Translation

Translation matrices are pretty straight-forward:

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

So that:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

Example - translating point [2, 2] by [1, 2] gives:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Rotation

Rotation matrix anti-clockwise about the origin is:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

So that:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example - rotate [2, 2] 90° AC about the origin:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Multiple Transformations

It's common to want to perform multiple transforms

For example a rotation followed by another rotation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

What about 20 rotations ? That's 20 multiplications

What if we wanted to rotate a whole model ?

Say 100k [x,y] points

That's a whole lot of multiplications !

Combining Transformations

Luckily we can join together multiple transforms

We can combine 2 rotations into a single matrix:

```
T  =  [ cos 90 -sin 90 ] [ cos -45 -sin -45 ]
      [ sin 90  cos 90 ] [ sin -45  cos -45 ]

T  =  [ 0 -1 ] [  0.7  0.7 ]
      [ 1  0 ] [ -0.7  0.7 ]

T  =  [ 0.7  -0.7 ]
      [ 0.7   0.7 ]
```

So we could compact those 20 rotations down

Into a single transformation matrix

Then apply that 1 matrix to each of the 100k points

Problem

Unfortunately, due to the limitations of matrices

We can only do such combining with rotations

If we have any translations (which is likely !)

It throws a spanner in the works :o(

Luckily, there is a solution to this problem...

<u>Homogenous Coordinates</u>

Homogeneous Coordinates

HC matrices have an extra dimension !

So a point in 2D space has 3 elements
And a 2D transformation matrix is 3x3

And (as we shall see later):
A point in 3D space has 4 elements
And 3D transformation matrix is 4x4

Implications of Homogeneous Coordinates

The extra dimension allows us to store

BOTH Rotation AND Translation information

<u>IN A SINGLE MATRIX</u>

(Making composite transformation matrices possible)

Homogeneous Coordinate Examples

2D points are represented as:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad \text{for example} \quad \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

2D translation matrices are represented as:

$$\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for example} \quad \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

2D anti-clockwise rotation matrix is represented as:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for example} \quad \begin{bmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Coordinates Example

A "classic" use of HCs is rotation about centre:

RotateAboutCenter

We need to:

- Translate the shape to the origin
- Perform the rotation
- Translate the shape back to it's original location

IMPORTANT: Order of transforms is RIGHT-to-LEFT !

$$T = \begin{bmatrix} 1 & 0 & 50 \\ 0 & 1 & 50 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -50 \\ 0 & 1 & -50 \\ 0 & 0 & 1 \end{bmatrix}$$

Composite Transformation Matrix

The separate transformations:

$$T = \begin{bmatrix} 1 & 0 & 50 \\ 0 & 1 & 50 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -50 \\ 0 & 1 & -50 \\ 0 & 0 & 1 \end{bmatrix}$$

Combining RIGHT-to-LEFT:

$$T = \begin{bmatrix} 1 & 0 & 50 \\ 0 & 1 & 50 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & (-50\cos\theta+50\sin\theta) \\ \sin\theta & \cos\theta & (-50\sin\theta-50\cos\theta) \\ 0 & 0 & 1 \end{bmatrix}$$

Ending up with a single, composite matrix:

$$T = \begin{bmatrix} \cos\theta & -\sin\theta & (-50\cos\theta+50\sin\theta+50) \\ \sin\theta & \cos\theta & (-50\sin\theta-50\cos\theta+50) \\ 0 & 0 & 1 \end{bmatrix}$$

Applied Example

Taking this composite transformation matrix:

$$T = \begin{bmatrix} \cos\theta & -\sin\theta & (-50\cos\theta+50\sin\theta+50) \\ \sin\theta & \cos\theta & (-50\sin\theta-50\cos\theta+50) \\ 0 & 0 & 1 \end{bmatrix}$$

If angle were currently 45° then matrix would be:

$$T = \begin{bmatrix} 0.7 & -0.7 & 50 \\ 0.7 & 0.7 & -20 \\ 0 & 0 & 1 \end{bmatrix}$$

We then just apply this ONCE to each vertex

To achieve our rotation-about-center

Try it for yourself if you don't believe me !

Final Thoughts on Homogeneous Coords

You don't HAVE to use homogeneous coordinates
You can get by perfectly well without them

It's just that they offer a more efficient
(and more elegant ???)
way of performing multiple transformations

HCs have lots of other special properties...
But these are beyond the scope of this lecture