Final: Lecture 1 ~ END

EL9343

Data Structure and Algorithm

Lecture 12: Shortest Paths

Instructor: Yong Liu

Last Lecture

- Greedy Algorithm (cont.)
  - Huffman codes

- Minimum Spanning Trees
  - Prim's algorithm
  - Kruskal's algorithm

Today

- Single-Source Shortest Paths
  - Nonnegative edge weights: Dijkstra's algorithm
  - Unweighted graphs: BFS
  - General case: Bellman-Ford algorithm
  - DAG: Topological sort + one pass Bellman-Ford

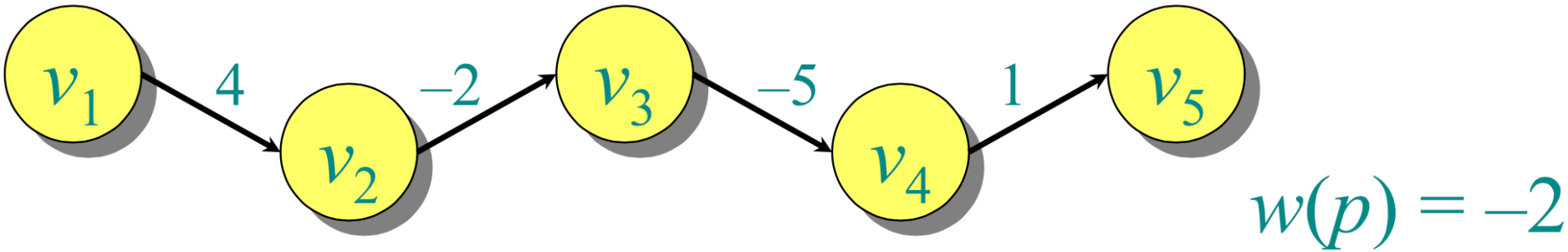
- All Pairs Shortest Paths
  - Nonnegative edge weights: |V| times of Dijkstra's algorithm
  - Unweighted graphs: |V| times of BFS
  - General case: Floyd-Warshall algorithm

Paths in graphs

Consider a digraph $G = (V, E)$ with edge-weight function $w : E \to \mathbb{R}$. The ***weight*** of path $p = v_1 \to v_2 \to \cdots \to v_k$ is defined to be

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}) .$$

Example:



$w(p) = -2$

Shortest paths

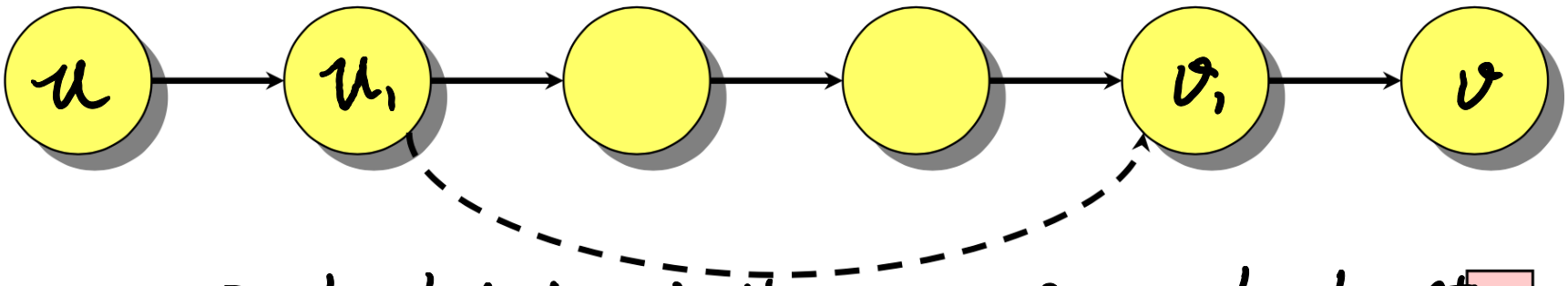
- A shortest path from **u** to **v** is a path of minimum weight from **u** to **v**.
- The shortest- path weight from **u** to **v** is defined as:

  $\delta(u, v) = \min\{w(p) : p$ is a path from **u** to **v**$\}$.

- Note: **$\delta(u, v) = \infty$** if no path from u to v exists.

Optimal substructure

- Theorem: A subpath of a shortest path is a shortest path.
- Proof: Cut and paste



⇒ if $u \to v$ is the shortest path, then $u_1 \to v_1$ must be shortest path. ⇒ Could use dynamic Programming

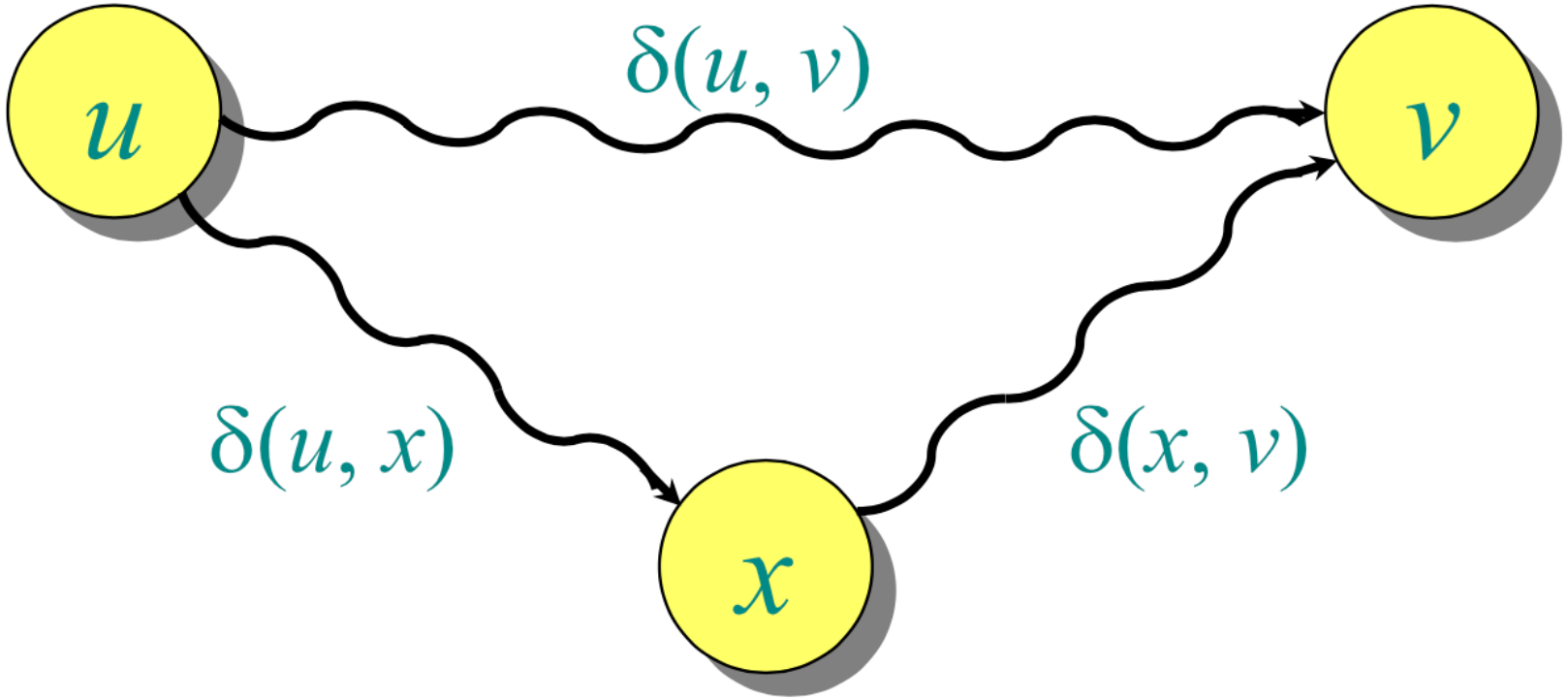
Triangle inequality

$\delta(u,v)$: shortest path from u to v

- **Theorem:** For all u, v, x ∈ V, we have δ(u, v) ≤ δ(u, x) + δ(x, v).

This Theorem is used for unweighted graph.

Not used for weighted graph. (negative situation)

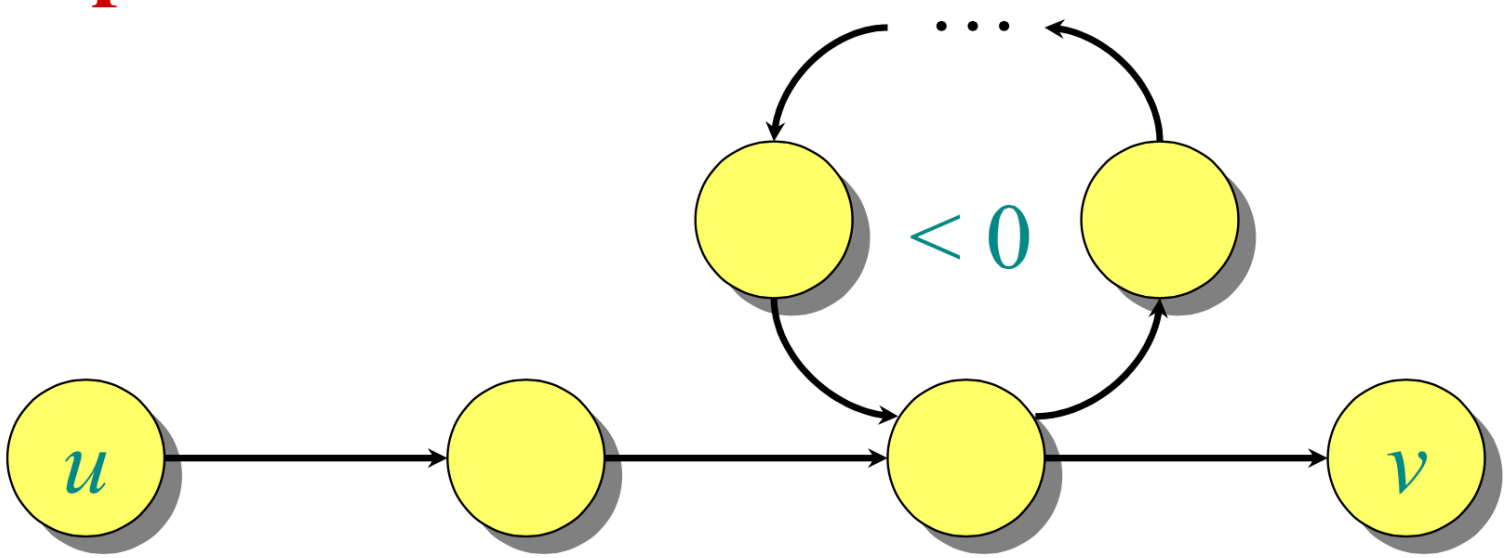


Negative-Weight Cycle

- If a graph G contains a negative-weight cycle, then some shortest paths may not exist.

There will be some Algs to detect negtive path.

Example:



Single-source shortest paths

- Problem. From a given source vertex $s \in V$, find the shortest-path weights $\delta(s, v)$ for all $v \in V$.
- If all edge weights $w(u, v)$ are nonnegative, all shortest-path weights must exist.

IDEA: Greedy. The idea is similar to MST (Prim's algorithm)

1. Maintain a set S of vertices whose shortest-path distances from s are known.
2. At each step add to S the vertex $v \in V - S$ whose distance estimate from s is minimal.
3. Update the distance estimates of vertices adjacent to v.

Dijkstra's algorithm

$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
    do $d[v] \leftarrow \infty$
$S \leftarrow \varnothing$
$Q \leftarrow V$ ▷ Q is a priority queue maintaining $V - S$

Dijkstra's algorithm

$d[s] \leftarrow 0$ Initialize to 0.
for each $v \in V - \{s\}$
  do $d[v] \leftarrow \infty$: Initialize all vertex
$S \leftarrow \emptyset$
$Q \leftarrow V$ ▷ Q is a priority queue maintaining $V - S$
while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q) Extract the min from Q. such that S→u (=u) is Min
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ ← The order doesn't matter.
      do if $d[v] > d[u] + w(u, v)$
        then $d[v] \leftarrow d[u] + w(u, v)$

The Extract-MIN add to S must be correct.

relaxation step = update.

Implicit DECREASE-KEY

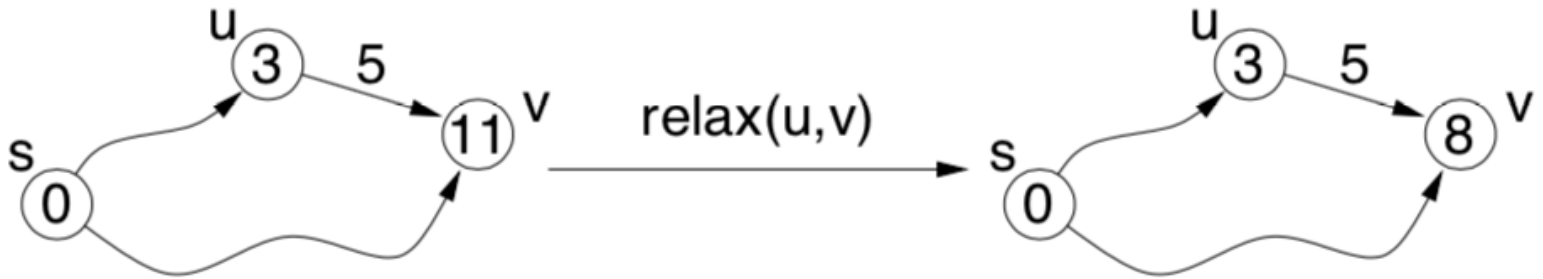
Shortest Paths and Relaxation

The basic structure of Dijkstra's algorithm is to maintain an *estimate* of the shortest path for each vertex, call this d[v].

- Algorithm works in iterations
- At each iteration, d[v] will be the length of the shortest path *that the algorithm knows of* from s to v. This value will always greater than or equal to the true shortest path distance from s to v.
- Initially d[s] = 0, no paths to all other vertices, so all other d[v] values are set to ∞.
- As the algorithm goes on, and sees more and more vertices, it attempts to update d[v] for each vertex in the graph, until all the d[v] values converge to the true shortest distances.

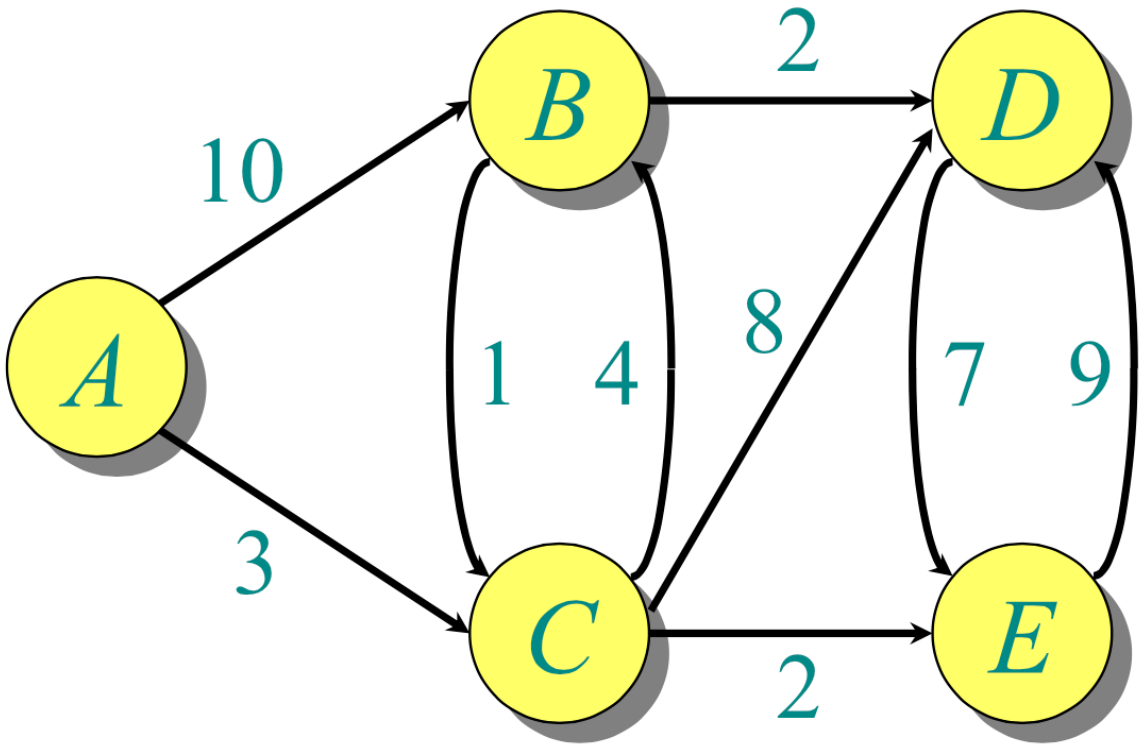
Shortest Paths and Relaxation

The process by which an estimate is updated is called *relaxation*. Here is how relaxation works. Intuitively, if you can see that your solution is not yet reached an optimum value, then push it a little closer to the optimum. In particular, if you discover a path from s to v shorter than d[v], then you need to update d[v].



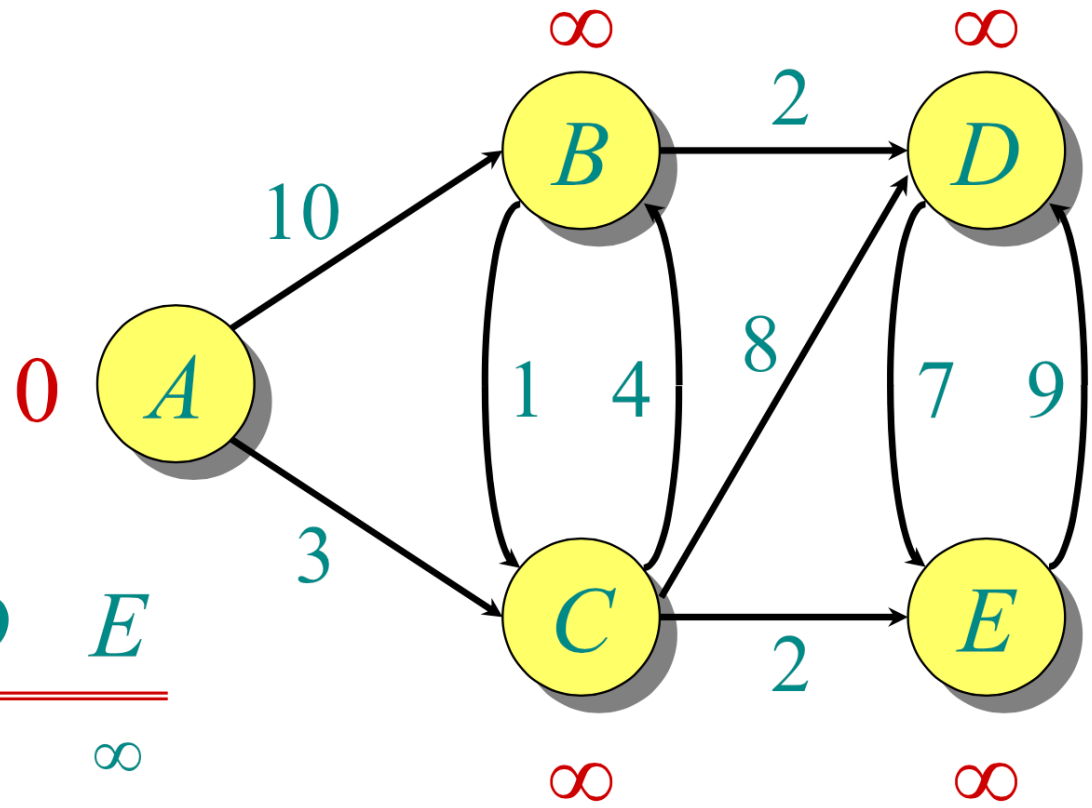
Example of Dijkstra's algorithm

Graph with nonnegative edge weights



Example of Dijkstra's algorithm

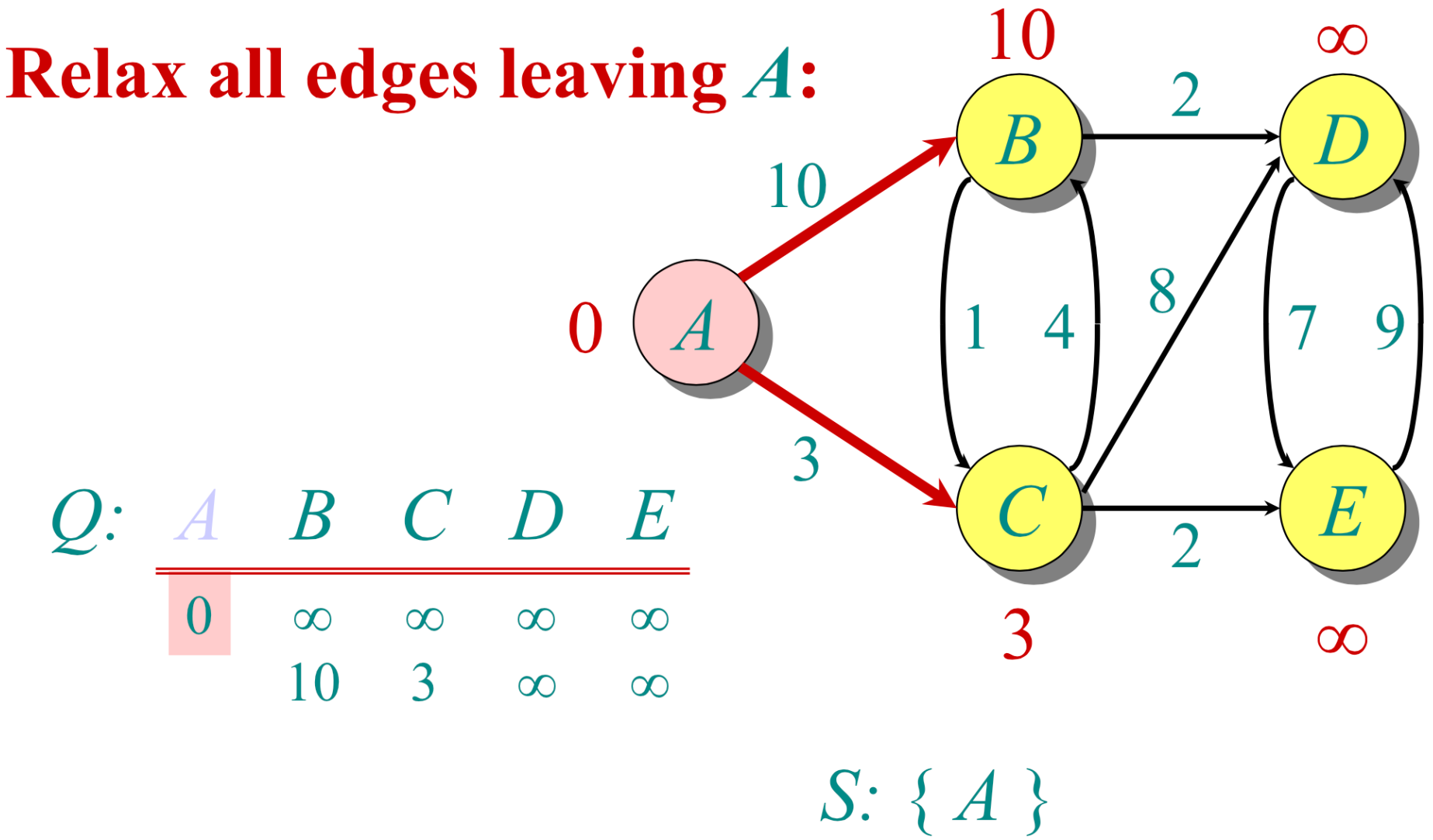
Initialize:



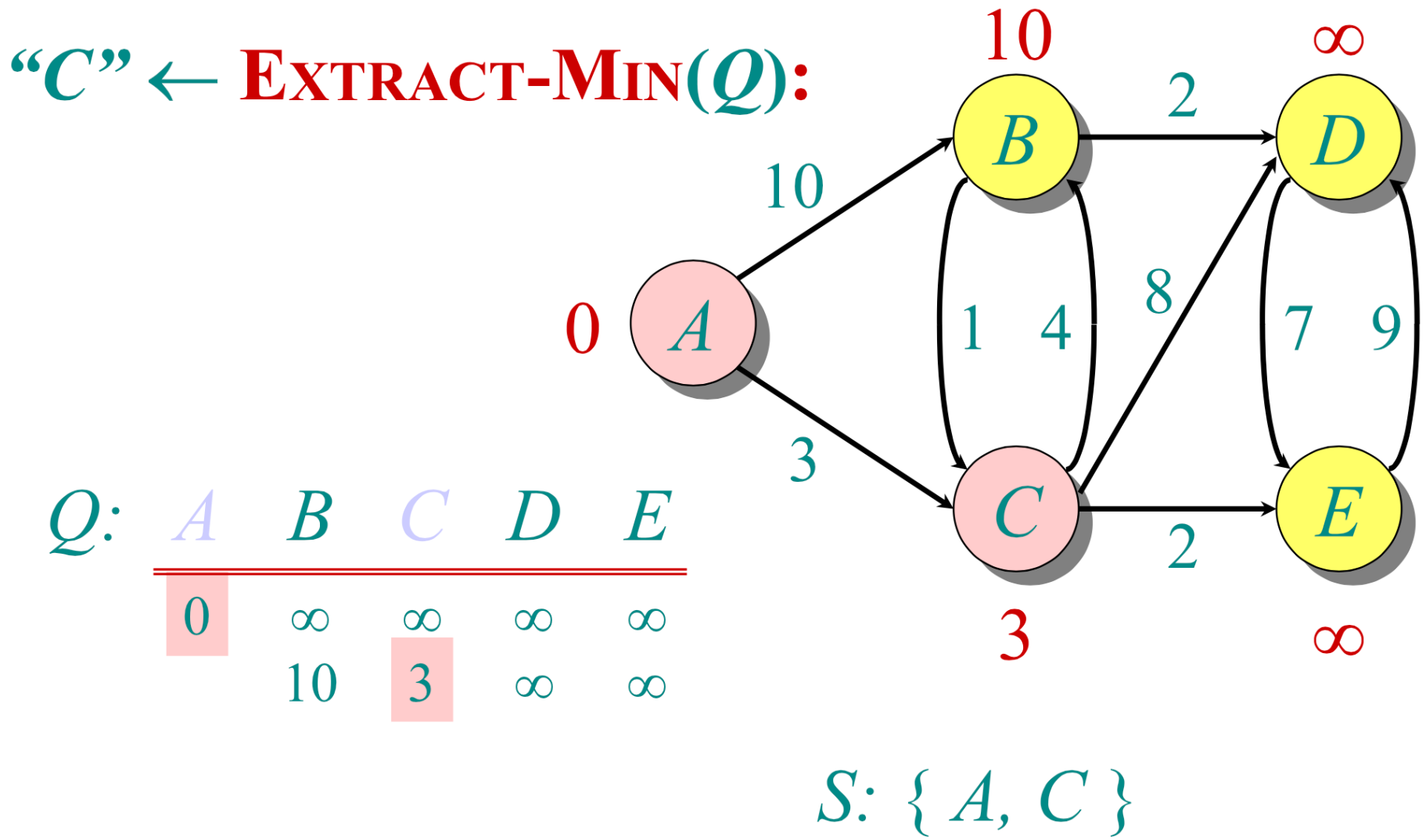
Q:	A	B	C	D	E
	0	∞	∞	∞	∞

S: {}

Example of Dijkstra's algorithm

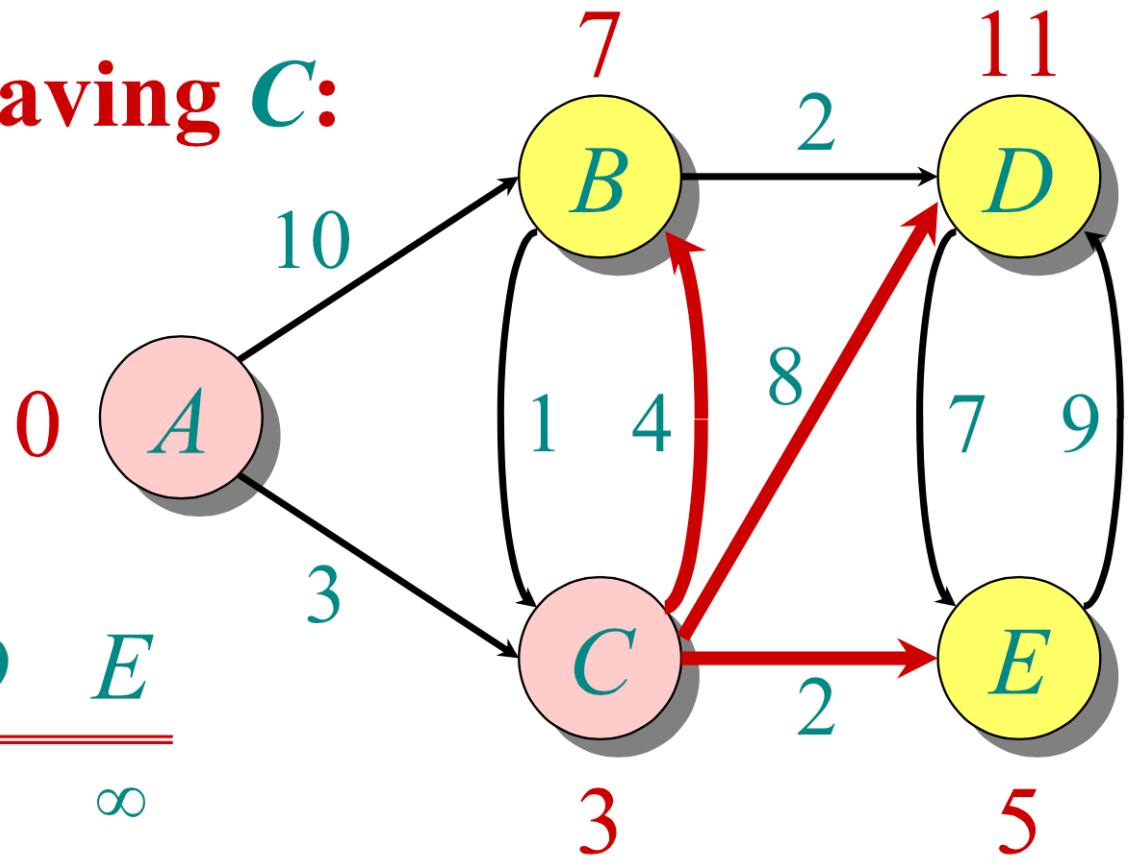


Example of Dijkstra's algorithm



Example of Dijkstra's algorithm

Relax all edges leaving *C*:

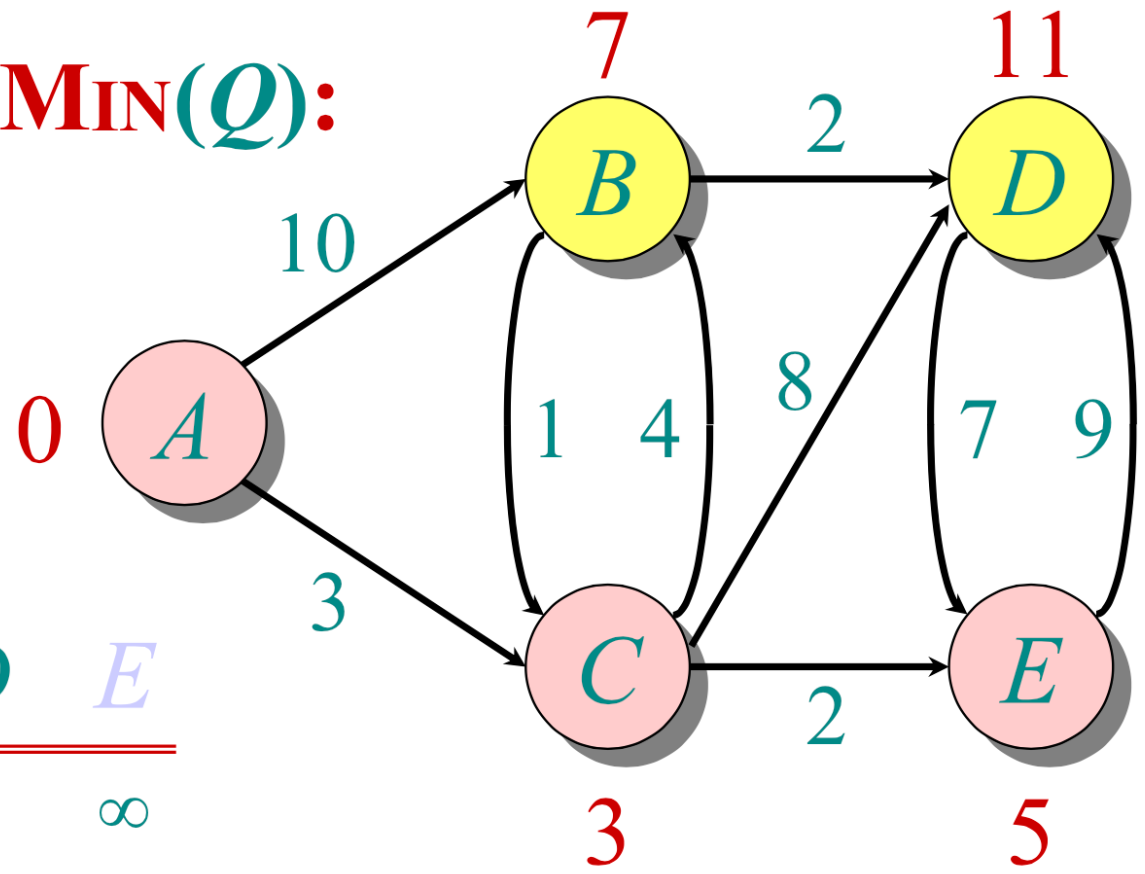


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5

S: { *A*, *C* }

Example of Dijkstra's algorithm

"E" ← EXTRACT-MIN(Q):

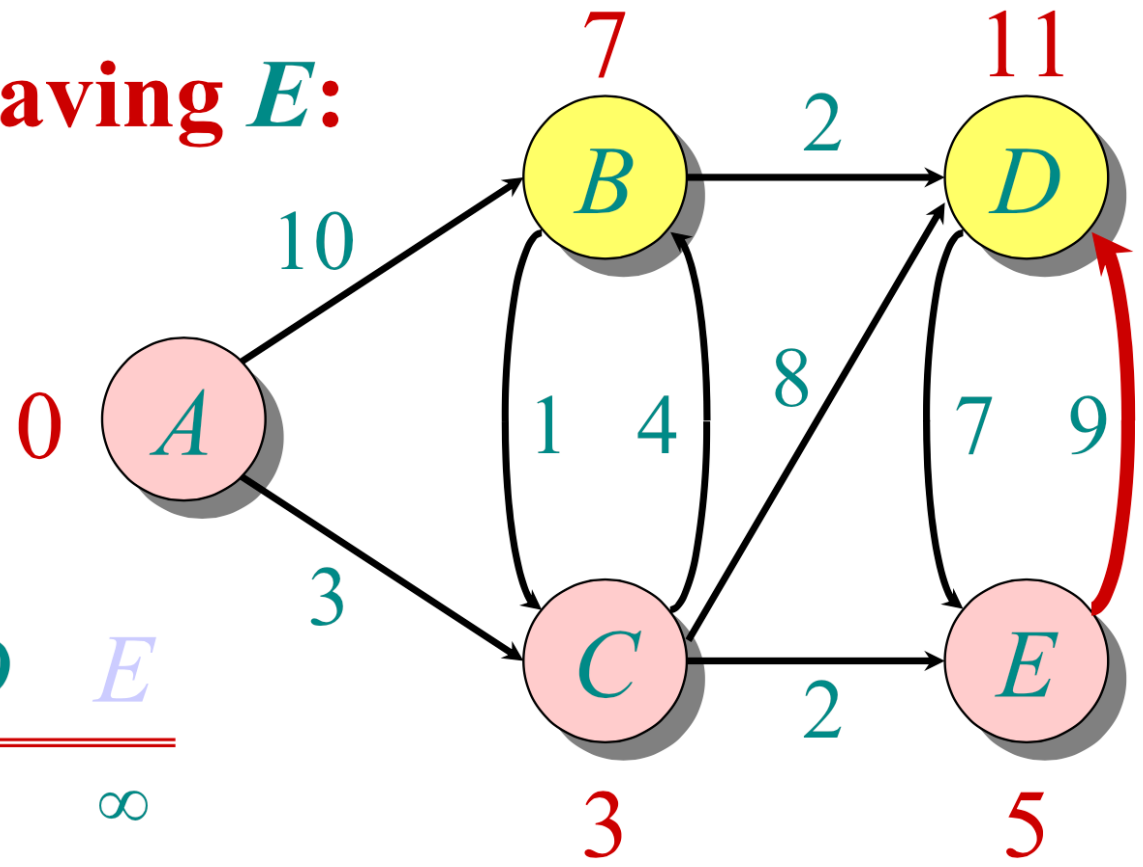


Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5

S: { A, C, E }

Example of Dijkstra's algorithm

Relax all edges leaving *E*:

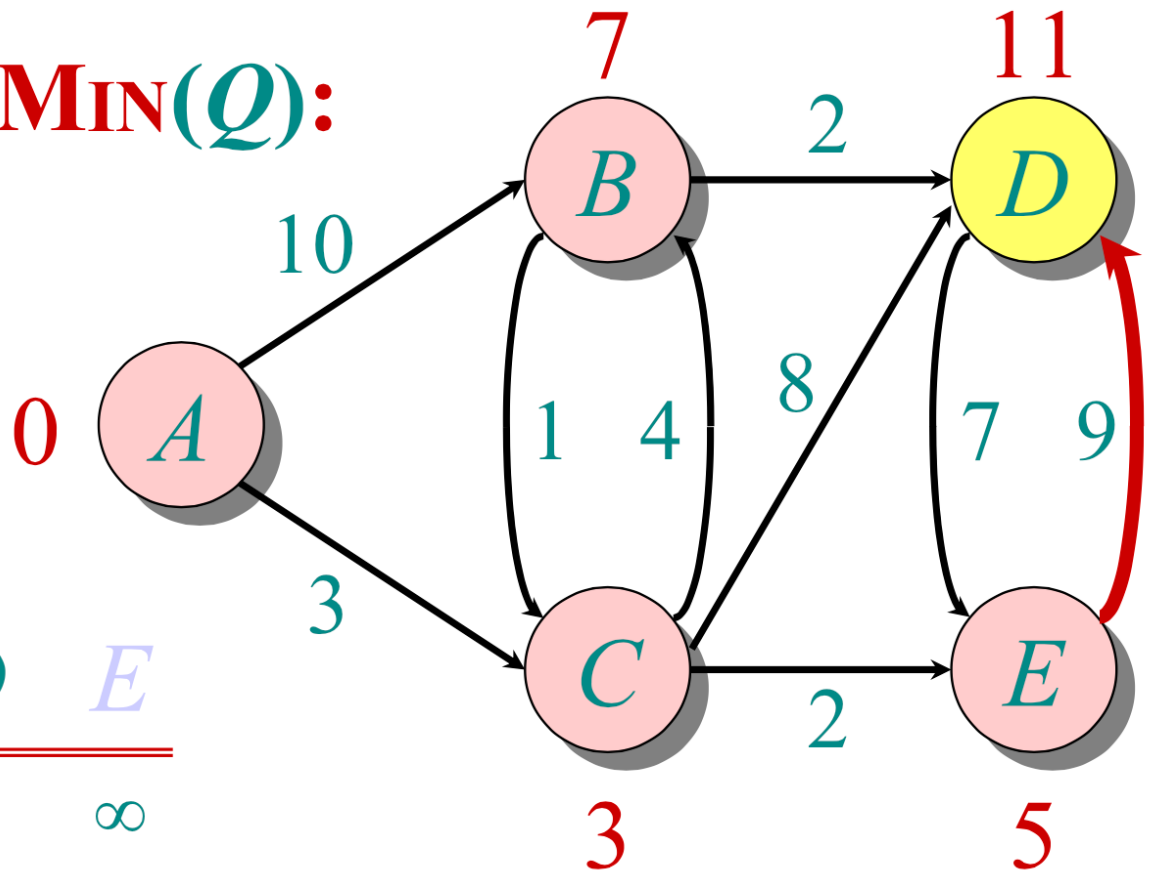


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { *A*, *C*, *E* }

Example of Dijkstra's algorithm

"*B*" ← EXTRACT-MIN(*Q*):

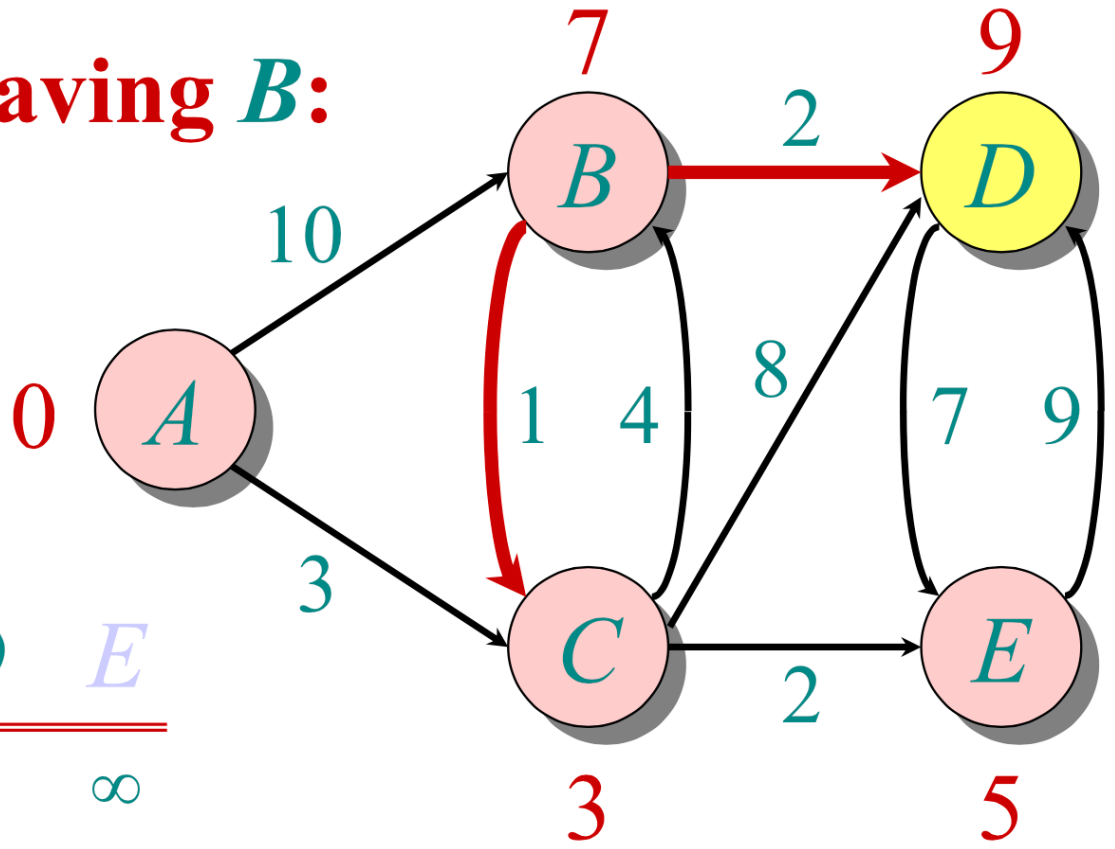


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { *A*, *C*, *E*, *B* }

Example of Dijkstra's algorithm

Relax all edges leaving *B*:

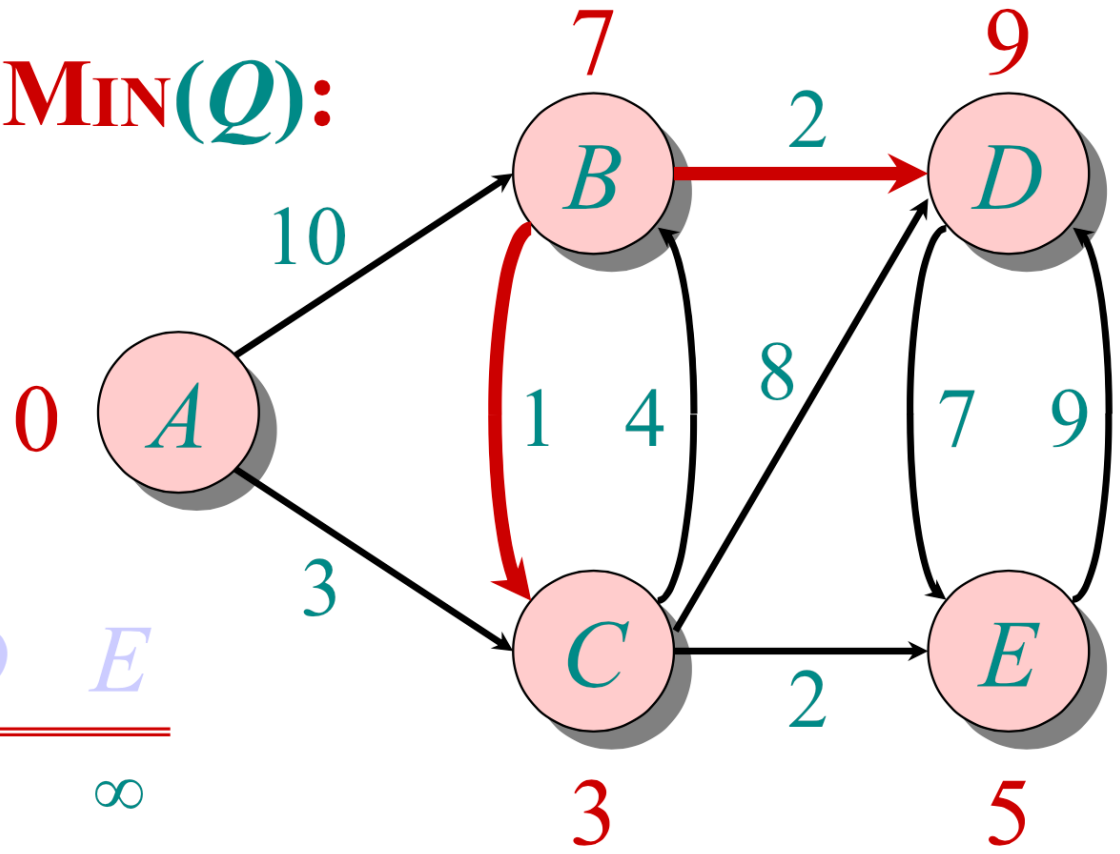


Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

S: { A, C, E, B }

Example of Dijkstra's algorithm

"*D*" ← EXTRACT-MIN(*Q*):

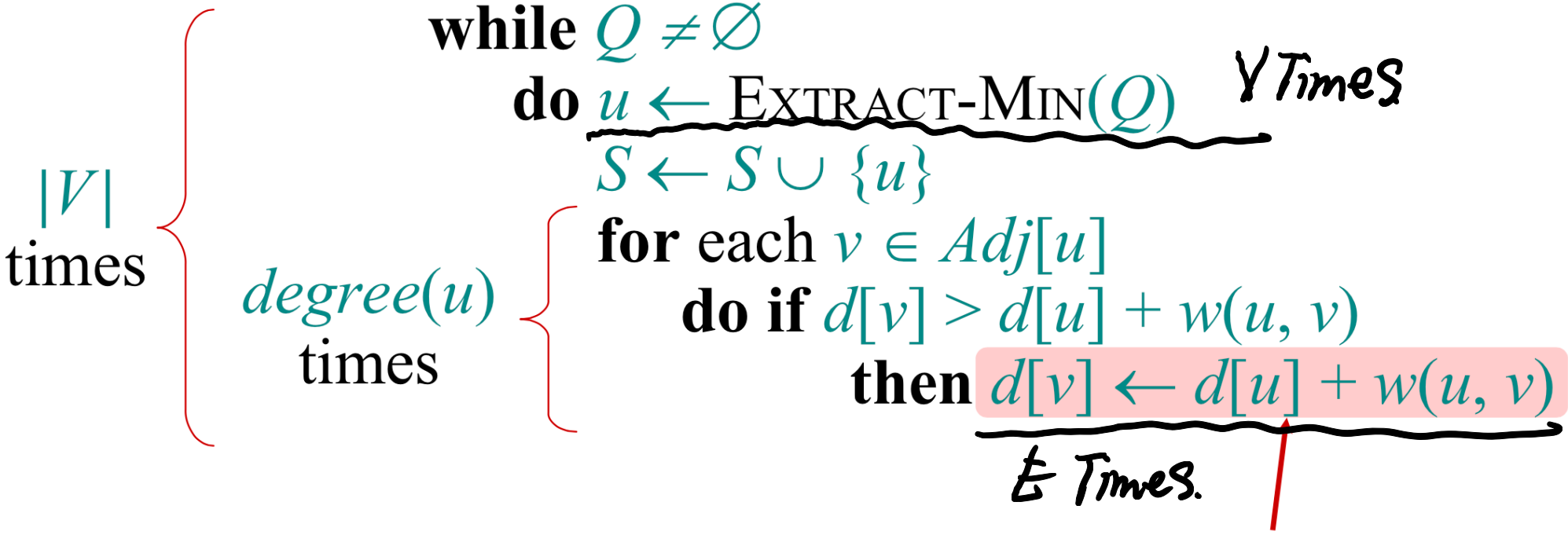


Q:

A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5
	7		11	
			9	

S: { *A*, *C*, *E*, *B*, *D* }

Analysis of Dijkstra's algorithm



Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.

$$\text{Time} = \Theta(V \cdot T_{\text{EXTRACT-MIN}} + E \cdot T_{\text{DECREASE-KEY}})$$

- Same formula as in the analysis of Prim's minimum spanning tree algorithm.

Analysis of Dijkstra's algorithm

The same with Prim's Algorithm: Running Time depends on the Data Structure.

$$\text{Time} = \Theta(V)\cdot T_{\text{EXTRACT-MIN}} + \Theta(E)\cdot T_{\text{DECREASE-KEY}}$$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$
Fibonacci heap	$O(\lg V)$ amortized	$O(1)$ amortized	$O(E + V \lg V)$ worst case

- Same formula as in the analysis of Prim's minimum spanning tree algorithm.

Unweighted graphs

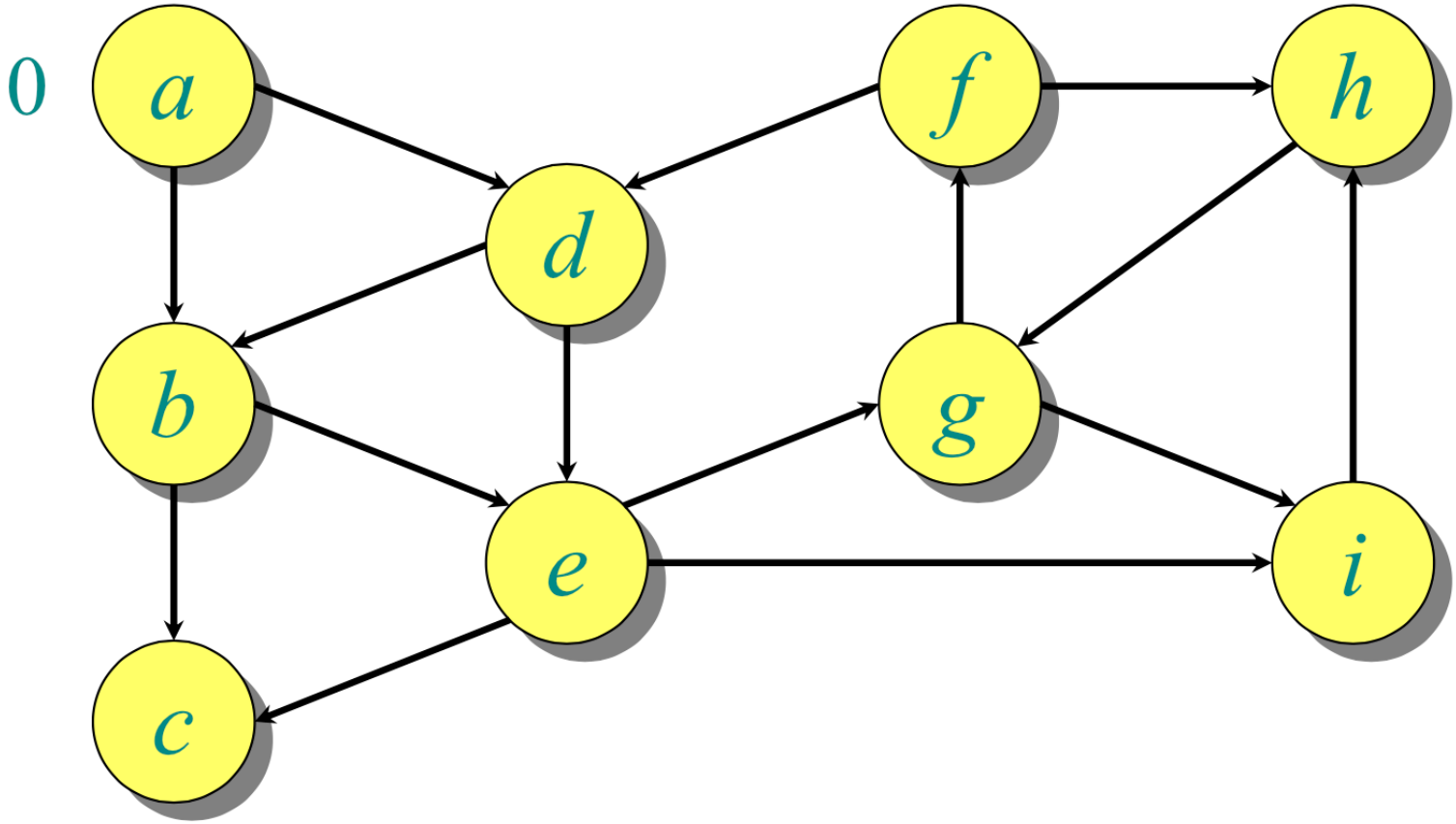
Suppose that **w(u, v) = 1** for all **(u, v) ∈ E**. Can Dijkstra's algorithm be improved?

- **Breadth-first search!** BFSTree is Shortest Path Tree.

```
while Q ≠ ∅
    do u ← DEQUEUE(Q)
        for each v ∈ Adj[u]
            do if d[v] = ∞        ← color[v] = white.
                then d[v] ← d[u] + 1
                     ENQUEUE(Q, v)
```

Analysis: Time = O(V + E). We don't need to maintain Extract-Min heap DS

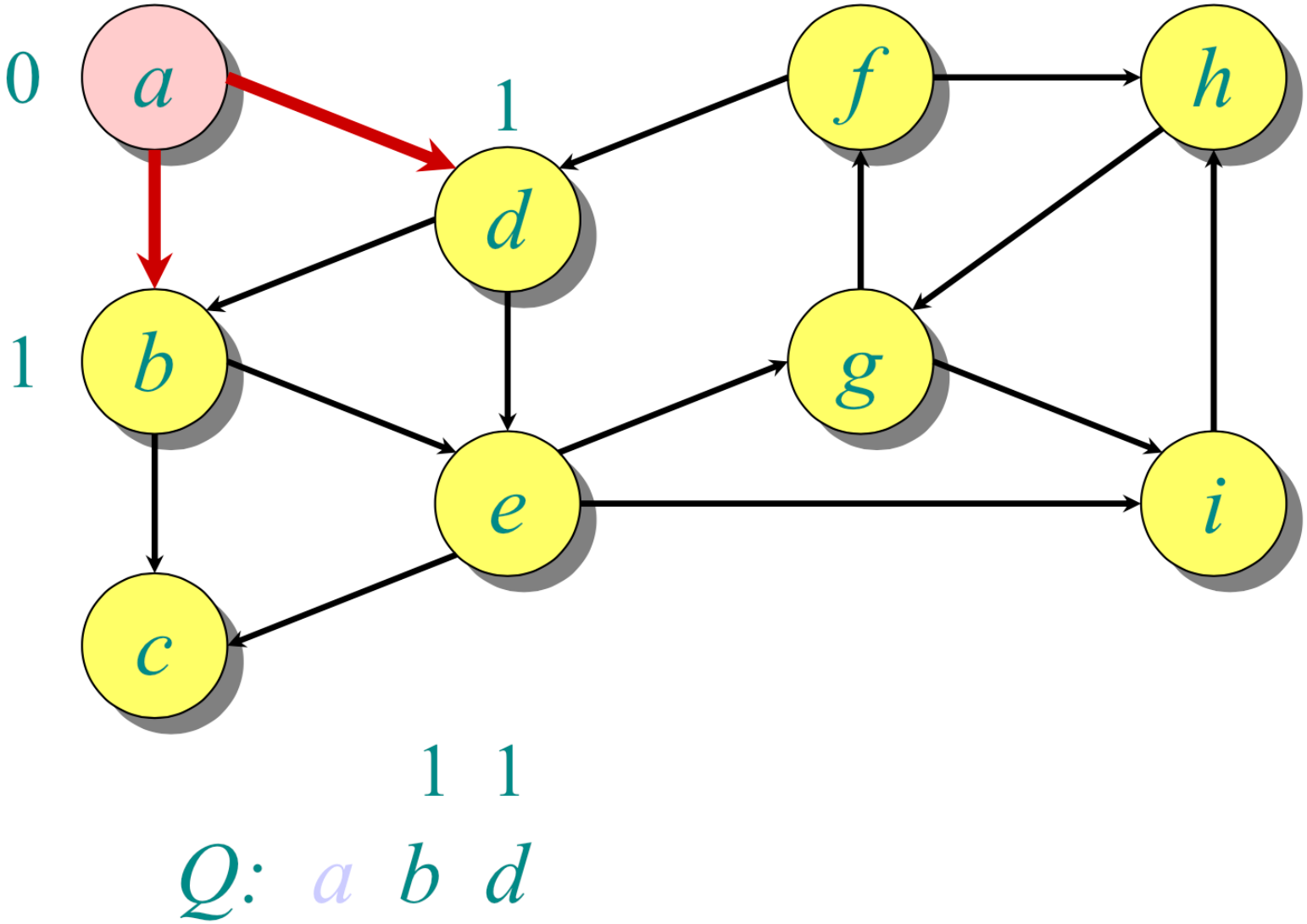
Example of breadth-first search



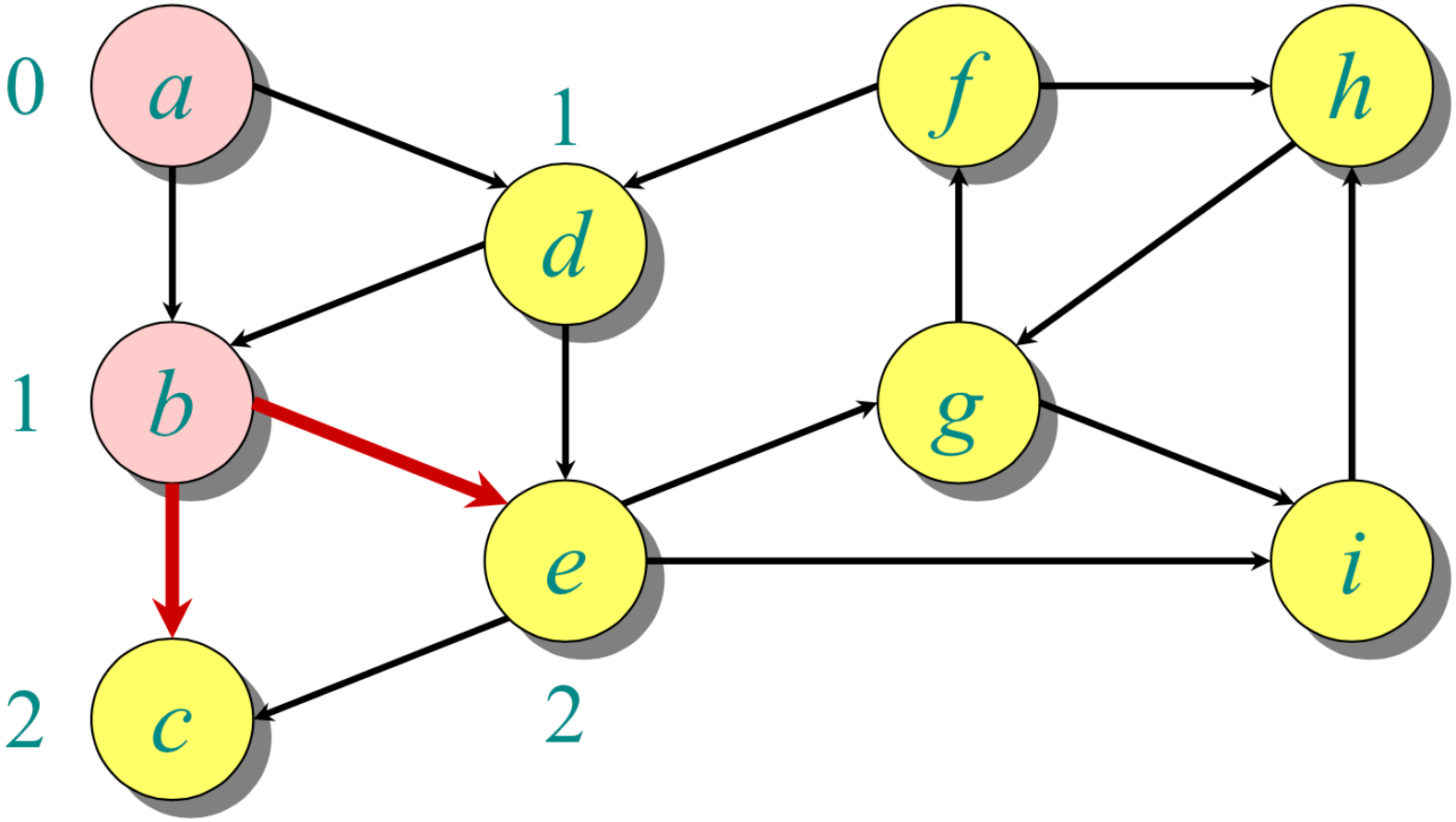
0

Q: a

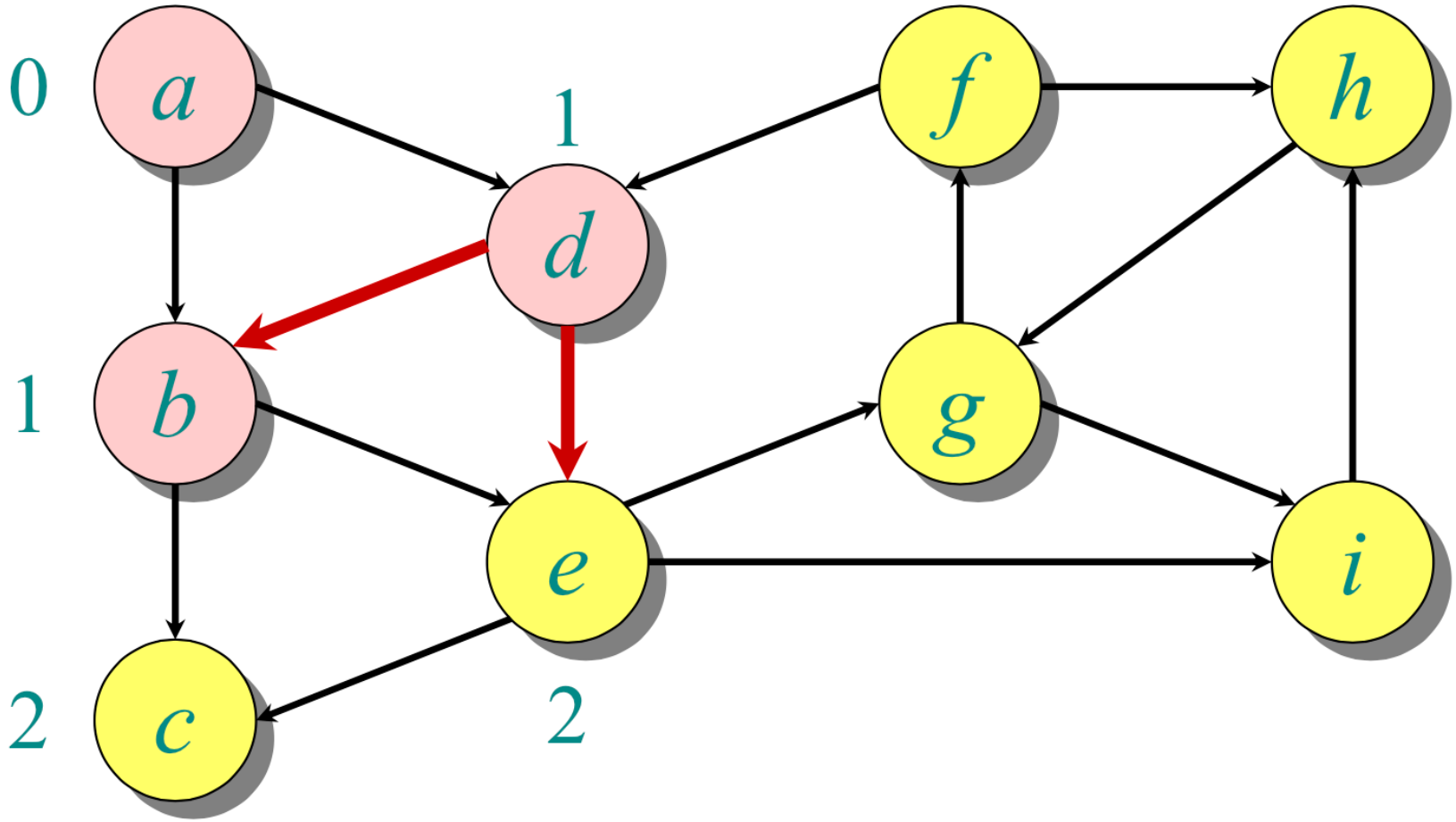
Example of breadth-first search



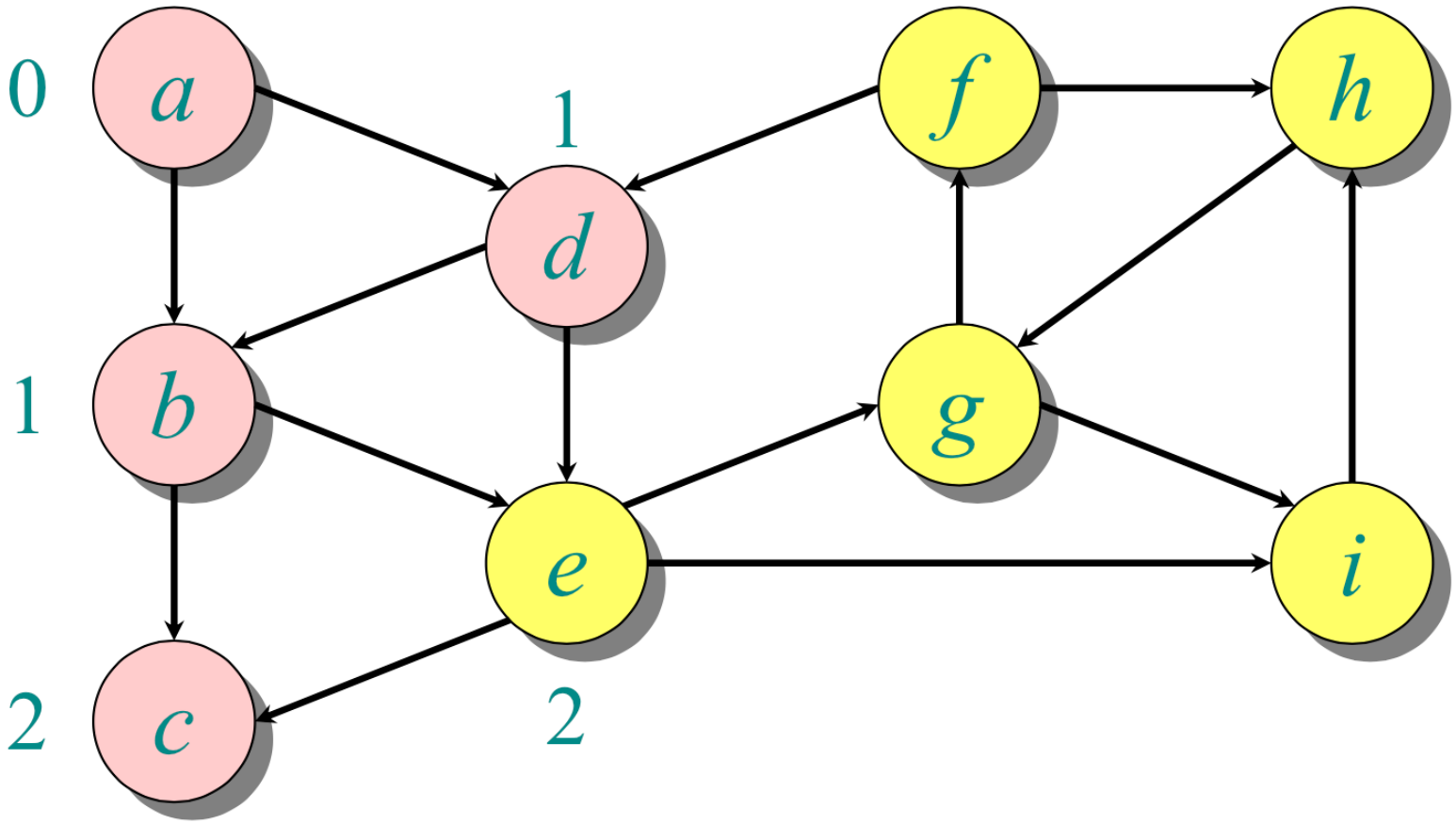
Example of breadth-first search



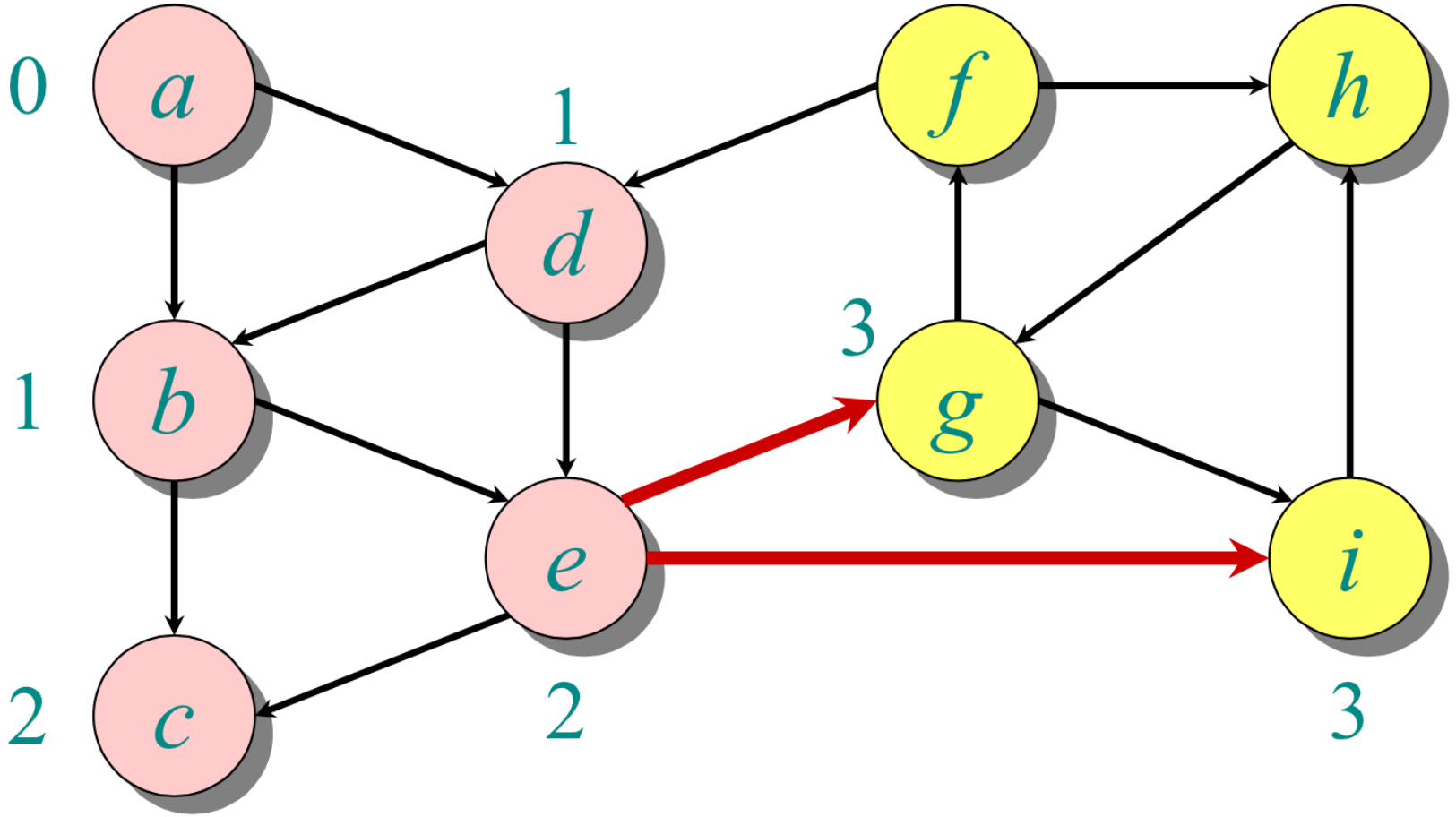
Example of breadth-first search



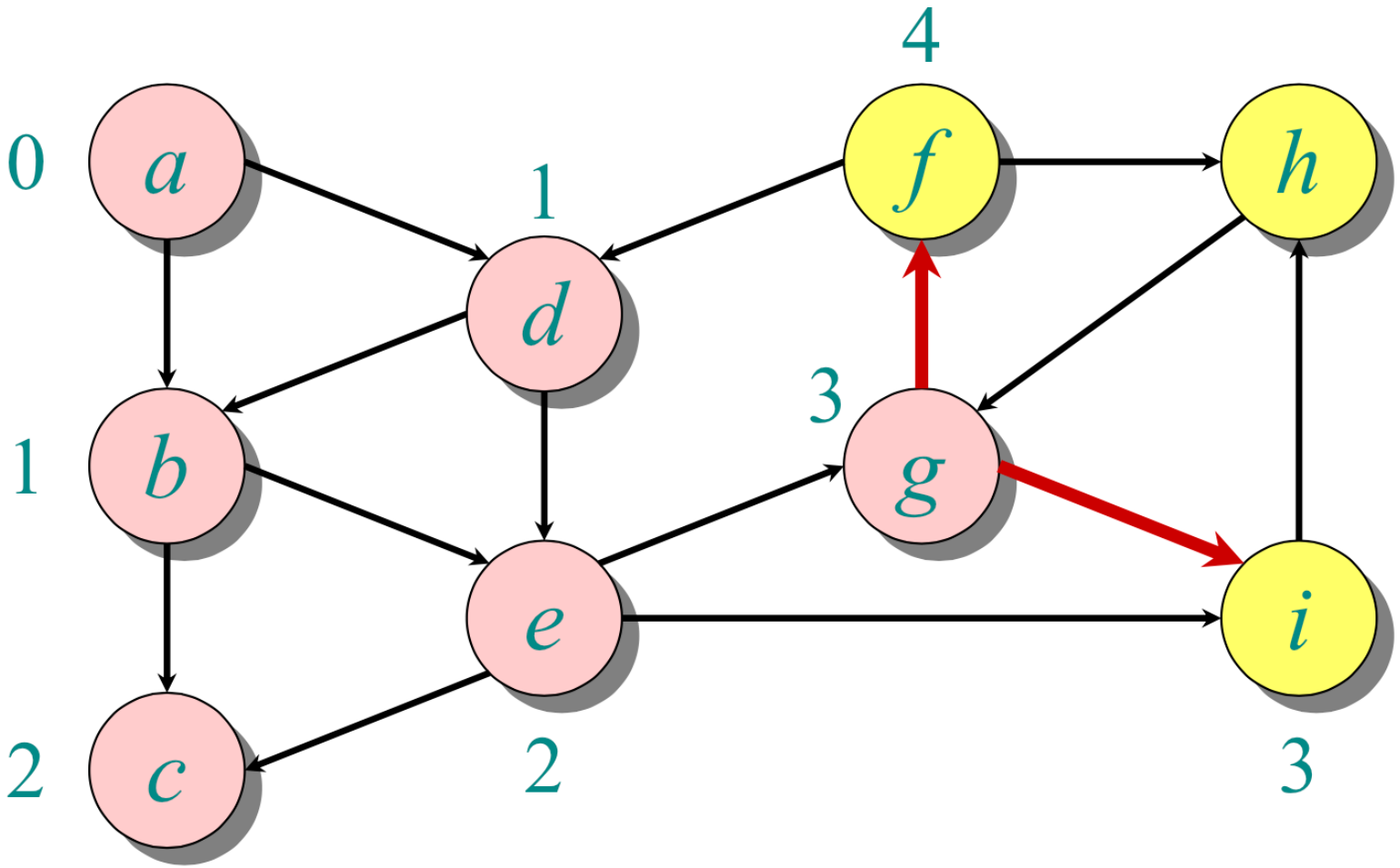
Example of breadth-first search



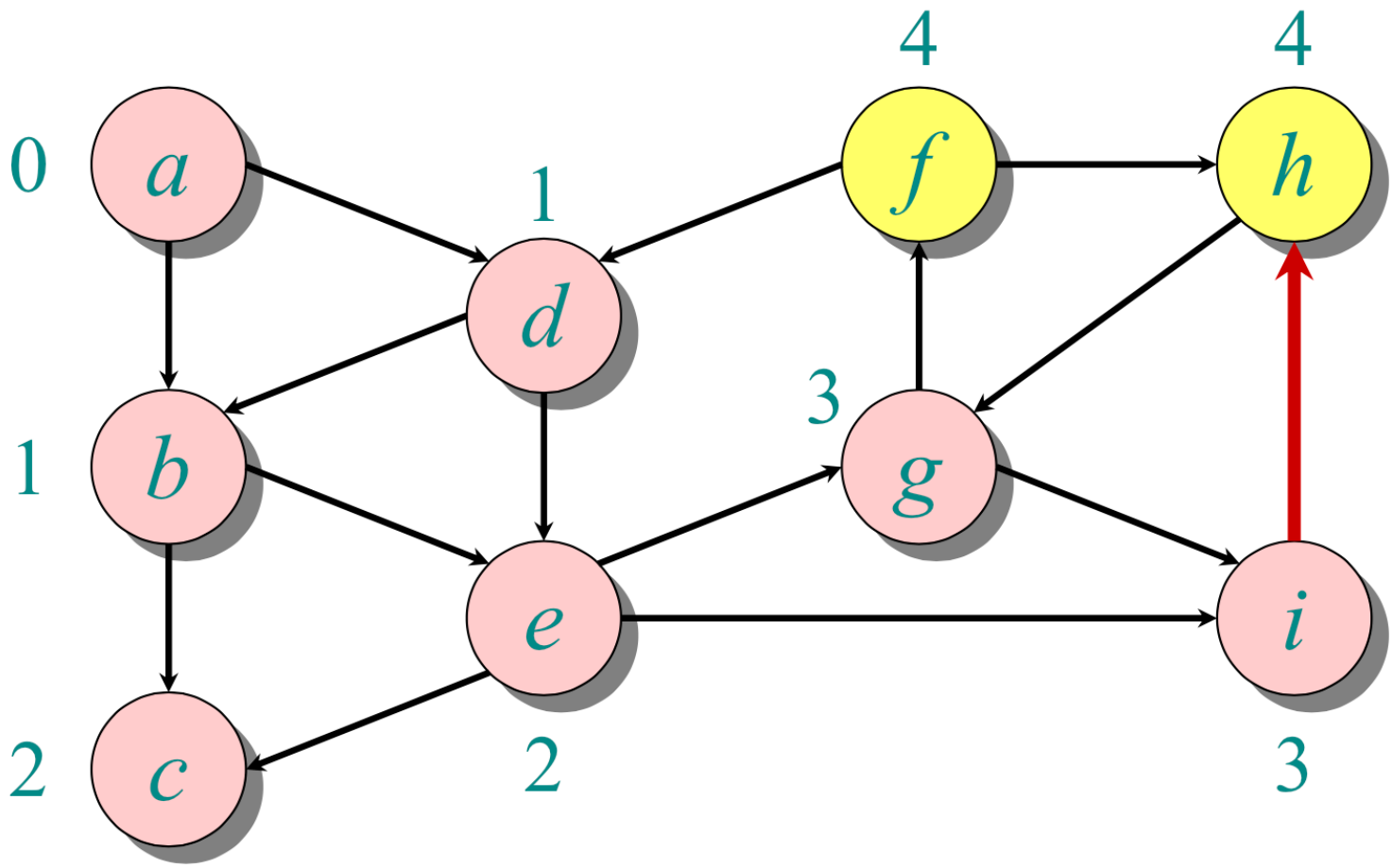
Example of breadth-first search



Example of breadth-first search



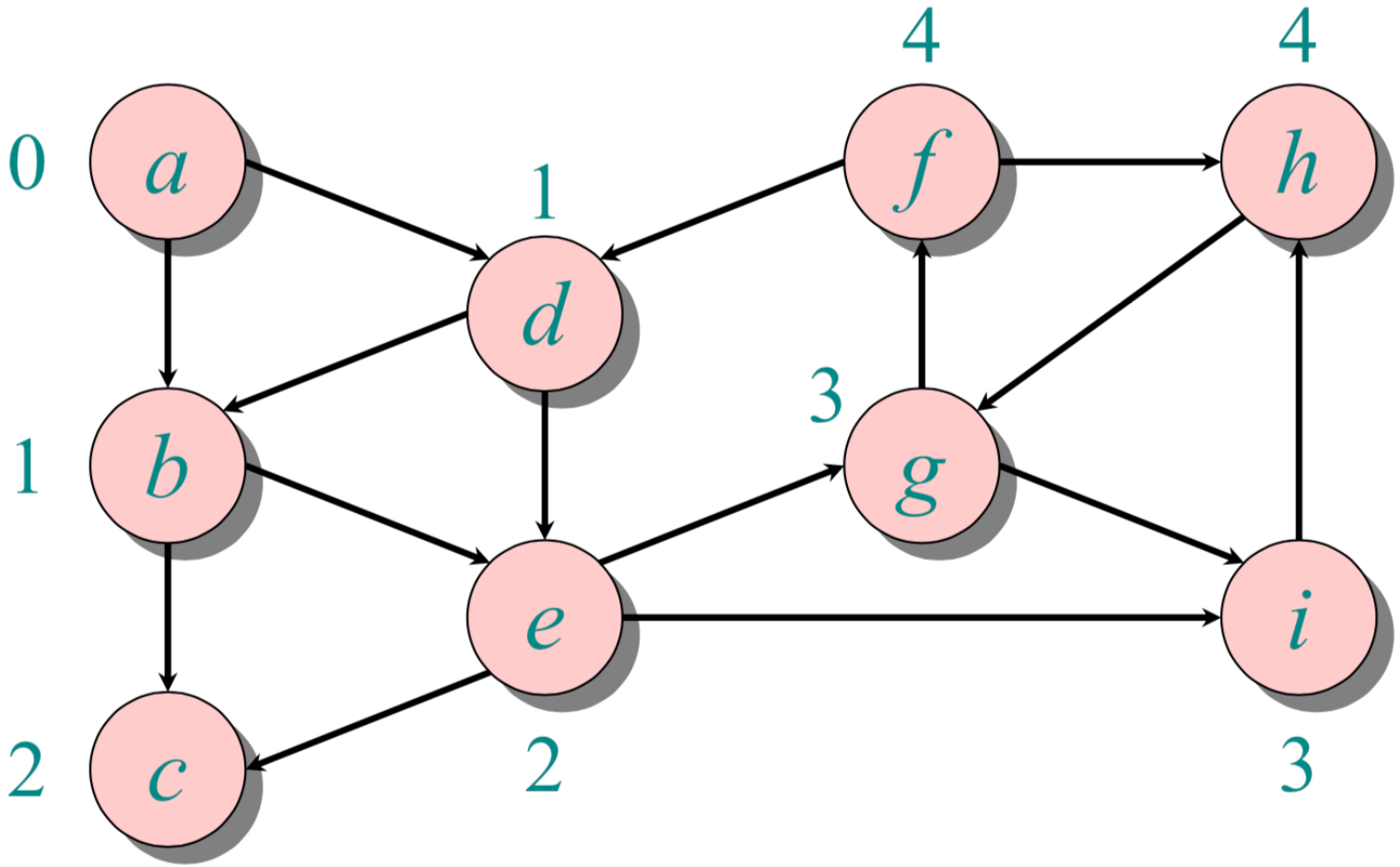
Example of breadth-first search



4 4

Q: a b d c e g i f h

Example of breadth-first search

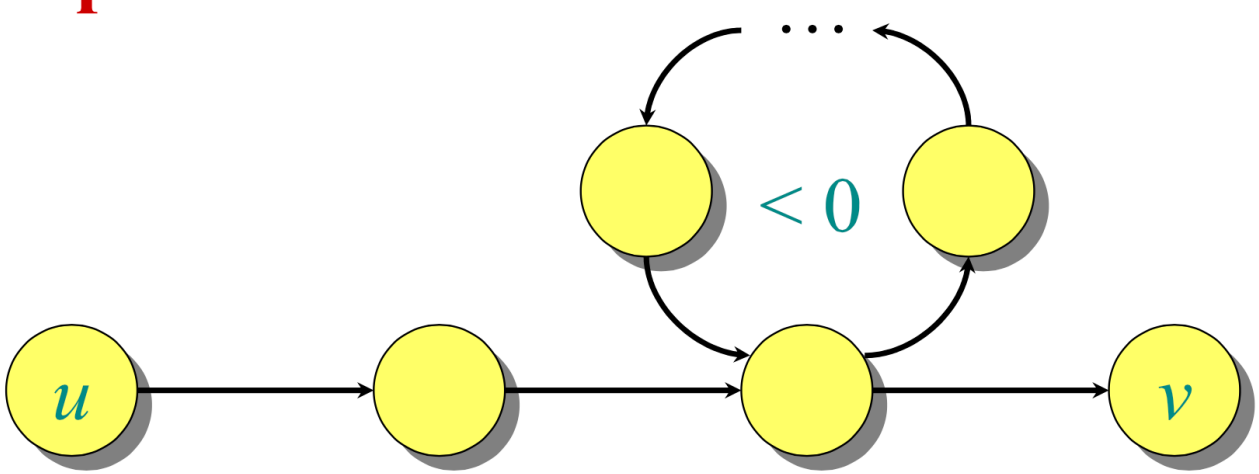


Q: a b d c e g i f h

Negative-Weight Cycle

- Recall: If a graph G contains a negative-weight cycle, then some shortest paths may not exist.

Example:



Bellman-Ford algorithm: Finds all shortest-path lengths from a source $s \in V$ to all $v \in V$ or determines that a negative-weight cycle exists.

Can't be a greedy Algorithm.

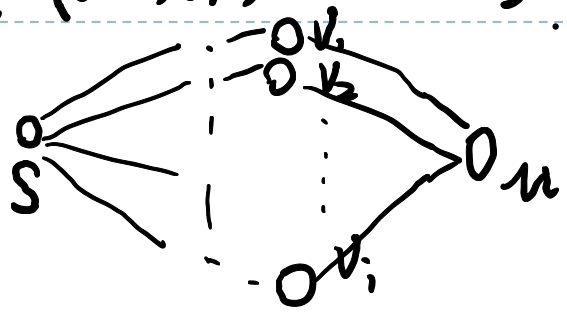
maintain 2 tables: D value / π value. HW12 A1

See the note of lecture12

Bellman-Ford algorithm

update after each iteration according to value from last iteration

$d^{(k)}(s,u) = \min_i \left(d^{(k-1)}(s,v_i), w(v_i,u) \right)$




```
d[s] ← 0                                          } initialization
for each v ∈ V − {s}                              } to value from last iteration
    do d[v] ← ∞
for i ← 1 to |V| − 1                              } at most |V|−1 : |V| vertex involved
    do for each edge (u, v) ∈ E                   } relaxation step
        do if d[v] > d[u] + w(u, v)
            then d[v] ← d[u] + w(u, v)
for each edge (u, v) ∈ E
    do if d[v] > d[u] + w(u, v)                   } after |V|−1 still has update.
        then report that a negative-weight cycle exists
```

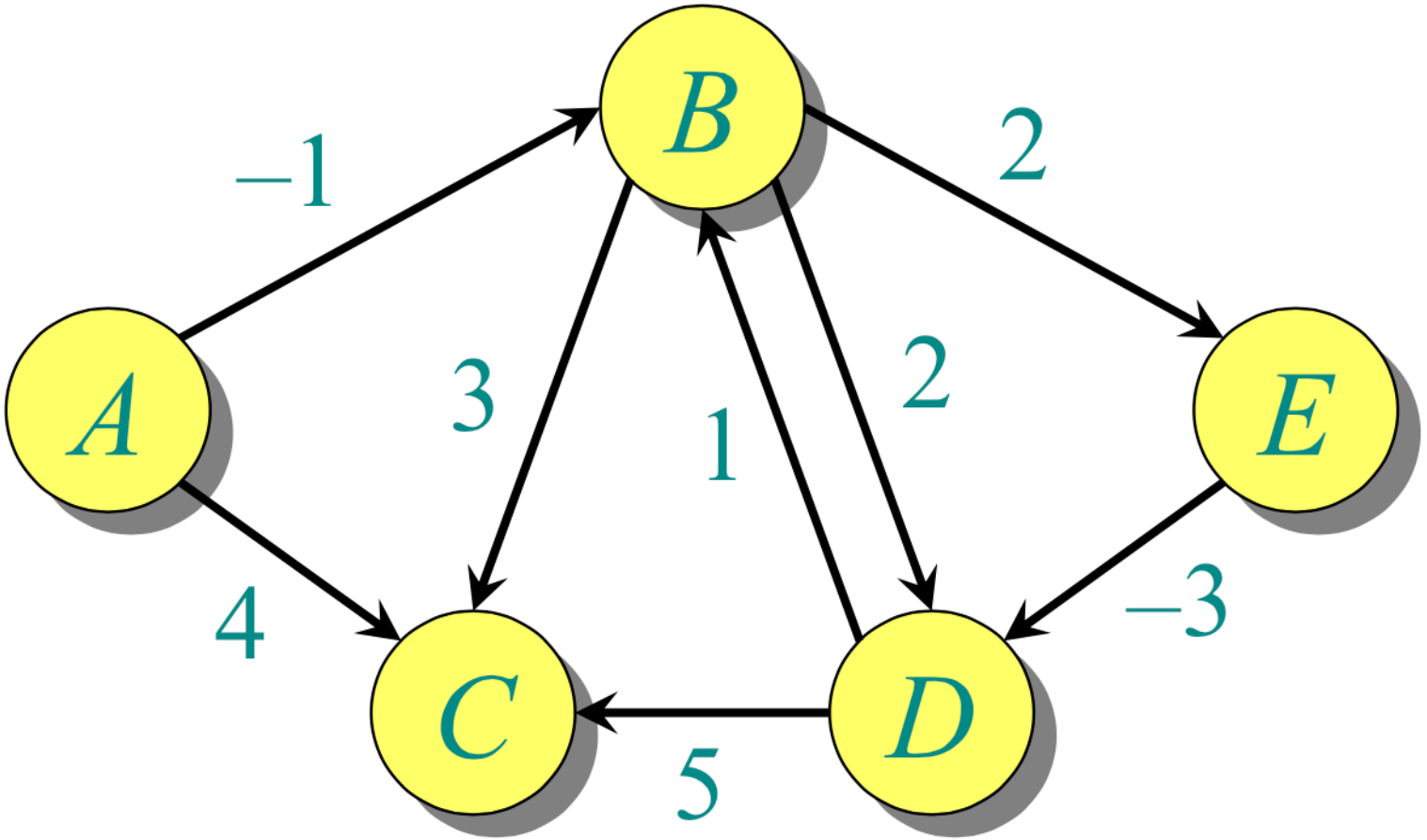
$d^{i}(v) = d^{i-1}[u] + w(u,v)$

V ← for, E ← do for each edge, $O(|V| \cdot E)$

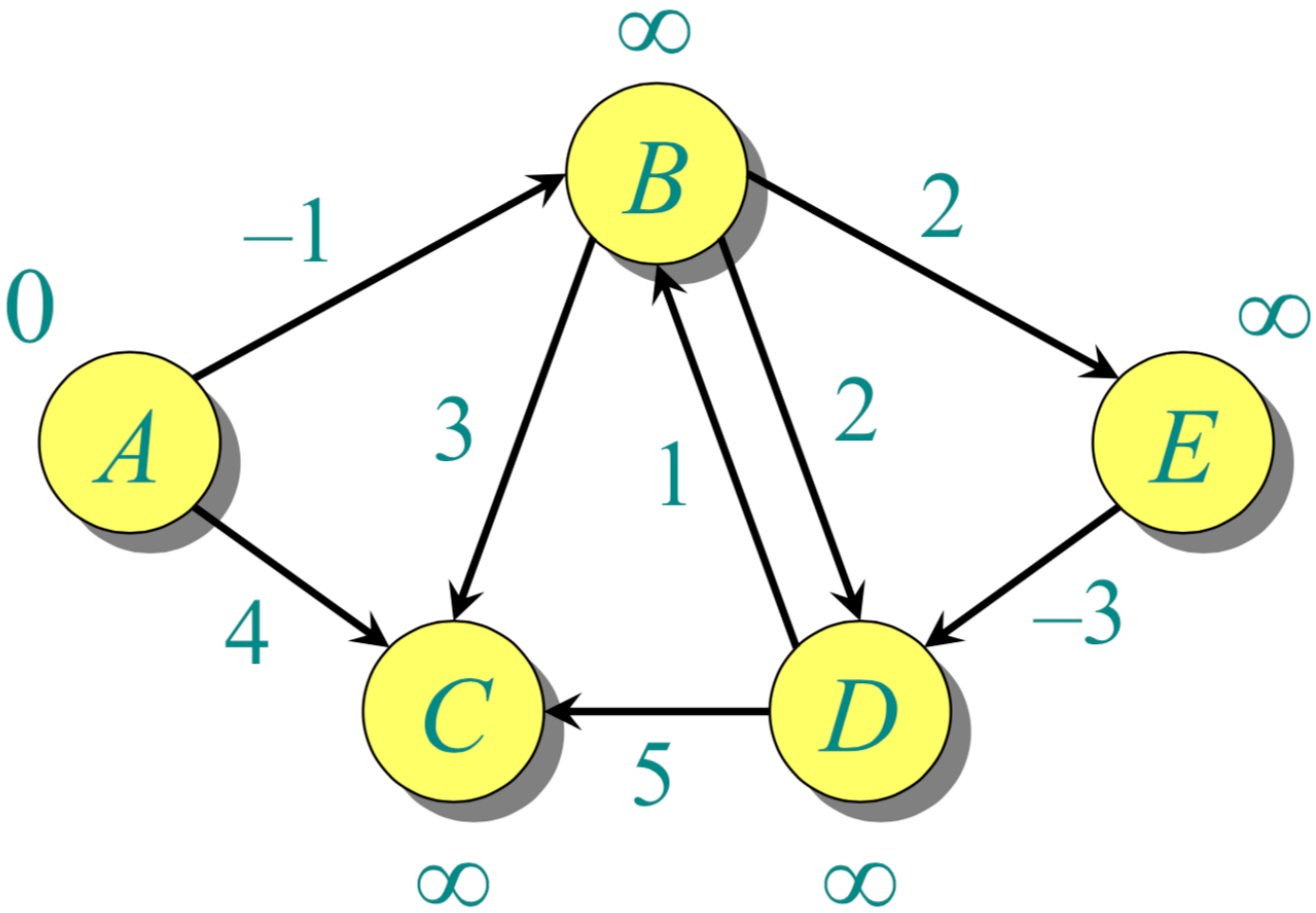
Time = O(VE).

At the end, **d[v] = δ(s, v)**, if no negative-weight cycles.

Example of Bellman-Ford

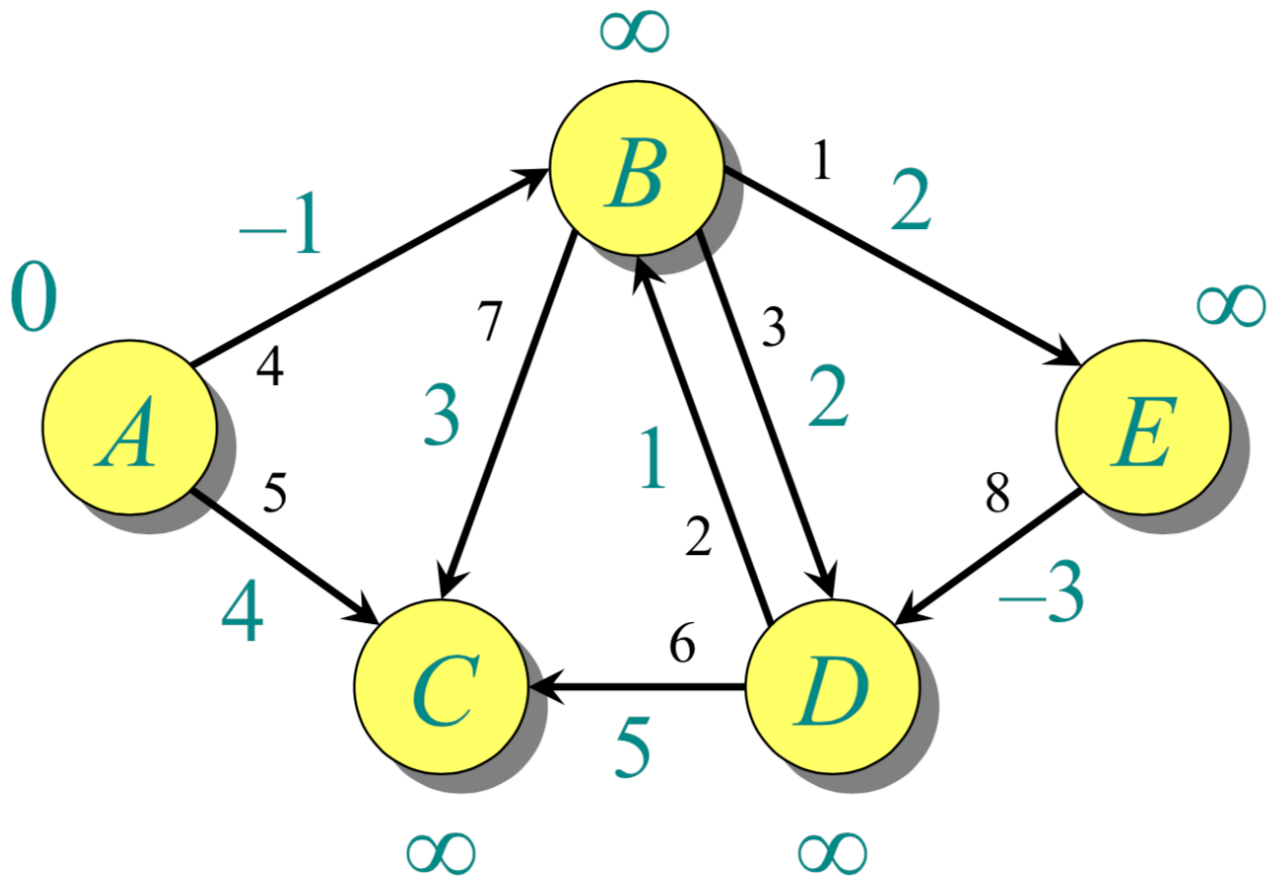


Example of Bellman-Ford



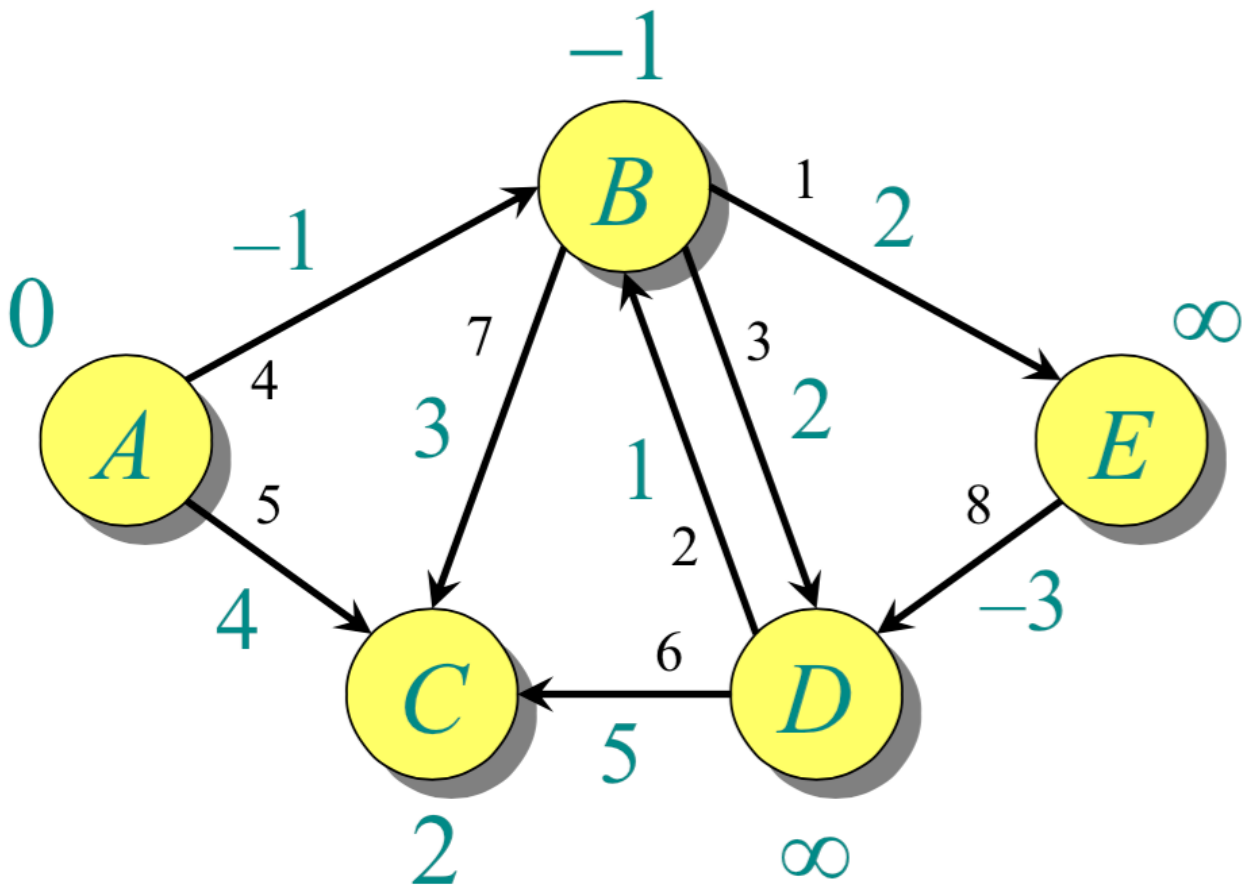
Initialization.

Example of Bellman-Ford



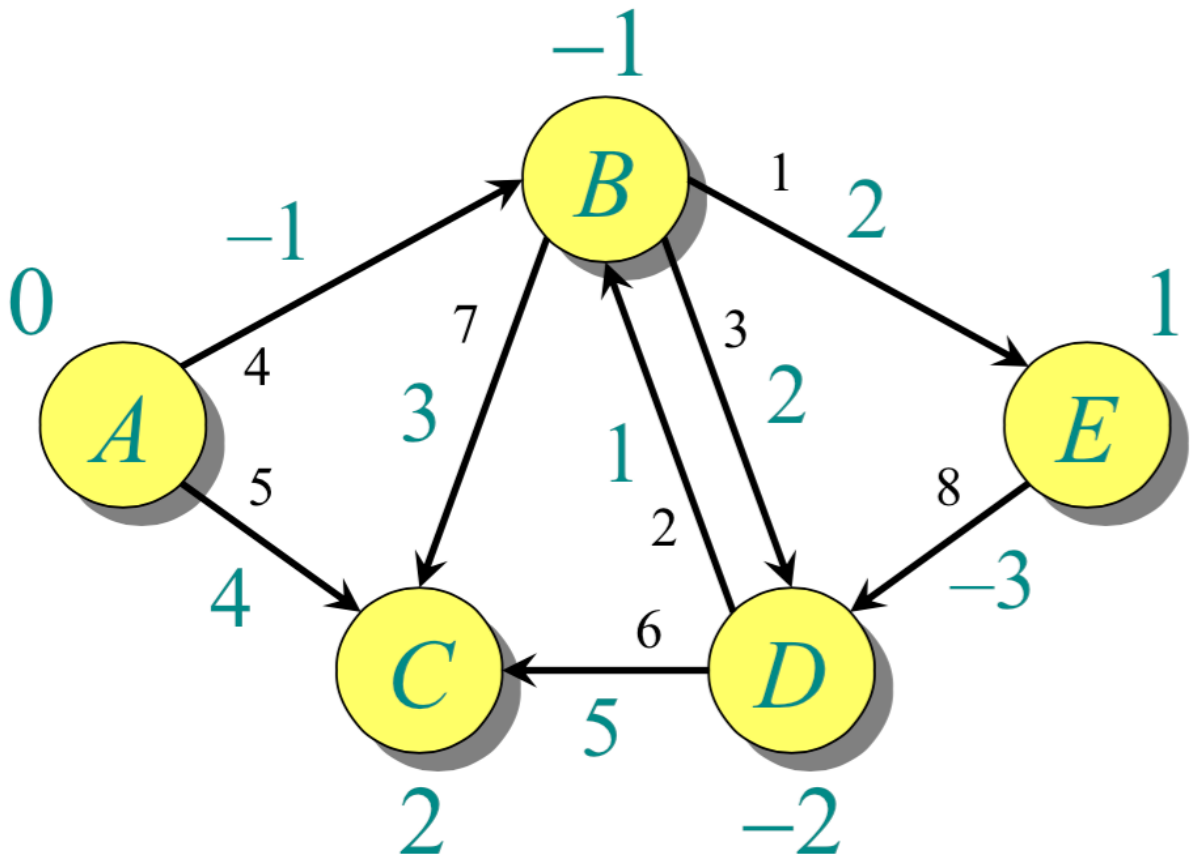
Order of edge relaxation.

Example of Bellman-Ford



End of pass 1.

Example of Bellman-Ford



End of pass 2 (and 3 and 4).

Detection of negative-weight cycles

- Corollary: If a value **d[v]** fails to converge after **|V| – 1** passes, there exists a negative-weight cycle in **G** reachable from **s**.

Shortest Paths in Directed Acyclic Graphs

- Shortest path are always well defined in a DAG: even if there are negative weight edges, no negative-weight cycle can exist.

- By relaxing the edges of weighted DAG G=(V, E) according to a topological sort of its vertices, we can compute shortest paths from a single source in $\Theta(V+E)$ time
  - Topological sort on the DAG to impose linear ordering on vertices
  - Make one pass relaxation over the vertices in the topological sort order. As we process each vertex, relax each edge that leaves the vertex.

Shortest Paths in DAG

- Topological sort + one pass Bellman-Ford

```
DAG-SHORTEST-PATHS(G, w, s)
1  topologically sort the vertices of G
2  INITIALIZE-SINGLE-SOURCE(G, s)
3  for each vertex u, taken in topologically sorted order
4      for each vertex v ∈ G.Adj[u]
5          RELAX(u, v, w)
```

From left to right according to the topo order. One vertex at a time to update.

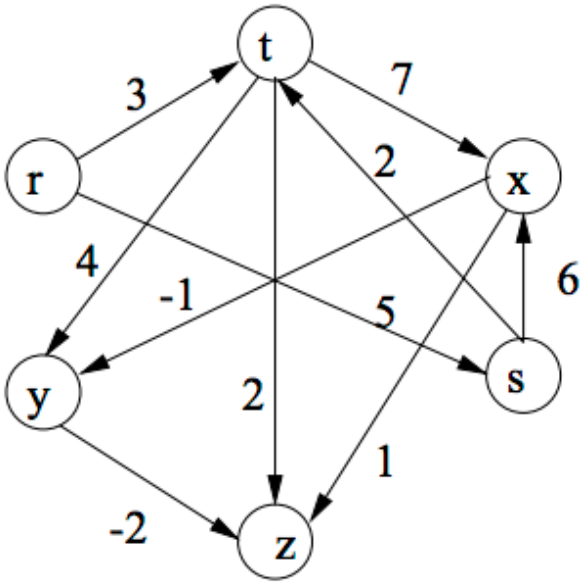
⇒ update:

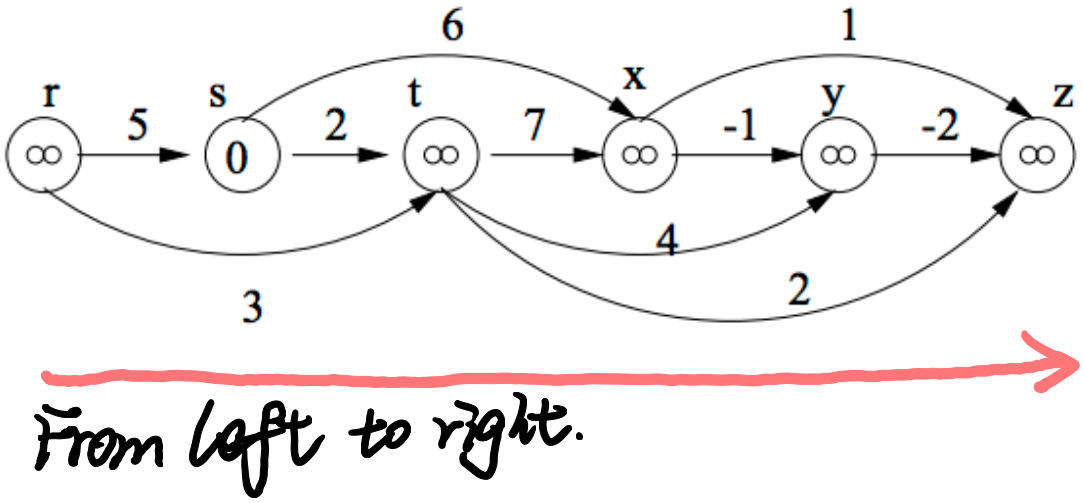
d[v] = d[u] + w(u, v)

⇒ if:

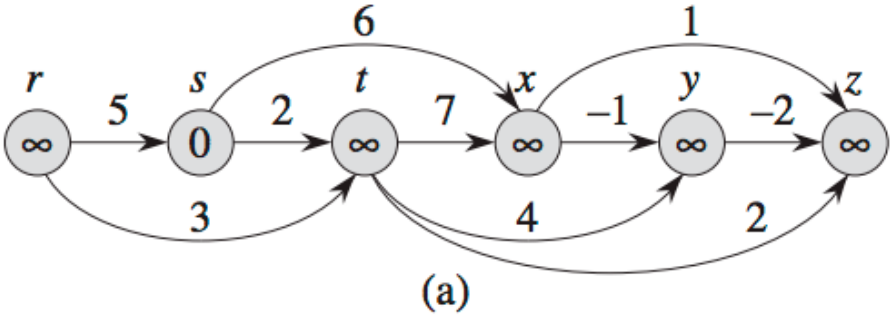
d[v] > d[u] + w(u, v)

Example of Shortest Paths in DAG

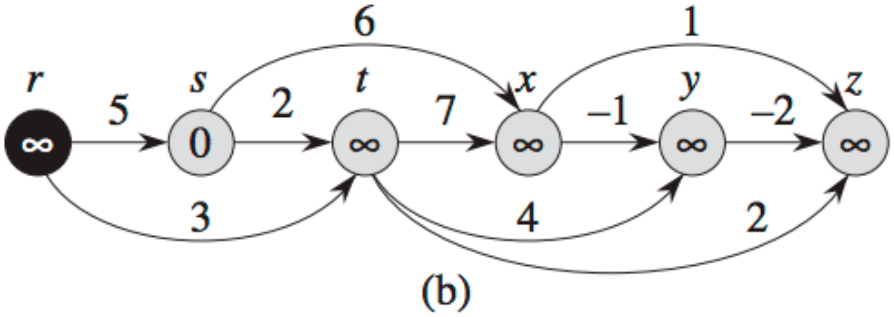




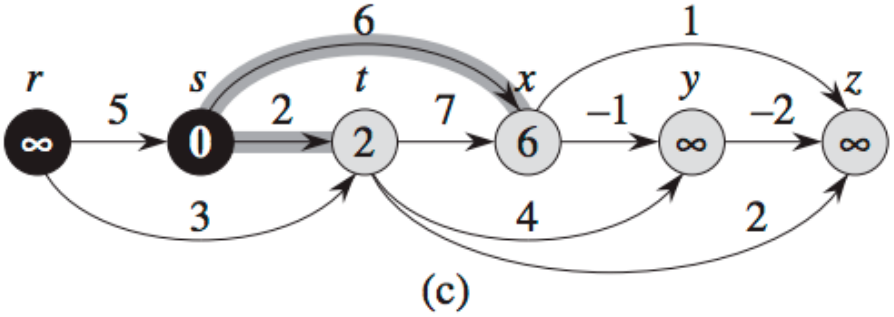
From left to right.


r
s
t
x
y
z
∞
0
∞
∞
∞
∞
5
6
2
7
1
−1
−2
3
4
2

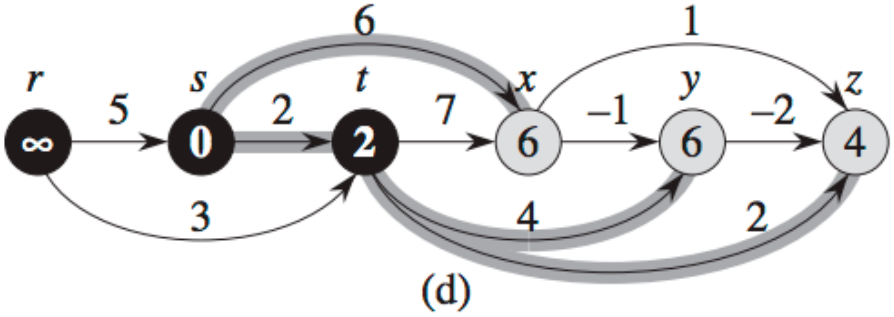
(a)


r
s
t
x
y
z
∞
0
∞
∞
∞
∞
5
6
2
7
1
−1
−2
3
4
2

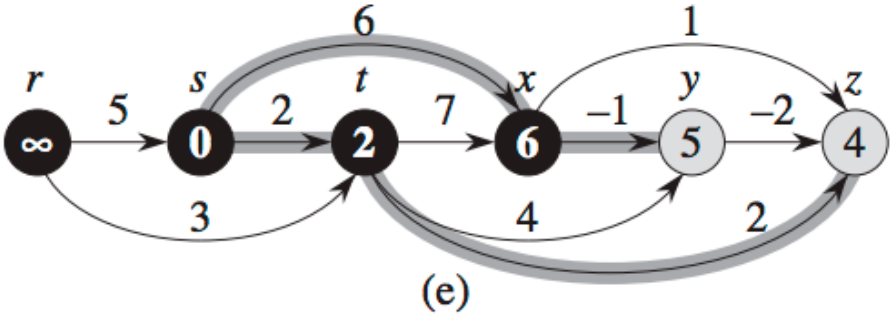
(b)


r
s
t
x
y
z
∞
0
2
6
∞
∞
5
6
2
7
1
−1
−2
3
4
2

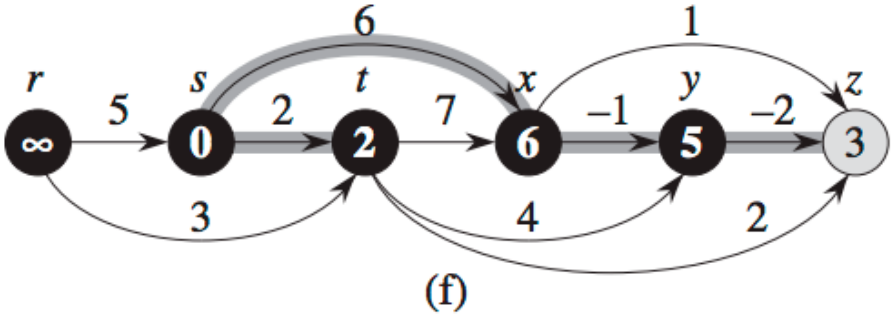
(c)


r
s
t
x
y
z
∞
0
2
6
6
4
5
6
2
7
1
−1
−2
3
4
2

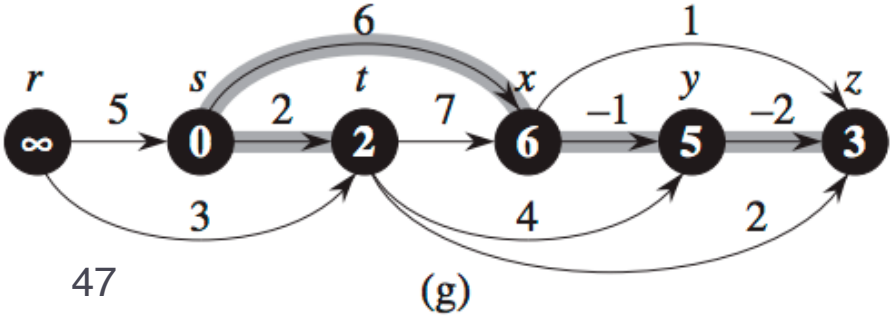
(d)


r
s
t
x
y
z
∞
0
2
6
5
4
5
6
2
7
1
−1
−2
3
4
2

(e)


r
s
t
x
y
z
∞
0
2
6
5
3
5
6
2
7
1
−1
−2
3
4
2

(f)


r
s
t
x
y
z
∞
0
2
6
5
3
5
6
2
7
1
−1
−2
3
4
2

(g)

All-pairs shortest paths

Given a weighted digraph G = (V, E) with weight function w: E → R (R is the set of real numbers), determine the length of the shortest path (i.e., distance) between all pairs of vertices in G.

1. Nonnegative edge weights: |V| times of Dijkstra's algorithm: $O(VE + V^2 \lg V)$
2. Unweighted graphs: |V| times of BFS: $O(VE + V^2)$
3. General case: Bellman-Ford algorithm: $O(V^2E)$.
   - Dense graph ($|E| = |V|^2$ edges) $\Rightarrow \Theta(V^4)$ time in the worst case.
4. Better approach? Floyd-Warshall algorithm

Input and Output Formats

To simplify the notation, we assume that $V = \{1, 2, \ldots, n\}$.

Assume that the graph is represented by an $n \times n$ matrix with the weights of the edges:

Initialize the matrix

$$w_{ij} = \begin{cases} 0 & \text{if } i = j, \\ w(i,j) & \text{if } i \neq j \text{ and } (i,j) \in E, \\ \infty & \text{if } i \neq j \text{ and } (i,j) \notin E. \end{cases}$$

∵ adjascent

Output Format: an $n \times n$ matrix $D = [d_{ij}]$ where d_{ij} is the length of the shortest path from vertex i to j.

The Floyd-Warshall Algorithm

Dynamic Programming Approach: The Floyd-Warshall algorithm considers the intermediate vertices of a shortest path.

Definition: The vertices $v_2, v_3, \ldots, v_{l-1}$ are called the *intermediate vertices* of the path $p = \langle v_1, v_2, \ldots, v_{l-1}, v_l \rangle$.

Decomposition with Intermediate Vertices

Let $d_{ij}^{(k)}$ be the length of the shortest path from i to j such that *all* intermediate vertices on the path (if any) are in set $\{1, 2, \ldots, k\}$. intermediate vertices number are included in $\{1,2,\cdots,k\}$.

$D^0 = W$

$d_{ij}^{(0)}$ is set to be w_{ij}, i.e., no intermediate vertex.

Let $D^{(k)}$ be the $n \times n$ matrix $[d_{ij}^{(k)}]$.

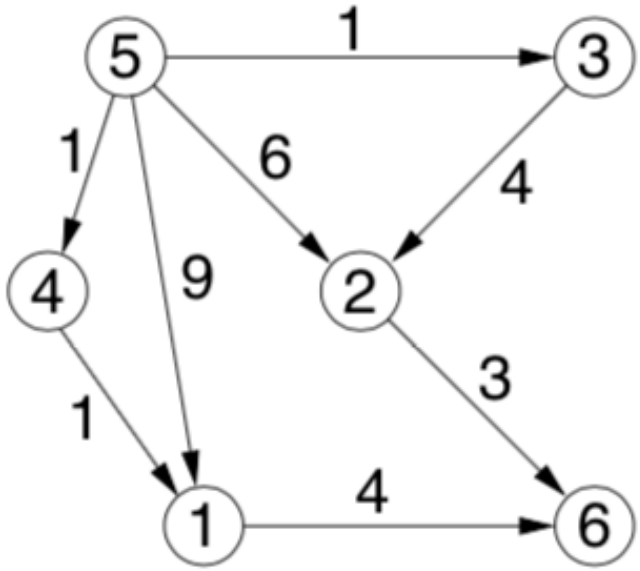
Claim: $d_{ij}^{(n)}$ is the distance from i to j. So our aim is to compute $D^{(n)}$.

$d_{ij}^{(k)} \le d_{ij}^{(l)}$ if $k > l$

Subproblems: compute $D^{(k)}$ for $k = 0, 1, \cdots, n$.

Decomposition with Intermediate Vertices

Example: how the value of $d_{5,6}^{(k)}$ changes as k varies



$d_{5,6}^{(0)} = \text{INF}$ (no path)

$d_{5,6}^{(1)} = 13$ (5,1,6) Number of intermediate node = 1

$d_{5,6}^{(2)} = 9$ (5,2,6) Number of intermediate node = 2

$d_{5,6}^{(3)} = 8$ (5,3,2,6)

$d_{5,6}^{(4)} = 6$ (5,4,1,6)

Key: How to build $d_{i,j}^{(k)}$ from $d_{i,j}^{(k-1)}$

Structure of shortest paths

Observation 1: A shortest path does not contain the same vertex twice.

Proof: A path containing the same vertex twice contains a cycle. Removing cycle gives a shorter path.

Observation 2: For a shortest path from i to j such that any intermediate vertices on the path are chosen from the set {1,2,...,k}, there are two possibilities:

1. k is not a vertex on the path; The shortest such path has length $d_{ij}^{(k-1)}$.
2. k is a vertex on the path. The shortest such path has length $d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$.

If k is on the shortest path from i to j

If k is not on the shortest path from i to j.

Structure of shortest paths

Consider a shortest path from i to j containing the vertex k. It consists of a subpath from i to k and a subpath from k to j.

Each subpath can only contain intermediate vertices in $\{1, \ldots, k-1\}$, and must be as short as possible, namely they have lengths $d_{ik}^{(k-1)}$ and $d_{kj}^{(k-1)}$.

Hence the path has length $d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$.

Combining the two cases we get On the shortest path.

$$d_{ij}^{(k)} = \min\left\{ d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \right\}.$$

No on the shortest path

The Bottom-up Computation

A recursive solution to the all-pairs shortest-paths problem:

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0\,, \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1\,. \end{cases}$$

- Bottom: $D^{(0)} = [w_{ij}]$, the weight matrix.

- Compute $D^{(k)}$ from $D^{(k-1)}$ using

$$d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

for $k = 1, \ldots, n$.

The Floyd-Warshall Algorithm

```
FLOYD-WARSHALL(W)
1  n = W.rows                                  ← number of vertices
2  D^(0) = W
3  for k = 1 to n                              dynamic programming
4      let D^(k) = (d_ij^(k)) be a new n × n matrix
5      for i = 1 to n
6          for j = 1 to n
7              d_ij^(k) = min(d_ij^(k-1), d_ik^(k-1) + d_kj^(k-1))
8  return D^(n)
```

The algorithm's running time is clearly $\Theta(n^3)$

Extracting Shortest Paths

To extract the final shortest paths, we compute the predecessor matrices: $\Pi^{(0)}, \Pi^{(1)}, \ldots, \Pi^{(n)}$ and $\Pi = \Pi^{(n)}$ where $\pi_{ij}^{(k)}$ is the predecessor of vertex j on the shortest path from vertex i with all intermediate vertices in the set of {1,2, …, k)

Compute Predecessor matrices

- Initialization:

Same vertex

Adjascent vertex

$$\pi_{ij}^{(0)} = \begin{cases} \text{NIL} & \text{if } i = j \text{ or } w_{ij} = \infty , \\ i & \text{if } i \neq j \text{ and } w_{ij} < \infty . \end{cases}$$

- Update:

★ update $\pi_{ij}^{(k)}$ with $d_{ij}^{(k)}$ together.

k is not on the shortest path

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)} , \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)} . \end{cases}$$

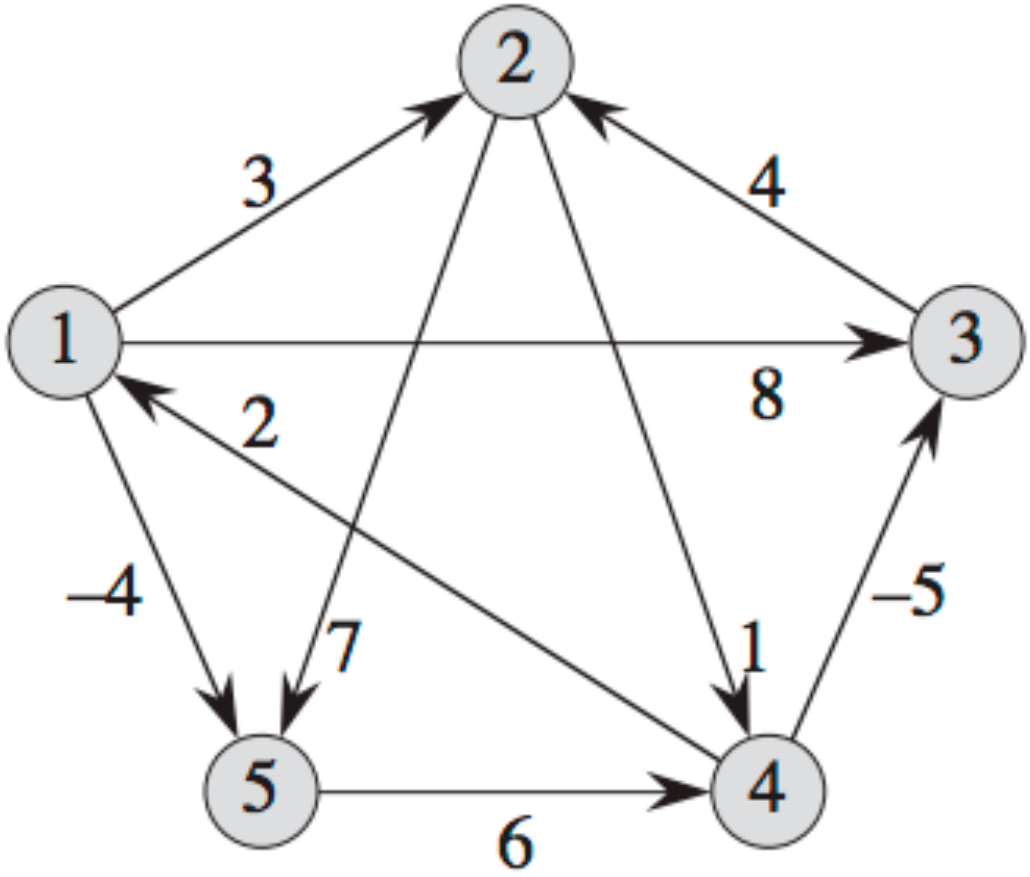
- Termination: $pred(i,j) \triangleq \pi_{ij} = \pi_{ij}^{(n)}$

k is on the shortest path

Extracting the Shortest Paths

```
Path(i, j)
{
    if ( pred(i, j) = i )    single edge
        output (i, j);
    else          compute the two parts of the path
    {
        Path(i, pred[i, j]);
        Path(pred[i, j], j);
    }
}
```

Example of Floyd-Warshall



Example of Floyd-Warshall

$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(0)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & \text{NIL} & 4 & \text{NIL} & \text{NIL} \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(1)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(2)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

Example of Floyd-Warshall

$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(3)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \qquad \Pi^{(4)} = \begin{pmatrix} \text{NIL} & 1 & 4 & 2 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \qquad \Pi^{(5)} = \begin{pmatrix} \text{NIL} & 3 & 4 & 5 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$

Transitive Closure of a Directed Graph

Given a directed graph $G = (V, E)$ with vertex set $V = \{1, 2, \ldots, n\}$, we might wish to determine whether G contains a path from i to j for all vertex pairs $i, j \in V$. We define the ***transitive closure*** of G as the graph $G^* = (V, E^*)$, where

$$E^* = \{(i, j) : \text{there is a path from vertex } i \text{ to vertex } j \text{ in } G\} .$$

One way to compute the transitive closure of a graph in $\Theta(n^3)$ time is to assign a weight of 1 to each edge of E and run the Floyd-Warshall algorithm. If there is a path from vertex i to vertex j, we get $d_{ij} < n$. Otherwise, we get $d_{ij} = \infty$.

Transitive Closure of a Directed Graph

Another Solution:

Compute $t_{ij} = \begin{cases} 1 & \text{if there exists a path from } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$

IDEA: Use Floyd-Warshall, but with $(\vee, \wedge)$ instead of $(\min, +)$:

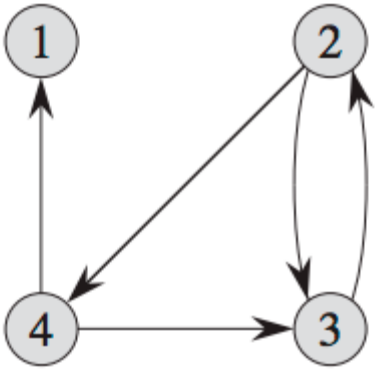
Recurrence:

$$t_{ij}^{(0)} = \begin{cases} 0 & \text{if } i \neq j \text{ and } (i,j) \notin E\,, \\ 1 & \text{if } i = j \text{ or } (i,j) \in E\,, \end{cases}$$

and for $k \geq 1$,

$$t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee \left(t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)}\right)\,.$$

Example of Transitive Closure



$$T_{43}^{(2)} = T_{43}^{(1)} \vee (T_{42}^{(1)} \wedge T_{23}^{(1)})$$

$$= 0 \vee (1 \wedge 1) = 0 \vee 1 = 1$$

$$T^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad T^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

$$T^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad T^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Next Lecture

- Introduction to NP-Completeness

- Final Review