Advanced Studies In Mathematics Exercise

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Definition 1. *We say a random variable X has a chi-squared distribution with degree of freedom ν if its p.d.f. is given by*

$$\chi^2(x;\nu) = \begin{cases} \frac{x^{\nu/2-1}e^{-x/2}}{2^{\nu/2}\Gamma(\nu/2)}, & x > 0, \\ 0, & otherwise. \end{cases}$$

1. Let $X \sim \mathcal{N}(0,1)$. Show that $Y = X^2$ has a chi-squared distribution with 1 degree of freedom.

2.Let X have the probability distribution

$$f(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & elsewhere. \end{cases}$$

Show that the random variable $Y = -2\ln X$ has a chi-squared distribution with 2 degrees of freedom.

3. A random variable X has the Poisson distribution $p(x;\mu) = e^{-\mu}\frac{\mu^x}{x!}$ for $x = 0,1,2,\ldots$. Show that the moment-generating function of X is

$$M_X(t) = e^{\mu(e^t-1)}.$$

Using $M_X(t)$, find the mean and variance of the Poisson distribution.

4. (Python) Write a Python program that calculates the sample mean and sample variance of given random samples. Use this program to test whether these estimates converge to their true values as $n \to \infty$, specifically for uniform and normal distributions.

5. (Python) Write a Python program to verify the results of the Central Limit Theorem. (Hint: The central limit theorem is about the **distribution** of the sample mean $\bar{X}_n$.)