EL9343

Data Structure and Algorithm

Lecture 11: Greedy Algorithm, Minimum Spanning Tree

Instructor: Yong Liu

Last Lecture

- Dynamic Programming
  - Rod Cutting Problem
  - Longest Common Subsequence Problem

- Introduction to Greedy Algorithm
  - An Activity-Selection Problem
  - Knapsack Problem

Today

- Greedy Algorithm (cont.)
  - Huffman codes

- Algorithm & It's Application in Network
  - Minimum Spanning Trees
    - Prim's algorithm
    - Kruskal's algorithm

Huffman Coding

- **Coding** is used for data compression
- **Binary character code:** character is represented by a unique binary string
  - Fixed-length code (block code)
    - a: 000, b: 001, …, f: 101
    - ace: 000 010 100
  - Variable-length code
    - frequent characters: short codeword
    - infrequent characters: long codeword

	a	b	c	d	e	f	cost / 100 characters
Frequency	45	13	12	16	9	5	
Fixed–length codeword	000	001	010	011	100	101	300
Variable–length codeword	0	101	100	111	1101	1100	224

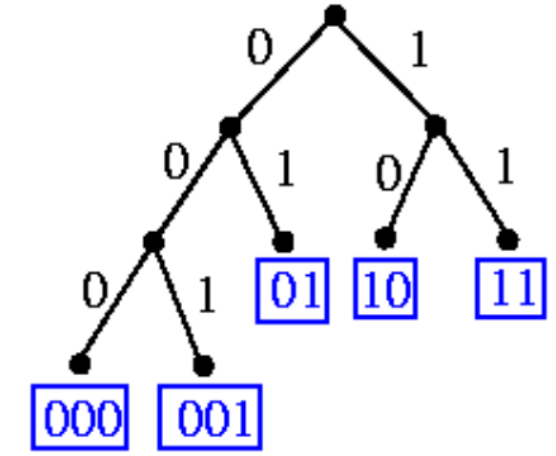
Prefix codes

- Prefix codes
  - one code per input symbol
  - no code is a prefix of another

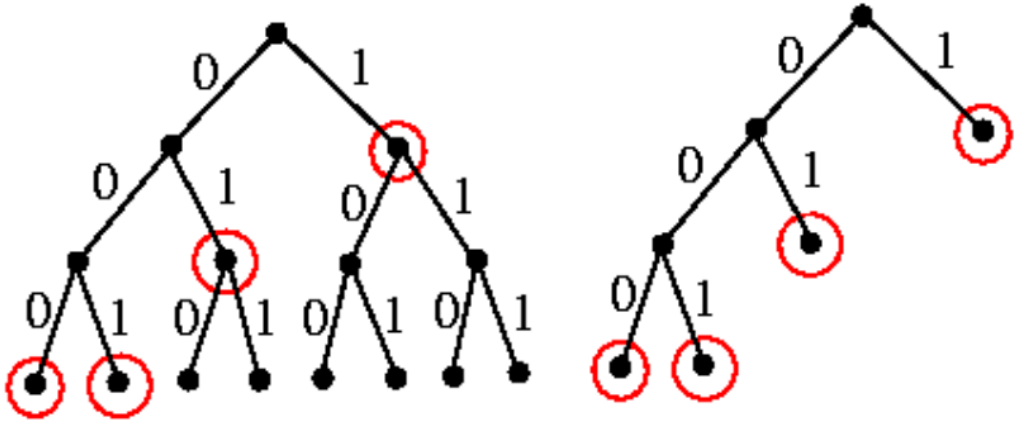
if 10110 stands for C
Then 1, 10, 101, 1011 can not stands for any other letter.

- Why prefix codes?
  - Easy decoding
  - Since no codeword is a prefix of any other, the codeword that begins an encoded file is unambiguous
  - Identify the initial codeword, translate it back to the original character, and repeat the decoding process on the remainder of the encoded file

Binary Tree vs. Prefix Code

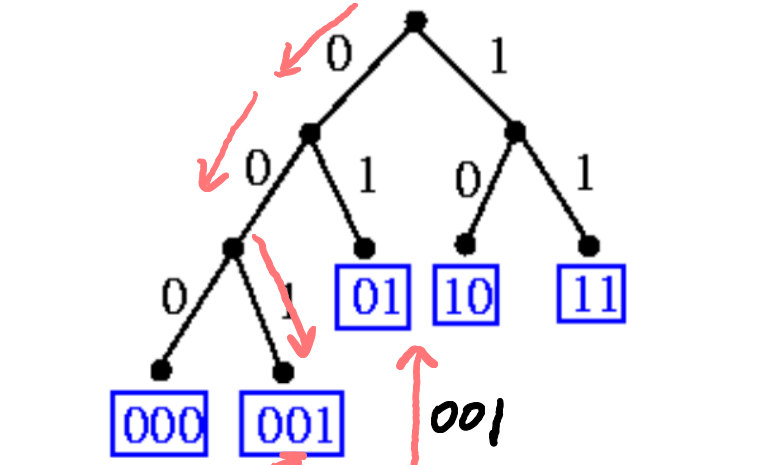


binary tree –> prefix code



prefix code {1, 01, 000, 001}–> binary tree

- Any binary prefix coding can be described by a binary tree in which the codewords are the leaves of the tree, and where a left branch means "0" and a right branch means "1".

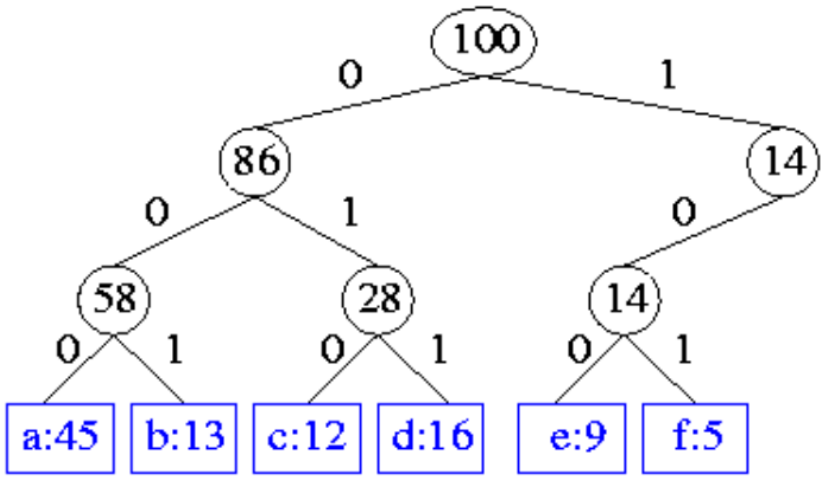


decoding: 01 10 000 000 001 11 11

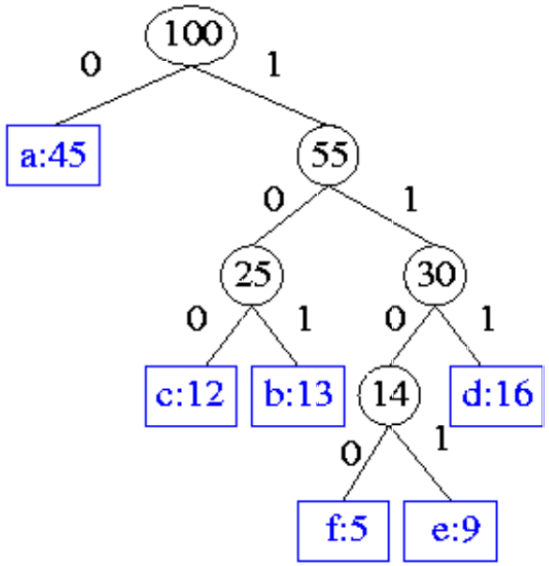
arrive at a leaf, start at root

Optimal Prefix Code Design

- Coding Cost of T: $B(T)=\sum_{c \in C} c.freq \cdot d_T(c)$ = length(c)
  - c: character in the alphabet C
  - c.freq: frequency of c
  - $d_T(c)$: depth of c's leaf (length of the codeword of c)
- Code design: Given $c_1.freq, c_2.freq, \ldots, c_n.freq$, construct a binary tree with n leaves such that B(T) is minimized.
  - Idea: more frequently used characters use shorter depth.

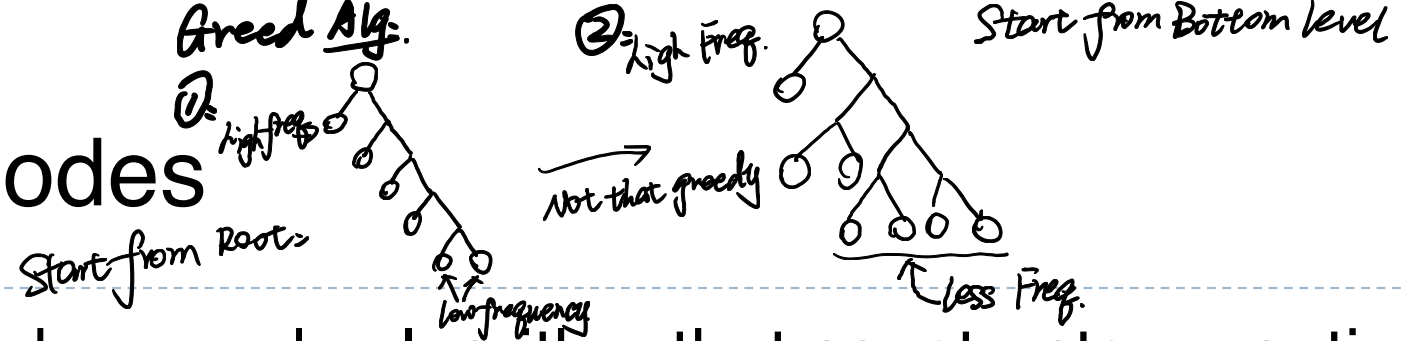


Fixed–length cost: 3 * 100 = 300



Variable–length cost = 224

Huffman Codes

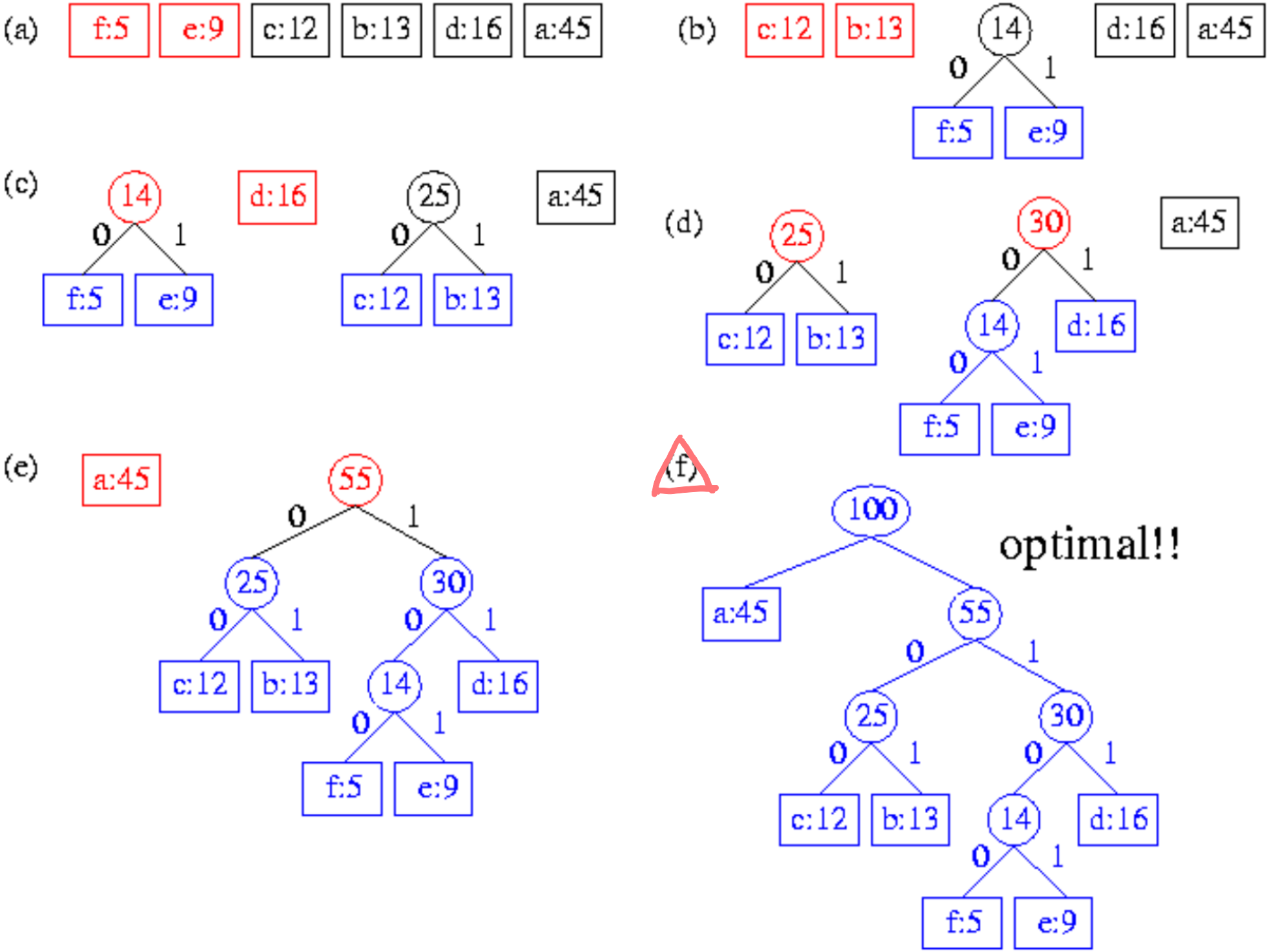


Huffman invented a greedy algorithm that constructs an optimal prefix code called a Huffman code. The algorithm builds the tree T corresponding to the optimal code in a bottom-up manner.

- Create tree (leaf) node for each symbol that occurs with nonzero frequency
  - Node weights = frequencies
- Find two nodes with smallest frequency
- Create a new node with these two nodes as children, and with weight equal to the sum of the weights of the two children
- Continue until have a single tree

Huffman's Procedure

- 1. Place the elements into minimum heap (by frequency).
- 2. Remove the first two elements from the heap.
- 3. Combine these two elements into one.
- 4. Insert the new element back into the heap.


(a)
f:5
e:9
c:12
b:13
d:16
a:45
(b)
c:12
b:13
14
0
1
f:5
e:9
d:16
a:45
(c)
14
0
1
f:5
e:9
d:16
25
0
1
c:12
b:13
a:45
(d)
25
0
1
c:12
b:13
30
0
1
14
0
1
f:5
e:9
d:16
a:45
(e)
a:45
55
0
1
25
0
1
c:12
b:13
30
0
1
14
0
1
f:5
e:9
d:16
(f)
100
a:45
55
0
1
25
0
1
c:12
b:13
30
0
1
14
0
1
f:5
e:9
d:16
optimal!!

Huffman's Algorithm

```
Huffman(C)
1. n = |C|
2. Q = C
3. for i = 1 to n -1
4.     Allocate a new node z
5.     z.left = x = Extract-Min(Q)
6.     z.right = y = Extract-Min(Q)
7.     z.freq = x.freq + y.freq
8.     Insert(Q, z)
9. return Extract-Min(Q)  //return the root of the tree
```

logn = Build-the heap

logn = Extract-min.

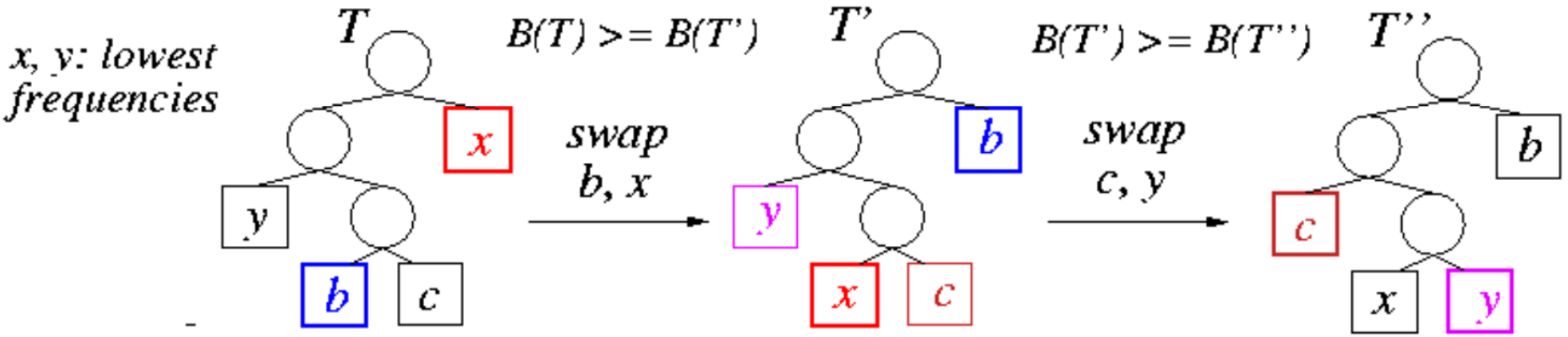
Time complexity: O(nlgn).

- Extract-Min(Q) needs O(lg n) by a heap operation.
- Requires initially O(n) time to build a binary heap.

Huffman's Algorithm: Greedy Choice

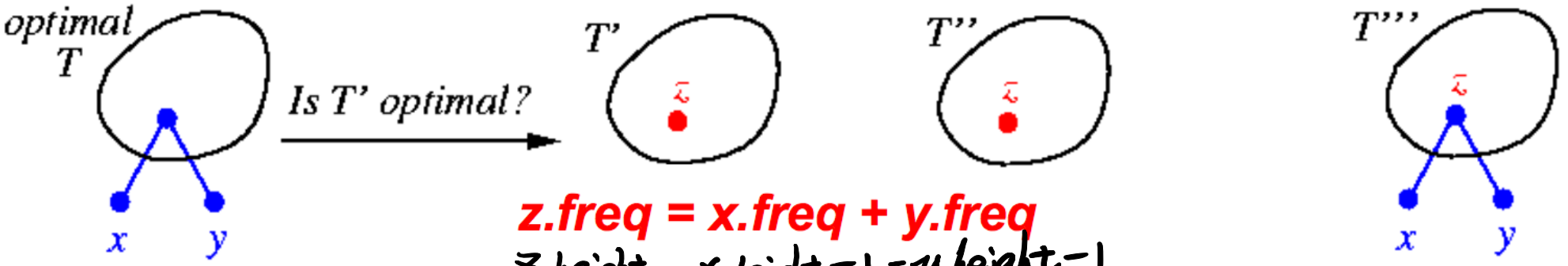
Greedy choice: The binary tree for optimal prefix code must be full, and the two characters x and y with the lowest frequencies must have the same longest length and differ only in the last bit.

have the same parent.



Huffman's Algorithm: Optimum Substructure

Optimal substructure: Let T be a full binary tree for an optimal prefix code over C. Let z be the parent of two leaf characters x and y. If z.freq = x.freq + y.freq, tree T' = T - {x, y} represents an optimal prefix code for C'= C - {x, y} U {z}.



z.freq = x.freq + y.freq

z.height = x.height − 1 = y.height − 1

B(T) = B(T')+x.freq+y.freq

($d_T(x)=d_T(y)=d_{T'}(z)+1$)

If T' is not optimal, find T'' s.t. B(T'')<B(T')

z is in C' => z is a leaf of T''

Add x, y as z's children to form T'''

=> B(T''') = B(T'')+x.freq+y.freq

< B(T')+x.freq+y.freq

= B(t) = B(T)

B(T'')<B(T')

Contradiction!!

Minimum Spanning Tree Problem

MST: The subset of edges that connected all vertices in the graph, and has minimum total weight

Greedy algorithm for solving MST

- Prim's algorithm
- Kruskal's algorithm

Minimum Spanning Trees

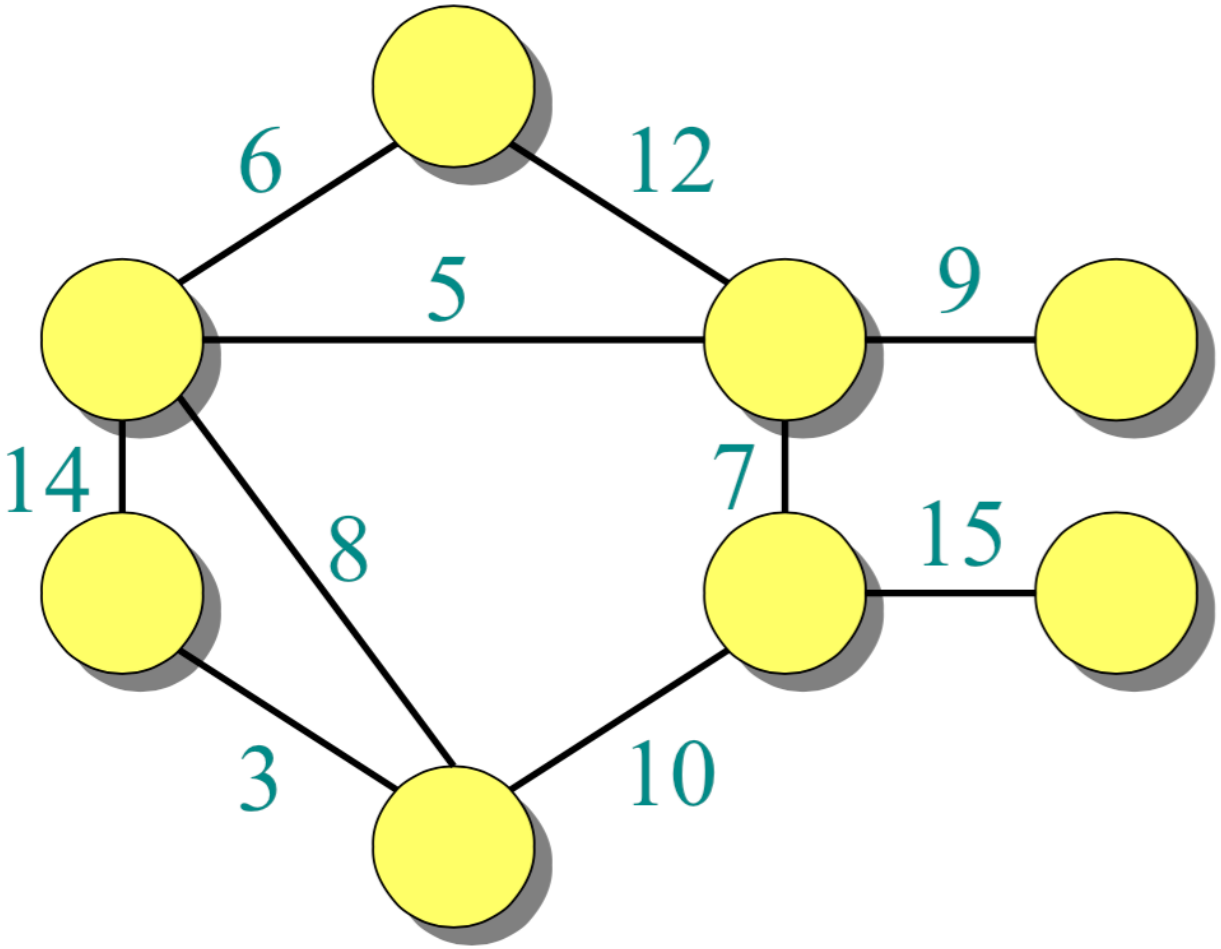
Input: A connected, undirected graph G = (V, E) with weight function w : E → R.

- For simplicity, assume that all edge weights are distinct. (CLRS covers the general case.)

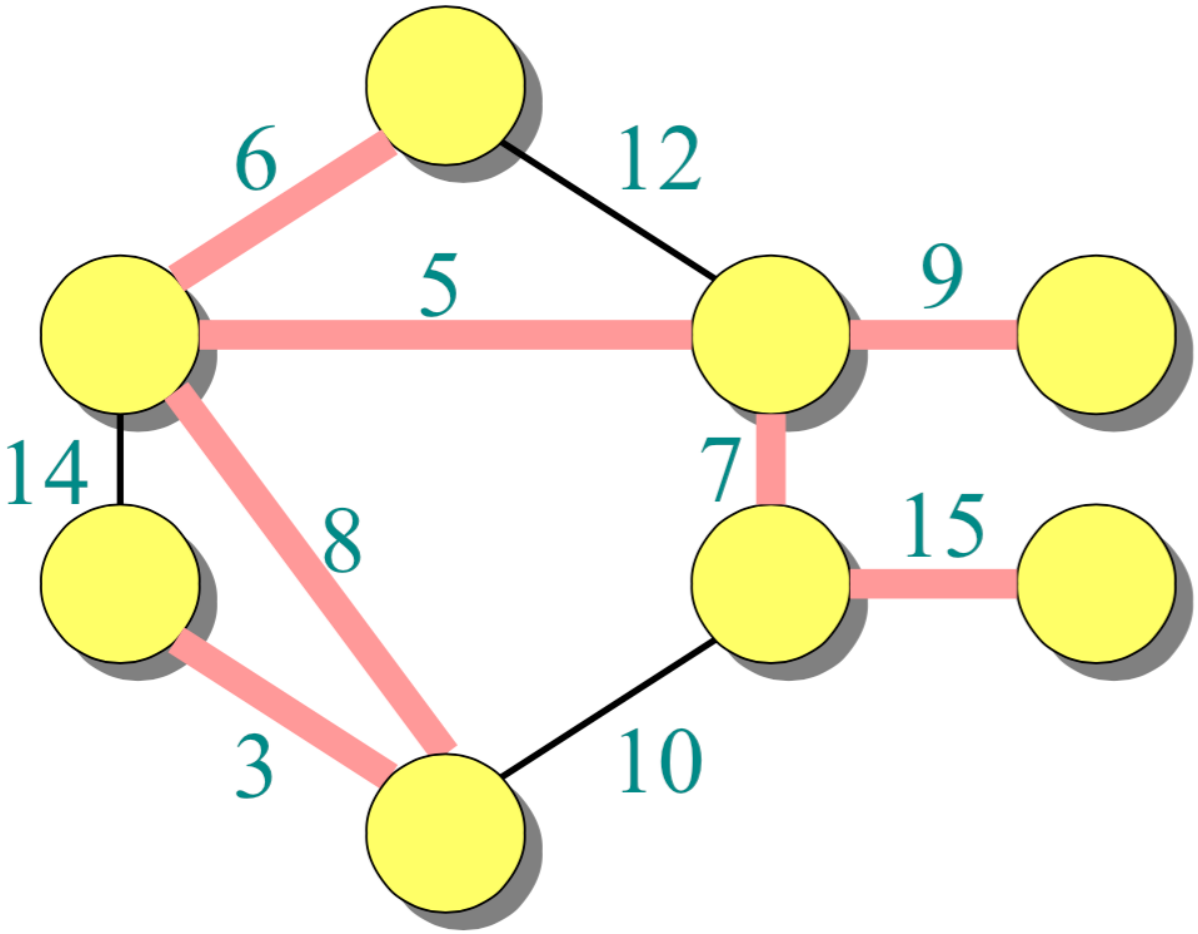
Output: A spanning tree T — a tree that connects all vertices — of minimum weight:

$$w(T) = \sum_{(u,v) \in T} w(u,v).$$

Example of MST



Example of MST



MST Algorithms

Greedy Algorithms: Prim, Kruskal

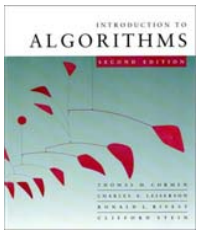
1. Start with initial tree/forest

2. Gradually grow it by adding the lowest weight edge

3. Finish until all nodes are connected in a tree

Prim's Algorithm

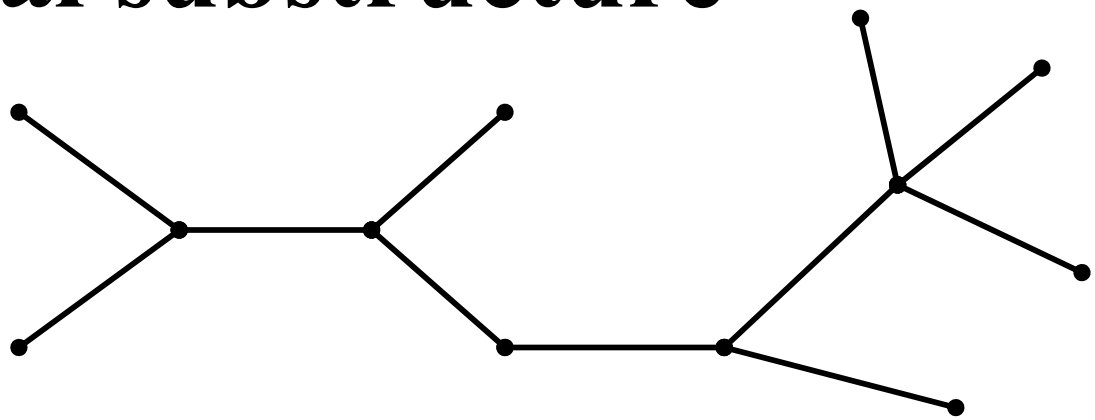
Slides from Demaine and Leiserson

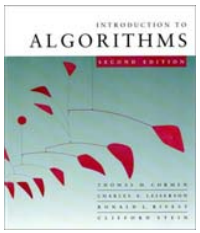


Optimal substructure

MST T:

(Other edges of G are not shown.)





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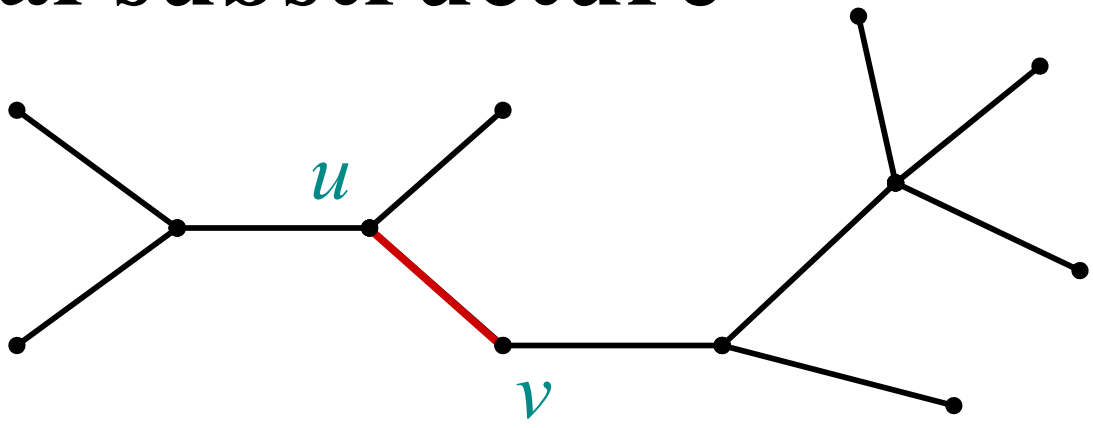
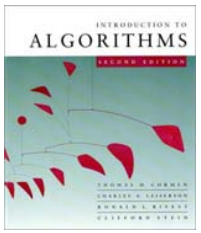


Remove any edge $(u, v) \in T$.



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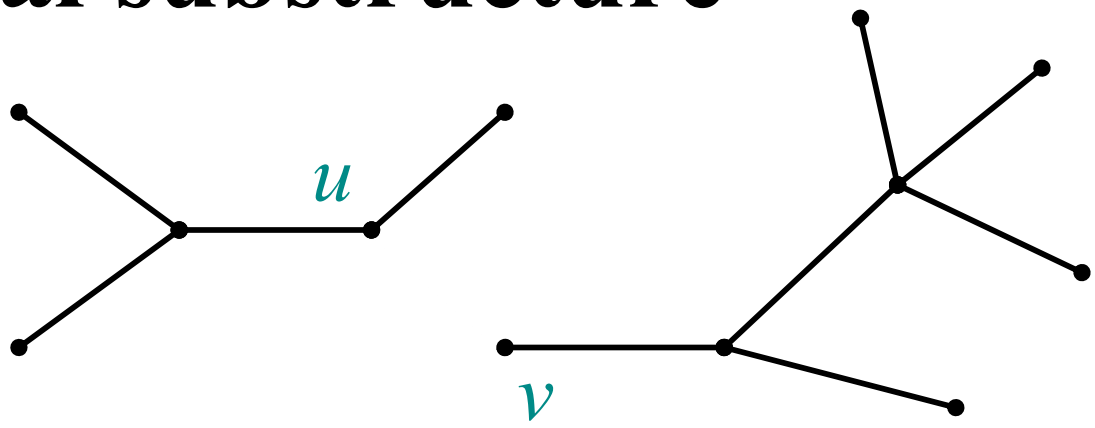

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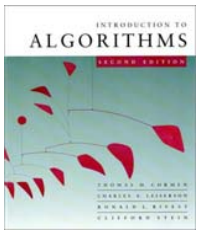

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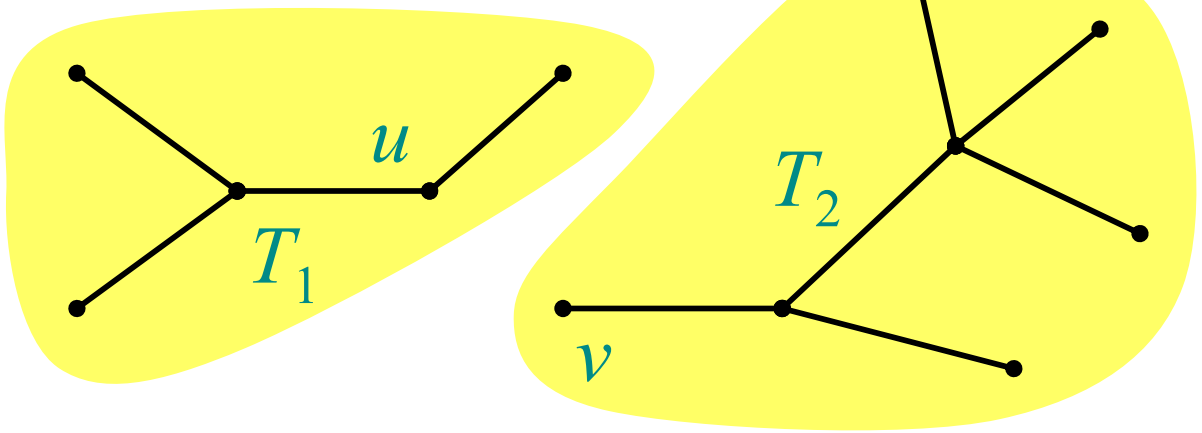
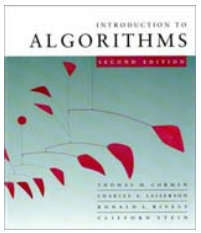


Remove any edge $(u, v) \in T$. Then, T is partitioned into two subtrees T_1 and T_2.



Optimal substructure

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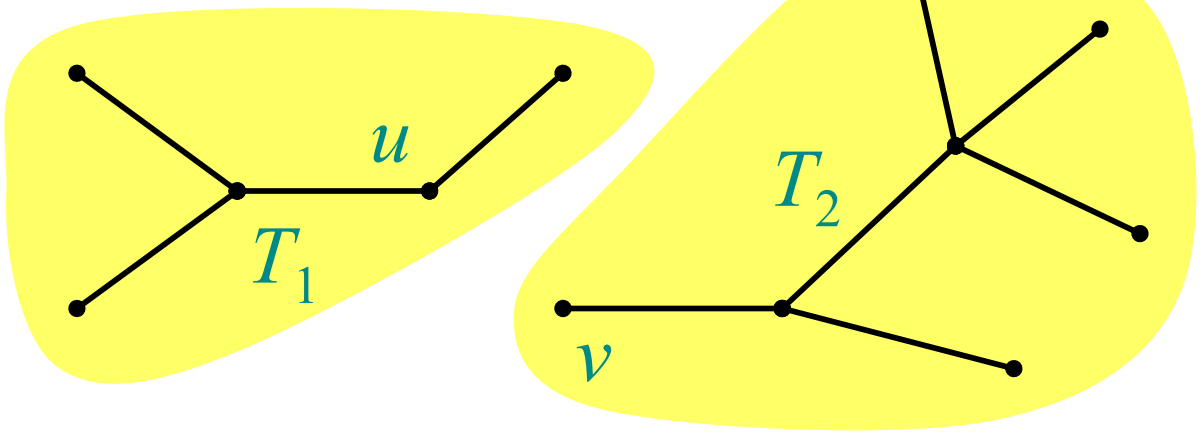
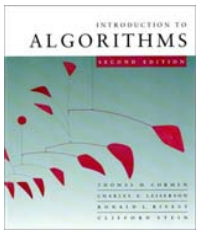


Remove any edge $(u, v) \in T$. Then, T is partitioned into two subtrees T_1 and T_2.

Theorem. The subtree T_1 is an MST of $G_1 = (V_1, E_1)$, the subgraph of G ***induced*** by the vertices of T_1:

$$V_1 = \text{vertices of } T_1,$$
$$E_1 = \{ (x, y) \in E : x, y \in V_1 \}.$$

Similarly for T_2.



Proof of optimal substructure

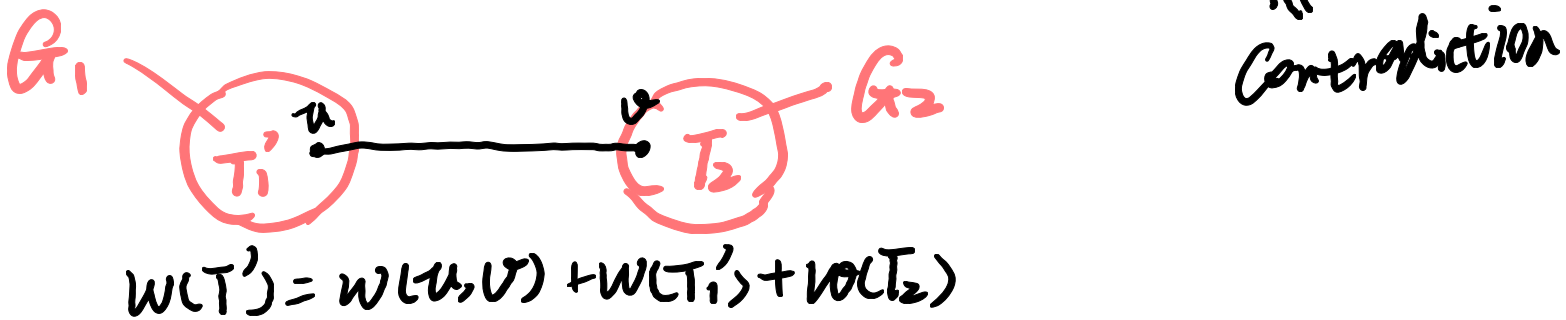
Proof. Cut and paste: $T = T_1 + T_2 + (u, v)$

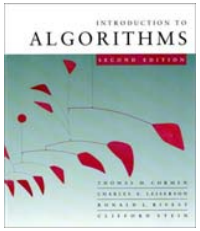
$$w(T) = w(u, v) + w(T_1) + w(T_2).$$

Assume:

If T_1' were a lower-weight spanning tree than T_1 for G_1, then $T' = \{(u, v)\} \cup T_1' \cup T_2$ would be a lower-weight spanning tree than T for G. □

Contradiction





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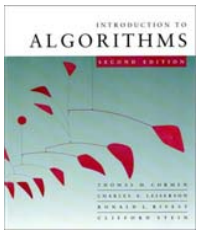
Do we also have overlapping subproblems?

- Yes.



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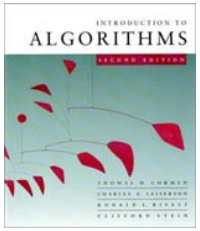
Do we also have overlapping subproblems?

- Yes.

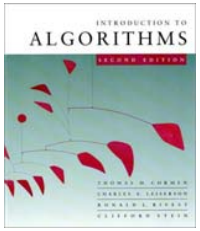
Great, then dynamic programming may work!

- Yes, but MST exhibits another powerful property which leads to an even more efficient algorithm.



Hallmark for "greedy" algorithms

Greedy-choice property
A locally optimal choice is globally optimal.



Hallmark for "greedy" algorithms

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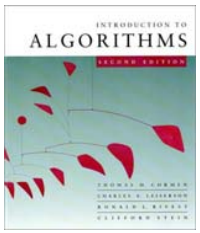
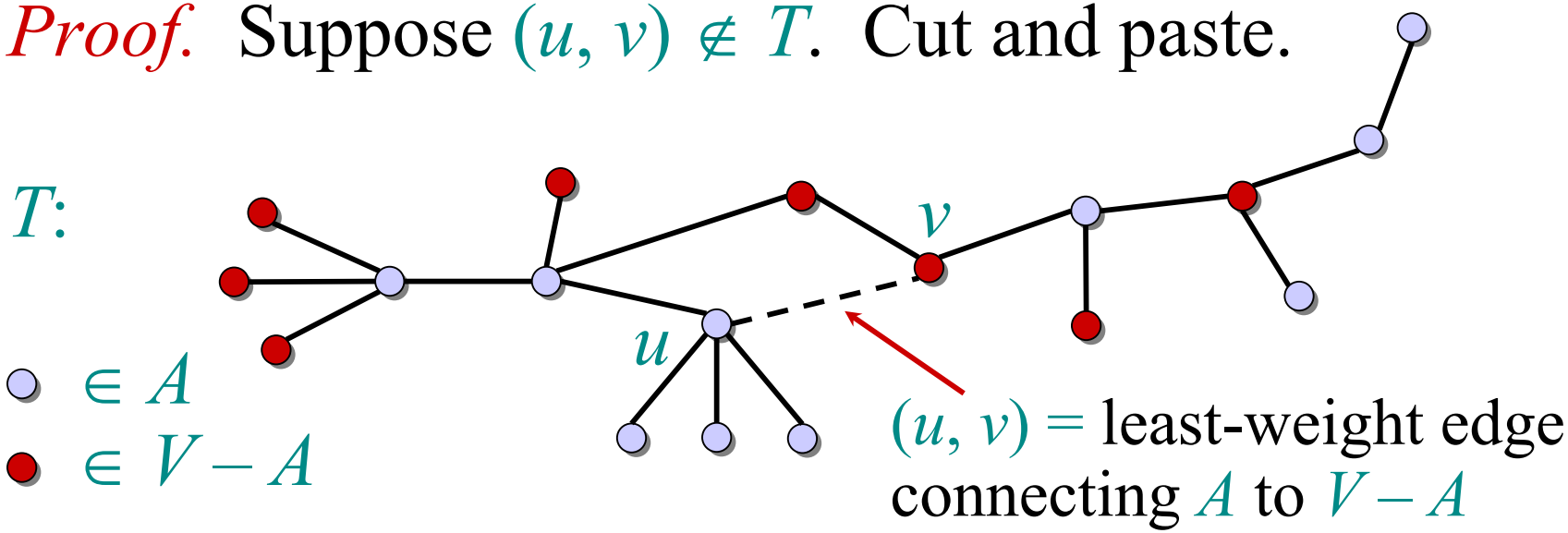
A locally optimal choice is globally optimal.

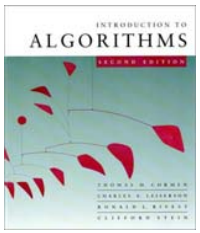
Theorem. Let T be the MST of $G = (V, E)$, and let $A \subseteq V$. Suppose that $(u, v) \in E$ is the least-weight edge connecting A to $V - A$. Then, $(u, v) \in T$.



Proof of theorem

Proof. Suppose $(u, v) \notin T$. Cut and paste.





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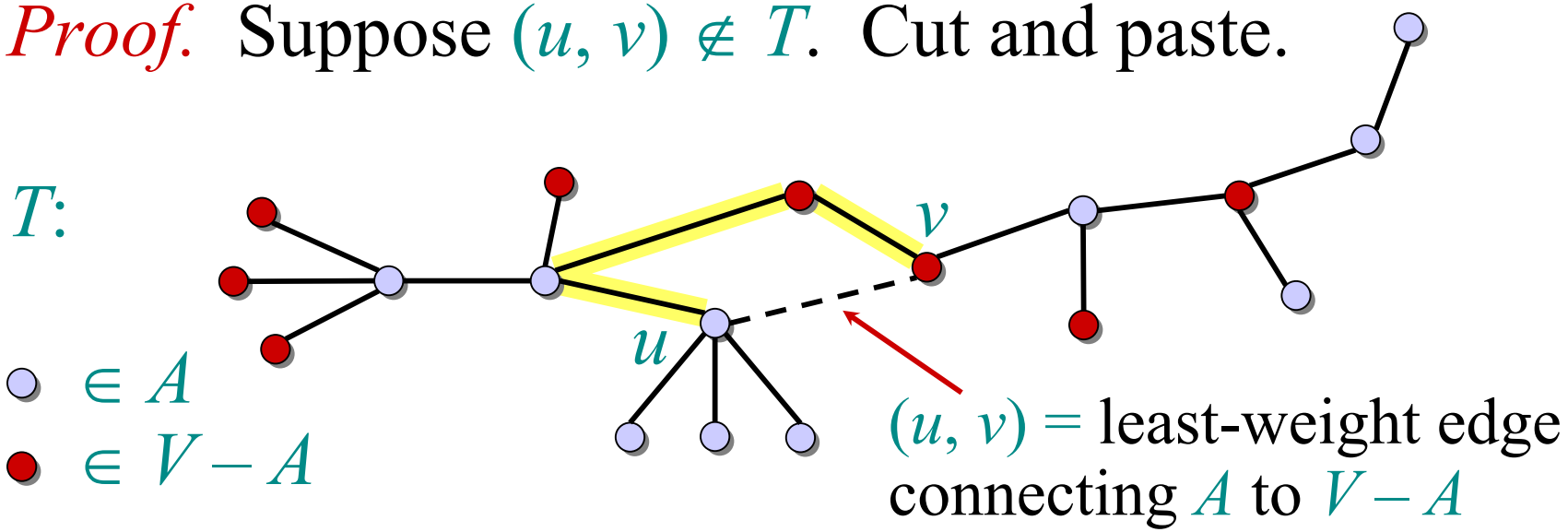
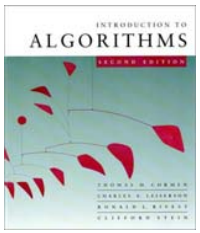


Consider the unique simple path from u to v in T.



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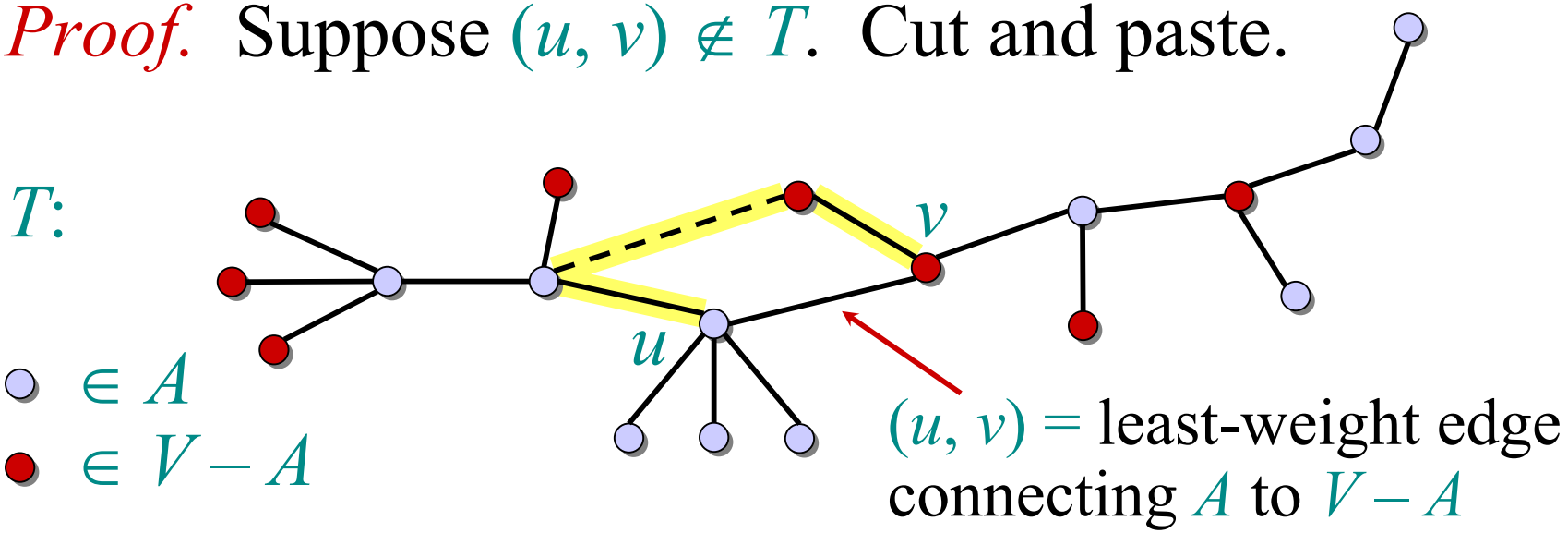
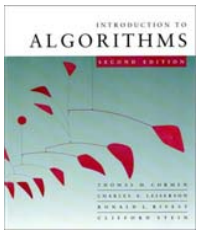


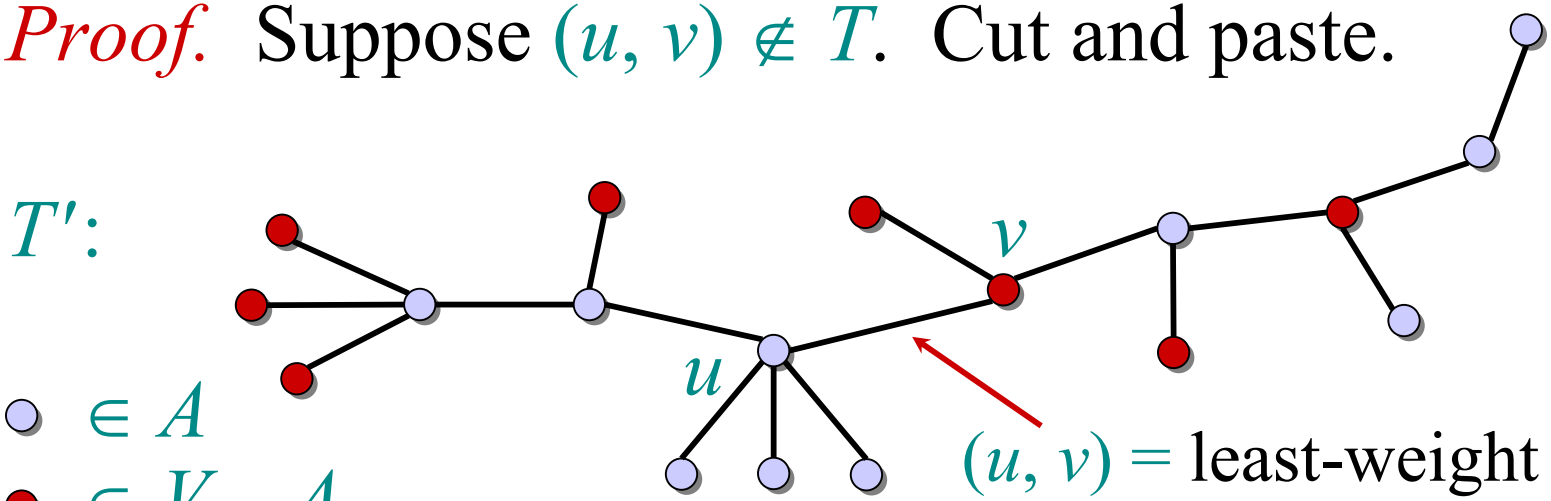
Consider the unique simple path from u to v in T.

Swap (u, v) with the first edge on this path that connects a vertex in A to a vertex in $V - A$.



Proof of theorem

Proof. Suppose $(u, v) \notin T$. Cut and paste.

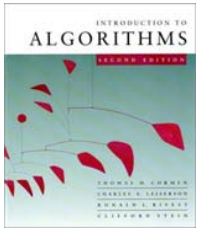


Consider the unique simple path from u to v in T.

Swap (u, v) with the first edge on this path that connects a vertex in A to a vertex in $V - A$.

A lighter-weight spanning tree than T results.

Contradiction. □



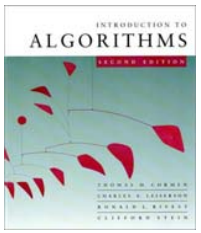
Prim's algorithm

IDEA: Maintain $V - A$ as a priority queue Q. Key each vertex in Q with the weight of the least-weight edge connecting it to a vertex in A.

$Q \leftarrow V$
$key[v] \leftarrow \infty$ for all $v \in V$
$key[s] \leftarrow 0$ for some arbitrary $s \in V$
while $Q \neq \varnothing$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    for each $v \in Adj[u]$
      do if $v \in Q$ and $w(u, v) < key[v]$
        then $key[v] \leftarrow w(u, v)$ ▷ DECREASE-KEY
          $\pi[v] \leftarrow u$

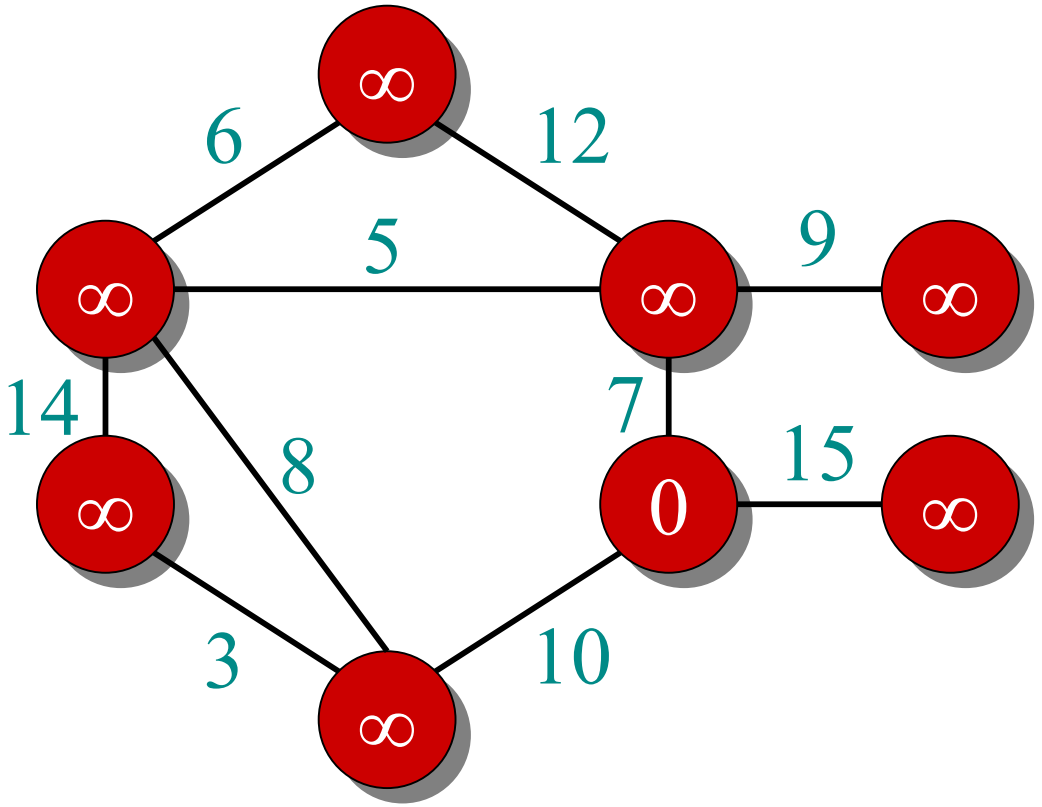
initialize
greedy. (u→A)
add u to A

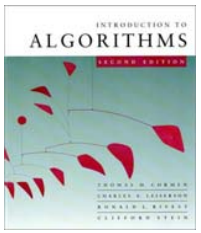
At the end, $\{(v, \pi[v])\}$ forms the MST.



Example of Prim's algorithm

- $\in A$
- $\in V - A$





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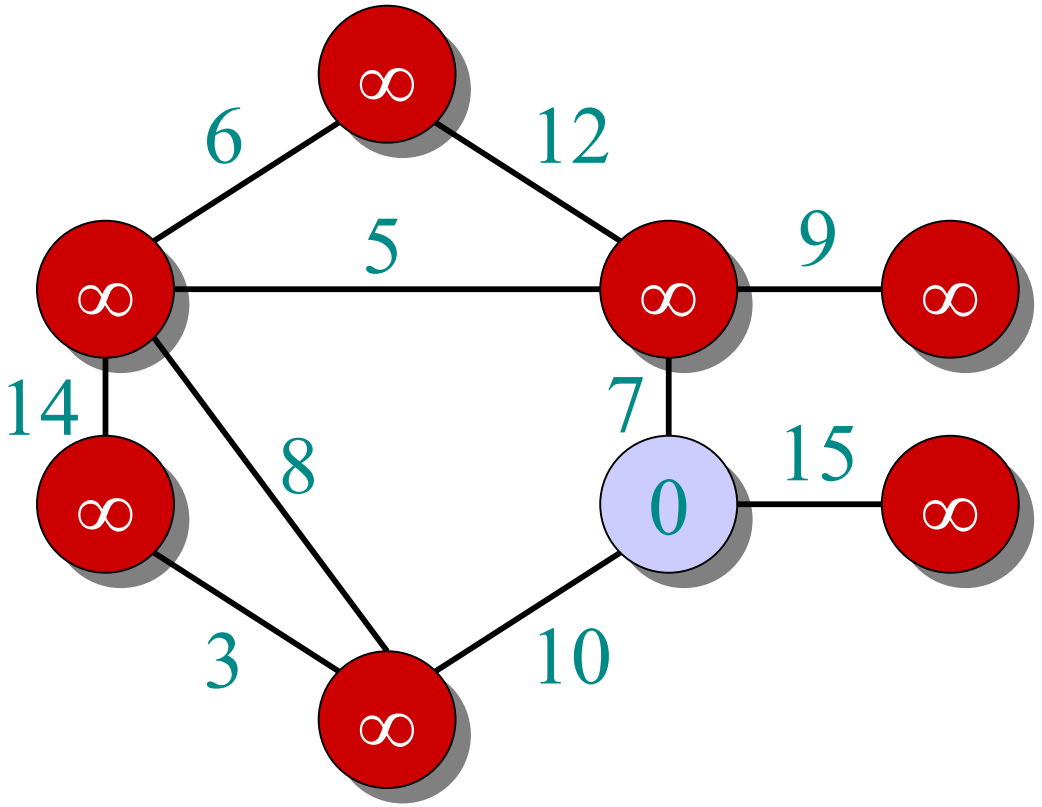
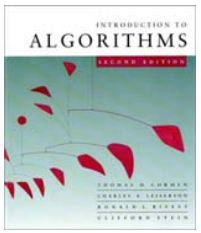
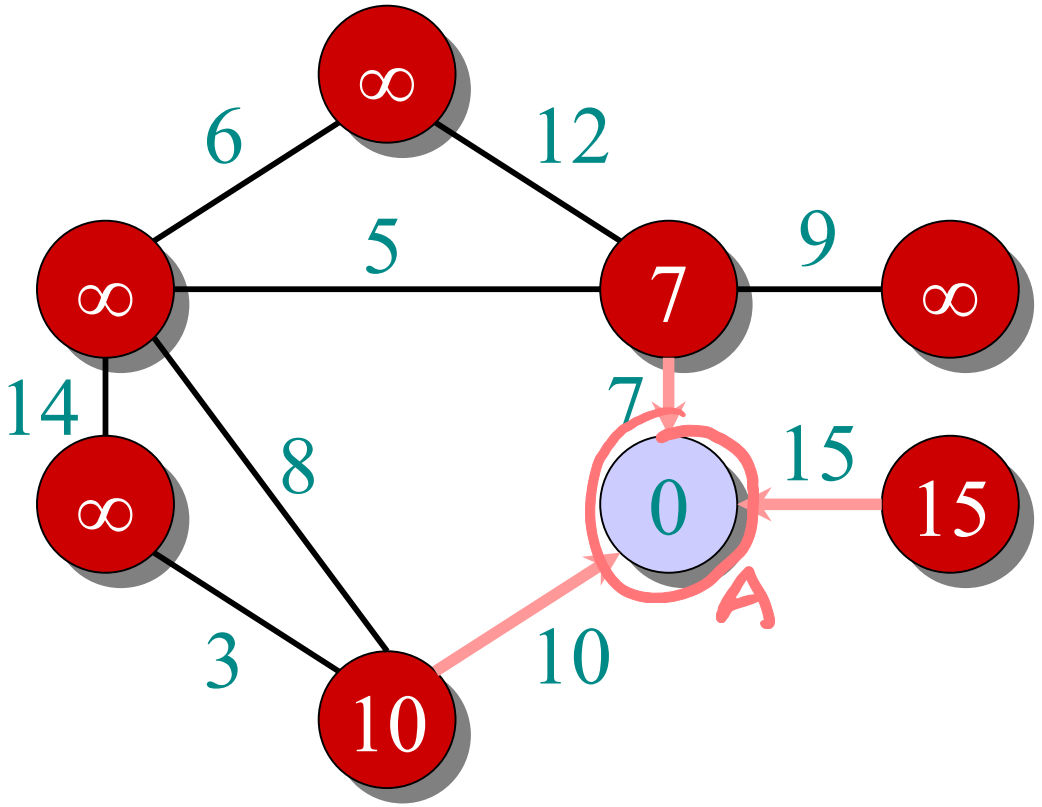


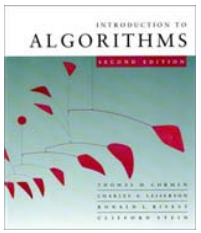


Example of Prim's algorithm

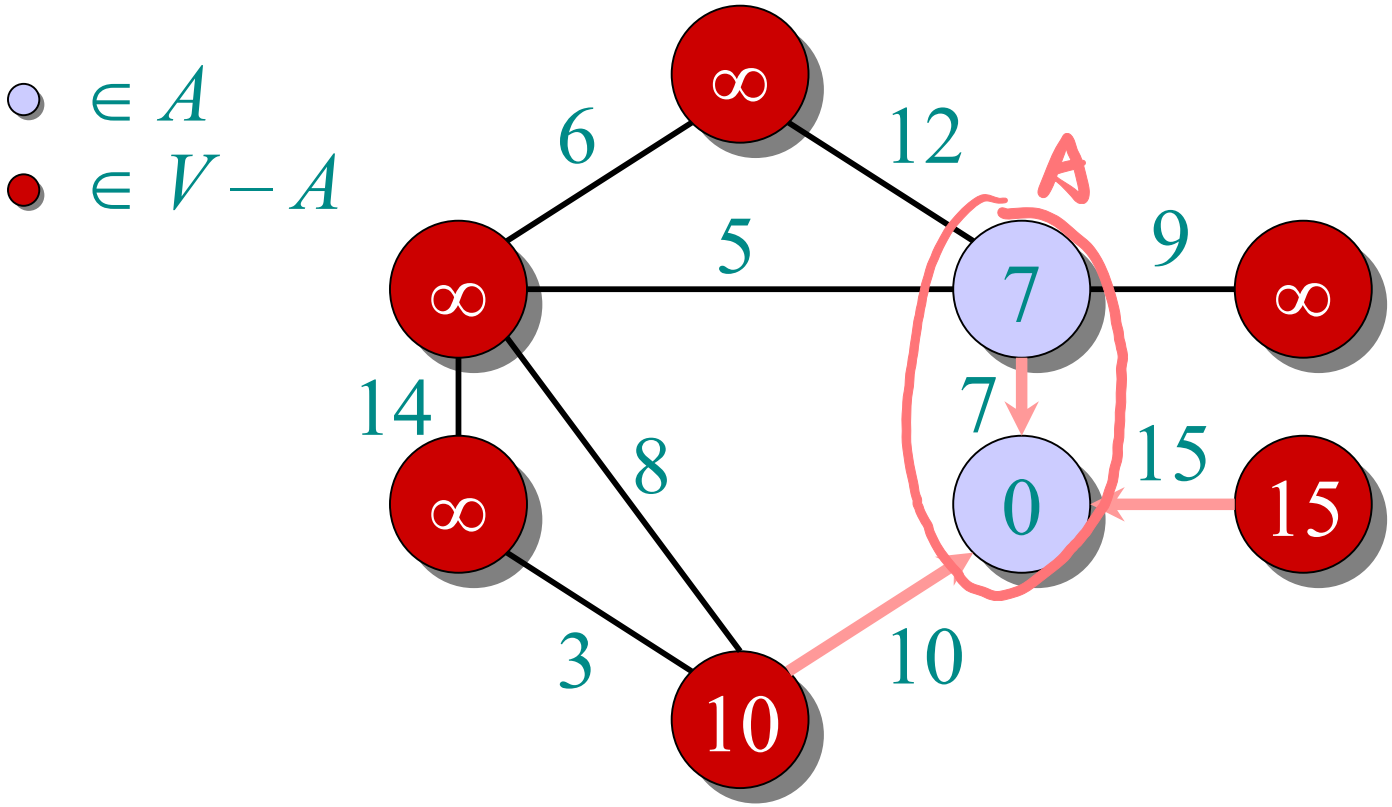
$\in A$

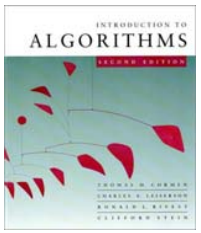
$\in V - A$





Example of Prim's algorithm

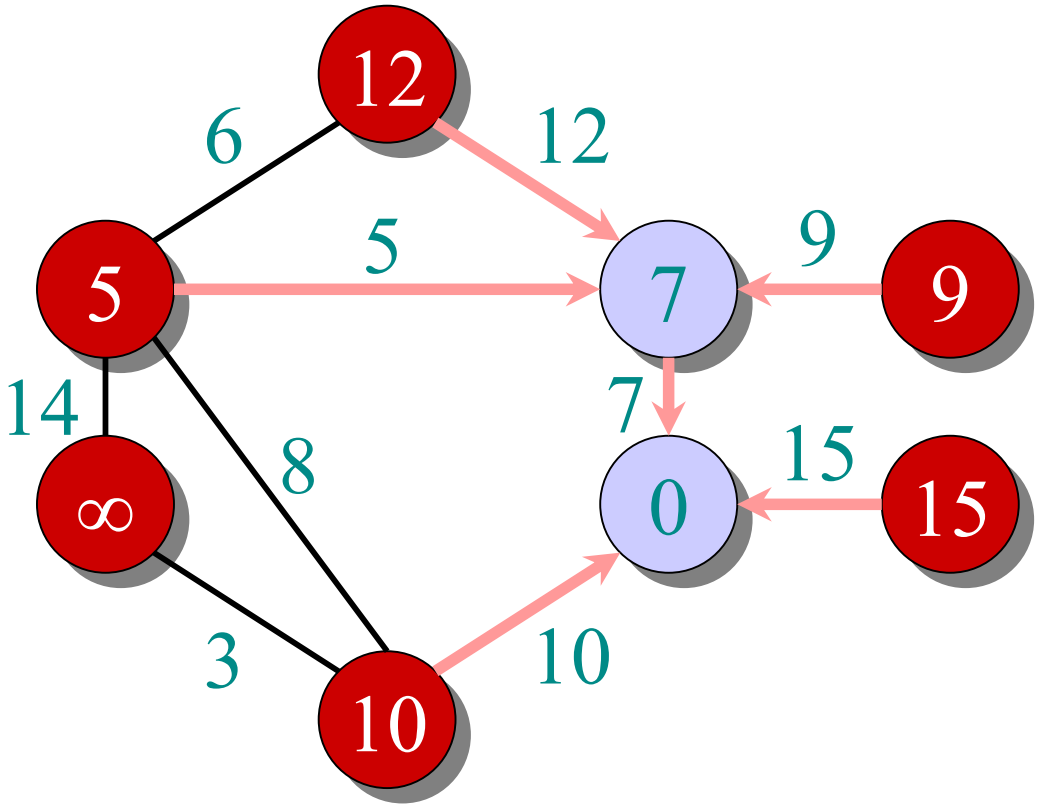


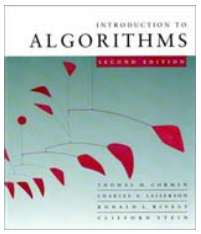


Example of Prim's algorithm

$\in A$

$\in V - A$

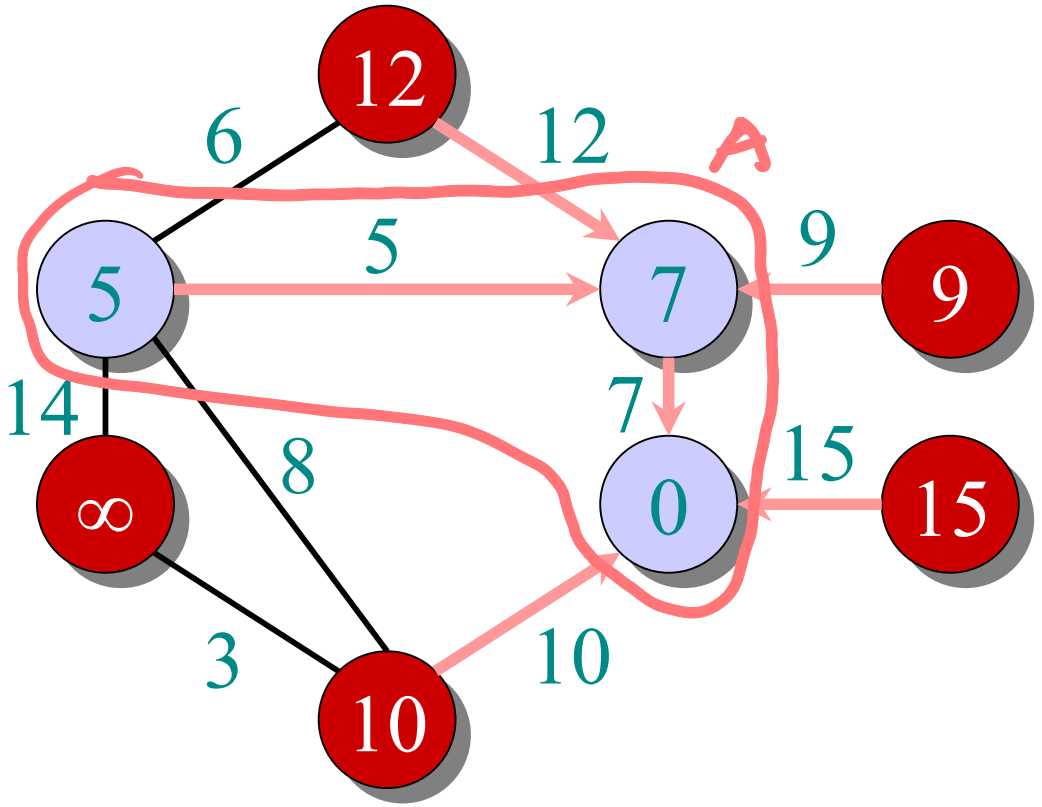


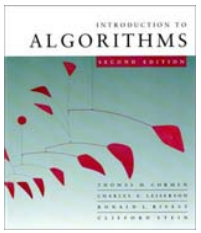


Example of Prim's algorithm

$\in A$

$\in V - A$





Example of Prim's algorithm

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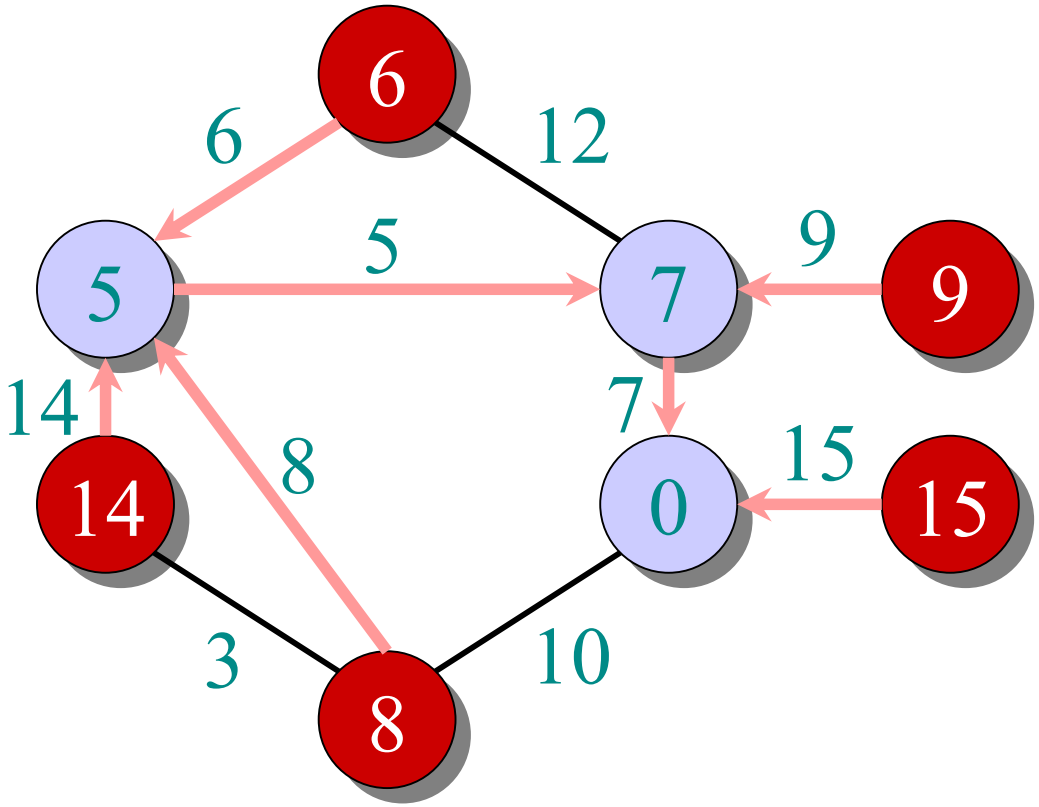
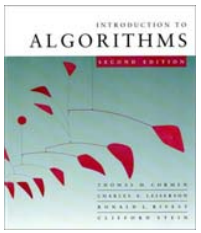
- $\in V - A$

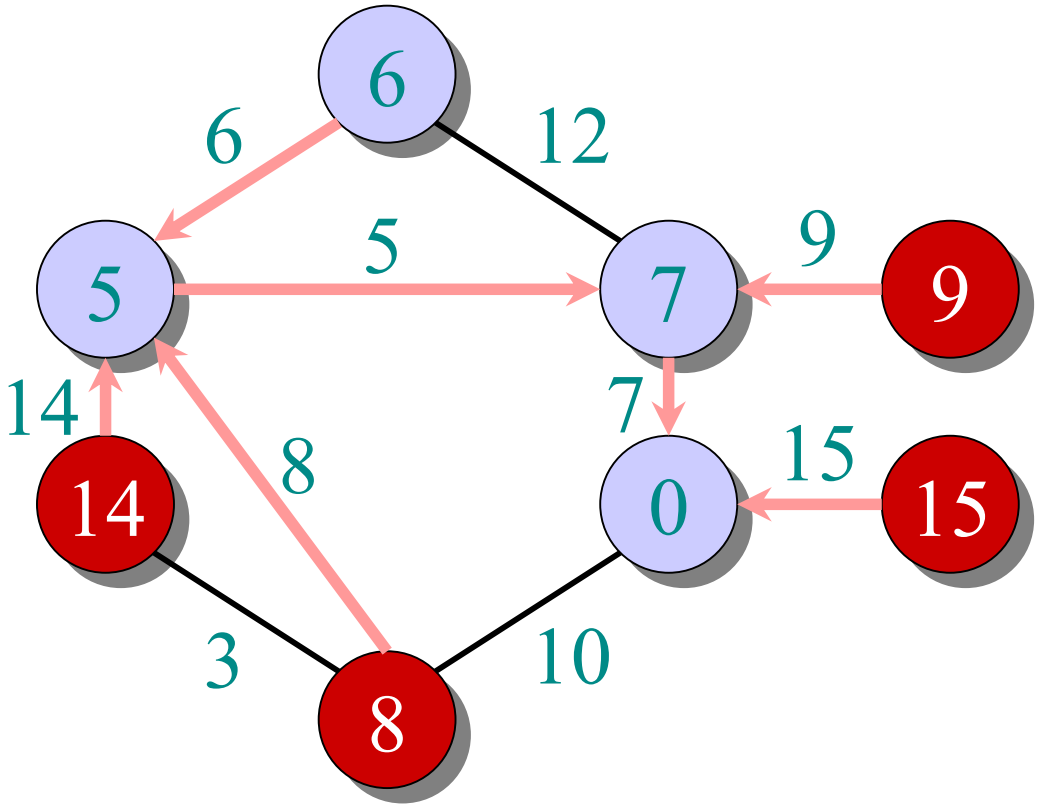


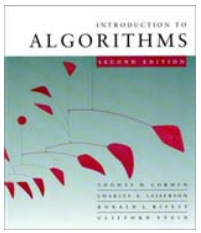


Example of Prim's algorithm

$\circ \in A$

$\bullet \in V - A$

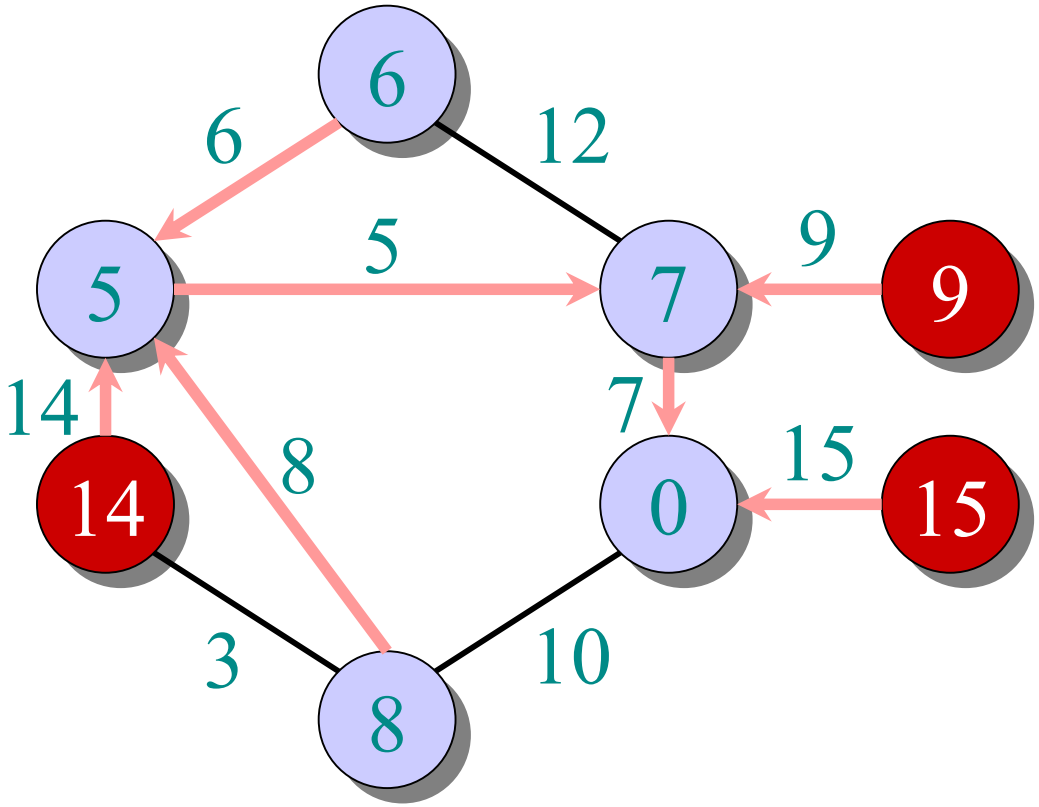


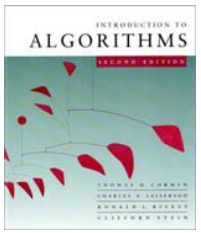


Example of Prim's algorithm

$\circ \in A$

$\bullet \in V - A$

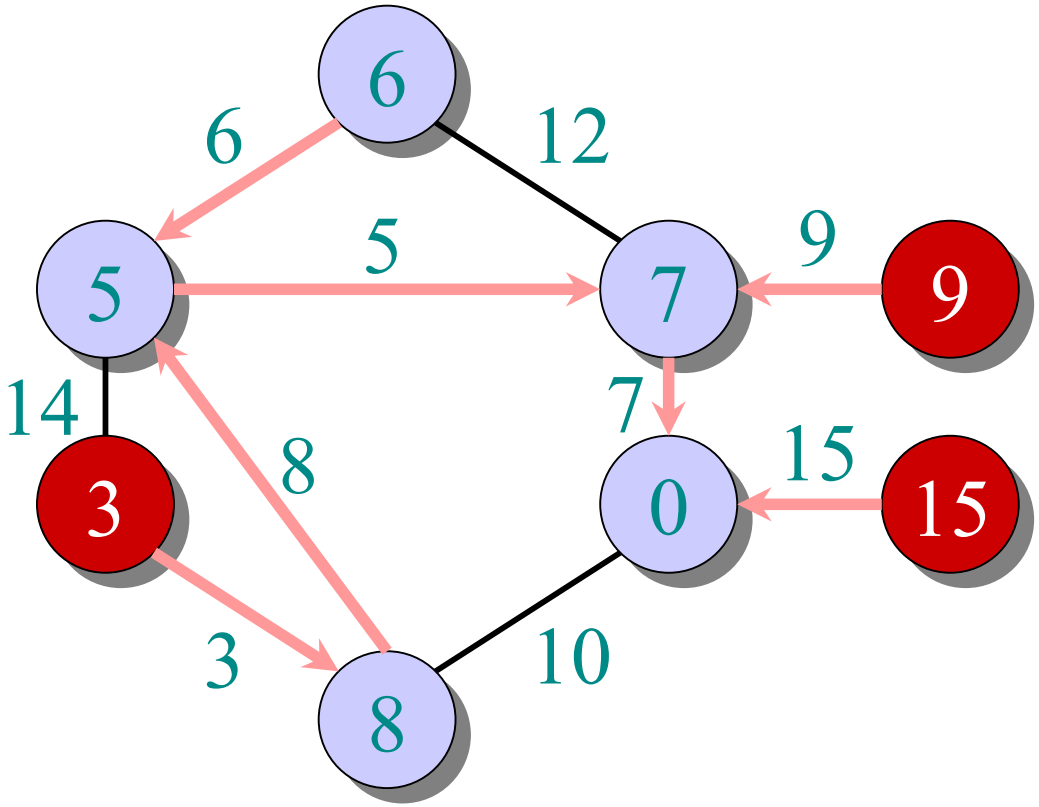


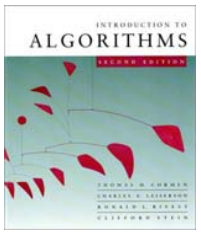


Example of Prim's algorithm

$\bullet \in A$

$\bullet \in V - A$





Example of Prim's algorithm

$\circ \in A$

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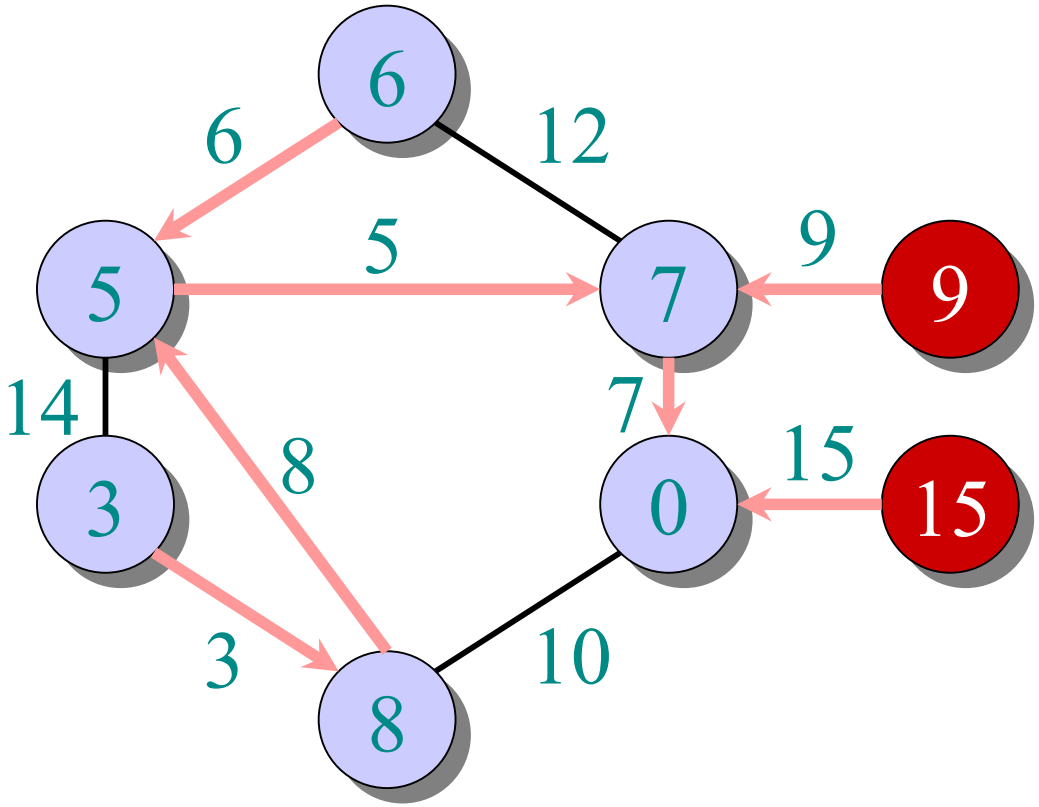

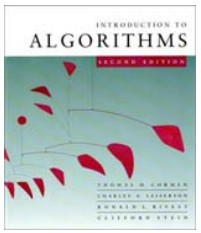

Example of Prim's algorithm

$\in A$

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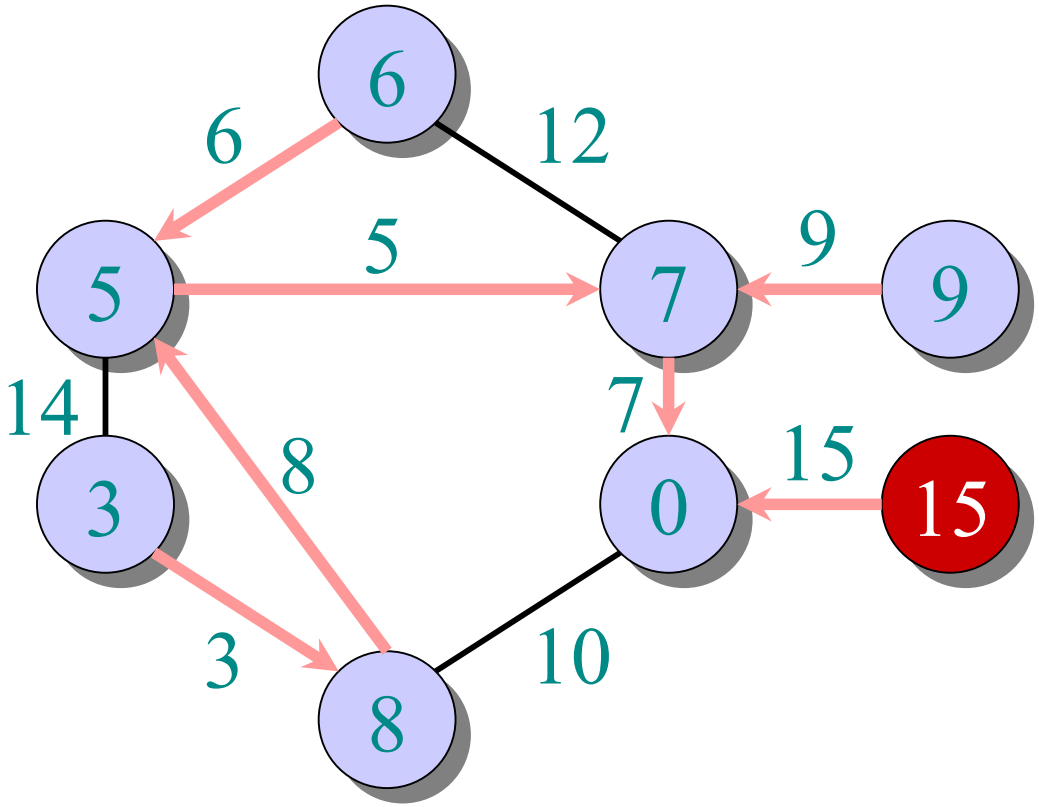

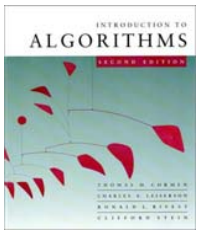

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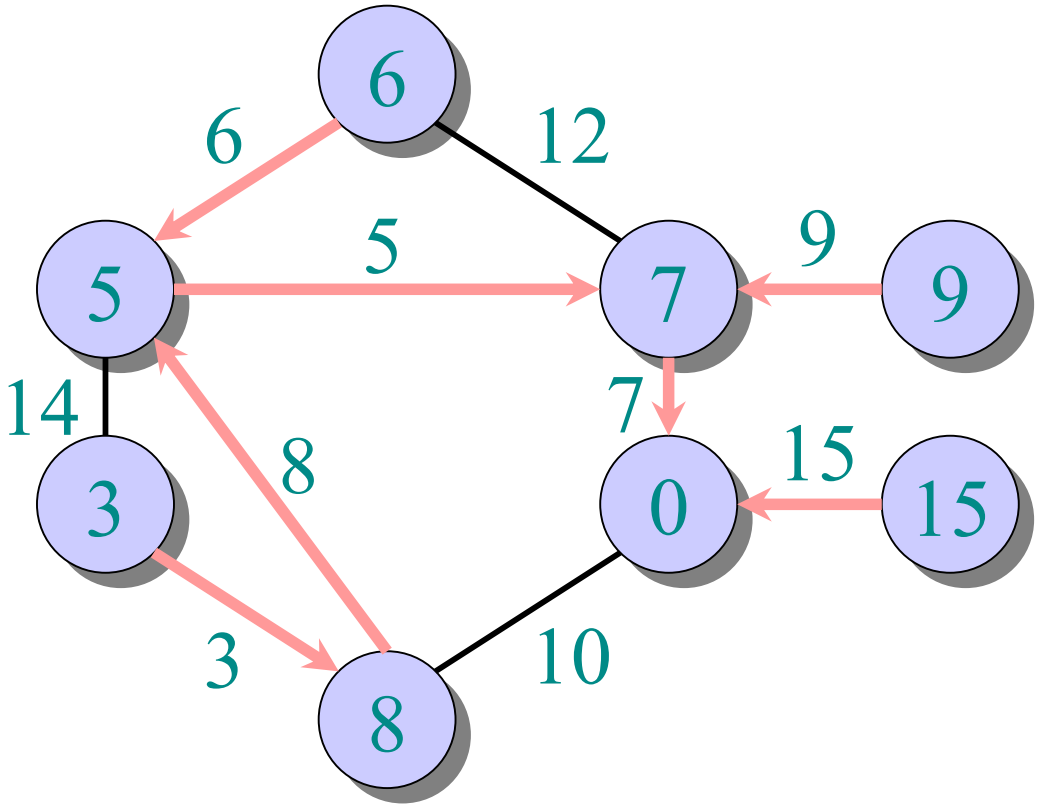
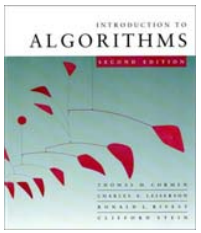




Analysis of Prim

Initialize:

$Q \leftarrow V$

$key[v] \leftarrow \infty$ for all $v \in V$

$key[s] \leftarrow 0$ for some arbitrary $s \in V$

---

while $Q \neq \varnothing$ — $|V|$ Times.

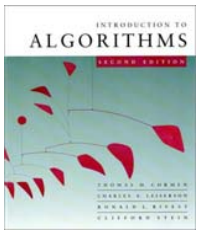
  do $u \leftarrow$ EXTRACT-MIN(Q)

    for each $v \in Adj[u]$ — degree(u)

      do if $v \in Q$ and $w(u, v) < key[v]$

        then $key[v] \leftarrow w(u, v)$

          $\pi[v] \leftarrow u$



Analysis of Prim

$\Theta(V)$ total

$Q \leftarrow V$
$key[v] \leftarrow \infty$ for all $v \in V$
$key[s] \leftarrow 0$ for some arbitrary $s \in V$
while $Q \neq \varnothing$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    for each $v \in Adj[u]$
      do if $v \in Q$ and $w(u, v) < key[v]$
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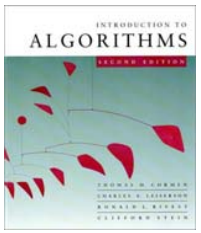

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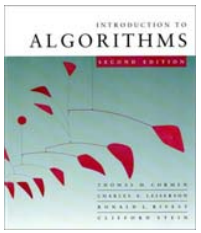

Analysis of Prim

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- $Q \leftarrow V$
- $key[v] \leftarrow \infty$ for all $v \in V$
- $key[s] \leftarrow 0$ for some arbitrary $s \in V$

while $Q \neq \varnothing$ — $|V|$ times

  do $u \leftarrow$ EXTRACT-MIN(Q) — $degree(u)$ times

    for each $v \in Adj[u]$

      do if $v \in Q$ and $w(u, v) < key[v]$

        then $key[v] \leftarrow w(u, v)$

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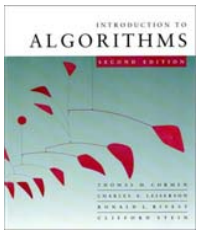

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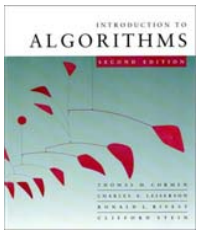
Handshaking Lemma $\Rightarrow$ $\Theta(E)$ implicit DECREASE-KEY's.



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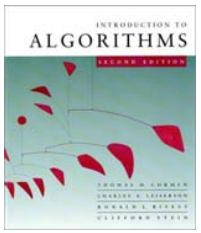
Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.

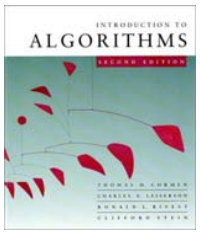
$$\text{Time} = \Theta(V)\cdot T_{\text{EXTRACT-MIN}} + \Theta(E)\cdot T_{\text{DECREASE-KEY}}$$



Analysis of Prim (continued)

$$\text{Time} = \Theta(V)\cdot T_{\text{EXTRACT-MIN}} + \Theta(E)\cdot T_{\text{DECREASE-KEY}}$$

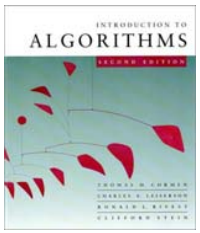
Complexity Depends on Operation



Analysis of Prim (continued)

$$\text{Time} = \Theta(V)\cdot T_{\text{EXTRACT-MIN}} + \Theta(E)\cdot T_{\text{DECREASE-KEY}}$$

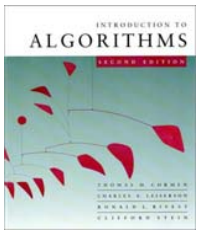
Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total



Analysis of Prim (continued)

$$\text{Time} = \Theta(V)\cdot T_{\text{EXTRACT-MIN}} + \Theta(E)\cdot T_{\text{DECREASE-KEY}}$$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$

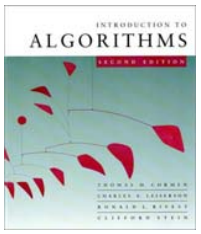


Analysis of Prim (continued)

$$\text{Time} = \Theta(V)\cdot T_{\text{EXTRACT-MIN}} + \Theta(E)\cdot T_{\text{DECREASE-KEY}}$$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$

Dense Graph or Sparse Graph



Analysis of Prim (continued)

$$\text{Time} = \Theta(V)\cdot T_{\text{EXTRACT-MIN}} + \Theta(E)\cdot T_{\text{DECREASE-KEY}}$$

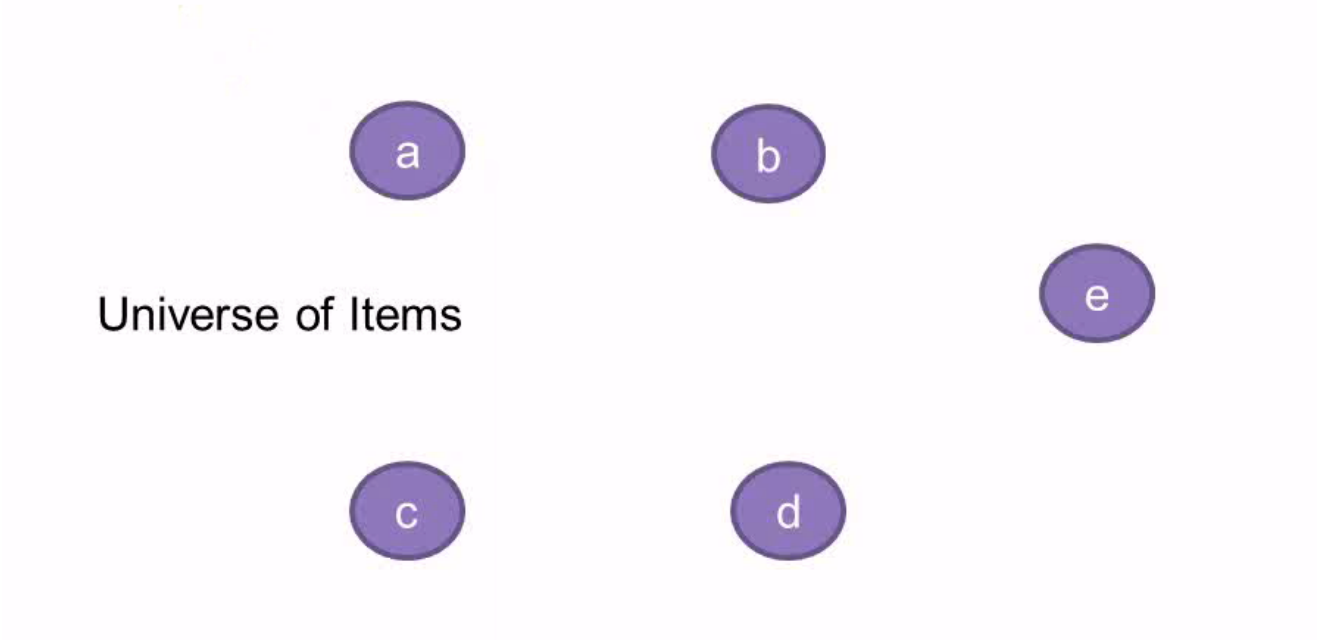
Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$
Fibonacci heap	$O(\lg V)$ amortized	$O(1)$ amortized	$O(E + V \lg V)$ worst case

MST Algorithms

- Kruskal's algorithm (see CLRS):
  - Uses the disjoint-set data structure
  - Running time = O(ElgV)

Disjoint Set

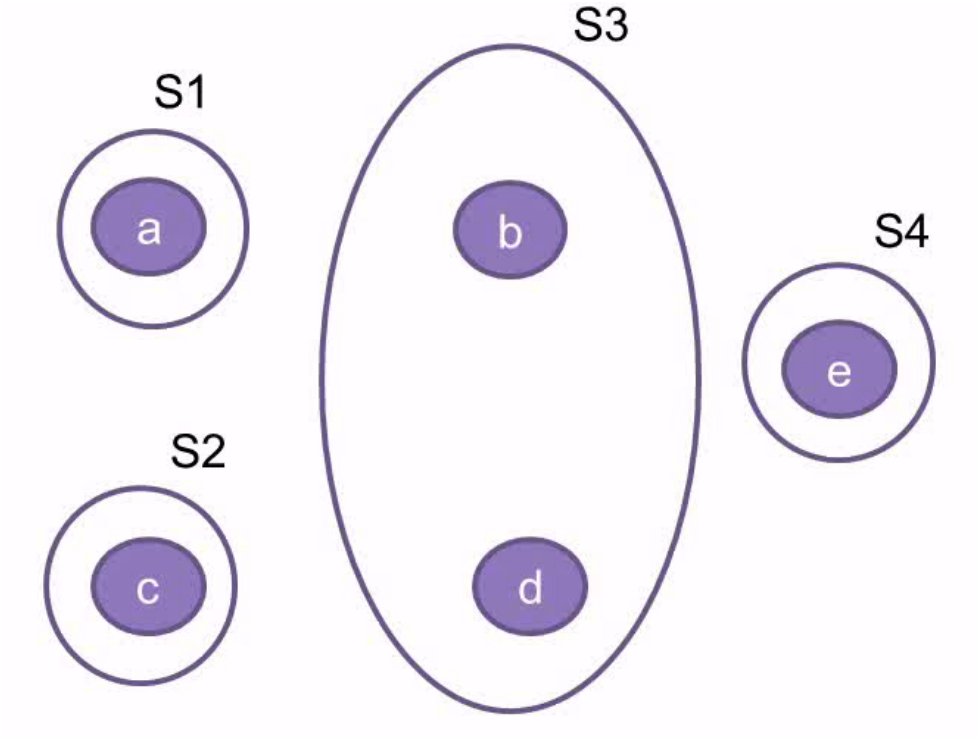
A group of sets where no item can be in more than one set



Disjoint Set

A group of sets where no item can be in more than one set

Supported Operations:
Find()
Union()

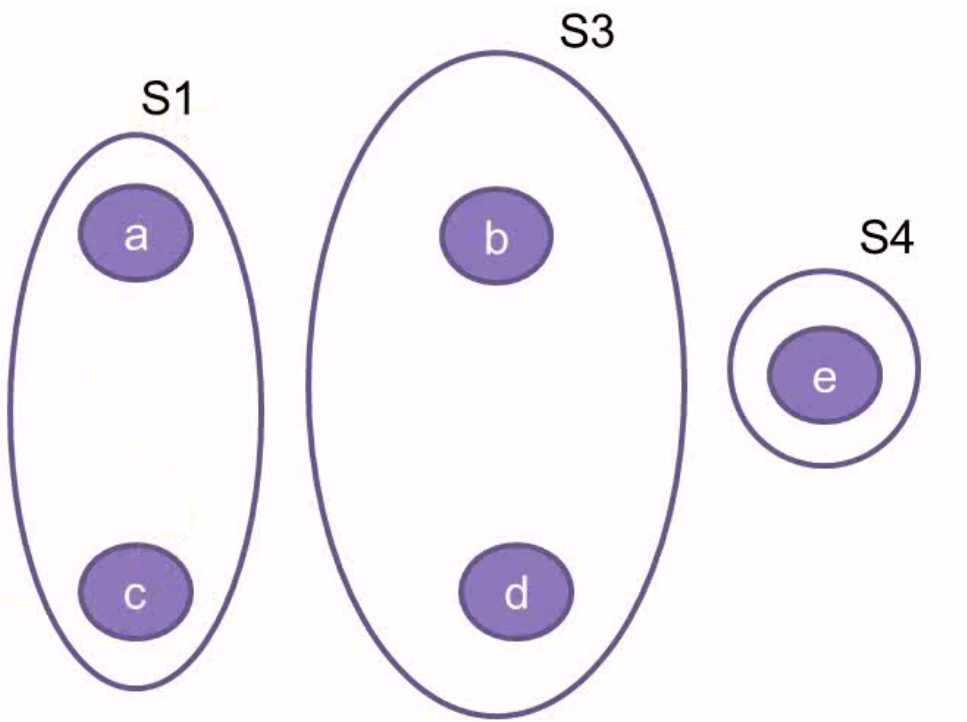


Find(d) => S3
Union(S2, S1)

Disjoint Set

A group of sets where no item can be in more than one set

Supported Operations:
Find()
Union()

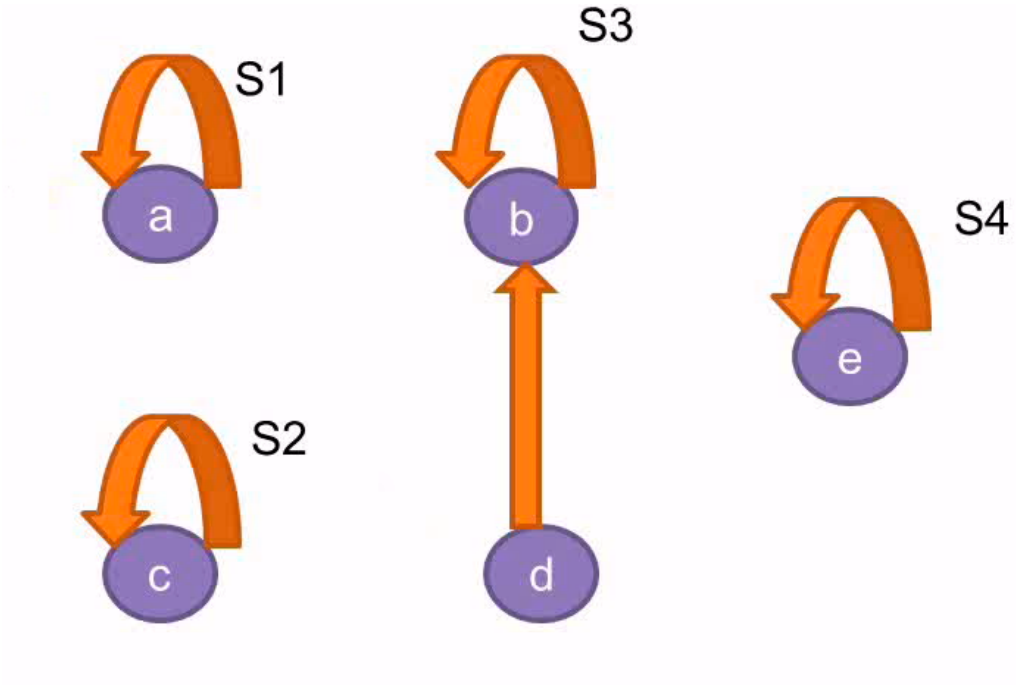


Find(d) => S3
Union(S2, S1)

Tree Based Disjoint Set

- Each set is a tree
- A set is identified by the root of the tree

Supported Operations:
Find()
Union()

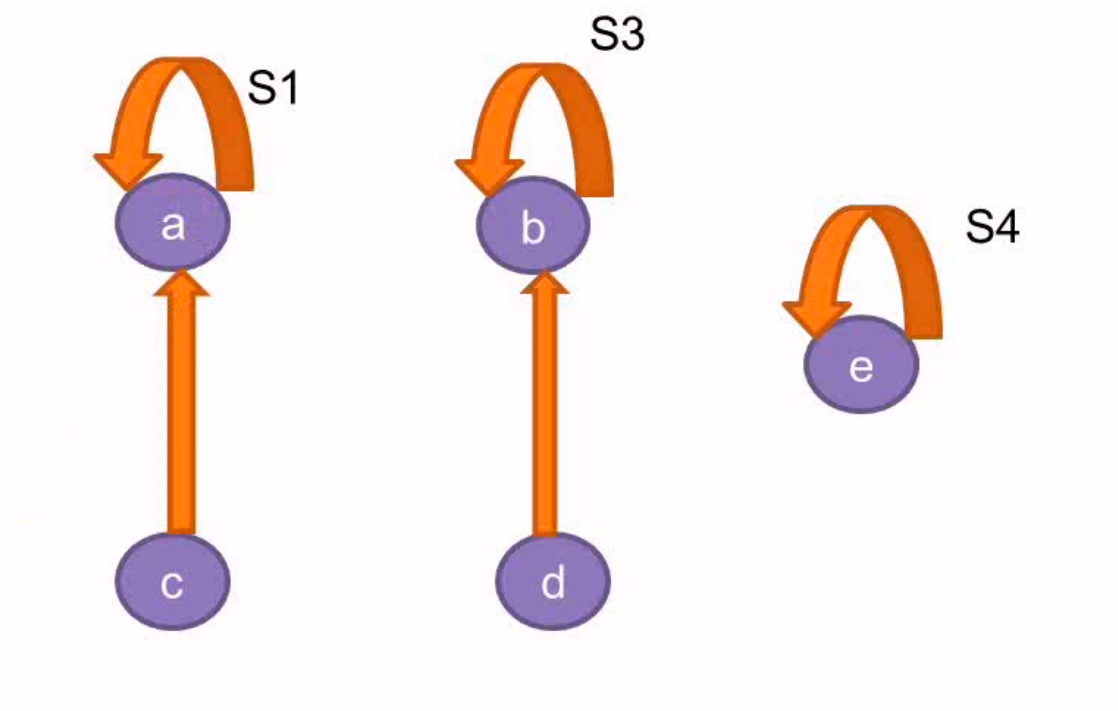


Find(d) => S3 or b
Union(S2, S1)

Tree Based Disjoint Set

- Each set is a tree
- A set is identified by the root of the tree

Supported Operations:
Find()
Union()



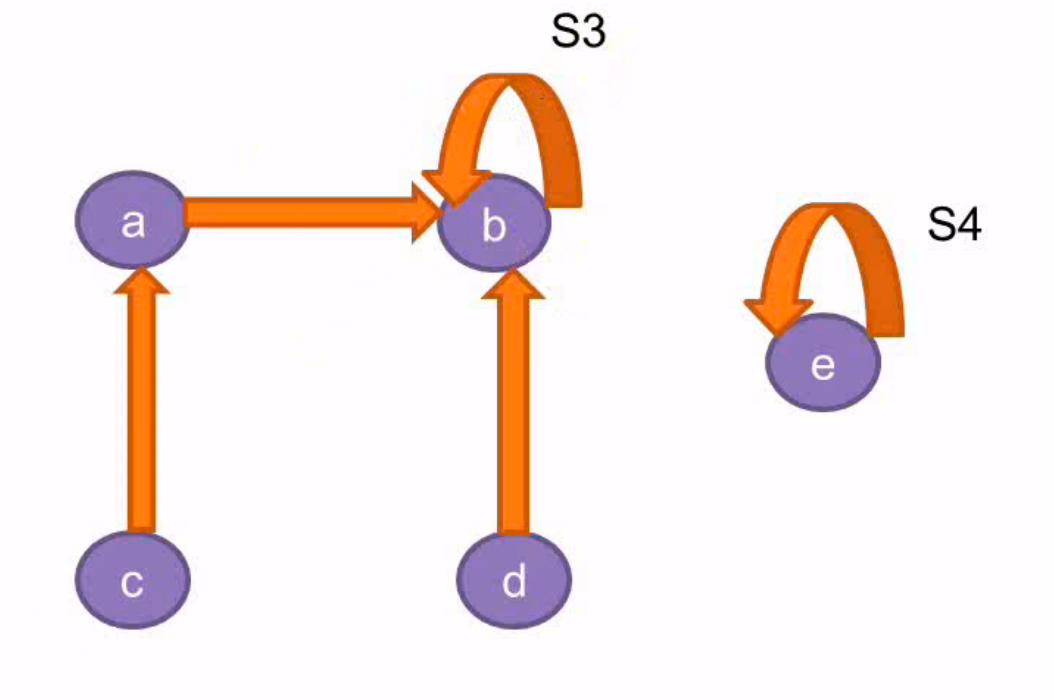
Union(S1, S3)

Tree Based Disjoint Set

- Each set is a tree
- A set is identified by the root of the tree

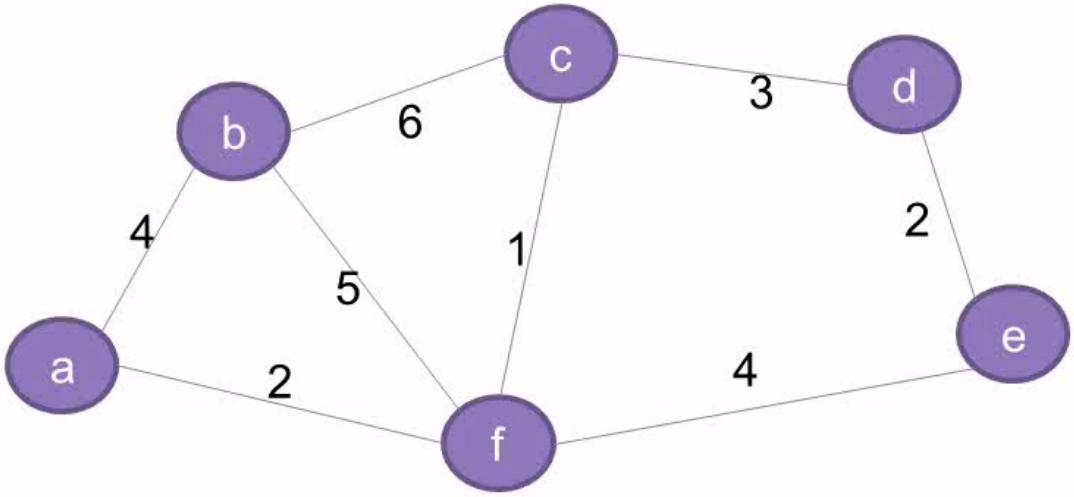
Supported Operations:
Find()
Union()

Complexity:
Find() - O (depth)
Union() - O(1)



Kruskal's algorithm

Uses the disjoint-set data structure



A = { }

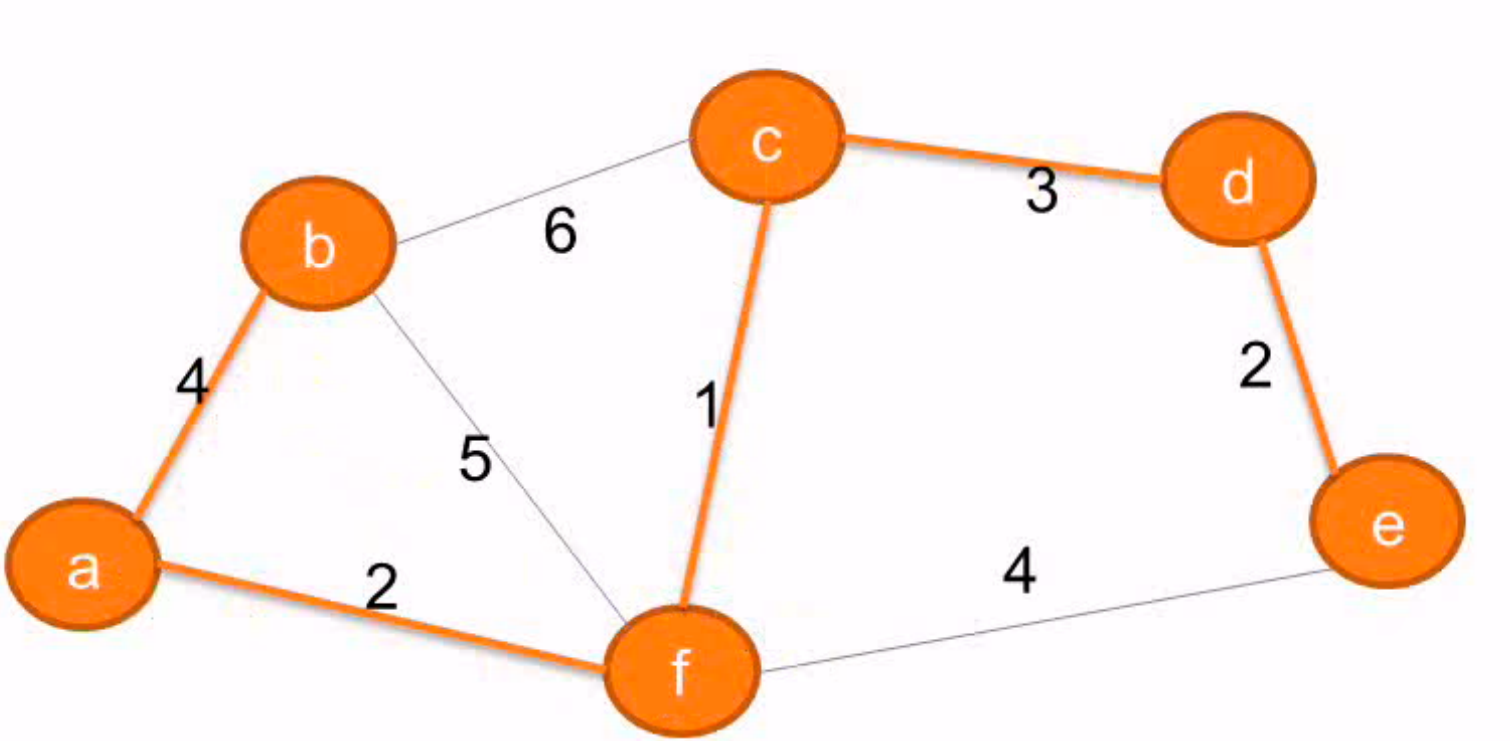
```
Kruskal(V, E)

A = ∅
foreach v ∈ V:
    Make-disjoint-set(v)
Sort E by weight increasingly
foreach (v1, v2) ∈ E:
    if Find(v1) ≠ Find(v2):
        A = A ∪ {(v1, v2)}
        Union(v1, v2)
return A
```

Separate

- Running time = O(ElgV)

Kruskal's algorithm



A = { (c, f), (a, f), (d, e), (c, d), (a, b) }

Next Lecture

- Shortest path
  - Single-source shortest paths
  - All-pairs shortest paths