

Astrophysical Fluid Dynamics

Example sheet 1

1. Determine the equation of a general streamline of the flow $u_\phi = a$, $u_R = b$, $u_z = 0$ in cylindrical polar coordinates, and sketch the flow. Include more than one streamline in your sketches. Repeat for the flow $u_\phi = aR^2$, $u_R = bR^2$, $u_z = 0$. If the flows are steady, and the density at a given radius is independent of ϕ , find the radial dependence of the density in both cases.

Consider what the difference is between the flows? If someone were to present both flows to you, could you tell a difference? What would you notice?

2. Show that for a steady flow with $\nabla \cdot \mathbf{u} = 0$, the density ρ is constant along the streamlines. Need ρ be constant throughout the medium?

Most students answer this properly by giving some sort of argument relating \mathbf{u} and ρ , can you figure out how to formally show that ρ along a streamline $\mathbf{r}(s)$ is constant?

3. If $\mathbf{R} = (x, y, 0)$ and $\hat{\mathbf{R}} = \frac{\mathbf{R}}{|\mathbf{R}|}$ and a flow velocity is given for $R \geq a$ by

$$\mathbf{u} = U\hat{\mathbf{x}}\left(1 + \frac{a^2}{R^2}\right) - 2U\frac{a^2x}{R^3}\hat{\mathbf{R}}$$

(where $\hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$) show that the streamlines obey

$$U\left(R - \frac{a^2}{R}\right)\sin\phi = \text{constant} \quad \left[\text{where } \phi = \tan^{-1}\frac{y}{x}\right].$$

Sketch the streamlines and explain what the flow represents physically. [Hint: consider which coordinate system is better suited for this problem.]

Think about what the flows would be at large distances away from $R = a$

4. A steady 2D flow is described by $u_x = \frac{2}{x}$, $u_y = 1$. Find and sketch the streamlines. Find also a general expression for the surface density of the flow $P(x, y)$ assuming it can be written as a separable function of x and y . Radioactive nuclei are introduced in a small patch at (x_0, y_0) so as to maintain a fixed concentration there. These nuclei decay such that their number per unit mass is given by $Q = Q_0 e^{-t}$ where t is the time since introduction into the flow. Show that the surface density of radioactive nuclei (i.e. number per unit area) attains a maximum along the radioactive streakline if x_0 is less than a critical value, and determine the coordinates of this maximum.

5. Use the summation convention to prove:

$$(a) \quad \mathbf{b} \times \nabla \times \mathbf{b} \equiv \nabla\left(\frac{1}{2}\mathbf{b} \cdot \mathbf{b}\right) - \mathbf{b} \cdot \nabla \mathbf{b},$$

$$(b) \quad \nabla \times (\nabla a) \equiv \mathbf{0},$$

$$(c) \quad \nabla \times (a\mathbf{b}) \equiv a \nabla \times \mathbf{b} - \mathbf{b} \times \nabla a.$$

Using the above identities and the curl of the momentum equation, show that if $\nabla \times \mathbf{u} = \mathbf{0}$ everywhere at time $t = t_0$, then it remains so provided that the pressure is a function of the density only.

6. A static infinite slab of incompressible self-gravitating fluid of density ρ occupies the region $|z| < a$. Find the gravitational field everywhere and the pressure distribution within the slab. [Hint: check the limits when integrating the pressure gradient.]

If a galactic disk is approximated by a uniform density slab with density $10^{-18} \text{ kg m}^{-3}$ and $a = 10^{18} \text{ m}$, determine the velocity of a star at the midplane if it starts from rest at $z = a$, and the period of its oscillation.

7. A stellar wind behaves as a steady adiabatic spherical outflow of a perfect monatomic gas (so $\gamma = c_p/c_V = 5/3$) from the surface of the star, so at radius a the density is ρ_0 , temperature T_0 , and outflow velocity u_0 . If the fluid motions are dominated by the star's gravitational potential, determine the temperature as a function of the radius from the star centre.

If the flow velocity u_0 at radius a is just the gravitational escape velocity from that point do pressure effects ever become significant?

How do we start this question? What does it mean by 'the fluid motions are dominated by the star's gravitational potential'. Instead of jumping straight to our fluid equations, can we take a more elementary route?

8. A particle is released at rest at radius R_0 from the centre of a body mass M . Compute

- (a) its initial acceleration
- (b) the time it takes to reach the centre of the body

for the two cases

- (i) that the body is a point mass
- (ii) that the body is a uniform sphere radius R_0 .

A cluster consists initially of stars at rest, distributed in a uniform sphere. Find how long it takes a star to reach the centre as a function of its initial radius in the cluster and comment on your results.

Most students answer this using an integral. However, imagine we had given the particle some initial angular momentum wrt the body mass. What would happen here? Could we somehow treat this question as a limit where angular momentum approaches 0?

9. (i) If the Earth's atmosphere can be approximated by a perfect static gas at constant temperature of 300 K subject to a uniform gravitational field, find the variation in number density (molecules per cubic metre) with height above the Earth's surface. If the number density of molecules at the Earth's surface is $3 \times 10^{25} \text{ m}^{-3}$, estimate the height above the Earth where the fluid approximation breaks down, and compare it with the height at which the assumption of constant gravity breaks down.
- (ii) The Earth runs into a cloud (made of hydrogen) which is stationary with respect to the Sun. Estimate the number density the cloud would have to have in order that it seriously disturbed the Earth's atmosphere. [You might take "seriously disturb" to mean that the ram pressure is comparable with the atmospheric pressure.]

[Some data possibly relevant to this question: $T \sim 300$ K, $\mu \sim 30$, $k_B = 1.381 \times 10^{-23}$ J K $^{-1}$, $m_p = 1.673 \times 10^{-27}$ kg, $g = 10$ m s $^{-2}$, $M_\odot = 2 \times 10^{30}$ kg, collisional cross-section $\sigma \sim 2 \times 10^{-19}$ m 2 .]