COPENHAGEN BUSINESS SCHOOL

EXAMINATION

Python for the Financial Economist

Examiner

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Student ID

S144474 & S145085

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Introduction

This assignment focuses on the application of Python to quantitative analysis, with an emphasis on managing currency risk. The objective is to implement mathematical and statistical models to simulate and optimize a portfolio for a EUR-based investor investing in both EUR and USD equities and zero-coupon bonds. The analysis includes simulating market dynamics, deriving distributions, and presenting results through visualizations and tables. Where the entire Jupyter notebook is attached with all calculations. The coding and tasks were primarily solved collaboratively but have been divided based on student numbers.

1 Part I

1.1 Analysis of the Distribution and Simulation of $\boldsymbol{X}_t$, S144474

1.1.1 What is the distribution of $\boldsymbol{X}_1$?

We are assuming that we are standing at time $t = 0$, with the initial values $\boldsymbol{X}_0$ as specified in the file *init values.xlsx*. Here it is assumed that $t = 1$ year (52 weeks). The distribution of $\boldsymbol{X}_1$ can be derived from the given information that the market invariants are normally distributed:

$$\Delta \boldsymbol{X}_t \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

where $\boldsymbol{\mu} = \left(0, 0.07 \cdot \Delta t, 0.06 \cdot \Delta t, \mathbf{0}^\top, \mathbf{0}^\top\right)^\top$ is the expectation vector and $\boldsymbol{\Sigma}$ is the covariance matrix given in the file *covariance matrix.xlsx* and $\Delta t = 1/52$ year. Here the expectation vector and covariance matrix is in weekly terms. Since $\boldsymbol{X}_1 = \boldsymbol{X}_0 + \Delta \boldsymbol{X}_1$, and the sum of normally distributed random variables is also normally distributed, $\boldsymbol{X}_1$ follows a multivariate normal distribution:

$$\boldsymbol{X}_1 \sim \mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{X}_1}, \boldsymbol{\Sigma}_{\boldsymbol{X}_1}),$$

where:

$$\boldsymbol{\mu}_{\boldsymbol{X}_1} = \boldsymbol{X}_0 + \boldsymbol{\mu} \cdot 52 \quad \wedge \quad \boldsymbol{\Sigma}_{\boldsymbol{X}_1} = \boldsymbol{\Sigma} \cdot 52$$

The distribution can be verified through simulations with weekly time steps and a comparison between the analytical and empirical results.

1.1.2 Simulate the evolution of $\boldsymbol{X}_t$ and visualize the evolution of log FX_t

The evolution of $\boldsymbol{X}_t$ was simulated over 52 weekly time steps (from time zero to the horizon) where we sat `np.random.seed(2000)` to ensure reproducibility. Using initial values $\boldsymbol{X}_0$ from *init_values.xlsx*, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ as defined in part 1.1.1 and $\boldsymbol{X}_t$ was iteratively updated using:

$$\boldsymbol{X}_t = \boldsymbol{X}_{t-1} + \Delta \boldsymbol{X}_t,$$

where $\Delta \boldsymbol{X}_t \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The simulation was performed for 10.000 independent paths to capture the stochastic variability inherent in the model

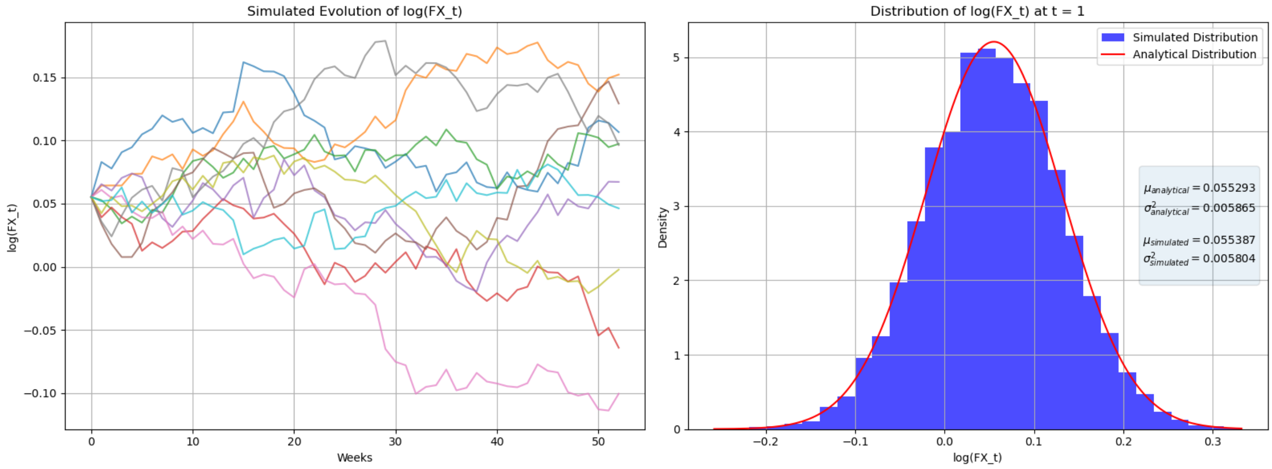


Figure 1: Simulation and evolution of $\log(FX_t)$

Figure 1 presents two visualizations. The left plot shows the evolution of $\log(FX_t)$ for 10 randomly selected paths over the investment horizon of 52 weeks, reflecting stochastic fluctuations around a mean trend. The right plot compares the simulated distribution of $\log(FX_t)$ at the horizon ($t = 1$) with the analytical normal distribution, confirming alignment with model assumptions.

Metric	**Formula**	**Value**
Analytical Expectation of $\log(FX_1)$	$\mathbb{E}[\log(FX_1)] = 0.055292786 + 0 \cdot 52 = 0.055293$	0.055293
Analytical Variance of $\log(FX_1)$	$\mathrm{Var}[\log(FX_1)] = 0.00011278 \cdot 52 = 0.005865$	0.005865
Simulated Expectation of $\log(FX_1)$	$\mathbb{E}[\log(FX_1)] = 0.055387$	0.055387
Simulated Variance of $\log(FX_1)$	$\mathrm{Var}[\log(FX_1)] = 0.005804$	0.005804

Table 1: Analytical and simulated values for $\log(FX_1)$ after one year (52 weeks)

The results from the simulation indicate that the expected value of $\log(FX_t)$ after one year is approximately $\mathbb{E}[\log(FX_1)] = 0.055387$, while the variance is approximately $\mathrm{Var}[\log(FX_1)] = 0.005804$. These are consistent with the analytical values: $\mathbb{E}[\log(FX_1)] = 0.055293$ and $\mathrm{Var}[\log(FX_1)] = 0.005865$, confirming the reliability of the simulation.

1.2 Analysis of the distribution and simulation of $V_1^{\text{US,local}}$, S145085

1.2.1 Obtain the distribution of $V_1^{\text{US,local}}$

Firstly the distribution of $\Delta \log V_1^{\text{US,local}}$ is derived from $\Delta \boldsymbol{X}$:

$$\Delta \log V_1^{\text{US,local}} \sim \mathcal{N}(\mathbb{E}[\Delta \log V_1^{\text{US,local}}], \mathrm{Var}[\Delta \log V_1^{\text{US,local}}])$$

where:

$$\mathbb{E}[\Delta \log V_1^{\text{US,local}}] = \mu[2], \quad \wedge \quad \mathrm{Var}[\Delta \log V_1^{\text{US,local}}] = \Sigma_{2,2}$$

Here $\log V_1^{\text{US,local}} = X_0[2] + \Delta \log V_1^{\text{US,local}}$. From part 1.1.1, we know that $\boldsymbol{X}_1$ follows a multivariate normal distribution. Since $\log V_1^{\text{US,local}}$ is the second component of $\boldsymbol{X}_1$, it follows a one-dimensional normal distribution, which means the distribution of $\log V_1^{\text{US,local}}$ is:

$$\log V_1^{\text{US,local}} \sim \mathcal{N}(\mu_{V_1^{\text{US}}}, \sigma^2_{V_1^{\text{US}}})$$

where:

$$\mu_{V_1^{\text{US}}} = X_0[2] + \mu[2] \cdot 52 \quad \wedge \quad \sigma^2_{V_1^{\text{US}}} = \Sigma_{2,2} \cdot 52$$

Here, $X_0[2]$ is pseudo-code to indicate the second element of the vector. In Python, this corresponds to $X_0[1]$, since Python uses zero-based indexing.

The distribution of $V_1^{\text{US,local}}$ is obtained by exponentiating $\log V_1^{\text{US,local}}$. Therefore, $V_1^{\text{US,local}}$ follows a log-normal distribution:

$$V_1^{\text{US,local}} \sim \text{LogN}(\mu_{V_1^{\text{US}}}, \sigma^2_{V_1^{\text{US}}}),$$

where:

$$\mathbb{E}[V_1^{\text{US,local}}] = \exp\left(\mu_{V_1^{\text{US}}} + \frac{\sigma^2_{V_1^{\text{US}}}}{2}\right)$$

and:

$$\text{Var}[V_1^{\text{US,local}}] = \left(\exp(\sigma^2_{V_1^{\text{US}}}) - 1\right) \cdot \exp\left(2\mu_{V_1^{\text{US}}} + \sigma^2_{V_1^{\text{US}}}\right).$$

These results confirm that $V_1^{\text{US,local}}$ is log-normally distributed as expected from the transformation of the normal distribution of $\log V_1^{\text{US,local}}$, and this will be visualized in the next part.

1.2.2 Simulate the evolution of $V_1^{\text{US,local}}$ and compare with the true analytical distribution

To validate the analytical distribution, the evolution of $\log V_1^{\text{US,local}}$ was already simulated as part of the simulation of $\boldsymbol{X}_1$ in part 1.1.2. The second component of $\boldsymbol{X}_1$ from each simulation represents $\log V_1^{\text{US,local}}$. Alternatively, $V_1^{\text{US,local}}$ could have been simulated directly by exponentiating $\log V_1^{\text{US,local}}$ during the simulation process. The simulation parameters are consistent with those used previously.

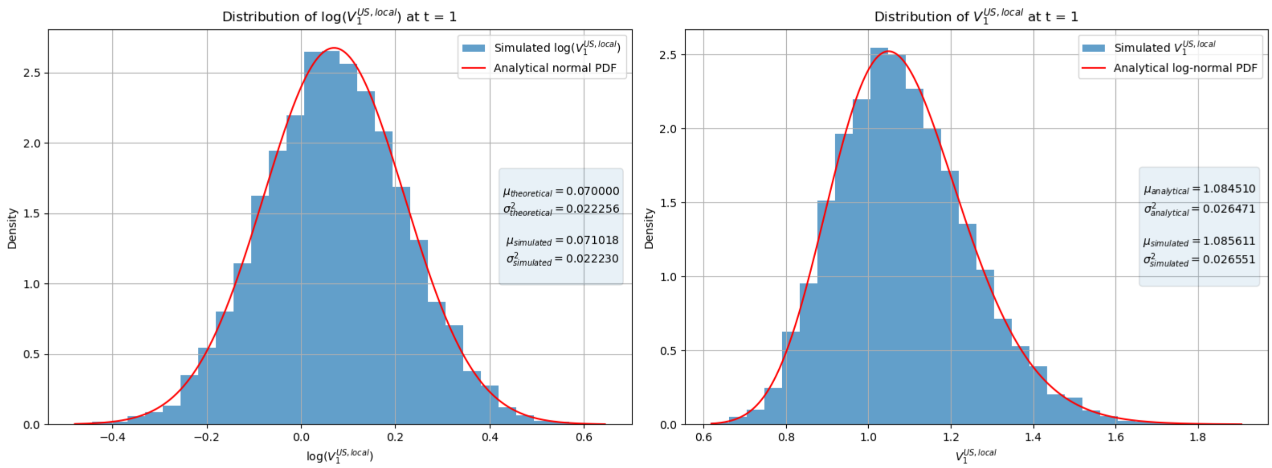


Figure 2: Simulated and analytical distribution of $\log(V_1^{\text{US,local}})$ and $V_1^{\text{US,local}}$

In Figure 2 the left plot shows the histogram of $\log(V_1^{\text{US,local}})$ alongside the analytical normal PDF (probability density function), while the right plot shows the corresponding results for $V_1^{\text{US,local}}$ and its analytical log-normal PDF, which shows right skewness, a typical feature of log-normal distributions.

Metric	Analytical Value	Simulated Value
Mean of $\log V_1^{\text{US,local}}$	0.070000	0.071018
Variance of $\log V_1^{\text{US,local}}$	0.022256	0.022230
Mean of $V_1^{\text{US,local}}$	1.084510	1.085611
Variance of $V_1^{\text{US,local}}$	0.026471	0.026551

Table 2: Comparison of Analytical and Simulated Values for $\log V_1^{\text{US,local}}$ and $V_1^{\text{US,local}}$

As shown in Figure 2 and in Table 2, the analytical and simulated values of both the mean and variance for $\log(V_1^{\text{US,local}})$ and $V_1^{\text{US,local}}$ closely align. The minimal differences between the simulated and analytical results validate the correctness of the simulation and the assumptions of the model.

1.3 Calculation of the Value of a EUR Zero-Coupon Bond (5Y EUR chosen), S144474

A zero-coupon bond is a bond that makes a single payment (principal) at maturity. The current value (B_t^T) of a zero-coupon bond can be calculated using continuously compounded yields (spot rates) as follows:

$$B_t^T = e^{-y_t^T \cdot (T-t)}$$

where:

- y_t^T: The cumulative zero-coupon yield (spot rate) for maturity $T - t$.
- $T - t$: The bond's time to maturity in years.

For instance, for a 5-year zero-coupon bond with a cumulative spot rate $y_t^T = 0.02$, the value is:

$$B_t^T = e^{-0.02 \cdot 5} \approx 0.9048$$

The vector of yields provided in this assignment represents cumulative zero-coupon yields (spot rates) for various maturities. These values are directly used to compute bond prices and construct yield curves using methods such as **linear interpolation**, **cubic splines**, or **bootstrapping**. If the interest rate is not fixed but varies over time as in this simulation case, a **discount curve** or a series of spot rates for each period is applied. For a zero-coupon bond, the value is calculated as:

$$PV = \frac{F}{(1 + r_1)(1 + r_2) \cdots (1 + r_t)}$$

Where $r_1, r_2, \ldots, r_t$ are the annual rates for each period until maturity.

Market Relationship

1. **Longer maturity means greater effect:** Zero-coupon bonds with long maturities are more sensitive to interest rate changes than those with short maturities. This is called *interest rate duration.*
2. **Interest rate changes affect returns:** If interest rates change after the purchase date, it will impact the investor's realized return if the bond is sold before maturity.

Linear Interpolation

Linear interpolation estimates spot rates for intermediate maturities by calculating a weighted average between two known rates. For example, given the spot rates for 3 years ($y_{3Y} = 0.02$) and 5 years ($y_{5Y} = 0.025$), the spot rate for 4 years (y_{4Y}) is:

$$y_{4Y} = y_{3Y} + \frac{4-3}{5-3} \cdot (y_{5Y} - y_{3Y}) = 0.02 + 0.5 \cdot (0.025 - 0.02) = 0.0225.$$

Figure 3 demonstrates the use of linear interpolation to estimate yields for both EUR and USD.

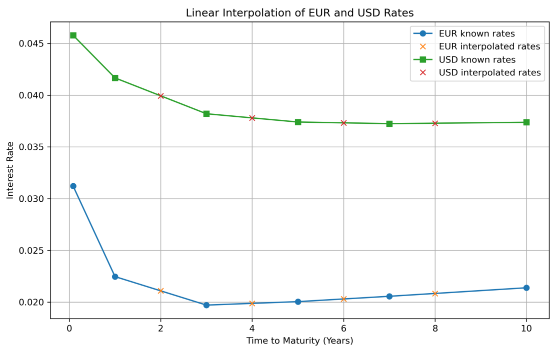


Figure 3: Linear interpolation of yields (EUR and USD)

This can also be done through a set of methods that can be seen in Appendix 7.1 and Figure 16 visualize Cubic Splines and the Bootstrapping method.

1.3.1 Distribution of an Initial 5-Year Zero-Coupon Bond at the Horizon, S145085

To derive the distribution for a zero coupon bond we used inspiration from (Meucci, A. (2005)). At the horizon $t = 1$, the time to maturity for a bond that initially had a 5-year maturity will be $T = 4$. The yield at the horizon, $y_{t=1}$, is the 4-year yield at $t = 1$, denoted as $y_{4,1}$. The bond price at the horizon can be expressed as:

$$B_{\text{horizon}} = F \cdot e^{-y_{\text{horizon}} \cdot T_{\text{horizon}}} \quad \Longleftrightarrow \quad B_1 = 1 \cdot e^{-y_{4,1} \cdot 4}$$

Changes in yields ($\Delta y_{4,t}$) are part of the market invariants ($\Delta \boldsymbol{X}_t$), which are normally distributed:

$$\Delta \boldsymbol{X}_t \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

Specifically, the change in the 4-year yield, $\Delta y_{4,1}$, is also normally distributed:

$$\Delta y_{4,1} \sim \mathcal{N}(\mu_4, \sigma_4^2),$$

where μ_4 and σ_4^2 are derived from the expectation vector $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$. The yield at the horizon is therefore:

$$y_{4,1} = y_{4,0} + \Delta y_{4,1}$$

Since $y_{4,1}$ is a linear transformation of a normally distributed variable, $y_{4,1}$ is also normally distributed:

$$y_{4,1} \sim \mathcal{N}(y_{4,0} + \mu_4, \sigma_4^2)$$

As we know the bond price at the horizon, B_1, is an exponential transformation of $y_{4,1}$:

$$B_1 = e^{-4 \cdot y_{4,1}}$$

As $y_{4,1}$ is normally distributed, then B_1 follows a log-normal distribution:

$$B_1 \sim \text{LogN}(-4 \cdot (y_{4,0} + \mu_4 \cdot 52), 16 \cdot \sigma_4^2 \cdot 52)$$

The expectation and variance of B_1 are given by:

$$\mathbb{E}[B_1] = \exp\left(-4 \cdot (y_{4,0} + \mu_4) + 8 \cdot \sigma_4^2\right) \quad \wedge \quad \text{Var}[B_1] = \left(\exp(16 \cdot \sigma_4^2) - 1\right) \cdot \exp\left(-8 \cdot (y_{4,0} + \mu_4) + 16 \cdot \sigma_4^2\right)$$

To generalize the bond price transformation for multiple bonds which we will need later, we introduce a bond matrix $\boldsymbol{B}$ that facilitates the transformation of both the covariance matrix $\boldsymbol{\Sigma}$ and the expectation vector $\boldsymbol{\mu}$. Specifically, the transformed covariance matrix and expectation vector for the log of bond prices at the horizon are given by:

$$\boldsymbol{\Sigma}_{\boldsymbol{P}_{1,\log}} = \boldsymbol{B} \cdot \boldsymbol{\Sigma}_1 \cdot \boldsymbol{B}^\top \quad \wedge \quad \boldsymbol{\mu}_{\boldsymbol{P}_{1,\log}} = \boldsymbol{\mu}_1 \cdot \boldsymbol{B}^\top$$

where $\boldsymbol{B}$ is a diagonal matrix with diagonal entries $\text{diag}(\boldsymbol{B}) = [1, 1, 1, -4, -4]$. For a more detailed explanation of these transformations and the derivation of analytical distributions, see Appendix 7.2.

1.3.2 Simulation of the Evolution of the Initial 5-Year Zero-Coupon Bond and Distribution Analysis, S144474

To evaluate the performance and distribution of the 5-year zero-coupon bond at the horizon, a simulation approach was employed to model the stochastic evolution of the yield curve. The following methodology was applied:

1. **Yield Dynamics Simulation:** The weekly evolution of the yields (y_t) was modeled using the multivariate normal distribution of market invariants ($\Delta X_t \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$), where the simulation from part 1 again was used, here in this case to propagate yields over the investment horizon.

2. **Bond Price Computation:** The bond price (B_t) at each simulation step was calculated using the continuously compounded yield formula:

$$B_t = F \cdot e^{-y_t \cdot T},$$

   where F is the face value (assumed to be 1), y_t is the simulated yield, and T is the time to maturity.

At the horizon ($t = 1$), the time to maturity for the bond was reduced to 4 years. The analytical distribution of the bond prices as derived in part 1.3.1 was compared with the simulated values:.

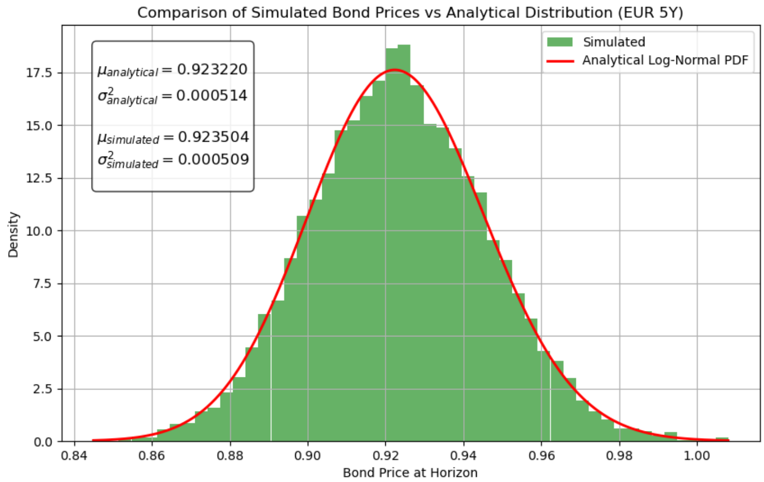


Figure 4: Simulated vs Analytical Distribution of 5-Year Zero-Coupon Bond.

Figure 4 presents a histogram of the simulated bond prices overlaid with the analytical log-normal probability density function (PDF). The close alignment validates the simulation accuracy and confirms the analytical assumptions. The analytical and simulated metrics of the bond prices at the horizon are summarized in the Table below.

Metric	**Simulated Values**	**Analytical Values**
Mean (μ)	0.923504	0.923220
Variance (σ^2)	0.000509	0.000514
Distribution	Log-normal	Log-normal

The results validate that the bond price at the horizon follows a log-normal distribution, as derived in part 1.3.1. This alignment confirms the robustness of the simulation and analytical modeling assumptions.

1.4 Joint Distribution of $\mathbf{P}_1$, S145085

The vector $\mathbf{P}_1$ is defined as: $\mathbf{P}_1 = \left(FX_1, V_1^{\text{US,local}}, V_1^{\text{EUR}}, Z_1^{\text{USD 4Y, local}}, Z_1^{\text{EUR 4Y}}\right)$. Where each element of $\mathbf{P}_1$ corresponds to a market variable at the investment horizon $t = 1$:

1. FX_1: Denotes the EUR/USD exchange rate, the number of USD required to by one EUR.
2. $V_1^{\text{US,local}}$: Value of US equities in USD.
3. V_1^{EUR}: Value of EUR equities in EUR.
4. $Z_1^{\text{USD,4Y,local}}$: Value of a 4-year USD zero-coupon bond.
5. $Z_1^{\text{EUR,4Y}}$: Value of a 4-year EUR zero-coupon bond.

In part 1.2.1 we already derived the distribution of $V_1^{\text{US,local}}$ and that is the same logic regarding FX_1 and V_1^{EUR}. In part 1.3.1 we derived the distribution of an initial 5-year zero-coupon bond at the horizon, which are very similar to $Z_1^{\text{USD,4Y,local}}$ and $Z_1^{\text{EUR,4Y}}$. Therefore from the simulation results and analytical derivations, the joint distribution of $\mathbf{P}_1$ is found to be a multivariate log-normal distribution:

$$\mathbf{P}_1 \sim \text{LogN}(\boldsymbol{\mu}_{P_{1,\log}}, \boldsymbol{\Sigma}_{P_{1,\log}}),$$

where $\boldsymbol{\mu}_{P_{1,\text{log}}}$ is the expectation vector, and $\boldsymbol{\Sigma}_{P_{1,\text{log}}}$ is the covariance matrix based on the log-values. The expectation vector and covariance matrix of $\mathbf{P}_1$ could be estimated numerically based on the simulated data or analytically derived as seen in appendix 7.2, where the analytical results are highligted here:

$$\mathbb{E}[\mathbf{P}_1] = \begin{pmatrix} 1.059954 \\ 1.084510 \\ 1.074765 \\ 0.861442 \\ 0.923220 \end{pmatrix}, \mathrm{Cov}[\mathbf{P}_1] = \begin{pmatrix} 6.61\cdot10^{-3} & 2.30\cdot10^{-3} & -2.79\cdot10^{-4} & 5.99\cdot10^{-4} & 7.90\cdot10^{-5} \\ 2.30\cdot10^{-3} & 2.65\cdot10^{-2} & 2.05\cdot10^{-2} & -3.24\cdot10^{-4} & 6.56\cdot10^{-4} \\ -2.79\cdot10^{-4} & 2.05\cdot10^{-2} & 2.83\cdot10^{-2} & -6.63\cdot10^{-4} & 3.68\cdot10^{-4} \\ 5.99\cdot10^{-4} & -3.24\cdot10^{-4} & -6.63\cdot10^{-4} & 7.22\cdot10^{-4} & 4.33\cdot10^{-4} \\ 7.90\cdot10^{-5} & 6.56\cdot10^{-4} & 3.68\cdot10^{-4} & 4.33\cdot10^{-4} & 5.14\cdot10^{-4} \end{pmatrix}$$

These analytical values are verified through simulations in Appendix 7.3 where each of the marginal distributions from $\mathbf{P}_1$ are also checked and plotted in Figure 18, which illustrates the marginal distributions of each component of $\mathbf{P}_1$ along with the analytical log-normal fits. The simulated results align closely with the theoretical distributions, further validating the numerical implementation.

1.5 Distribution of $\mathbf{P}_1^{\mathbf{EUR}}$, S145085 & S144474

The vector $\mathbf{P}_1^{\text{EUR}}$ is defined as: $\mathbf{P}_1^{\text{EUR}} = \left(1/FX_1, V_1^{\text{US}}, V_1^{\text{EUR}}, Z_1^{\text{USD 4Y}}, Z_1^{\text{EUR 4Y}}\right)$. Where all components are expressed in the domestic currency (EUR). The transformation is applied to convert USD-based values into EUR using the exchange rate FX_1. The difference from $\mathbf{P}_1$ are these components: $1/FX_1, V_1^{\text{US}}$ and $Z_1^{\text{USD 4Y}}$. Here $V_1^{\text{US}} = V_1^{\text{US, local}} \cdot 1/FX_1$ and $Z_1^{\text{USD 4Y}} = Z_1^{\text{USD 4Y, local}} \cdot 1/FX_1$. The distribution of $\mathbf{P}_1^{\text{EUR}}$ is also found to be a multivariate log-normal distribution:

$$\mathbf{P}_1^{\text{EUR}} \sim \mathrm{LogN}(\boldsymbol{\mu}_{P_{1,\text{log, dom}}}, \boldsymbol{\Sigma}_{P_{1,\text{log, dom}}})$$

where $\boldsymbol{\mu}_{P_{1,\text{log, dom}}}$ is the expectation vector, and $\boldsymbol{\Sigma}_{P_{1,\text{log, dom}}}$ is the covariance matrix in log-values and transformed into domestic values. The expectation vector and covariance matrix of $\mathbf{P}_1^{\text{EUR}}$ are analytically derived as seen in Appendix 7.2, where the analytical results are highligted here:

$$\mathbb{E}[\mathbf{P}_1^{\text{EUR}}] = \begin{pmatrix} 0.948987 \\ 1.027131 \\ 1.074765 \\ 0.816961 \\ 0.923220 \end{pmatrix}, \mathrm{Cov}[\mathbf{P}_1^{\text{EUR}}] = \begin{pmatrix} 5.30\cdot10^{-3} & 3.78\cdot10^{-3} & 2.50\cdot10^{-4} & 4.05\cdot10^{-3} & -7.10\cdot10^{-5} \\ 3.78\cdot10^{-3} & 2.58\cdot10^{-2} & 1.97\cdot10^{-2} & 2.41\cdot10^{-3} & 5.45\cdot10^{-4} \\ 2.50\cdot10^{-4} & 1.97\cdot10^{-2} & 2.83\cdot10^{-2} & -4.14\cdot10^{-4} & 3.68\cdot10^{-4} \\ 4.05\cdot10^{-3} & 2.41\cdot10^{-3} & -4.14\cdot10^{-4} & 3.70\cdot10^{-3} & 3.49\cdot10^{-4} \\ -7.10\cdot10^{-5} & 5.45\cdot10^{-4} & 3.68\cdot10^{-4} & 3.49\cdot10^{-4} & 5.14\cdot10^{-4} \end{pmatrix}$$

The red components are still the same values as we had in $\mathbf{P}_1$ in part 1.4, as they are the only components that remains the same after we transformed the rest of the variables into domestic currency as they already were in domestic currency. These analytical values are also verified through simulations at the end in this part.

To highlight the true analytical distribution for V_1^{US} we derived this in Appendix 7.2 and the exact calculations can be seen in Appendix 7.4, where V_1^{US} follows a log-normal distribution:

$$V_1^{\text{US}} \sim \mathrm{LogN}(\mu_{V1}, \sigma_{V1}^2)$$

were the true analytical mean and variance are derived with the following standard formulas (tricks) we know:

$$\mathbb{E}[V_1^{\text{US}}] = \exp(\mu_{V1} + \frac{\sigma_{V1}^2}{2}) = 1.02713138521889$$

$$\mathrm{Var}[V_1^{\text{US}}] = (\exp(\sigma_{V1}^2) - 1) \cdot (\exp(\mu_{V1} + \sigma_{V1}^2)) = 0.025761646547379086$$

Figure 5 illustrates the comparison between the simulated and analytical distribution for V_1^{US}, where the analytical distribution is derived under the assumption of a log-normal distribution. The results reveal a close match between the simulated data and the theoretical model, confirming the accuracy of the simulation process.

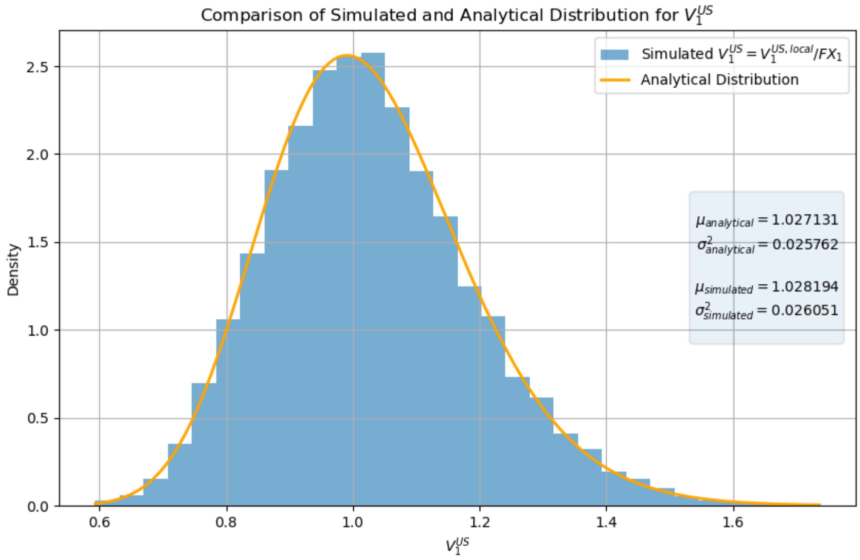


Figure 5: Comparison of Simulated and Analytical Distribution for V_1^{US}

Key statistical metrics are presented within Figure 5. The simulated mean and variance of V_1^{US} are calculated as $\mu_{\text{simulated}} = 1.028194$ and $\sigma^2_{\text{simulated}} = 0.026051$. These are based on on 10.000 simulations, the more simulations the closer we get to the true analytical values. Moreover each of the marginal distributions from $\mathbf{P}_1^{\mathrm{EUR}}$ were also checked and plotted in Figure 6 along with the analytical log-normal fits, which again confirms the numerical implementation. Through 10.000 simulation we get the estimated simulated expectation vector and covariance-matrix:

$$\mathbb{E}[\mathbf{P}_{1,\text{sim}}^{\mathrm{EUR}}] = \begin{pmatrix} 0.9489 \\ 1.0219 \\ 1.0742 \\ 0.8171 \\ 0.9235 \end{pmatrix}, \mathrm{Cov}[\mathbf{P}_{1,\text{sim}}^{\mathrm{EUR}}] = \begin{pmatrix} 5.24 \cdot 10^{-3} & 3.85 \cdot 10^{-3} & 2.66 \cdot 10^{-4} & 4.01 \cdot 10^{-3} & -5.87 \cdot 10^{-5} \\ 3.85 \cdot 10^{-3} & 2.61 \cdot 10^{-2} & 1.99 \cdot 10^{-2} & 2.44 \cdot 10^{-3} & 5.61 \cdot 10^{-4} \\ 2.66 \cdot 10^{-4} & 1.99 \cdot 10^{-2} & 2.84 \cdot 10^{-2} & -4.05 \cdot 10^{-4} & 4.02 \cdot 10^{-4} \\ 4.01 \cdot 10^{-3} & 2.44 \cdot 10^{-3} & -4.05 \cdot 10^{-4} & 3.68 \cdot 10^{-3} & 3.53 \cdot 10^{-4} \\ -5.87 \cdot 10^{-5} & 5.61 \cdot 10^{-4} & 4.02 \cdot 10^{-4} & 3.53 \cdot 10^{-4} & 5.09 \cdot 10^{-4} \end{pmatrix}.$$

The purple components are still the same components as we had in the simulation for $\mathbf{P}_1$ in Appendix 7.3, as they already were in domestic currency. As we see the simluated expectation vector and simulated covariance matrix are close to the analytical expectation vector and analytical covariance matrix.

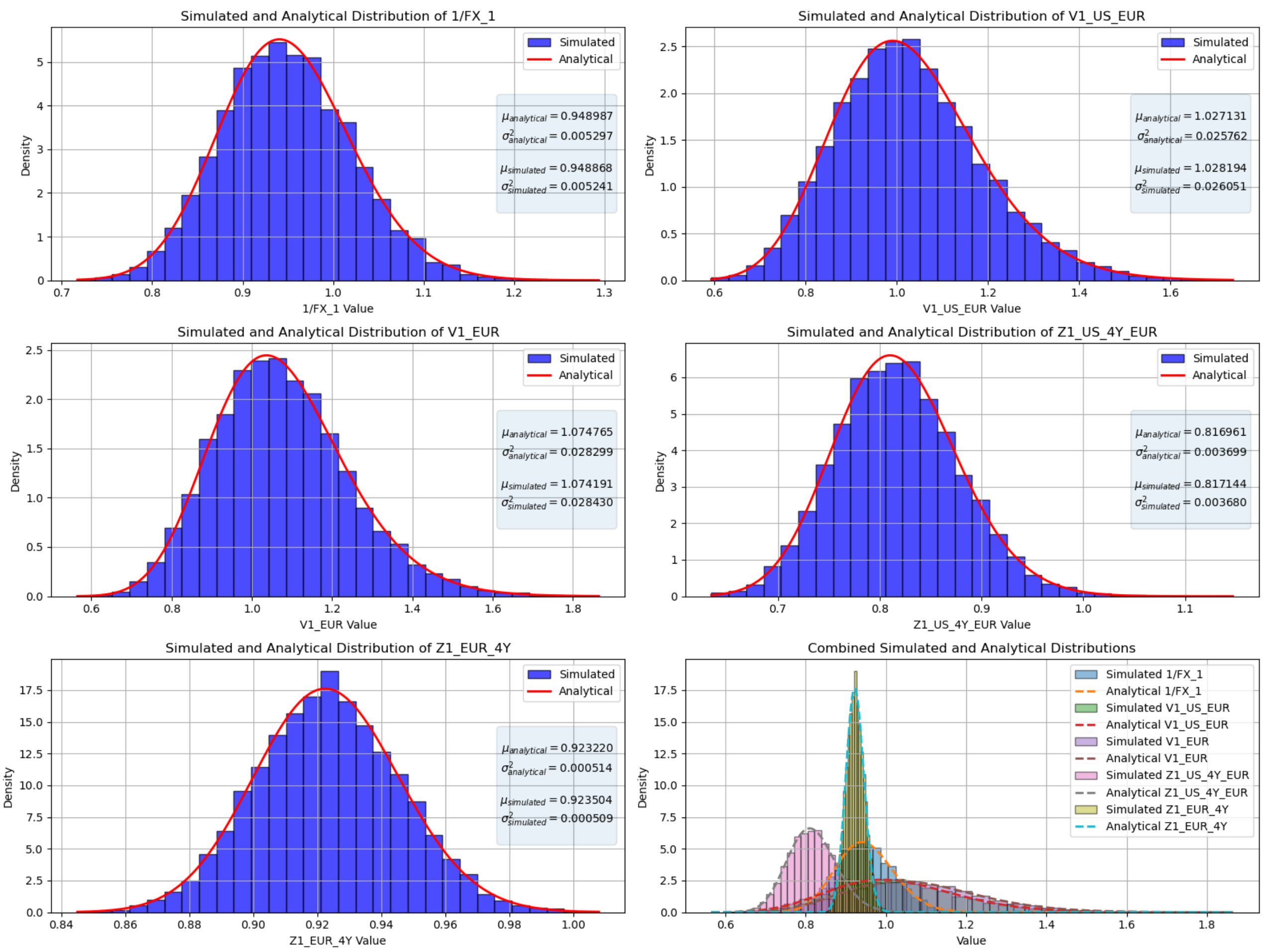


Figure 6: Simulated and Analytical Marginal Distributions of the Components of $\mathbf{P}_1^{\text{EUR}}$.

Figure 6 again highlight the robustness of the numerical implementation and the correctness of the theoretical derivations.

2 Part II - PNL

As stated in the assignment the portfolio PnL of the investor is given by the product of the holding vector and the vector of PnLs:

$$PnL_1 = \mathbf{h}^\top \mathbf{PnL}_1$$

where

$$\mathbf{PnL}_1 = \begin{pmatrix} \frac{1}{F_0^1} - \frac{1}{FX_1} \\ V_1^{\text{US}} - V_0^{\text{US}} \\ V_1^{\text{EUR}} - V_0^{\text{EUR}} \\ Z_1^{\text{USD 4Y}} - Z_0^{\text{USD 5Y}} \\ Z_1^{\text{EUR 4Y}} - Z_0^{\text{EUR 5Y}} \end{pmatrix}$$

Here the one year FX forward price is given by:

$$F_0^1 = FX_0 e^{1 \cdot \left(y_0^{USD,1} - y_0^{EUR,1}\right)}$$

2.1 Expectation, Covariance Matrix and distribution of the PnL Vector, S145085

To calculate the expectation of the PnL vector, we compute the mean of each entry. The analytical expectation of the PnL vector is given by:

$$\mathbb{E}[\mathbf{PnL}_1] = \begin{pmatrix} \frac{1}{F_0^1} - \mathbb{E}\left[\frac{1}{FX_1}\right] \\ \mathbb{E}[V_1^{\text{US}}] - V_0^{\text{US}} \\ \mathbb{E}[V_1^{\text{EUR}}] - V_0^{\text{EUR}} \\ \mathbb{E}[Z_1^{\text{USD 4Y}}] - Z_0^{\text{USD 5Y}} \\ \mathbb{E}[Z_1^{\text{EUR 4Y}}] - Z_0^{\text{EUR 5Y}} \end{pmatrix} = \begin{pmatrix} 0.928208 - 0.948987 \\ 1.027131 - 0.946208 \\ 1.074765 - 1.000000 \\ 0.816961 - 0.784793 \\ 0.923220 - 0.904623 \end{pmatrix} = \begin{pmatrix} -0.0208 \\ 0.0809 \\ 0.0748 \\ 0.0322 \\ 0.0186 \end{pmatrix},$$

where the blue components are initial values (constants) while the remaining terms are are the same analytical components we calculated from $\mathbf{P}_1^{\text{EUR}}$ in part 1.5. This expectation can also be approximated using simulated data from part 1, as demonstrated in the provided code. For the analytical approach we used that each components log-normally distributed. For log-normal distributions, the analytical mean can be calculated as:

$$\mathbb{E}[X] = \exp(\mu_X + \frac{1}{2}\sigma_X^2), \quad \text{given that} \quad X \sim \text{LogN}(\mu_X, \sigma_X^2)$$

To calculate the covariance matrix of the PnL vector, we note that it is nearly identical to the covariance matrix of $\mathbf{P}_1^{\text{EUR}}$, except for adjustments in the first row and first column. Specifically, the term $-\frac{1}{FX_1}$ introduces a scaling factor of -1 to the first row and column (excluding the diagonal). Thus, the analytical covariance matrix of $\mathbf{PnL}_1$ becomes:

$$\text{Cov}[\mathbf{PnL}_1] = \begin{pmatrix} 5.30 \cdot 10^{-3} & -3.78 \cdot 10^{-3} & -2.50 \cdot 10^{-4} & -4.05 \cdot 10^{-3} & 7.10 \cdot 10^{-5} \\ -3.78 \cdot 10^{-3} & 2.58 \cdot 10^{-2} & 1.97 \cdot 10^{-2} & 2.41 \cdot 10^{-3} & 5.45 \cdot 10^{-4} \\ -2.50 \cdot 10^{-4} & 1.97 \cdot 10^{-2} & 2.83 \cdot 10^{-2} & -4.14 \cdot 10^{-4} & 3.68 \cdot 10^{-4} \\ -4.05 \cdot 10^{-3} & 2.41 \cdot 10^{-3} & -4.14 \cdot 10^{-4} & 3.70 \cdot 10^{-3} & 3.49 \cdot 10^{-4} \\ 7.10 \cdot 10^{-5} & 5.45 \cdot 10^{-4} & 3.68 \cdot 10^{-4} & 3.49 \cdot 10^{-4} & 5.14 \cdot 10^{-4} \end{pmatrix}$$

where the red components denote the adjustments introduced by $-\frac{1}{FX_1}$. For simulated data, this adjustment is straightforward. For the analytical approach, we again assume log-normal distributions, where the variance can be calculated as:

$$\text{Var}[X] = [\exp(\sigma_X^2) - 1] \cdot \exp(2\mu_X + \sigma_X^2) \quad \text{given that} \quad X \sim \text{LogN}(\mu_X, \sigma_X^2)$$

For the covariance between any two elements of $\mathbf{PnL}_1$, say $\text{PnL}_1[i]$ and $\text{PnL}_1[j]$, depends on the covariance of their underlying variables. For example:

$$\text{Cov}\left(\frac{1}{FX_1}, V_1^{US} - V_0^{US}\right) = \text{Cov}\left(\frac{1}{FX_1}, V_1^{US}\right).$$

What is the distribution of the PnL vector?

The distribution of the PnL vector, $\mathbf{PnL}_1$, is closely related to the distribution of $\mathbf{P}_1^{\text{EUR}}$, as seen in part 1.5. However, for $\mathbf{PnL}_1$, initial values are subtracted from the components, introducing a shift. A visualization of this relationship based on simulated data is provided below:

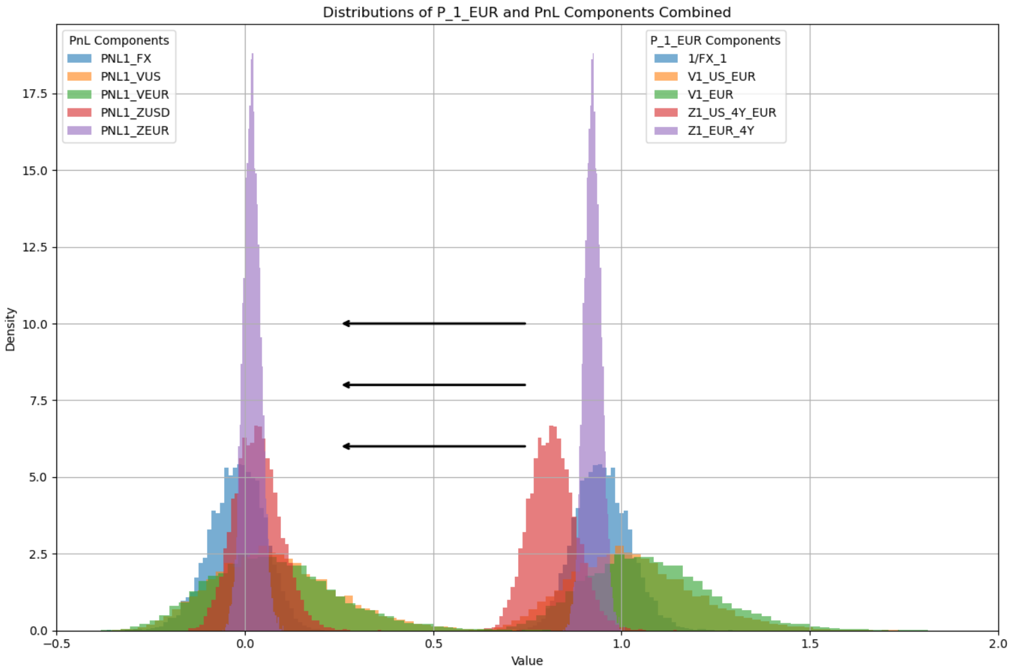


Figure 7: Comparison of components in $\mathbf{PnL}_1$ and $\mathbf{P}_1^{\mathrm{EUR}}$

From this, we observe that $\mathbf{PnL}_1$ follows a shifted multivariate lognormal distribution:

$$\mathbf{PnL}_1 \sim \mathrm{LogN}(\boldsymbol{\mu}_{P_{1,\mathrm{log,\ dom}}}, \boldsymbol{\Sigma}_{P_{1,\mathrm{log,\ dom}}})$$

where the excpectation vector, $\mathbb{E}[\mathbf{PnL}_1]$, and the covariance matrix of $\mathbf{PnL}_1$, $\mathrm{Cov}[\mathbf{PnL}_1]$, are as derived above. The shift in distribution arises due to the subtraction of initial values, which does not affect the underlying lognormal property but adjusts the location of the distribution. This ensures that while $\mathbf{PnL}_1$ inherits the multivariate lognormal structure of $\mathbf{P}_1^{\mathrm{EUR}}$, it is shifted according to the initial values.

2.2 Expectation and Covariance Matrix of the PnL of the portfolio, S144474

2.2.1 Can we derive the distribution of the PnL of the portfolio?

The portfolio PnL_1, defined as:

$$PnL_1 = \mathbf{h}^\top \mathbf{PnL}_1,$$

is a weighted sum of lognormal random variables. However, because the components of $\mathbf{PnL}_1$ are *dependent*, their sum cannot be expressed analytically as lognormal, and special care must be taken to include their covariance structure. For small variances and moderate dependency, the Central Limit Theorem (CLT) allows us to approximate the distribution of PnL_1 as normal:

$$PnL_1 \sim \mathcal{N}(\mathbb{E}[PnL_1], \mathrm{Var}[PnL_1]).$$

However, this approximation may not hold if the variances are large or the dependency between components is strong. In this case, simulation provides the most reliable approach for determining the distribution of PnL_1. While the exact distribution of PnL_1 cannot be expressed analytically due to dependency between components and negative values in the shiftet lognormal distributions, its first two moments (expectation and variance) can be derived as seen in the next part. A normal approximation may be valid under certain conditions, but simulation is necessary for a precise result when variances or dependencies are significant.

2.2.2 What is the expectation and variance of the portfolio PnL?

Obiously the expected value and the variance of the portfolio depends heavily on the 5 x 1 holding vector $\mathbf{h}$, however the calculations is fairly simple: The expected value of the portfolio PnL is a linear combination of the expectations of the individual PnL components. Let $\mathbf{h}$ be the vector of weights and $\mathbf{PnL}_1$ the vector of individual PnL components at the horizon. Then:

$$\mathbb{E}[\text{PnL}_{\text{portfolio}}] = \mathbf{h}^\top \mathbb{E}[\mathbf{PnL}_1]$$

Here the $\mathbb{E}[\mathbf{PnL}_1]$ is as calculated in part 2.1. The variance can be computed if we know the covariance matrix of the PnL vector, then:

$$\text{Var}(\text{PnL}_{\text{portfolio}}) = \mathbf{h}^\top \text{Cov}[\mathbf{PnL}_1]\mathbf{h}.$$

Here $\text{Cov}[\mathbf{PnL}_1]$ is as calculated in part 2.1.

2.3 Consider 3 portfolios, S145085

As stated in the assigment, the optimal number of FX forward contracts, h_1, to minimize the PnL variance for a fixed allocation to the remaining assets $\mathbf{h}_2$ is given by:

$$h_1 = \frac{-\Sigma_{12}^{PnL}\mathbf{h}_2}{\Sigma_{11}^{PnL}}, \quad \text{where} \quad \Sigma^{PnL} = \begin{pmatrix} \Sigma_{11}^{PnL} & \Sigma_{12}^{PnL} \\ \Sigma_{12}^{PnL} & \Sigma_{22}^{PnL} \end{pmatrix}$$

In the analysis of the portfolios, it is assumed that fractional ownership of assets is possible, allowing for precise allocation of investments across different asset classes. The holdings in each portfolio have been calculated based on the amount invested in euros and the corresponding prices of the assets. Where the prices are:

$$(V_0^{\text{US}}, V_0^{\text{EUR}}, Z_0^{\text{USD 5Y}}, Z_0^{\text{EUR 5Y}}) = (0.94620807, 1, 0.78479266, 0.90462341)$$

The table below presents the allocation and holdings for each portfolio.

Portfolio	**US Eq**	**EUR Eq**	**USD 5Y**	**EUR 5Y**
1 EUR in US Equities	1	0	0	0
Holdings in Assets	1.05685	0	0	0
1 EUR in 5-Year USD Zero-Coupon Bond	0	0	1	0
Holdings in Assets	0	0	1.27422191	0
Mixed Portfolio	0.2	0.2	0.3	0.3
Holdings in Assets	0.21137	0.2	0.38227	0.33163

Table 3: Allocation and Calculated Holdings in Euros for Each Portfolio

The calculated holdings for each portfolio reflect the exact allocation of assets based on the investments and their respective prices. For instance, in the first portfolio, the entire investment is allocated to US equities, resulting in a holding of 1.05685 units of the asset when purchased at the given price by time zero.

PnL of the Portfolio

The portfolio's total PnL, including the forward contract and the other assets, is given by, $\mathbf{PnL}_1$:

$$h_1\left(\frac{1}{F_0^1} - \frac{1}{FX_1}\right) + h_2\left(V_1^{US} - V_0^{US}\right) + h_3\left(V_1^{EUR} - V_0^{EUR}\right) + h_4(Z_1^{USD4Y} - Z_0^{USD5Y}) + h_5(Z_1^{EUR4Y} - Z_0^{EUR5Y})$$

So, to sum up, this gives us the following perspective: $h_1 > 0$: Short USD (hedged) and $h_1 < 0$: Short EUR.

The USD exposure of a portfolio can be calculated by summing the USD-denominated holdings. In this case, the USD exposure is derived from the holdings in US equities and US zero-coupon bonds. After converting all prices to USD, the exposure can be computed as:

$$\text{USD Exposure} = \text{Holdings in US Equities} + \text{Holdings in US Zero-Coupons}$$

The hedge percentage (hp) is then defined as:

$$\text{hp} = \frac{\text{h1}}{\text{USD Exposure}} * 100$$

hp $= 100\%$ means the exposure is fully hedged, hp $> 100\%$ means over-hedging, and hp $< 100\%$ indicates partial or no hedging.

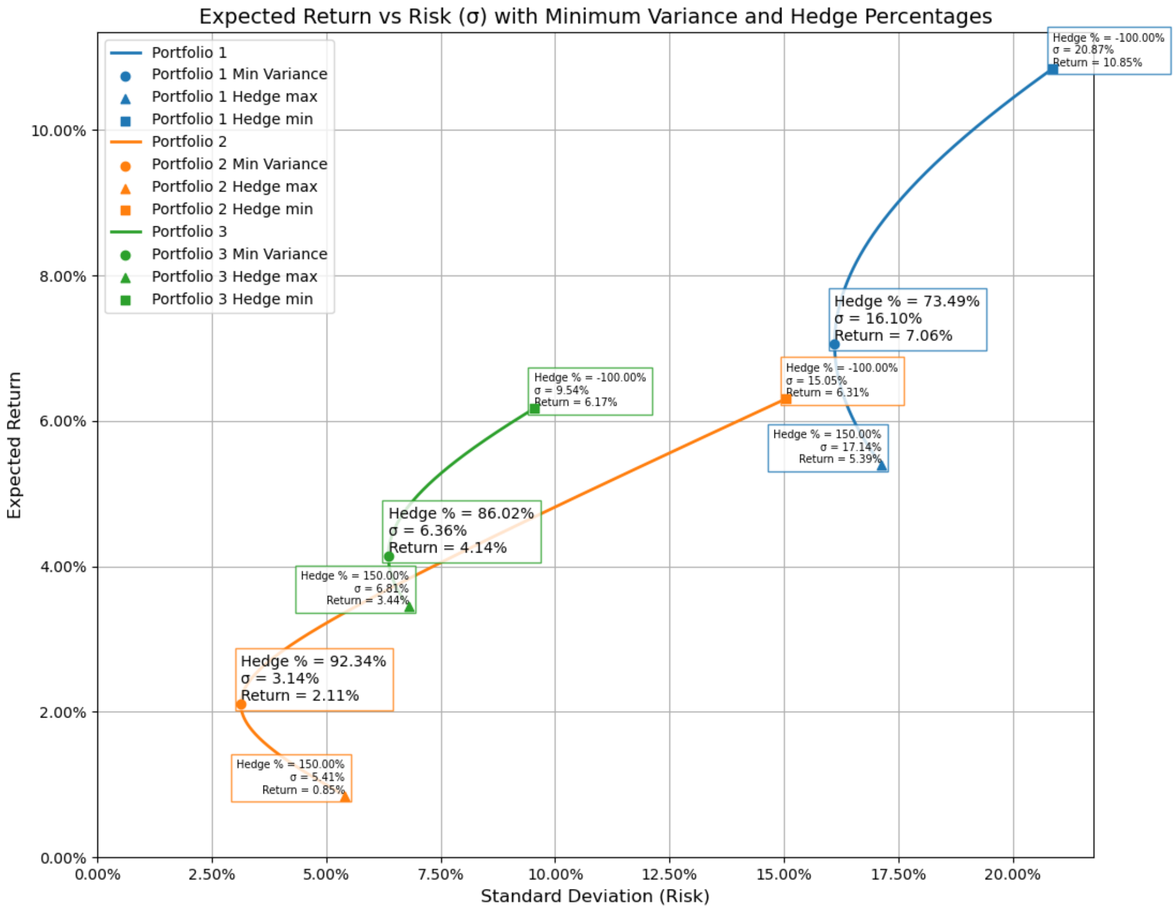


Figure 8: Expected Return vs Risk (σ) with Minimum Variance Points

Figure 8 shows the relationship between expected return (μ) and standard deviation (σ) for three portfolios across varying hedge ratios from -100% to 150%, . The key observations are as follows:

Table 4: Expected Return (μ) and Risk (σ) for Portfolios.

Portfolio	**Hedge** (hp)	**Risk** (σ)	**Expec Return** (μ)
Portfolio 1	−100%	20.87%	10.85%
Portfolio 1	**73.49%**	**16.10%**	**7.06%**
Portfolio 1	150%	17.14%	5.39%
Portfolio 2	−100%	15.05%	6.31%
Portfolio 2	**92.34%**	**3.14%**	**2.11%**
Portfolio 2	150%	5.41%	0.85%
Portfolio 3	−100%	9.54%	6.17%
Portfolio 3	**86.02%**	**6.36%**	**4.14%**
Portfolio 3	150%	6.81%	3.44%

The analysis of the three portfolios demonstrates the impact of varying hedge ratios on risk and return. Hedging is primarily designed to reduce risk by mitigating the portfolio's exposure to exchange rate volatility. For each portfolio, the hedge ratio corresponding to the minimum variance point provides the optimal balance between risk and return. This is evident from the observations where the portfolios achieve their lowest standard deviation at these specific hedge ratios.

However, while hedging minimizes risk, it also negatively affects the expected return. This is because the mean of the exchange rate movement is assumed to be zero. By fully or partially hedging, the portfolio eliminates the possibility of benefiting from favorable exchange rate movements, thereby reducing the expected profit and loss (PnL). For example, in Portfolio 1, which invests entirely in US equities, a hedge ratio close to the minimum variance point significantly reduces risk but also leads to a lower return compared to the unhedged or over-hedged positions, beacuase of the cost of hedging. Similarly, for Portfolio 2, which is fully invested in US zero-coupon bonds, hedging close to the optimal ratio minimizes risk while reducing return. The effect is particularly pronounced in Portfolio 3, which is more diversified, as it invests in both US and EUR assets. While hedging reduces the exposure to USD volatility, it also decreases the potential gains from favorable currency movements.

Overall, the results highlight the trade-off between risk and return when implementing hedging strategies. A fully hedged position minimizes risk but also limits return potential, as seen in the reduced PnL at the minimum variance points. Conversely, a less hedged or over-hedged position increases both risk and return, as the portfolio becomes more sensitive to exchange rate fluctuations. The choice of hedge ratio should therefore align with the investor's risk tolerance and return objectives, taking into consideration the currency exposure and diversification of the portfolio.

2.4 Finding the Minimum 5% CVaR Hedge Ratio for the multi asset portfolio, S144474

To identify the minimum 5% Conditional Value at Risk (CVaR) hedge ratio for the multi-asset portfolio, CVaR is calculated for a range of hedge ratios from -1 to 1.5. CVaR is used as a measure of extreme risk, capturing the average loss in the worst 5% of outcomes. The optimal hedge ratio is determined as the point where CVaR reaches its minimum. From the calculations, it is evident that the hedge ratio of approximately hp = 61.19% minimizes the 5% CVaR, achieving a value of −7.3054%, while maintaining an expected return of 4.4115% as seen in Figure 9:

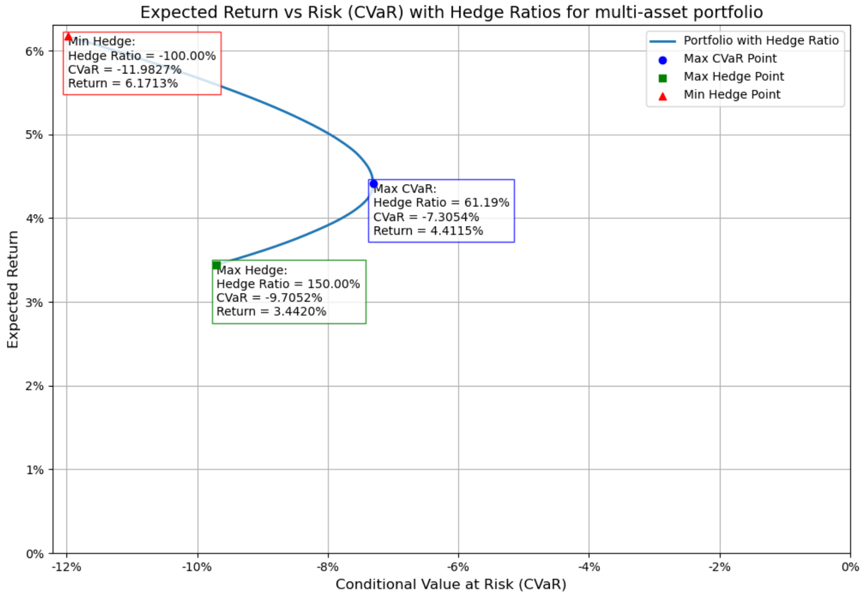


Figure 9: Portfolio Expected Return vs CVaR for Hedge Ratios $h_1 \in [-1, 1.5]$

The results from Figure 9 reveal key patterns in the relationship between hedge ratios, CVaR, and expected PnL. At hp $= -100\%$, the portfolio is more than fully exposed to USD risk, effectively going long on the USD, and this results in a CVaR of -11.9827%. While this hedge ratio offers the highest expected return at 6.1713%, it also introduces significant tail risk due to the more than fully exposure to unfavorable currency movements. On the other hand, an over-hedged position, such as hp $= 150\%$, reduces CVaR to -9.7052% but leads to a lower expected return of 3.4420%. This emphasizes the diminishing benefits of over-hedging, as it not only eliminates potential currency gains but also reduces returns due to the associated costs.

The findings emphasize the importance of calibrating hedge ratios based on the portfolio's composition and risk tolerance. While minimizing CVaR reduces exposure to extreme negative outcomes, it is crucial to consider the associated impact on expected returns. For the multi-asset portfolio, the hedge ratio of hp $= 61.19\%$ demonstrates the most efficient trade-off between reducing extreme risk and maintaining attractive returns, making it the preferred strategy under the given conditions. When the hedge ratio is reduced from the level that minimizes standard deviation, the portfolio may become more exposed to average fluctuations (standard deviation increases), but it can also achieve better protection against extreme losses. This occurs because the optimization that minimizes CVaR prioritizes the worst-case scenarios, which may be more influenced by asymmetry or specific negative movements in exchange rates. This difference highlights an important trade-off: portfolios optimized for low standard deviation may not achieve the same robustness against extreme events as those optimized for low CVaR. In this case, a lower hedge ratio can help create a more balanced portfolio in extreme scenarios, but this comes at the cost of slightly higher average risk. Furthermore, the pnl distribution for the minimum CVAR and minimum std mixed portfolio can be seen in here.

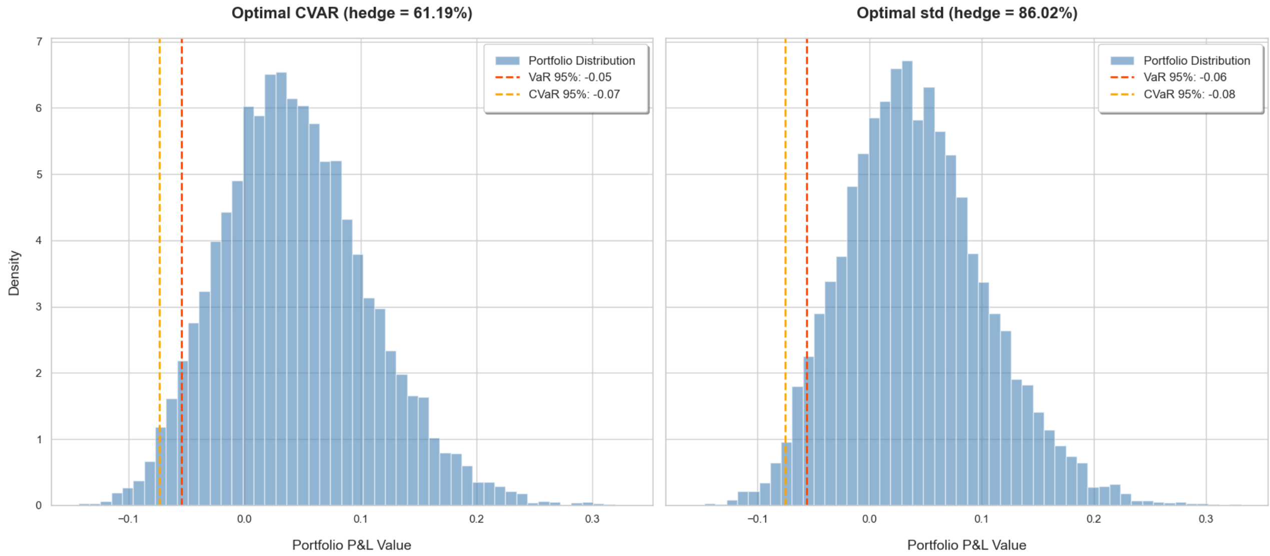


Figure 10: Distribution of portfolio value changes under two optimal hedge ratios.

The left plot in Figure 10 illustrates the portfolio distribution for a hedge ratio of 61.19%, optimized for Conditional Value-at-Risk (CVaR), while the right plot corresponds to a hedge ratio of 86.02%, optimized for standard deviation. Both plots indicate the 95% Value-at-Risk (VaR) and CVaR thresholds, highlighting risk levels associated with different hedging strategies. It is evident that as the hedge percentage increases, the distribution narrows, thereby reducing the variance and thereby standard deviation. However, due to the cost associated with hedging, the Conditional Value at Risk (CVaR) increases. Consequently, it makes sense that the optimal CVaR portfolio would have a lower hedge percentage.

3 Part III - Simulation Study

In this part, we conduct a simulation study to analyze the impact of estimation uncertainty in the covariance matrix on the optimal hedge ratio.. As a basis for this simulation study we here have the true analytical expectation vector and covariance matrix for $\mathbf{PnL}_2$ for $t = 2$:

$$\mathbb{E}[\mathbf{PnL}_2] = \begin{pmatrix} -0.0412 \\ 0.1688 \\ 0.1551 \\ 0.0657 \\ 0.0373 \end{pmatrix}, \quad \mathrm{Cov}(\mathbf{PnL}_2) = \begin{pmatrix} 1.07 \cdot 10^{-2} & -8.23 \cdot 10^{-3} & -5.39 \cdot 10^{-4} & -8.74 \cdot 10^{-3} & 1.08 \cdot 10^{-4} \\ -8.23 \cdot 10^{-3} & 6.15 \cdot 10^{-2} & 4.65 \cdot 10^{-2} & 5.93 \cdot 10^{-3} & 9.05 \cdot 10^{-5} \\ -5.39 \cdot 10^{-4} & 4.65 \cdot 10^{-2} & 6.62 \cdot 10^{-2} & -5.74 \cdot 10^{-4} & 6.05 \cdot 10^{-5} \\ -8.74 \cdot 10^{-3} & 5.93 \cdot 10^{-3} & -5.74 \cdot 10^{-4} & 7.89 \cdot 10^{-3} & 3.93 \cdot 10^{-4} \\ 1.08 \cdot 10^{-4} & 9.05 \cdot 10^{-5} & 6.05 \cdot 10^{-5} & 3.93 \cdot 10^{-4} & 6.02 \cdot 10^{-4} \end{pmatrix}$$

3.1 Describe a simulation study with a two year sample of the market invariants, S145085

The purpose of this simulation study is to examine how estimation uncertainty from using the sample covariance matrix $\hat{\mathbf{\Sigma}}$ impacts the optimal hedge ratio, expected PnL, standard deviation, and risk metrics such as the 5% CVaR for the multi-asset portfolio, while keeping $\boldsymbol{\mu}$ fixed.

The first step is to make the covariance matrix simulation. We wil be using af Wishart distribution $(\mathcal{W}(\hat{\mathbf{\Sigma}}, \nu))$, where we generate 10.000 simulated covariance matrices. The degrees of freedom are set to 103 (corresponding to 104 weeks minus one week). The reduced covariance matrix is used, where we only focus on the five selected risk drivers that are relevant for the PnL. The second step would be to make some transformations, where assets prices are transformed using a bond price adjusment matrix $\boldsymbol{B}$ and converted to domestic currency (EUR) using a weighted matrix $\boldsymbol{W}$:

$$\boldsymbol{\Sigma}_{\text{Price}} = \boldsymbol{B} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{B}^\top \quad \wedge \quad \boldsymbol{\Sigma}_{\text{dom}} = \boldsymbol{W} \cdot \boldsymbol{\Sigma}_{\text{Price}} \cdot \boldsymbol{W}^\top$$

where:

$$\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \quad \wedge \quad \boldsymbol{W} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The third setp would be for each simulated covariance-matrix to compute the optimal number of FX forward contracts, given by: $h_1 = \frac{-\Sigma_{12}^{PnL}\mathbf{h}_2}{\Sigma_{11}^{PnL}}$, which obvioulsy depends heavily on the covariance-matrix and $\mathbf{h}_2$ is fixed for the multi-asset portfolio. Then we will evalute the hedge ratio, with:

$$\text{Hedge Ratio} = \frac{h_1}{\text{USD Exposure}} \cdot 100$$

The final step would be for each simulated optimal hedge ratio (each simulated portfolio) to evalute the expected PnL, standard deviation (risk), and 5% CVaR are with the true distributional parameters which was presented in the introduction to this part. The results will be visualized with the following plots: Hedge ratio distribution, expected PnL distribution (μ_{PnL_2}), the risk distribution (σ_{PnL_2}) and the 5% CVaR distribution. The key Assumptions are:

- $\boldsymbol{\mu}$ is fixed and known (analytically derived for $t = 2$)
- The true weekly covariance matrix $\boldsymbol{\Sigma}$ is used to simulate initial market invariants
- Simulations assume independence of estimation errors across covariance elements.

3.2 Based on simulated covariance matrices, calculate optimal hedge ratios, S144474

As described in part 3.1, a simulation study was conducted where 10.000 covariance matrices were generated using the Wishart distribution. For each simulated covariance matrix, the optimal number of FX forward contracts was calculated to minimize risk, followed by the corresponding hedge ratio for each simulation. The distribution of hedge ratios is illustrated in Figure 11 below:

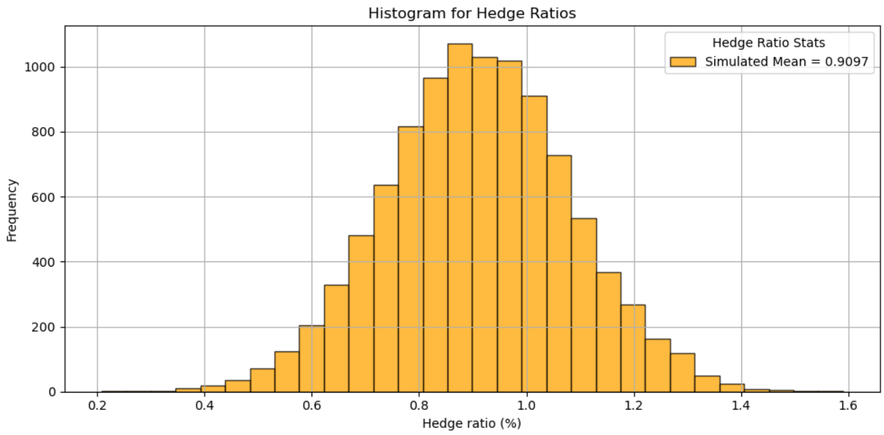


Figure 11: Histogram over hedge ratios for simulated covariance-matrices

As seen in Figure 11, the mean of the optimal hedge ratios is approximately 0.9097. This indicates that, on average, the simulated covariance matrices suggest hedging around 91% of the USD exposure to minimize portfolio risk on a two year basis.

3.3 Evaluate the Simulated Optimal Hedge Ratios, S145085 & S144474

The histogram of the simulated hedge ratios is presented in Figure 11. To evaluate these hedge ratios, key portfolio metrics were calculated for each simulated covariance matrix. These metrics include Expected PnL, Standard Deviation, and 5% Conditional Value at Risk (CVaR). The results are visualized in Figure 12 below:

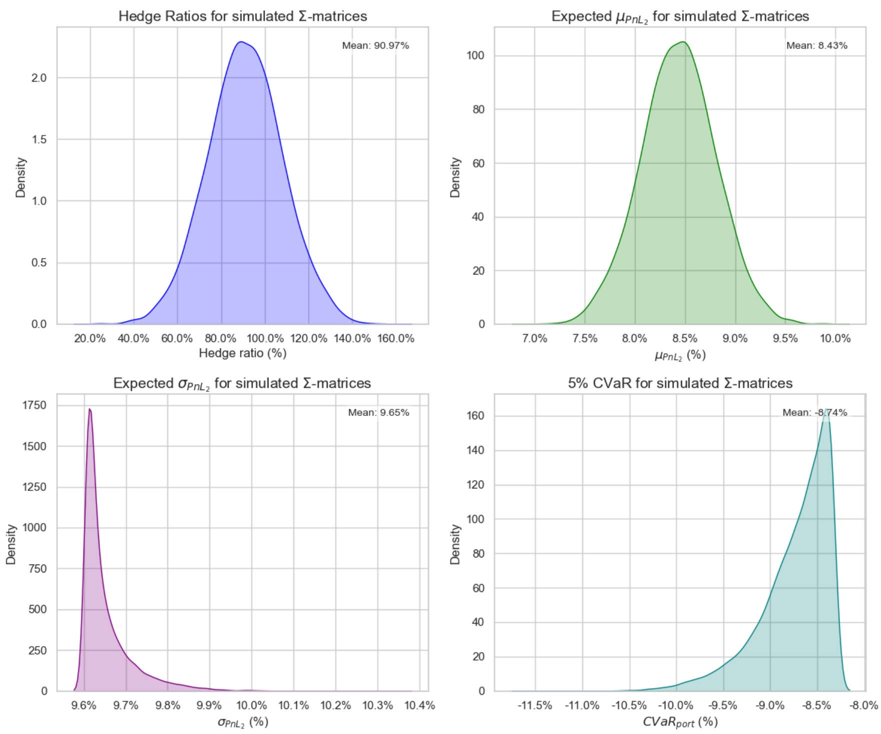


Figure 12: Visualizations of hedge ratio, expected PnL, risk, and 5% CVaR for simulated covariance matrices.

The results emphasize the influence of estimation uncertainty on the portfolio's performance. The simulated hedge ratios display noticeable dispersion, reflecting the variability introduced by the estimation of $\boldsymbol{\Sigma}$. However, the mean values for Expected PnL, Standard Deviation, and 5% CVaR closely align with the true distributional parameters, confirming the robustness of the simulation approach. This alignment suggests that the methodology is reliable for capturing the underlying portfolio dynamics despite the inherent uncertainty in covariance matrix estimation. In addition, the Sharpe Ratio calculation, assuming a risk-free rate of 4%, is presented in Appendix 7.5.

4 Part IV - Implement the three portfolio optimization strategies

In this part, we describe the mathematical framework for constructing a minimum-variance portfolio while accounting for different approaches for hedge ratios. The price vector is a 4-dimensional vector containing the prices of the four assets. By adding a price of 0 as the first element, we obtain a price vector with the correct dimensionality of five. This will be the price vector in the following sections. The mathematical models are structured to determine the lowest possible variance for each level of return. This approach constructs the efficient frontier by identifying portfolios with the minimum achievable variance. In the code, we also include a function to specifically find the minimum-variance portfolio for each strategy.

4.1 1. Mean-Variance Portfolio Optimization with Pre-Specified Hedge Ratios, S144474

Here we set up the mean-variance portfolio optimization with pre-specified hedge ratios. We aim to minimize the portfolio variance by choosing appropriate weights for $\boldsymbol{w_{assets}}$, because w_1 is given from the optimal individually hedge ratios, subject to two constraints:

1. The portfolio weights times prices must sum to 1. (budget constraint)
2. The portfolio weights for all assets, excluding the hedge asset, must be non-negative (no shorting).

Let the covariance matrix of the asset returns be denoted as $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$, where n includes the hedge asset. The expected returns for all assets are given by $\boldsymbol{\mu} \in \mathbb{R}^n$, and the initial market values of the assets are $\boldsymbol{v} \in \mathbb{R}^n$. The target portfolio return is μ_{target}. The weight vector is defined as: $\boldsymbol{w} \in \mathbb{R}^n$ as:

$$\boldsymbol{w} = \begin{bmatrix} w_1 \\ \boldsymbol{w}_{\text{assets}} \end{bmatrix},$$

where w_1 is the weight for the hedge asset, and $\boldsymbol{w}_{\text{assets}} \in \mathbb{R}^{n-1}$ represents the weights of the remaining assets. To account for the hedge asset, we calculate the individual hedge forwards contracts for each asset using:

$$\text{hr}_i = -\frac{\Sigma_{1i}}{\Sigma_{11}} \quad \wedge \quad w_1 = \sum_{i=2}^{n} w_i \cdot \text{hr}_i.$$

where hr_i represents the hedge ratio for asset i and Σ_{1i} represents the covariance between the hedge and asset i, and Σ_{11} is the variance of the hedge. The full mathematical optimization problem is:

$$
\begin{aligned}
\text{Minimize} \quad & \text{Var}(\boldsymbol{w}) = \boldsymbol{w}^\top \Sigma \boldsymbol{w} && \text{Minimizing the total variance} \\
\text{s.t.} \quad & \\
\boldsymbol{w}^\top \boldsymbol{\mu} &= \mu_{\text{target}} && \text{Return target } \mu_{\text{portfolio}} = \mu_{\text{target}} \\
\text{hr}_i &= -\frac{\Sigma_{1i}}{\Sigma_{11}} \quad i = 2, \dots, n && \text{Hedge ratio for asset } i \\
w_1 &= \sum_{i=2}^{n} w_i \cdot \text{hr}_i && \text{Weight for } w_1 \\
\sum_{i=1}^{n} w_i v_i &= 1, && \text{Budget constraint} \\
w_i &\geq 0 \quad i = 2, \dots, n && \text{The portfolio weights must be non-negative (no shorting)}
\end{aligned}
$$

We solve this constrained optimization problem using numerical methods. The result is the optimal weight vector $\boldsymbol{w}$ that minimizes portfolio variance while satisfying the constraints. This ensures getting the entire Efficient Frontier by varrying the μ_{target}. Without the target constraint we will found the minimum varians portfolio.

4.2 2. Initial Mean-Variance Portfolio Optimization with Hedge Ratio Constrained to Zero, S145085

In this part, we describe the process of constructing a mean-variance portfolio in two steps: 1) Optimizing the portfolio with the hedge ratio constrained to zero, and 2) Choosing the hedge ratio as the minimum-variance hedge ratio.

4.2.1 Step 1: Initial Portfolio Optimization (Hedge Ratio Constrained to Zero)

The first step is to perform a mean-variance portfolio optimization while setting the hedge ratio (w_1) to zero. This simplifies the optimization problem, as the hedge asset is excluded. The weight vector $\boldsymbol{w}_{\text{assets}} \in \mathbb{R}^{n-1}$ contains the weights for all non-hedge assets. The objective function is to minimize the portfolio variance defined as: $\text{Var}(\boldsymbol{w}_{\text{assets}}) = \boldsymbol{w}_{\text{assets}}^\top \Sigma_{\text{assets}} \boldsymbol{w}_{\text{assets}}$, where $\Sigma_{\text{assets}} \in \mathbb{R}^{(n-1)\times(n-1)}$ is the covariance matrix for the non-hedge assets. Once again we will have the weight-sum constraint and a non-shorting constraint on the 4 assets. If we would like to find the efficient frontier we need to include the hedge ratio asweell to include in the target-return. The optimization problem for Step 1 is formulated as:

$$
\begin{aligned}
\text{Minimize} \quad & \text{Var}(\boldsymbol{w}_{\text{assets}}) = \boldsymbol{w}_{\text{assets}}^\top \Sigma_{\text{assets}} \boldsymbol{w}_{\text{assets}} && \text{Minimizing the total variance} \\
\text{s.t.} \quad & \\
\boldsymbol{w}^\top \boldsymbol{\mu} &= \mu_{\text{target}} && \text{Return target } \mu_{\text{portfolio}} = \mu_{\text{target}} \\
\sum_{i=1}^{n} w_i v_i &= 1, && \text{Budget constraint} \\
w_i &\geq 0 \quad i = 2, \dots, n && \text{The portfolio weights must be non-negative (no shorting)}
\end{aligned}
$$

The result is the initial optimal weight vector $\boldsymbol{w}_{\text{assets}}$. Afterwards we are finding the optimal number of hedge contracts to lower the risk as much as possible.

4.2.2 Step 2: Adding the Hedge Ratio (Minimum-Variance Hedge Ratio)

In the second step, the hedge ratio is introduced and optimized to minimize the portfolio variance while keeping the asset weights $\boldsymbol{w}_{\text{assets}}$ fixed from Step 1. The minimum-variance hedge number of contrats is defined as:

$$w_1 = -\frac{\sum_{i=2}^{n} w_i \Sigma_{1i}}{\Sigma_{11}}$$

where Σ_{1i} is the covariance between the hedge asset and asset i, and Σ_{11} is the variance of the hedge asset. The total portfolio variance is given by:

$$\text{Var}(\boldsymbol{w}) = \boldsymbol{w}^\top \Sigma \boldsymbol{w} \quad \text{where} \quad \boldsymbol{w} = \begin{bmatrix} w_1 \\ \boldsymbol{w}_{\text{assets}} \end{bmatrix}$$

4.3 3. Full-Scale Mean-Variance Portfolio Optimization with Direct Hedge Ratio Calculation, S144474

In this part, we describe the full-scale mean-variance portfolio optimization, where the hedge weight (w_1) is calculated directly as part of the optimization process. This allows for simultaneous optimization of all portfolio components, including the hedge asset. The goal is to minimize the portfolio variance while ensuring the portfolio meets specific constraints related to the weight-sum and target return. The portfolio variance, considering all assets including the hedge, is given by: $\text{Var}(\boldsymbol{w}) = \boldsymbol{w}^\top \Sigma \boldsymbol{w}$ where $\boldsymbol{w} \in \mathbb{R}^n$ represents the full weight vector, including the hedge weight w_1 and the asset weights $\boldsymbol{w}_{\text{assets}}$. Once again we will have the weight-sum constraint, the non-shorting constraint on the 4 assets, and find the target return by the full portfolio to find the efficient frontier. The full-scale optimization problem can be written as:

$$\begin{aligned}
&\text{Minimize} \quad \text{Var}(\boldsymbol{w}) = \boldsymbol{w}^\top \Sigma \boldsymbol{w} && \text{Minimizing the total variance} \\
&\text{s.t.} && \\
&\boldsymbol{w}^\top \boldsymbol{\mu} = \mu_{\text{target}} && \text{Return target } \mu_{\text{portfolio}} = \mu_{\text{target}} \\
&\sum_{i=1}^{n} w_i v_i = 1, && \text{Budget constraint} \\
&w_i \geq 0 \quad i = 2, \ldots, n && \text{The portfolio weights must be non-negative (no shorting)}
\end{aligned}$$

4.4 Choosing an optimizer and results, S145085

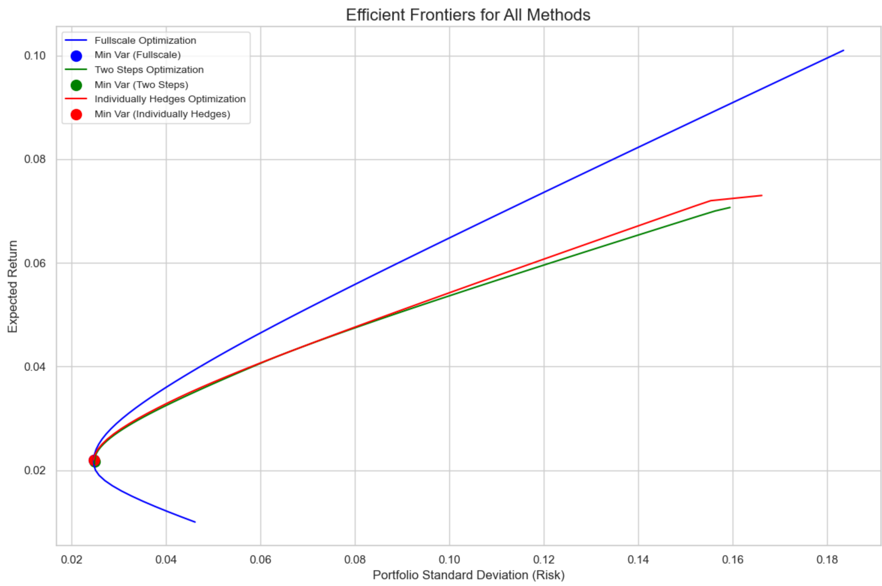
The `trust-constr` method is designed to handle both linear and nonlinear constraints more robustly by using a trust-region framework, which allows for better convergence properties in cases where the problem exhibits small gradients or near-linear behavior. In contrast, `SLSQP` relies on quasi-Newton approximations, which can struggle with stability and accuracy when the gradient changes are minimal. This issue becomes particularly pronounced in portfolio optimization problems with tight constraints, especially when dealing with forward contracts, as the optimization landscape can exhibit sharp transitions that `trust-constr` manages more effectively in this case.

Metric	Two-Step Method	Individually Hedge Method	Full-Scale Method
Portfolio Type	Minimum Variance	Minimum Variance	Minimum Variance
Standard Deviation (%)	2.4862	2.4692	2.4692
Expected Return (%)	2.1631	2.1778	2.1778
Weights	0.0155	0.1426	0.1426
	0.0028	0.0004	0.0002
	0.0072	0.0124	0.0125
	0.0327	0.1984	0.1985
	1.0662	0.9192	0.9191

Table 5: Comparison of Portfolio Optimization Methods

From Table 5, we observe that the Full-Scale and Individually Hedge Methods achieve identical expected returns and standard deviations for the minimum variance portfolio, both outperforming the Two-Step Method. Notably, portfolio weights differ very little. The efficient frontiers in Figure 13 illustrates that the Full-Scale Method consistently dominates at higher levels of return, aligning with its superior optimization flexibility. Conversely, the Two-Step Method underperforms, reflected in its lower curve.

Figure 13: The Efficient Frontiers for all three optimizing strategies



In the Two-Step Method, the minimum variance portfolio is first determined using the assets alone. The hedge ratio is then optimized in a subsequent step. While this approach is the least efficient on the efficient frontier, it can offer greater stability in the presence of estimation risk due to its sequential nature.

The Individually Hedge Method incorporates individual hedge ratios for each asset during the optimization process. Here, the number of forward contracts is directly tied to the portfolio weights, allowing for a more refined solution. As a result, this method produces a slightly better efficient frontier compared to the Two-Step Method. Furthermore, the individual hedge ratio stops earlier because the optimal hedge for the US equities is positive, leading to shorting USD, which reduces the expected return and prevents reaching the US equities' expected return. The final strategy, the Full-Scale Method, simultaneously optimizes the portfolio weights and the hedge ratio. This method delivers the most optimal portfolio configuration, especially at higher return levels, where it outperforms both the Two-Step and Individually Hedge Methods. Its flexibility in choosing optimal hedge ratios enhances its performance. However, this approach is notably sensitive to estimation errors in the covariance matrix and return vector, which can impact its robustness.

5 Part V - Perform a simulation study that examines the effect of estimation uncertainty from estimating $\boldsymbol{\Sigma}$

In part 3 we made a similar simulation study where we considered a two year sample of the market invariants. In this very similar simulation study we will use a one year sample of the markets invariants which is the primary difference, we still have the same key assumptions where the main ones is that $\boldsymbol{\mu}$ is fixed and known (analytically derived for $t = 1$) and the true weekly covariance matrix $\boldsymbol{\Sigma}$ is used to simulate inital market invariants. As a basis for this simulation study we have the true analytical expactation vector and covariance-matrix for $\mathbf{PnL}_1$ for $t = 1$ as given in part 2.1.

5.1 Simulation study: The minimum-variance portfolio for each strategy, S144474 & S145085

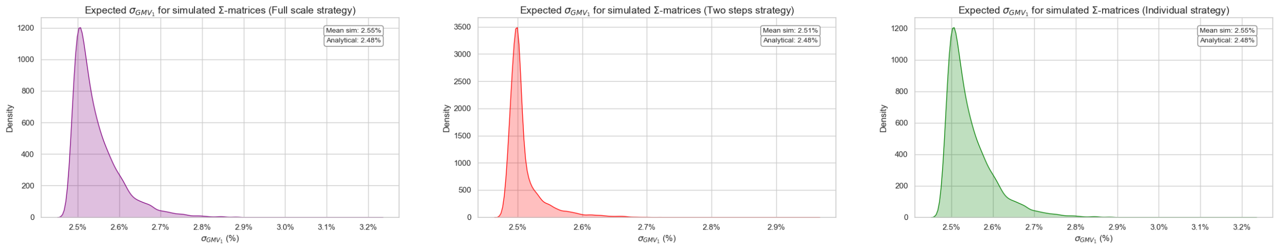
In this analysis, 10.000 minimum variance portfolios are generated for each of the three optimization strategies using 10.000 simulated covariance matrices. For each portfolio, the standard deviation is calculated based on the true analytical covariance matrix to evaluate how the different strategies perform under varying covariance conditions. The results are visualized as distributions of the standard deviation for the optimal portfolios, providing insights into the variation and robustness of each method. The mean of these distributions serves as an indicator of which strategy achieves the lowest average variance and thus best balances risk across different scenarios. The findings are summarized in a table to facilitate the comparison between the strategies.

Optimization Strategy	**Analytical Mean**	**Simulated Mean**
Two-Step Method	2.48%	2.51%
Individually Hedge Method	2.48%	2.55%
Full-Scale Method	2.48%	2.55%

Table 6: Comparison of Analytical and Simulated Means of std for Optimization Strategies

Although previous results indicated that the Full-Scale and Individually Hedge Methods identified the best portfolio for a given covariance matrix, this analysis reveals that they actually perform worse than the Two-Step Method. This discrepancy arises because the Full-Scale and Individually Hedge Methods are likely overfitting to the specific covariance matrix used in the optimization, making them more sensitive to estimation uncertainty. In contrast, the Two-Step Method achieves a more balanced portfolio by avoiding overfitting through the separation of portfolio optimization and hedge ratio determination.

This finding in Table 6 and visualized in Figure 14 is critically important in the context of portfolio optimization, as it highlights the trade-off between achieving precision under idealized conditions and ensuring robustness across varying scenarios. While the Full-Scale and Individually Hedge Methods may perform better when the covariance matrix is known with certainty or perfectly estimated, this situation is nearly unattainable in real-world applications, where estimation errors and uncertainty are inevitable.



(a) Full Scale (b) Two Steps (c) Pre Specified Hedge Ratios

Figure 14: Simulation for the three strategies for the gmv-portfolio

To demonstrate that the other two methods are indeed better at identifying the best portfolios when the covariance matrix used for optimization matches the one used to evaluate the portfolios, see Figure 15. The plot clearly shows that these methods yield portfolios with the lowest variance under such conditions. This highlights their effectiveness in scenarios where the covariance matrix is known with certainty or perfectly estimated, emphasizing their potential advantages in idealized environments.

Figure 15: Risk Distribution for Simulated Portfolios Using Different Optimization Methods

Figure 15 shows the risk distribution (σ_{GMV}) for simulated portfolios across the three optimization methods: Individual Strategy, Two-Step Strategy, and Full-Scale Strategy. The plot reveals that the Two-Step Strategy (orange line) has a slightly higher mean variance (2.44%) compared to the Individual Strategy (2.37%) and Full-Scale Strategy (2.37%). The distribution for the Full-Scale and Individual Strategies appears more concentrated around lower variance values, suggesting better performance when evaluated on the same covariance matrix used for portfolio estimation.

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7 Appendix

7.1 Appendix A: Cubic Splines, Boostrapping, and Nelson-Siegel Yield Curves

Cubic Splines

Cubic splines provide a smooth curve through all known spot rates and maturities. This method ensures continuity in both the yield curve's value and slope, which is essential for accurate pricing and risk management. Figure 16 shows cubic splines applied to EUR and USD rates. The advantage of splines is their ability to ensure continuity in both the value and the slope of the yield curve, which is crucial for pricing and risk management applications.

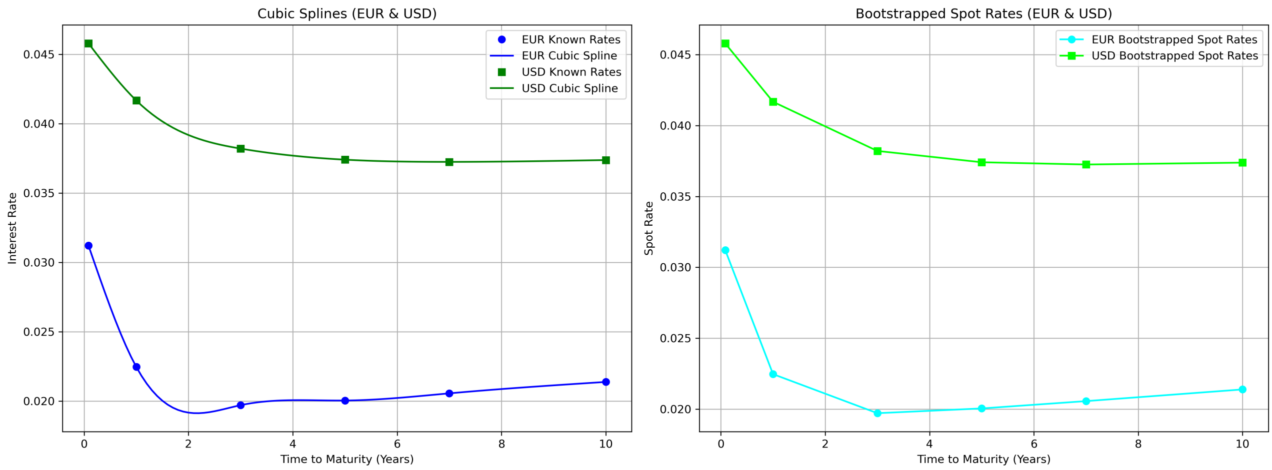

Figure 16: Cubic splines and bootstrapped of yields (EUR and USD)

Bootstrapping and cubic splines

Bootstrapping recalculates discount factors and ensures consistency across maturities. By validating or adjusting the yield curve, bootstrapping aligns spot rates with the given yield vector. Figure 16 (right-hand side) shows bootstrapped spot rates for EUR and USD.

Nelson-Siegel Parameters

The Nelson-Siegel model can also be used to estimate the yield curves for EUR and USD based on observed data. The estimated parameters are as follows:

- **EUR Nelson-Siegel Parameters:**
  - β_0: 0.028317
  - β_1: 0.002809
  - β_2: -0.035510
  - λ: 0.502044
- **USD Nelson-Siegel Parameters:**
  - β_0: 0.038986

– β_1: 0.007159

– β_2: -0.015243

– λ: 0.500682

Plot of the Yield Curves

The plot below illustrates the observed and fitted yield curves for EUR and USD based on the Nelson-Siegel model. The blue points represent the observed EUR yields, while the red points represent the observed USD yields. The smooth lines show the Nelson-Siegel fitted curves for each currency.

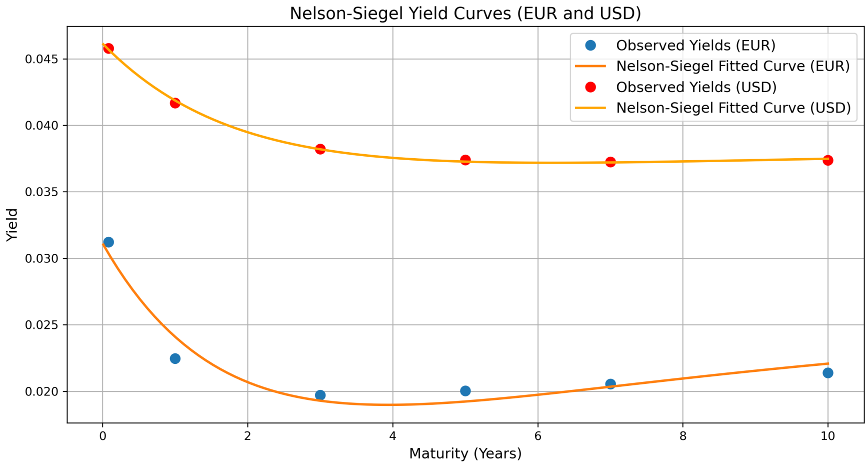

Figure 17: Nelson-Siegel Yield Curves for EUR and USD

The EUR yield curve exhibits a steeper decline for short maturities and a gradual increase for longer maturities, reflecting a relatively stable long-term interest rate environment. The USD yield curve starts higher and flattens out more quickly, consistent with expectations of higher short-term interest rates and inflation in the US market. The parameter λ determines the speed of the transition from short-term to long-term yields and is similar for both EUR and USD, indicating comparable dynamics in the adjustment of yield curves for these currencies.

7.2 Appendix B: Every analytical $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ for $t = 1$

The central elements are the market invariants in this order: $\log FX_t, \log V_t^{\text{US, local}}, \log V_t^{\text{EUR}}, y_t^{\text{USD, 5}}, y_t^{\text{EUR, 5}}$ with the following weekly expectation vector $\boldsymbol{\mu}$, covariance matrix $\boldsymbol{\Sigma}$ and initial values $\boldsymbol{X}_0$ as given in the assignment. For $t = 1$ we have:

$$\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma} \cdot 52 \quad \wedge \quad \boldsymbol{\mu}_1 = \boldsymbol{X}_0 + \boldsymbol{\mu} \cdot 52$$

At $t = 1$ and taking the bond transformation into account we have the following relevant elements:

$$\boldsymbol{P}_{1,\log} = (\log FX_1, \log V_1^{\text{US, local}}, \log V_1^{\text{EUR}}, Z_1^{\text{USD 4Y, local}}, Z_1^{\text{EUR 4Y}})$$

where:

$$\boldsymbol{\Sigma}_{\boldsymbol{P}_{1,\log}} = \boldsymbol{B} \cdot \boldsymbol{\Sigma}_1 \cdot \boldsymbol{B}^\top \quad \wedge \quad \boldsymbol{\mu}_{\boldsymbol{P}_{1,\log}} = \boldsymbol{\mu}_1 \cdot \boldsymbol{B}^\top$$

where

$$\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix}$$

The analytical expectation vector and covariance matrix for $\mathbf{P}_{1,\log}$:

$$\mathbb{E}[\mathbf{P}_{1,\log}] = \begin{pmatrix} 0.0553 \\ 0.0700 \\ 0.0600 \\ -0.1496 \\ -0.0802 \end{pmatrix}, \mathrm{Cov}[\mathbf{P}_{1,\log}] = \begin{pmatrix} 5.87 \cdot 10^{-3} & 1.99 \cdot 10^{-3} & -2.45 \cdot 10^{-4} & -6.56 \cdot 10^{-4} & 8.10 \cdot 10^{-5} \\ 1.99 \cdot 10^{-3} & 2.23 \cdot 10^{-2} & 1.74 \cdot 10^{-2} & -3.47 \cdot 10^{-4} & 6.55 \cdot 10^{-4} \\ -2.45 \cdot 10^{-4} & 1.74 \cdot 10^{-2} & 2.42 \cdot 10^{-2} & -7.17 \cdot 10^{-4} & 3.71 \cdot 10^{-4} \\ -6.56 \cdot 10^{-4} & -3.47 \cdot 10^{-4} & -7.17 \cdot 10^{-4} & 9.73 \cdot 10^{-4} & 5.44 \cdot 10^{-4} \\ 8.10 \cdot 10^{-5} & 6.55 \cdot 10^{-4} & 3.71 \cdot 10^{-4} & 5.44 \cdot 10^{-4} & 6.03 \cdot 10^{-4} \end{pmatrix}$$

We use standard tricks to go from $\mathbf{P}_{1,\log}$ to $\mathbf{P}_1$. The analytical expectation vector and covariance matrix for $\mathbf{P}_1$:

$$\mathbb{E}[\mathbf{P}_1] = \begin{pmatrix} 1.059954 \\ 1.084510 \\ 1.074765 \\ 0.861442 \\ 0.923220 \end{pmatrix}, \mathrm{Cov}[\mathbf{P}_1] = \begin{pmatrix} 6.61 \cdot 10^{-3} & 2.30 \cdot 10^{-3} & -2.79 \cdot 10^{-4} & 5.99 \cdot 10^{-4} & 7.90 \cdot 10^{-5} \\ 2.30 \cdot 10^{-3} & 2.65 \cdot 10^{-2} & 2.05 \cdot 10^{-2} & -3.24 \cdot 10^{-4} & 6.56 \cdot 10^{-4} \\ -2.79 \cdot 10^{-4} & 2.05 \cdot 10^{-2} & 2.83 \cdot 10^{-2} & -6.63 \cdot 10^{-4} & 3.68 \cdot 10^{-4} \\ 5.99 \cdot 10^{-4} & -3.24 \cdot 10^{-4} & -6.63 \cdot 10^{-4} & 7.22 \cdot 10^{-4} & 4.33 \cdot 10^{-4} \\ 7.90 \cdot 10^{-5} & 6.56 \cdot 10^{-4} & 3.68 \cdot 10^{-4} & 4.33 \cdot 10^{-4} & 5.14 \cdot 10^{-4} \end{pmatrix}.$$

At last for $\boldsymbol{P}_1^{\mathrm{EUR}}$ we need the transformation for the domestic currency which only effect element 1, 2, 4:

$$\boldsymbol{\Sigma}_{\boldsymbol{P}_{1,\mathrm{dom}}} = \boldsymbol{W} \cdot \boldsymbol{\Sigma}_{\boldsymbol{P}1,\log} \cdot \boldsymbol{W}^\top \quad \wedge \quad \boldsymbol{\mu}_{\boldsymbol{P}_{1,\mathrm{dom}}} = \boldsymbol{\mu}_{\boldsymbol{P}_{1,\log}} \cdot \boldsymbol{W}^\top$$

where

$$\boldsymbol{W} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The analytical expectation vector and covariance matrix for $\mathbf{P}_1^{\mathrm{EUR}}$:

$$\mathbb{E}[\mathbf{P}_1^{\mathrm{EUR}}] = \begin{pmatrix} 0.948987 \\ 1.027131 \\ 1.074765 \\ 0.816961 \\ 0.923220 \end{pmatrix}, \mathrm{Cov}[\mathbf{P}_1^{\mathrm{EUR}}] = \begin{pmatrix} 5.30 \cdot 10^{-3} & 3.78 \cdot 10^{-3} & 2.50 \cdot 10^{-4} & 4.05 \cdot 10^{-3} & -7.10 \cdot 10^{-5} \\ 3.78 \cdot 10^{-3} & 2.58 \cdot 10^{-2} & 1.97 \cdot 10^{-2} & 2.41 \cdot 10^{-3} & 5.45 \cdot 10^{-4} \\ 2.50 \cdot 10^{-4} & 1.97 \cdot 10^{-2} & 2.83 \cdot 10^{-2} & -4.14 \cdot 10^{-4} & 3.68 \cdot 10^{-4} \\ 4.05 \cdot 10^{-3} & 2.41 \cdot 10^{-3} & -4.14 \cdot 10^{-4} & 3.70 \cdot 10^{-3} & 3.49 \cdot 10^{-4} \\ -7.10 \cdot 10^{-5} & 5.45 \cdot 10^{-4} & 3.68 \cdot 10^{-4} & 3.49 \cdot 10^{-4} & 5.14 \cdot 10^{-4} \end{pmatrix}.$$

The covariance matrix for $\mathbf{PnL}_1$ almost remains the same as for $\mathbf{P}_1^{\mathrm{EUR}}$, except the first row and first column because of $-1/FX_1$, where in the expectation vector we substract the relevant intiatial values (constants), which means the analytical expectation vector and covariance matrix for $\mathbf{PnL}_1$:

$$\mathbb{E}[\mathbf{PnL}_1] = \begin{pmatrix} -0.0208 \\ 0.0809 \\ 0.0748 \\ 0.0322 \\ 0.0186 \end{pmatrix}, \mathrm{Cov}[\mathbf{PnL}_1] = \begin{pmatrix} 5.30 \cdot 10^{-3} & -3.78 \cdot 10^{-3} & -2.50 \cdot 10^{-4} & -4.05 \cdot 10^{-3} & 7.10 \cdot 10^{-5} \\ -3.78 \cdot 10^{-3} & 2.58 \cdot 10^{-2} & 1.97 \cdot 10^{-2} & 2.41 \cdot 10^{-3} & 5.45 \cdot 10^{-4} \\ -2.50 \cdot 10^{-4} & 1.97 \cdot 10^{-2} & 2.83 \cdot 10^{-2} & -4.14 \cdot 10^{-4} & 3.68 \cdot 10^{-4} \\ -4.05 \cdot 10^{-3} & 2.41 \cdot 10^{-3} & -4.14 \cdot 10^{-4} & 3.70 \cdot 10^{-3} & 3.49 \cdot 10^{-4} \\ 7.10 \cdot 10^{-5} & 5.45 \cdot 10^{-4} & 3.68 \cdot 10^{-4} & 3.49 \cdot 10^{-4} & 5.14 \cdot 10^{-4} \end{pmatrix}.$$

7.3 Appendix C: Simulating marginal distributions of $\mathbf{P}_1$ and checking the joint distribution

Through 10.000 simulation we get the estimated simulated expectation vector and covariance-matrix:

$$\mathbb{E}[\mathbf{P}_{1,\text{sim}}] = \begin{bmatrix} 1.0600 \\ 1.0856 \\ 1.0742 \\ 0.8617 \\ 0.9235 \end{bmatrix}, \text{Cov}[\mathbf{P}_{1,\text{sim}}] = \begin{bmatrix} 6.54 \cdot 10^{-3} & 2.12 \cdot 10^{-3} & -3.04 \cdot 10^{-3} & 5.89 \cdot 10^{-4} & 6.53 \cdot 10^{-5} \\ 2.12 \cdot 10^{-3} & 2.66 \cdot 10^{-2} & 2.06 \cdot 10^{-2} & -3.86 \cdot 10^{-4} & 6.58 \cdot 10^{-4} \\ -3.04 \cdot 10^{-3} & 2.06 \cdot 10^{-2} & 2.84 \cdot 10^{-2} & -6.71 \cdot 10^{-4} & 4.02 \cdot 10^{-4} \\ 5.89 \cdot 10^{-4} & -3.86 \cdot 10^{-4} & -6.71 \cdot 10^{-4} & 7.24 \cdot 10^{-4} & 4.26 \cdot 10^{-4} \\ 6.53 \cdot 10^{-5} & 6.58 \cdot 10^{-4} & 4.02 \cdot 10^{-4} & 4.26 \cdot 10^{-4} & 5.09 \cdot 10^{-4} \end{bmatrix}$$

Here each of the marginal distributions are plotted and checked with each marginal distribution:

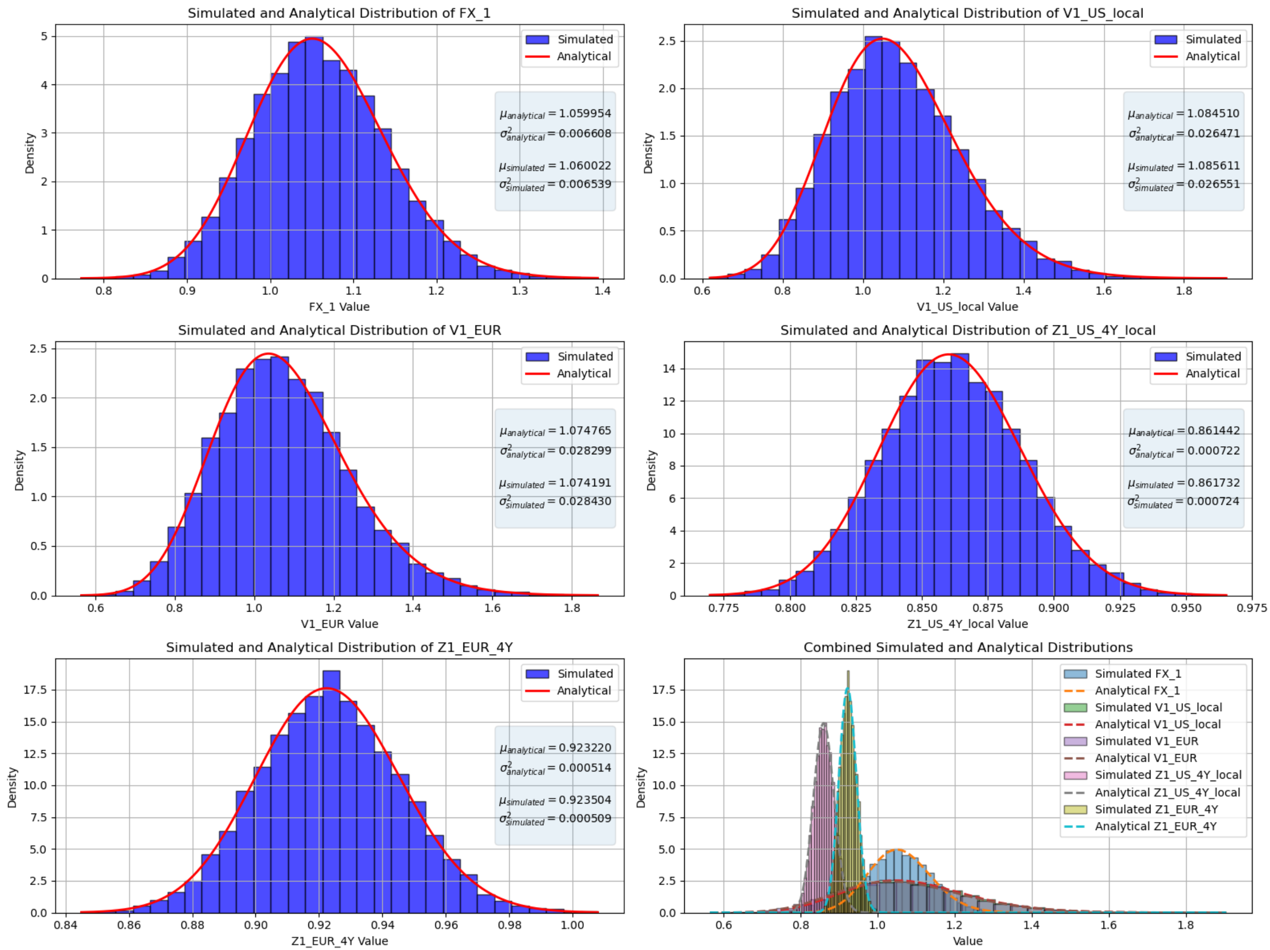

Figure 18: Simulated and Analytical Marginal Distributions of the Components of $\boldsymbol{P}_1$.

7.4 Appendix D: Distribution for V_1^{US}

To highlight the true analytical distribution for V_1^{US} we derive it here as well:

$$V_1^{\text{US}} \sim \text{LogN}(\mu_{V1}, \sigma^2_{V1})$$

where:

$$\mu_{V1} = \mu_{\log V_1^{\text{US, local}}} - \mu_{\log FX_1} = 0.014707214251291248$$

$$\sigma^2_{V1} = \sigma^2_{\log V_1^{\text{US, local}}} - \sigma^2_{\log FX_1} - 2 \cdot \rho \cdot \sigma_{\log V_1^{\text{US, local}}} \cdot \sigma_{\log FX_1} = 0.024125279184768228$$

$$\rho = \frac{\text{Cov}(\log V_1^{\text{US, local}}, \log FX_1)}{\sigma_{\log V_1^{\text{US, local}}} \cdot \sigma_{\log FX_1}} = 0.17487182735590576$$

$$\mu_{\log V_1^{\text{US, local}}} = xo[1] + \mu[1] \cdot 52 = 0.07$$

$$\sigma^2_{\log V_1^{\text{US, local}}} = \Sigma_{1,1} \cdot 52 = 0.022256461169803902$$

$$\mu_{\log FX_1} = xo[0] + \mu[0] \cdot 52 = 0.05529278574870876$$

$$\sigma^2_{\log FX_1} = \Sigma_{0,0} \cdot 52 = 0.005864534599574461$$

$$\text{Cov}(\log V_1^{\text{US, local}}, \log FX_1) = \Sigma_{1,0} \cdot 52 = \Sigma_{0,1} \cdot 52 = 0.0019978582923050672$$

where the true analytical mean and variance are derived with the foillowing formula as we know it:

$$\mathbb{E}[V_1^{\text{US}}] = \exp(\mu_{V1} + \frac{\sigma^2_{V1}}{2}) = 1.02713138521889$$

$$\text{Var}[V_1^{\text{US}}] = (\exp(\sigma^2_{V1}) - 1) \cdot (\exp(\mu_{V1} + \sigma^2_{V1})) = 0.025761646547379086$$

7.5 Appendix E: Visualization of Sharpe Ratio of 2-Year Portfolios with Different Hedge Ratios on Simulations

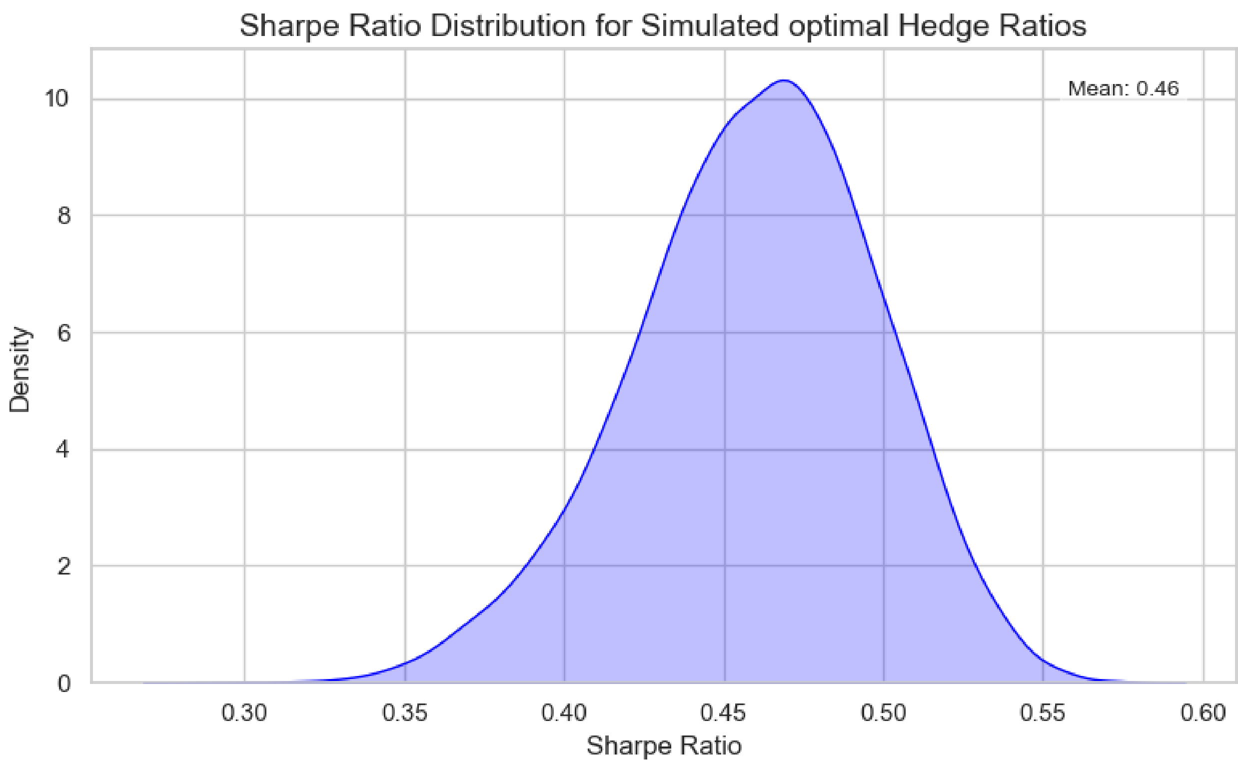

Figure 19: Sharpe Ratio Distribution for Simulated Portfolios with a 2-Year Risk-Free Rate of 4%

The traditional Sharpe Ratio is based on standard deviation as a measure of risk; however, it can also be calculated using Conditional Value at Risk (CVaR). This alternative approach provides a more conservative perspective on risk-adjusted returns, appealing particularly to risk-averse investors.

For this analysis, the risk-free rate is set at 4% over the 2-year period. This assumption adjusts the Sharpe Ratio calculations, reflecting a higher baseline for assessing portfolio performance relative to risk. Such a setting aligns with prevailing economic conditions and provides a realistic benchmark for evaluating optimal hedge ratios.