A Split Solver for Elastoplasticity under the GPR Model

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1 The GPR Model

The GPR model takes the following form:

$$\frac{\partial \rho}{\partial t}+\frac{\partial\left(\rho v_k\right)}{\partial x_k}=0 \tag{1a}$$

$$\frac{\partial\left(\rho v_i\right)}{\partial t}+\frac{\partial(\rho v_i v_k+p\delta_{ik}-\sigma_{ik})}{\partial x_k}=0 \tag{1b}$$

$$\frac{\partial A_{ij}}{\partial t}+\frac{\partial\left(A_{ik}v_k\right)}{\partial x_j}+v_k\left(\frac{\partial A_{ij}}{\partial x_k}-\frac{\partial A_{ik}}{\partial x_j}\right)=-\frac{\psi_{ij}}{\theta_1} \tag{1c}$$

$$\frac{\partial\left(\rho J_i\right)}{\partial t}+\frac{\partial\left(\rho J_i v_k+T\delta_{ik}\right)}{\partial x_k}=-\frac{\rho H_i}{\theta_2} \tag{1d}$$

$$\frac{\partial\left(\rho E\right)}{\partial t}+\frac{\partial\left(\rho E v_k+\left(p\delta_{ik}-\sigma_{ik}\right)v_i+q_k\right)}{\partial x_k}=0 \tag{1e}$$

where $\psi=\frac{\partial E}{\partial A}$ and $\boldsymbol{H}=\frac{\partial E}{\partial \boldsymbol{J}}$. Define $\operatorname{dev} X=X-\frac{\operatorname{tr} X}{3}I$. The EOS takes the following form, where E_1 is the internal energy (in terms of ρ,p) and c_s,α are variables related to the speed of characteristics of shear waves and heat waves:

$$E=E_1\left(\rho,p\right)+\frac{{c_s}^2}{4}\left\|\operatorname{dev}\left(A^TA\right)\right\|_F^2+\frac{\alpha^2}{2}\left\|\boldsymbol{J}\right\|^2+\frac{1}{2}\left\|\boldsymbol{v}\right\|^2 \tag{2}$$

The following forms are taken:

$$\theta_1=\frac{\tau_1 c_s^2}{3\left|A\right|^{\frac{5}{3}}} \tag{3a}$$

$$\theta_2=\tau_2\alpha^2\frac{\rho T_0}{\rho_0 T} \tag{3b}$$

where:

$$\tau_1 = \begin{cases} \frac{6\mu}{\rho_0 c_s^2} & viscous\,fluids \\ \tau_0 \left(\frac{\sigma_0}{\|\mathrm{dev}(\sigma)\|_F} \right)^n & elastoplastic\,solids \end{cases} \tag{4a}$$

$$\tau_2 = \frac{\rho_0 \kappa}{T_0 \alpha^2} \tag{4b}$$

Thus, defining $G = A^T A$, we have the following relations:

$$\sigma = -\rho c_s^2 G \,\mathrm{dev}\,(G) \tag{5a}$$

$$\boldsymbol{q} = \alpha^2 T \boldsymbol{J} \tag{5b}$$

$$-\frac{\psi}{\theta_1(\tau_1)} = -\frac{3}{\tau_1} |A|^{\frac{5}{3}} A \,\mathrm{dev}\,(G) \tag{5c}$$

$$-\frac{\rho \boldsymbol{H}}{\theta_2(\tau_2)} = -\frac{T \rho_0}{T_0 \tau_2} \boldsymbol{J} \tag{5d}$$

The following constraint also holds:

$$\det (A) = \frac{\rho}{\rho_0} \tag{6}$$

2 A Split Solver

The GPR system can be split into its homogeneous part, and the following temporal ODEs:

$$\frac{dA}{dt} = \frac{-3}{\tau_1} |A|^{\frac{5}{3}} A \,\mathrm{dev}\,(G) \tag{7a}$$

$$\frac{d\boldsymbol{J}}{dt} = -\frac{1}{\tau_2} \frac{T \rho_0}{T_0 \rho} \boldsymbol{J} \tag{7b}$$

A second-order Strang splitting is used to evolve the state variables from their initial state $\boldsymbol{Q_0}$ to a state time Δt later:

$$\boldsymbol{Q_{\Delta t}} = D^{\frac{\Delta t}{2}} T^{\frac{\Delta t}{2}} H^{\Delta t} T^{\frac{\Delta t}{2}} D^{\frac{\Delta t}{2}} \boldsymbol{Q_0} \tag{8}$$

where $D^{\delta t}, T^{\delta t}, H^{\delta t}$ are the operators solving the distortion and thermal impulse ODEs, and the homogeneous system respectively, over time step δt. $H^{\delta t}$ is performed using a second-order WENO method. $T^{\delta t}, D^{\delta t}$ are given below.

2.1 The Thermal Impulse ODEs

Denoting by E^A, E^J, E^v the components of E depending on $A, \boldsymbol{J}, \boldsymbol{v}$ respectively, we have:

$$\begin{aligned} T &= \frac{E_1}{c_v} \\ &= \frac{E - E^A(\rho, A) - E^v(\boldsymbol{v})}{c_v} - \frac{1}{c_v} E^J(\boldsymbol{J}) \\ &= c_1 - c_2 \|\boldsymbol{J}\|^2 \end{aligned} \tag{9}$$

where:

$$c_1 = \frac{E - E^A(\rho, A) - E^v(\boldsymbol{v})}{c_v} \tag{10a}$$

$$c_2 = \frac{\alpha^2}{2c_v} \tag{10b}$$

Over the time period of the ODE (7b), $c_1, c_2 > 0$ are constant. We have:

$$\frac{dJ_i}{dt} = -\left(\frac{1}{\tau_2}\frac{\rho_0}{T_0\rho}\right) J_i \left(c_1 - c_2 \|\boldsymbol{J}\|^2\right) \tag{11}$$

Therefore:

$$\frac{d}{dt}\left(J_i^2\right) = J_i^2 \left(-a + b\left(J_1^2 + J_2^2 + J_3^2\right)\right) \tag{12}$$

where

$$a = \frac{2\rho_0}{\tau_2 T_0 \rho c_v}\left(E - E_2^{(A)}(A) - E_3(\boldsymbol{v})\right) \tag{13a}$$

$$b = \frac{\rho_0 \alpha^2}{\tau_2 T_0 \rho c_v} \tag{13b}$$

Note that this is a generalized Lotka-Volterra system in $\left\{J_1^2, J_2^2, J_3^2\right\}$. It has the following analytical solution:

$$\boldsymbol{J}(t) = \boldsymbol{J}(0)\sqrt{\frac{1}{e^{at} - \frac{b}{a}\left(e^{at} - 1\right)\|\boldsymbol{J}(0)\|^2}} \tag{14}$$

2.2 The Distortion ODEs

Take the singular value decomposition $A = U\Sigma V^T$. Note that:

$$\sigma = -\rho c_s^2 A^T A \operatorname{dev}\left(A^T A\right) = -\rho c_s^2 V \Sigma^2 \operatorname{dev}\left(\Sigma^2\right) V^T \tag{15}$$

Thus:

$$\left\|\operatorname{dev}(\sigma)\right\|_F^n = \rho^n c_s^{2n} \left\|\operatorname{dev}\left(\Sigma^2 \operatorname{dev}\left(\Sigma^2\right)\right)\right\|_F^n \tag{16}$$

Using the fact that $d\Sigma = U^T dA V$, we have:

$$\frac{d\Sigma}{dt} = -\frac{3}{\tau_0}\left(\frac{\rho}{\rho_0}\right)^{\frac{5}{3}} \frac{\left(\frac{3}{2}\right)^{\frac{n}{2}} \rho^n c_s^{2n} \left\|\operatorname{dev}\left(\Sigma^2 \operatorname{dev}\left(\Sigma^2\right)\right)\right\|_F^n}{\sigma_0^n} \Sigma \operatorname{dev}\left(\Sigma^2\right) \tag{17}$$

Letting $x_i = \frac{a_i^2}{\det(A)^{\frac{2}{3}}} = \frac{a_i^2}{\left(\frac{\rho}{\rho_0}\right)^{\frac{2}{3}}}$ then $\Sigma^2 = \det(A)^{\frac{2}{3}} X$ where $X = \operatorname{diag}(x_1, x_2, x_3)$. Thus, we have:

$$\frac{dx_i}{d\tilde{t}} = -3 \left\|\operatorname{dev}\left(X \operatorname{dev}(X)\right)\right\|_F^n x_i \left(x_i - \bar{x}\right) \tag{18}$$

where:

$$\tilde{t} = \frac{2}{\tau_0}\left(\frac{\rho}{\rho_0}\right)^{\frac{4n+7}{3}} \left(\sqrt{\frac{3}{2}} \frac{\rho c_s^2}{\sigma_0}\right)^n t \tag{19}$$

Note that:

$$\begin{aligned}
\frac{27}{2} \left\|\operatorname{dev}\left(X \operatorname{dev}(X)\right)\right\|_F^2 = {} & 3\left(x_2 x_3 x_1^2 + x_2 x_3^2 x_1 + x_2^2 x_3 x_1\right) \\
& - 3\left(x_1^2 x_2^2 + x_3^2 x_2^2 + x_1^2 x_3^2\right) \\
& - 2\left(x_2 x_1^3 + x_3 x_1^3 + x_2^3 x_1 + x_3^3 x_1 + x_2 x_3^3 + x_2^3 x_3\right) \\
& + 4\left(x_1^4 + x_2^4 + x_3^4\right)
\end{aligned} \tag{20}$$

Define the following quantities:

$$m = \frac{x_1 + x_2 + x_3}{3} \tag{21a}$$

$$u = \frac{\left(x_1 - x_2\right)^2 + \left(x_2 - x_3\right)^2 + \left(x_3 - x_1\right)^2}{3} \tag{21b}$$

Note that, due to (6), $x_1 x_2 x_3 = 1$. Thus we have:

$$\|\mathrm{dev}\left(X\,\mathrm{dev}\left(X\right)\right)\|_F^2 = \frac{1}{6}u^2 + 4m^2u - 6m^4 + 6m \tag{22}$$

Thus, (18) leads to the following coupled system of ODEs:

$$\frac{du}{d\tilde{t}} = -18\left(\frac{1}{6}u^2 + 4m^2u - 6m^4 + 6m\right)^{\frac{n}{2}}\left(1 - m\left(m^2 - \frac{5}{6}u\right)\right) \tag{23a}$$

$$\frac{dm}{d\tilde{t}} = -\left(\frac{1}{6}u^2 + 4m^2u - 6m^4 + 6m\right)^{\frac{n}{2}} u \tag{23b}$$

Define the the variable τ by:

$$\frac{d\tau}{d\tilde{t}} = \left(\frac{1}{6}u^2 + 4m^2u - 6m^4 + 6m\right)^{\frac{n}{2}} \tag{24}$$

Then we have:

$$\frac{du}{d\tau} = -18\left(1 - m\left(m^2 - \frac{5}{6}u\right)\right) \tag{25a}$$

$$\frac{dm}{d\tau} = -u \tag{25b}$$

Combining these equations, we have:

$$\frac{d^2m}{d\tau^2} = -\frac{du}{d\tau} = 18\left(1 - m\left(m^2 - \frac{5}{6}u\right)\right) \tag{26}$$

Therefore:

$$\begin{cases} \frac{d^2m}{d\tau^2} + 15m\frac{dm}{d\tau} + 18\left(m^3 - 1\right) = 0 \\ m\left(0\right) = m_0 \\ m^{'}\left(0\right) = -u_0 \end{cases} \tag{27}$$

We make the following assumption, noting that it is true in all physical situations tested in this study:

$$m\left(t\right) = 1 + \eta\left(t\right), \quad \eta \ll 1 \;\forall t \geq 0 \tag{28}$$

Thus, we have the linearized ODE:

$$\begin{cases} \frac{d^2\eta}{d\tau^2} + 15\frac{d\eta}{d\tau} + 54\eta = 0 \\ \eta\left(0\right) = m_0 - 1 \\ \eta^{'}\left(0\right) = -u_0 \end{cases} \tag{29}$$

This is a Sturm-Liouville equation with solution:

$$\eta(\tau) = \frac{e^{-9\tau}}{3}\left((9m_0 - u_0 - 9)\,e^{3\tau} - (6m_0 - u_0 - 6)\right) \tag{30}$$

Thus, we also have:

$$u(\tau) = e^{-9\tau}\left(e^{3\tau}(18m_0 - 2u_0 - 18) - (18m_0 - 3u_0 - 18)\right) \tag{31}$$

Denoting $a = 9m_0 - u_0 - 9$, $b = 6m_0 - u_0 - 6$, it is straightforward to verify that:

$$\frac{d\tau}{d\tilde{t}} = \frac{1}{54^{\frac{n}{2}}}\left(\begin{array}{c} 108ae^{-6\tau} - 324be^{-9\tau} + 108a^2e^{-12\tau} \\ -396abe^{-15\tau} + 297b^2e^{-18\tau} - 24a^2be^{-21\tau} \\ +\left(48ab^2 - 4a^4\right)e^{-24\tau} + \left(16a^3b - 24b^3\right)e^{-27\tau} \\ -24a^2b^2e^{-30\tau} + 16ab^3e^{-33\tau} - 4b^4e^{-36\tau} \end{array}\right)^{\frac{n}{2}} \equiv \frac{f(\tau)^{\frac{n}{2}}}{54^{\frac{n}{2}}} \tag{32}$$

$f(\tau)$ is approximated by $g(\tau) \equiv ce^{-\lambda\tau}$, where:

$$c = 108a - 324b + 108a^2 - 396ab + 297b^2 - 24\left(a^2b - 2ab^2 + b^3\right) - 4(a-b)^4 \tag{33a}$$

$$\lambda = \frac{c}{18a - 36b + 9a^2 - \frac{132ab}{5} + \frac{33b^2}{2} - \frac{8a^2b}{7} + 2ab^2 - \frac{8b^3}{9} - \frac{a^4}{6} + \frac{16a^3b}{27} - \frac{4a^2b^2}{5} + \frac{16ab^3}{33} - \frac{b^4}{9}} \tag{33b}$$

Note that $f(0) = g(0)$ and $\int_0^\infty (f(\tau) - g(\tau))\,d\tau = 0$. Thus, we have:

$$\frac{d\tau}{d\tilde{t}} \approx \left(\frac{c}{54}\right)^{\frac{n}{2}} e^{-\frac{n\lambda}{2}\tau} \tag{34}$$

Therefore:

$$\begin{aligned} \tau &\approx \frac{2}{n\lambda}\log\left(\frac{n\lambda}{2}\left(\frac{c}{54}\right)^{\frac{n}{2}}\tilde{t} + 1\right) \\ &= \frac{2}{n\lambda}\log\left(\frac{n\lambda}{\tau_0}\left(\frac{\rho}{\rho_0}\right)^{\frac{4n+7}{3}}\left(\frac{\sqrt{c}}{6}\frac{\rho c_s^2}{\sigma_0}\right)^n t + 1\right) \end{aligned} \tag{35}$$

Thus, we have:

$$m_{\Delta t} = 1 + \eta\left(\frac{2}{n\lambda}\log\left(\frac{n\lambda}{\tau_0}\left(\frac{\rho}{\rho_0}\right)^{\frac{4n+7}{3}}\left(\frac{\sqrt{c}}{6}\frac{\rho c_s^2}{\sigma_0}\right)^n \Delta t + 1\right)\right) \tag{36a}$$

$$u_{\Delta t} = u\left(\frac{2}{n\lambda}\log\left(\frac{n\lambda}{\tau_0}\left(\frac{\rho}{\rho_0}\right)^{\frac{4n+7}{3}}\left(\frac{\sqrt{c}}{6}\frac{\rho c_s^2}{\sigma_0}\right)^n \Delta t + 1\right)\right) \tag{36b}$$

$$\frac{x_i + x_j + x_k}{3} = m_{\Delta t} \tag{37a}$$

$$\frac{(x_i - x_j)^2 + (x_j - x_k)^2 + (x_k - x_i)^2}{3} = u_{\Delta t} \tag{37b}$$

$$x_i x_j x_k = 1 \tag{37c}$$

This gives:

$$x_i = \frac{\sqrt[3]{6\left(\sqrt{81\Delta^2 - 6u_{\Delta t}^3} + 9\Delta\right)}}{6} + \frac{u_{\Delta t}}{\sqrt[3]{6\left(\sqrt{81\Delta^2 - 6u_{\Delta t}^3} + 9\Delta\right)}} + m_{\Delta t} \tag{38a}$$

$$x_j = \frac{1}{2}\left(\sqrt{\frac{x_i\left(3m_{\Delta t} - x_i\right)^2 - 4}{x_i}} + 3m_{\Delta t} - x_i\right) \tag{38b}$$

$$x_k = \frac{1}{x_i x_j} \tag{38c}$$

where

$$\Delta = -2m_{\Delta t}^3 + m_{\Delta t}u_{\Delta t} + 2 \tag{39}$$

Note that taking the real parts of the above expression for x_i gives:

$$x_i = \frac{\sqrt{6u_{\Delta t}}}{3}\cos\left(\frac{\theta}{3}\right) + m_{\Delta t} \tag{40a}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{6u_{\Delta t}^3 - 81\Delta^2}}{9\Delta}\right) \tag{40b}$$

At this point it is not clear which values of $\{x_i, x_j, x_k\}$ are taken by x_1, x_2, x_3. However, this can be inferred from the fact that any relation $x_i \geq x_j \geq x_k$ is maintained over the lifetime of the system. Thus, the stiff ODE solver has been obviated by a few arithmetic operations. The obtained values of x_1, x_2, x_3 are used to reconstruct A at time Δt.

3 Numerical Results

3.1 Strain Relaxation Test

Take initial data used by Barton:

$$A=\begin{pmatrix}1 & 0 & 0\\ -0.01 & 0.95 & 0.02\\ -0.015 & 0 & 0.9\end{pmatrix}^{-1} \tag{41}$$

The following parameter values were used: $\rho_0=1, c_s=0.219, n=4, \sigma_0=9\times10^{-4}, \tau_0=0.1$. As can be seen in figure 1, figure 2, and figure 3, the approximate analytic solver compares well with the exact solution for the distortion tensor A, and thus also the stress tensor and the energy.

3.2 Elastoplastic Piston Test

In this test, a piston with speed 0.002 is driven into copper initially at rest. An elastic shock wave develops, followed by a plastic shock wave. The following parameters were used: $\rho_0=8.93, c_s=0.219, n=10, \sigma_0=9\times10^{-4}, \tau_0=0.1$. The shock Mie-Gruneisen EOS is used for the internal energy, with $p_0=0, c_0=0.394, \Gamma_0=2, s=1.48$. figure 4 demonstrates the results using the split solver, compared with the ADER-WENO solution.

4 Discussion

I have demonstrated in my previous paper that the solver works well for Newtonian fluids. I believe this solver will also work for dilatants and pseudoplastics, as they follow the same power law used here:

$$\tau_1 \propto \frac{1}{\|\mathrm{dev}\,(\sigma)\|_F^n} \tag{42}$$

I am also working on an analytical solver for Bingham plastics, using:

$$\tau_1 \propto \frac{1}{1-\frac{f(|\sigma|)}{|\sigma|}} \tag{43a}$$

$$f\left(|\sigma|\right)=\frac{a}{\left(1+be^{-c|\sigma|}\right)^d}+\sigma_0-a \tag{43b}$$

where $a, b \gg 1$ and c, d are chosen such that:

$$\dot{\gamma}\approx\begin{cases}0 & |\sigma|\leq\sigma_0\\ \frac{1}{\mu_\infty}\left(\sigma-\sigma_0\right) & |\sigma|\geq\sigma_0\end{cases} \tag{44}$$

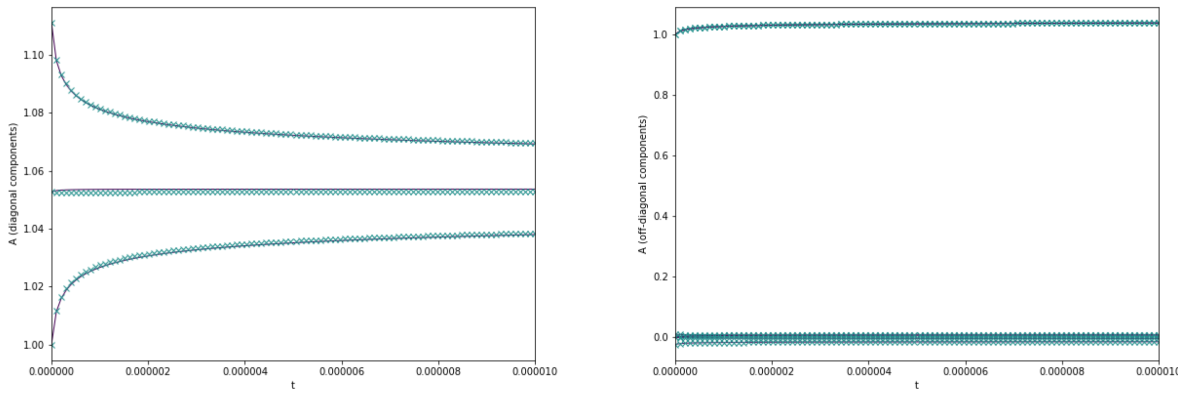

Figure 1: The distortion tensor during the Strain Relaxation Test

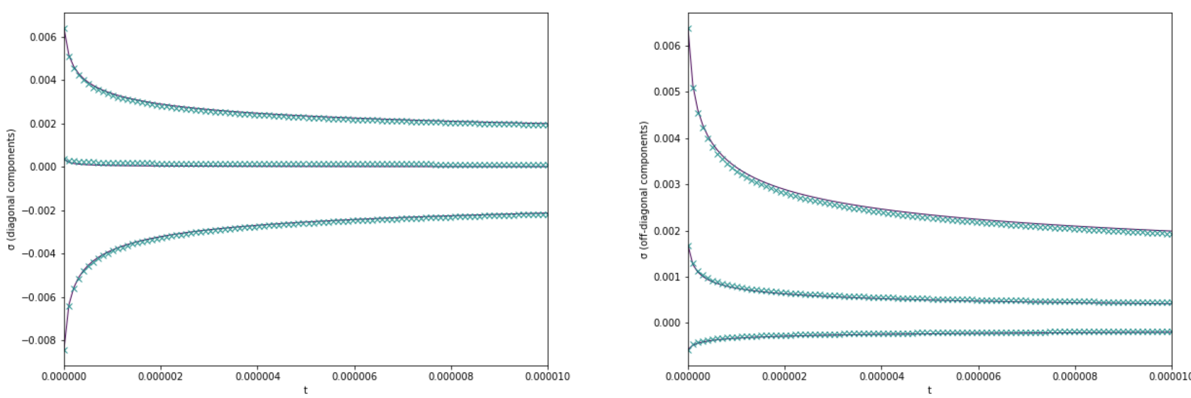

Figure 2: The stress tensor during the Strain Relaxation Test

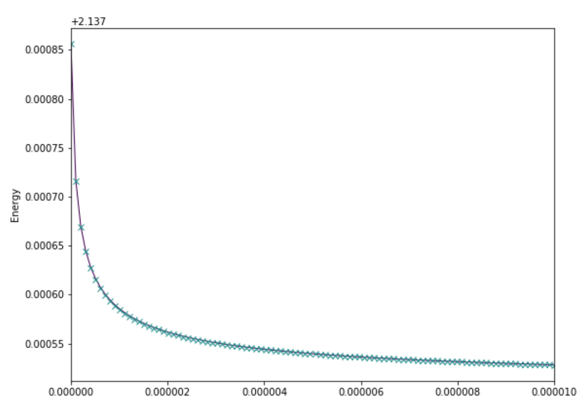

Figure 3: The total energy during the Strain Relaxation Test

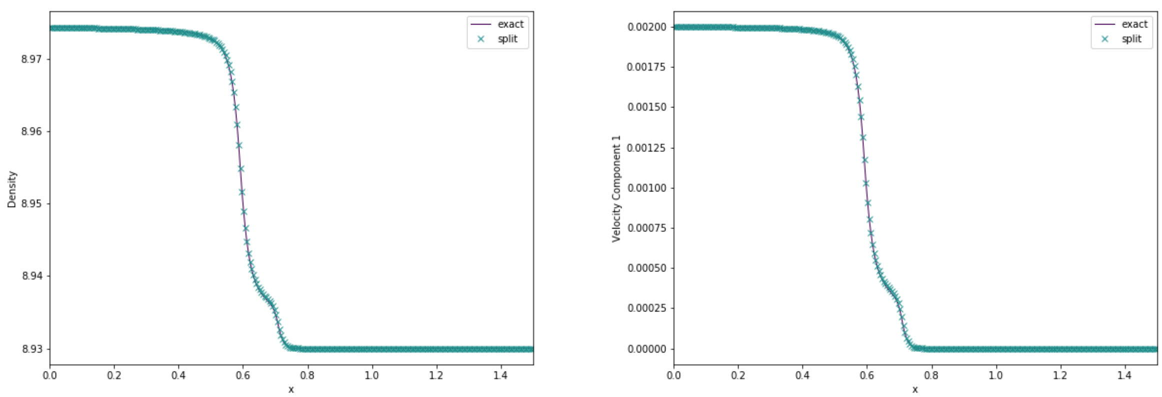

Figure 4: Density and velocity in the elastoplastic piston test