these problems and failing
- There is a large body of tools that often permit us to prove when a new problem is NP-complete.
- The problem of finding an efficient solution to an NP-Complete problem is known, in shorthand as P = NP? There is currently a US $1,000,000 award offered by the Clay Institute for its solution (https://www.claymath.org/millennium-problems)

Today:

NP / NP complete / Prove the problem is N-P complete

Optimization & Decision Problems

- **Decision problems**
  - Given an input and a question regarding a problem, determine if the answer is yes or no
- **Optimization problems**
  - Find a solution with the “best” value
- Optimization problems can be cast as decision problems that are easier to study
  - *E.g.:* Shortest path: G = unweighted directed graph
    - Find a path between u and v that uses the fewest edges
    - *Does a path exist from u to v consisting of at most k edges?*

Class of “P” Problems

- **Class P** consists of (decision) problems that are solvable in polynomial time
- Polynomial-time algorithms
  - Worst-case running time is $O(n^k)$, for some constant k
- Examples of polynomial time:
  - $O(n^2)$, $O(n^3)$, $O(1)$, $O(n \lg n)$
- Examples of non-polynomial time:
  - $O(2^n)$, $O(n^n)$, $O(n!)$
- Are non-polynomial algorithms always worse than polynomial algorithms?
  - - $n^{1,000,000}$ is *technically* $\xrightarrow{}$ polynomial time tractable, but really impossible - $n^{\log \log \log n}$ is *technically* intractable, but easy

$\searrow$ non-polynomial time.

Tractable/Intractable Problems

- Problems in P are also called **tractable**
- Problems **not** in P are **intractable or undecidable**
  - Can be solved in reasonable time only for small inputs, as they grow large, we are unable to solve them in reasonable time
  - Or, can not be solved at all (e.g, Halting Problem)

Example of Unsolvable Problem

- Turing discovered in the 1930's that there are problems **unsolvable/undecidable** by *any* algorithm.
- The most famous of them is the ***halting problem***
  - Given an arbitrary algorithm and its input, will that algorithm eventually halt, or will it continue forever in an "*infinite loop*?"
  - Decision problem: answer is yes or no
  - Uncomputable: no algorithm solves it (correctly in finite time on all inputs)

We don't care about such problems here; take a theory class

Intractable Problems

- Can be classified in various categories based on their degree of difficulty, e.g.,
  - NP
  - NP-complete
  - NP-hard
- Let's define NP algorithms and NP problems …

Nondeterministic and NP Algorithms

- **Nondeterministic algorithm** = two stage procedure:
- Nondeterministic (“guessing”) stage:
  - generate randomly an arbitrary string that can be thought of as a candidate solution (“certificate”)
- Deterministic (“verification”) stage:
  - take the certificate and the instance to the problem and returns YES if the certificate represents a solution
- **NP algorithms (Nondeterministic polynomial)**
  - verification stage is polynomial

Class of "NP" Problems

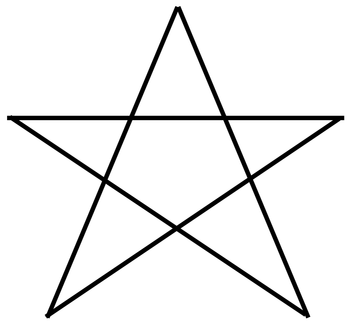
- **Class NP** consists of problems that could be solved by NP algorithms
  - i.e., verifiable in polynomial time
- If we were given a "certificate" of a solution, we could verify that the certificate is correct in time polynomial to the size of the input
- Warning: NP does **not** mean "non-polynomial"

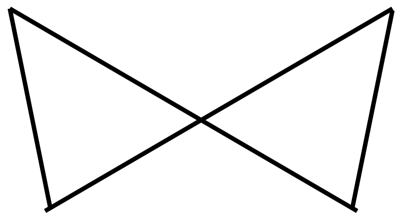
Example: Hamiltonian Cycle

- **Given:** a directed graph G = (V, E), determine a simple cycle that contains each vertex in V

  each cycle visit once only

  - Each vertex can only be visited once
- **Certificate**:
  - Sequence: $\langle v_1, v_2, v_3, \ldots, v_{|V|} \rangle$
- Cannot solve in polynomial time
- Can verify solution in polynomial time

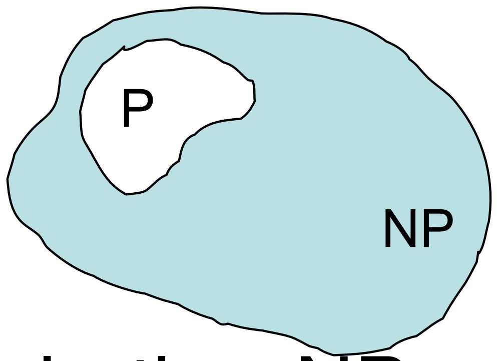




Is P = NP?

- Any problem in P is also in NP:

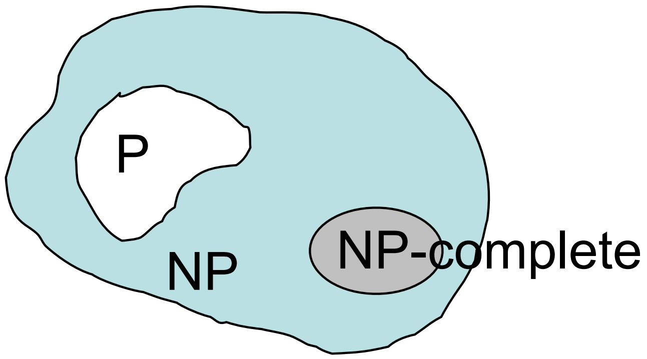
$$P \subseteq NP$$



- The big (and **open question**) is whether $NP \subseteq P$ or $P = NP$
  - i.e., if it is always easy to check a solution, should it also be easy to find a solution?
- Most computer scientists believe that this is false but we do not have a proof …

NP-Completeness (informally)

- **NP-complete** problems are defined as the hardest problems in NP: an interesting class of problems whose status is unknown



- Most practical problems turn out to be either P or NP-complete.

Proof NP complete.

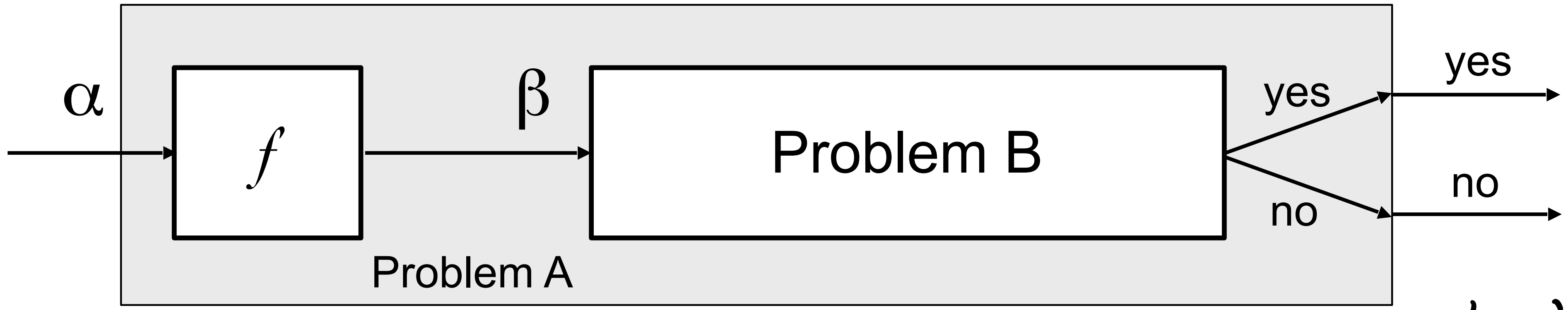
Reductions

- Reduction is a way of saying that one problem is "**easier**" than another.
- We say that problem A is easier than problem B, (i.e., we write "**$A \le B$**")

  if we can solve A using the algorithm that solves B.

Intuitively: If A reduces in polynomial time to B, A is "no harder to solve" than B

Idea: transform the inputs of A to inputs of B



transform problem A → problem B in polynomial time. A reduce in polynomial to B

Polynomial Reductions

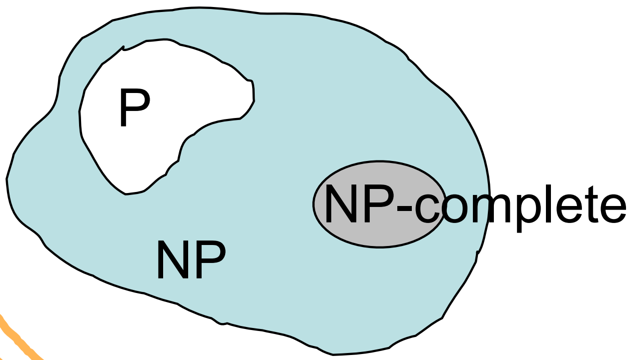
- Given two problems A, B, we say that A is polynomially **reducible** to B ($A \leq_p B$) if:
  - There exists a function *f* that converts the input of A to inputs of B in polynomial time
  - A(i) = YES ⇔ B(f(i)) = YES

NP-Completeness (formally)

- A problem B is **NP-complete** if:

  (1) B $\in$ **NP**

  (2) A $\leq_p$ B for all A $\in$ **NP**



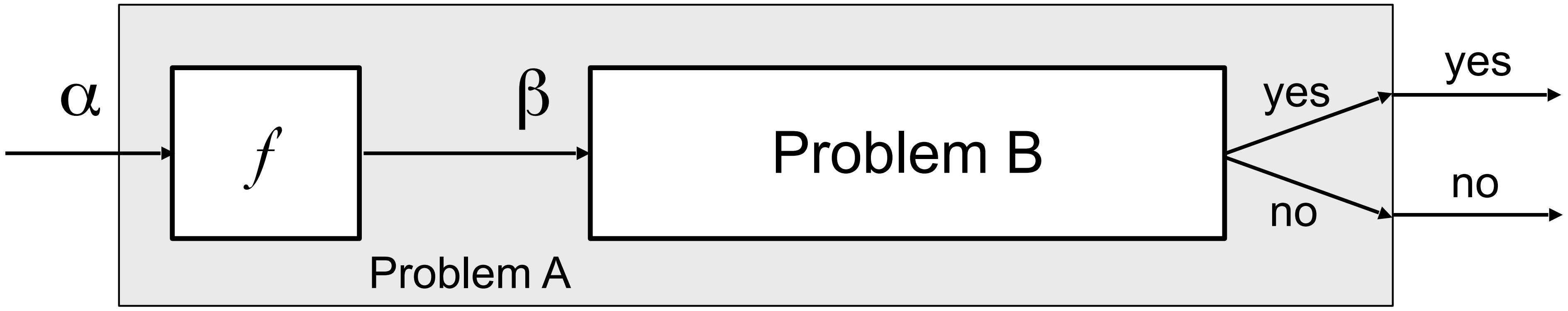
NOW!)

- If B satisfies only property (2) we say that B is **NP-hard**
- No polynomial time algorithm has been discovered for an **NP-Complete** problem
- No one has ever proven that no polynomial time algorithm can exist for any **NP-Complete** problem

NP-naming convention

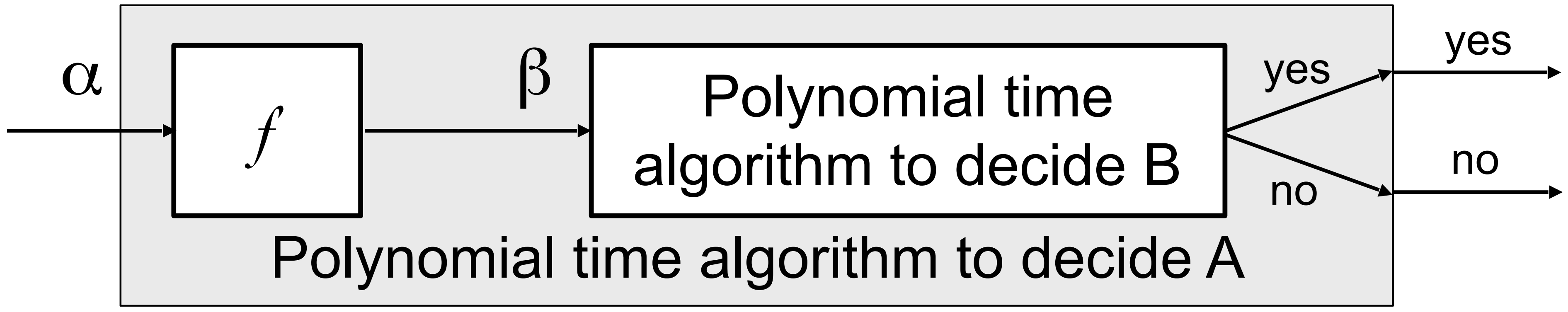
- **NP-complete - means problems that are 'complete' in NP, i.e. the most difficult to solve in NP**
- **NP-hard** $\geq NP$ - stands for 'at least' as hard as NP (but not necessarily **in** NP);
- **NP-easy** $\leq NP$ - stands for 'at most' as hard as NP (but not necessarily **in** NP);
- **NP-equivalent** $= NP$ - means equally difficult as NP, (but not necessarily **in** NP); But difficulty level maybe the same as NP.

Implications of Reduction



- If $A \leq_p B$ and $B \in P$, then $A \in P$
- If $A \leq_p B$ and $A \notin P$, then $B \notin P$
- If $A \leq_p B$ and A is NP-Complete, B is NP-Hard. In addition, if $B \in NP \Rightarrow$ B is NP-Complete

Proving Polynomial Time


α
f
β
Polynomial time algorithm to decide B
yes
no
yes
no
Polynomial time algorithm to decide A

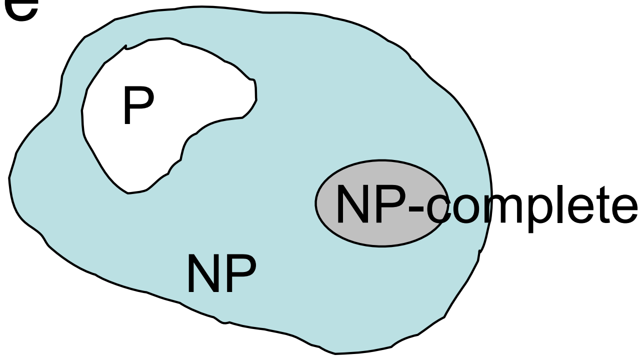
1. Use a **polynomial time** reduction algorithm to transform A into B
2. Run a known **polynomial time** algorithm for B
3. Use the answer for B as the answer for A

Proving NP-Completeness In Practice

- Prove that the problem B is in NP
  - A randomly generated string can be checked in polynomial time to determine if it represents a solution
- Pick a known NP-Complete problem A
- Reduce A to B: show that **one known** NP-Complete problem A can be transformed to B in polynomial time
  - No need to check that **all** NP-Complete problems are reducible to B

Revisit "Is P = NP?"

Theorem: If any NP-Complete problem can be solved in polynomial time ⇒ then P = NP.



- If any *one* NP-Complete problem can be solved in polynomial time…
- …then *every* NP-Complete problem can be solved in polynomial time…
- …and in fact *every* problem in **NP** can be solved in polynomial time (which would show **P = NP**)
- Thus: solve hamiltonian-cycle in $O(n^{100})$ time, you've proved that **P = NP**. Retire rich & famous.

Reductions Examples

Convert your problem into a problem you already know how to solve (instead of solving from scratch)

- unweighted shortest path (least-hops) → weighted (set weights = 1)
- min-product path → shortest path (take logs)

$\log W_p = \log(w_1 \times w_2 \cdots w_m)$
$= \sum_i \log w_i = \log \prod_{i=1}^{N} w_i$

- longest path in DAG → shortest path in DAG (negate weights)
    - longest path in general graph is NP-Complete

Satisfiability Problem (SAT)

Definition: A Boolean formula is a logical formula which consists of

boolean variables (0=false, 1=true),

logical operations

$\bar{x}$,	NOT,
$x \vee y$,	OR,
$x \wedge y$,	AND.

These are defined by:

x	y	$\bar{x}$	$x \vee y$	$x \wedge y$
0	0	1	0	0
0	1		1	0
1	0	0	1	0
1	1		1	1

- SAT problem: Determine whether an input Boolean formula is satisfiable. If an Boolean formula is satisfiable, it is a yes-input; otherwise, it is a no-input.
- SAT was the first problem shown to be NP-complete!

No reduction can be used.

Satisfiability Problem (SAT)

A given Boolean formula is *satisfiable* if there is a way to assign truth values (0 or 1) to the variables such that the final result is 1.

Example: $f(x, y, z) = (x \wedge (y \vee \bar{z})) \vee (\bar{y} \wedge z \wedge \bar{x})$.

x	y	z	$(x \wedge (y \vee \bar{z}))$	$(\bar{y} \wedge z \wedge \bar{x})$	$f(x, y, z)$
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	1	0	1
1	0	1	0	0	0
1	1	0	1	0	1
1	1	1	1	0	1

For example, the assignment $x = 1$, $y = 1$, $z = 0$ makes $f(x, y, z)$ true, and hence it is satisfiable.

Satisfiability Problem (SAT)

Example:

$$f(x,y) = (x \vee y) \wedge (\bar{x} \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{x} \vee \bar{y}).$$

All cases.

x	y	$x \vee y$	$\bar{x} \vee y$	$x \vee \bar{y}$	$\bar{x} \vee \bar{y}$	$f(x,y)$
0	0	0	1	1	1	0
0	1	1	1	0	1	0
1	0	1	0	1	1	0
1	1	1	1	1	0	0

There is no assignment that makes $f(x,y)$ true, and hence it is NOT satisfiable.

K-CNF-SAT Problem

For a fixed k, consider Boolean formulas in k-conjunctive normal form (k-CNF):

$$f_1 \wedge f_2 \wedge \cdots \wedge f_n$$

where each f_i is of the form

$$f_i = y_{i,1} \vee y_{i,2} \vee \cdots \vee y_{i,k}$$

where each $y_{i,j}$ is a variable or the negation of a vari-able.

- **3-SAT: NP Complete**
- **2-SAT: P**

An example of a 3-CNF formula is

$$(x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee x_4).$$

k-SAT problem: Determine whether an input Boolean k-CNF formula is satisfiable.

Other NP-Complete Problems

- Knapsack
- 3-Partition: given n integers, can you divide them into triples of equal sum?
- Traveling Salesman Problem: shortest path that visits all vertices of a given graph — decision version: is minimum weight ≤ x?
- Longest common subsequence of k strings
- Shortest paths amidst obstacles in 3D