

Advanced Studies In Mathematics Exercise

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1. Let X and Y be random variables with joint density function

$$f(x, y) = \begin{cases} 4xy, & 0 < x, y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of $Z = \sqrt{X^2 + Y^2}$.

2. The length of time, in minutes, for an airplane to obtain clearance for takeoff at a certain airport is given by a random variable $Y = 3X - 2$, where X has the density function

$$f(x) = \begin{cases} \frac{1}{4}e^{-x/4}, & x > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the mean and variance of the random variable Y .

3. Given a random variable X , with standard deviation σ_x , and a random variable $Y = a + bX$, show that if $b < 0$, the correlation coefficient $\rho_{XY} = -1$, and if $b > 0$, $\rho_{XY} = 1$.

4. Random variables X and Y follow a joint distribution

$$f(x, y) = \begin{cases} 2, & 0 < x \leq y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the correlation coefficient between X and Y .

5. If a random variable X is defined such that

$$\mathbb{E}[(X - 1)^2] = 10 \text{ and } \mathbb{E}[(X - 2)^2] = 6,$$

find μ and σ^2 .

6.

- (a) (Markov's Inequality) Show that if X is a nonnegative random variable and $a > 0$, then

$$P(X \geq a) \leq \frac{\mathbb{E}(X)}{a}.$$

- (b) (Chebyshev's Inequality) Show that

$$P(|X - \mathbb{E}(X)| \geq a) \leq \frac{\text{Var}(X)}{a^2}.$$