

## Astrophysical Fluid Dynamics

## Example sheet 4

1. A shocked gas is well described by the adiabatic jump conditions at the shock face, but gradually cools, becoming denser, downstream of the shock. Show that in the case of a strong shock (pre-shock pressure  $p_1 \ll p_2$ , the post-shock pressure) that the ratio of ram pressure to thermal pressure is always  $\leq \frac{1}{2}(\gamma - 1)$  if  $p \propto \rho^\gamma$ . Hence show that, for a monatomic gas which can cool back only to the original (unshocked) temperature, the thermal pressure in the shocked gas varies by no more than 33% as the gas cools.

A crude model for the structure of shocked gas as it cools employs the above result in order to approximate the gas as being at constant thermal pressure, so that the thermal equation may be written in the form

$$c_p \frac{dT}{dt} = -\mathcal{Q}^-,$$

where  $c_p$  is the specific heat at constant pressure,  $T$  the temperature,  $t$  the time, and  $\mathcal{Q}^-$  the cooling rate per unit mass. If  $\mathcal{Q}^- = KT^2$ , where  $K$  is a function of the pressure only, determine  $T(t)$  (where  $T(0) = T_2$ , the temperature just behind the shock). Show that in this model the velocity,  $u$ , in the shocked gas satisfies  $u = u_2 T(t)/T_2$  where  $u_2$  is the velocity just behind the shock. Hence, or otherwise, show that the variation of temperature in the shocked gas with distance,  $x$  from the shock front is given by

$$T = T_2 \exp\left(-\frac{xKT_2}{c_p u_2}\right).$$

(Based on a 1999 examination question.)

2. An incompressible fluid of density  $\rho$  with constant viscosity coefficient  $\eta$  flows along an annular pipe of length  $\ell$  in the region between the inner radius  $R_1$  and the outer radius  $R_2$ . Determine the mass flow rate  $\mathcal{Q}$  through the pipe if the pressure at one end of the pipe is  $p_1$  and the other end it is  $p_2$ .
3. A layer of incompressible fluid of thickness  $h$  is bounded above by a free surface and below by a fixed plane inclined at an angle  $\alpha$  to the horizontal in a uniform gravitational field with gravitational acceleration  $g$ . Show that the flow rate (per unit length perpendicular to the flow) due to gravity is  $\mathcal{Q} = \rho gh^3 \sin \alpha / 3\nu$ , where  $\nu$  is the kinematic viscosity and  $\rho$  the fluid density.
4. Suppose there is a unidirectional flow  $u_x(y, t = 0)$  in an infinite viscous fluid at time  $t = 0$ . Show that the flow remains unidirectional, and evolves with time as

$$u_x(y, t) = \frac{1}{2\sqrt{\pi\nu t}} \int_{-\infty}^{\infty} u_x(y', 0) \exp\left[-\frac{(y-y')^2}{4\nu t}\right] dy'$$

if there is no pressure gradient.

5. Show that for an axisymmetric thin disk where the surface density is  $\Sigma$  and variations in the  $z$  direction can be ignored, the continuity equation and the Navier-Stokes equation with constant viscosity coefficient  $\eta$  yield

$$\frac{\partial}{\partial t} (R^2 \Sigma \Omega) + \frac{1}{R} \frac{\partial}{\partial R} (\Sigma R^3 \Omega u_R) = \frac{\nu \Sigma}{R} \frac{\partial}{\partial R} \left( R^3 \frac{d\Omega}{dR} \right)$$

and

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (R \Sigma u_R) = 0$$

where  $\nu$  is the kinematic viscosity and  $\Omega(R)$  the angular velocity.

6. An infinite homogeneous stationary fluid has uniform density  $\rho_0$  and pressure  $p_0$ , and is permeated by a constant magnetic field  $\mathbf{B}_0 = (B_0, 0, 0)$  in Cartesian coordinates. A small velocity perturbation

$$\mathbf{u} = (0, u_1(y, t), 0)$$

gives rise to a perturbation of the magnetic field of the form

$$\mathbf{B}_0 = (B_0 + B_1(y, t), 0, 0)$$

with  $B_1 \ll B_0$ , and gives rise to adiabatic perturbations to pressure and density such that  $p = p_0 + p_1(y, t)$  and  $\rho = \rho_0 + \rho_1(y, t)$ . Show that to linear order

$$\frac{\partial^2 u_1}{\partial t^2} = (c_{s0}^2 + v_A^2) \frac{\partial^2 u_1}{\partial y^2}$$

where  $c_{s0}$  is the sound speed and  $v_A$  the Alfvén speed in the unperturbed fluid.

What is the physical meaning of this equation?

(From 2001 exam question)

7. A gas is predominantly supported against gravity by magnetic pressure which scales as  $B^2$  for magnetic flux density  $B$ . By comparing the gravitational collapse timescale to the propagation timescale for magnetic disturbances, show that the magnetic Jeans mass scales as  $B^3/\rho^2$ .

Hence show that if a uniform spherical cloud with a frozen in magnetic field contracts homogeneously, the number of magnetic Jeans masses it contains is constant.