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An Operator Splitting Solver for the GPR Model of Continuum Mechanics

Background

The GPR model is a hyperbolic set of PDEs, capable of describing the action of both fluids and solids. It is based on the solid model of Godunov & Romenski [4] and was first presented in [6] and expanded upon in [3] to include heat conduction:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v_k)}{\partial x_k}=0 \tag{1a}$$

$$\frac{\partial(\rho v_i)}{\partial t}+\frac{\partial(\rho v_i v_k+p\delta_{ik}-\sigma_{ik})}{\partial x_k}=0 \tag{1b}$$

$$\frac{\partial A_{ij}}{\partial t}+\frac{\partial(A_{ik}v_k)}{\partial x_j}+v_k\left(\frac{\partial A_{ij}}{\partial x_k}-\frac{\partial A_{ik}}{\partial x_j}\right)=-\frac{1}{\theta_1(\tau_1)}\frac{\partial E}{\partial A_{ij}} \tag{1c}$$

$$\frac{\partial(\rho J_i)}{\partial t}+\frac{\partial(\rho J_i v_k+T\delta_{ik})}{\partial x_k}=-\frac{\rho}{\theta_2(\tau_2)}\frac{\partial E}{\partial J_i} \tag{1d}$$

$$\frac{\partial(\rho E)}{\partial t}+\frac{\partial(\rho E v_k+(p\delta_{ik}-\sigma_{ik})v_i+q_k)}{\partial x_k}=0 \tag{1e}$$

$\rho, \mathbf{v}, p, \delta, \sigma, T, \boldsymbol{q}, E$ retain their usual meanings. A is the distortion tensor (containing information about the deformation and rotation of material elements), $\mathbf{J}$ is the thermal impulse vector (the conjugate of the entropy flux with respect to E), τ_1 is the strain dissipation time, τ_2 is the thermal impulse relaxation time. $\theta_1, \theta_2 > 0$ are chosen according to the properties of the medium and $E(\rho, p, A, J, v)$ depends on the equation of state chosen for the material.

The model was solved in [2, 3] using the ADER-WENO method. Whilst this method provides arbitrarily-high-order solutions, it is relatively computationally expensive, especially if only 2nd-order accurate solutions are required. A computationally cheaper 2nd-order method was developed in this study.

Formulation

The GPR model takes the following form:

$$\frac{\partial Q}{\partial t}+\nabla\cdot F(Q)+B(Q)\cdot\nabla Q=S(Q) \tag{2}$$

In the new method, the homogeneous system and temporal ODE system resulting from the source terms are solved separately:

$$\frac{\partial Q}{\partial t}+\nabla\cdot F(Q)+B(Q)\cdot\nabla Q=\mathbf{0} \tag{3a}$$

$$\frac{dQ}{dt}=S(Q) \tag{3b}$$

Denoting the discrete action of (3a) and (3b) over time step Δt by the operators $H^{\Delta t}$ and $S^{\Delta t}$ respectively, the discrete solution to the GPR system is advanced from t_n to t_{n+1} by the following second-order scheme, based on a Strang splitting [7]:

$$q_{n+1}=S^{\Delta t/2}H^{\Delta t}S^{\Delta t/2}q_n \tag{4}$$

$H^{\Delta t}$ is computed by performing a 2nd order WENO reconstruction of the data and advancing this reconstruction by $\frac{\Delta t}{2}$:

$$w_p^{n+\frac{1}{2}}=w_p^n-\frac{\Delta t}{2\Delta x}\left(F\left(w_k^n\right)\cdot\nabla\psi_k(\chi_p)+B\left(w_p^n\right)\cdot\left(w_k^n\nabla\psi_k(\chi_p)\right)\right) \tag{5}$$

where ψ_p is the pth basis polynomial, w_p^n is the WENO reconstruction coefficient of ψ_p at time t_n, and χ_p is interpolation point of ψ_p. Cell values are then updated using the finite volume formula given in [3], using the path-conservative Rusanov formulation for the interface jump terms.

In computing $S^{\Delta t}$, note that the ODEs governing J have the analytic solution:

$$J(t)=J(0)\sqrt{\frac{1}{e^{at}-\frac{b}{a}\left(e^{at}-1\right)\|J(0)\|^2}} \tag{6}$$

for constants a, b given in [5]. In the same paper, an approximate analytic solution for the ODEs governing A is derived, valid when the arithmetic mean of the singular values of A is close to the geometric mean, as is true for moderate flows. If this condition does not hold, a stiff ODE solver is used instead.

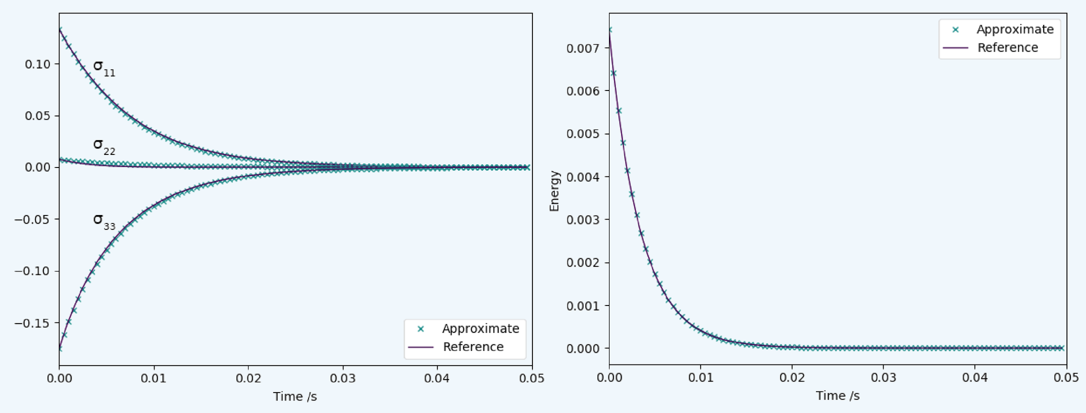
Thus, the computationally expensive Discontinuous Galerkin predictor required in the ADER-WENO method is avoided. The WENO matrices (given in the appendix of [5]) are precomputed and so the WENO reconstruction is relatively computationally cheap, and the analytical solution to (3b) is of negligible cost to calculate.

Results

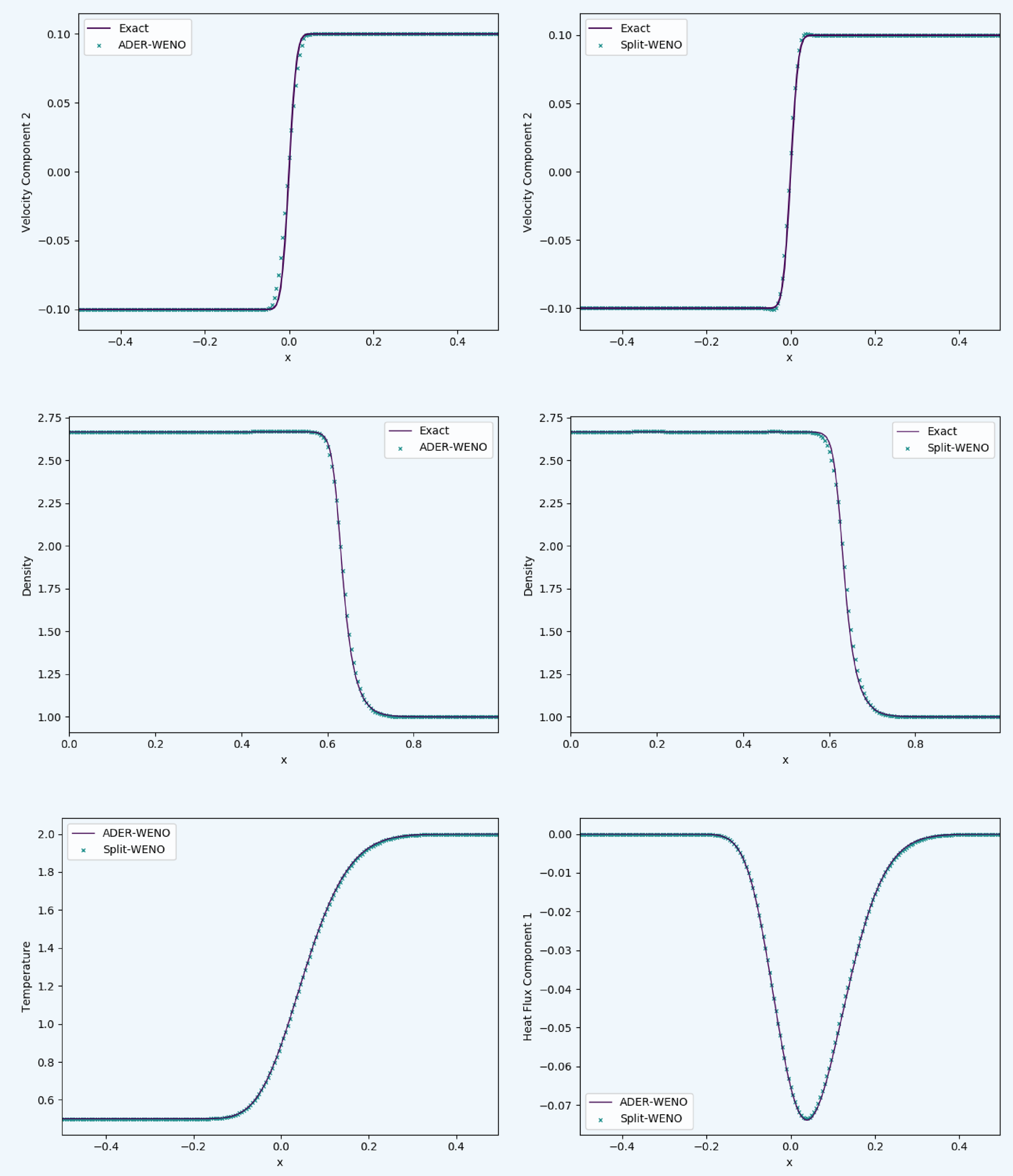
The approximate analytic solver for the ODEs governing A is compared here with a stiff numerical ODE solver. Initial data is taken from [1]:

$$A_0=\begin{pmatrix}1 & 0 & 0\\ -0.01 & 0.95 & 0.02\\ -0.015 & 0 & 0.9\end{pmatrix}^{-1} \tag{7}$$

The resulting evolution of the stress tensor and energy (the macroscopic variables depending on A) are shown below for the two methods. As can be seen, the approximate analytic solver performs well with this initial data, at a fraction of the computational cost.



The following tests are taken from [3]. The results below show the ADER-WENO method compared with the new Split-WENO method for: Stokes' First Problem with $\mu = 10^{-4}$ (top); the Viscous Shock Problem (middle); and the Heat Conduction Problem (bottom).



The tables below show: the run times for various tests (top); the convergence rates for the ADER-WENO method on the viscous shock test (middle); and the corresponding convergence rates for the new method (bottom). A significant speedup is attained with little loss of accuracy. In fact, convergence is better in all norms with the new method, for the test used.

	ADER-WENO	New Method	Speedup
Stokes' First Problem, $\mu = 10^{-2}$	265s	48s	5.5
Stokes' First Problem, $\mu = 10^{-3}$	294s	48s	6.1
Stokes' First Problem, $\mu = 10^{-4}$	536s	48s	11.1
Viscous Shock Problem	297s	70s	4.2
Heat Conduction Problem	544s	117s	4.6

Cells	$\epsilon(L_1)$	$\epsilon(L_2)$	$\epsilon(L_\infty)$	$O(L_1)$	$O(L_2)$	$O(L_\infty)$
50	3.40×10^{-4}	8.84×10^{-4}	3.67×10^{-3}			
100	5.42×10^{-5}	1.54×10^{-4}	7.45×10^{-4}	2.65	2.52	2.30
150	2.47×10^{-5}	6.43×10^{-5}	3.34×10^{-4}	1.94	2.16	1.98
200	1.58×10^{-5}	3.85×10^{-5}	1.97×10^{-4}	1.56	1.78	1.84

Cells	$\epsilon(L_1)$	$\epsilon(L_2)$	$\epsilon(L_\infty)$	$O(L_1)$	$O(L_2)$	$O(L_\infty)$
50	7.20×10^{-4}	1.59×10^{-3}	5.86×10^{-3}			
100	1.30×10^{-4}	3.36×10^{-4}	1.44×10^{-3}	2.47	2.24	2.02
150	4.68×10^{-5}	1.30×10^{-4}	5.74×10^{-4}	2.52	2.35	2.27
200	2.48×10^{-5}	6.81×10^{-5}	3.19×10^{-4}	2.20	2.24	2.04

Discussion

It has been demonstrated that this new method is able to produce results of high fidelity in a range of situations. It is significantly faster than the other currently available methods, and easier to implement. The author would however recommend that if very high order-of-accuracy is required, and computational cost is not of paramount importance, ADER-WENO methods may present a better option, as by design the Split-WENO method cannot achieve better than second-order accuracy.

It should be noted that the assumption used to derive the approximate analytical solver for the distortion ODEs may break down for situations where the flow is compressed heavily in one direction but not the others. The reason for this is that one of the singular values of the distortion tensor will be much larger than the others, and the mean of the squares of the singular values will not be close to its geometric mean, meaning that the subsequent linearization of the ODE governing the mean of the singular values fails. It should be noted that none of the situations covered in this study presented problems for the approximate analytical solver, and situations which may be problematic are in some sense unusual. In any case, a stiff ODE solver can be used to solve the system (3b) if necessary, and in these situations the Split-WENO method is still fast and accurate, though slightly slower.

It should be noted that both the ADER-WENO and Split-WENO methods, as described in this study, are trivially parallelizable on a cell-wise basis. Thus, given a large number of computational cores, deficiencies in the Split-WENO method in terms of its order of accuracy may be overcome by utilizing a larger number of computational cells and cores. The computational cost of each time step is significantly smaller than with the ADER-WENO method, and the number of grid cells that can be used scales roughly linearly with number of cores, at constant run time per iteration.

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