Advanced Studies In Mathematics Exercise

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1. Prove or disprove the following statements for convex functions f and g.

(a) $h(x) = f(x) + g(x)$ is convex.

(b) $h(x) = \max f(x), g(x)$ is convex.

(c) $h(x) = f(x)g(x)$ is convex.

(d) $h(x) = f(g(x))$ is convex.

2. (Strong convexity implies strict convexity)

(a) Let $f : \mathbb{R}^n \to \mathbb{R}$. Prove that if f is strongly convex then strictly convex.

(b) Let $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$. Prove that if f is strongly convex then strictly convex.

(c) Let $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$. Prove that if f is strongly convex then strictly convex.

3. Show that a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if $\forall x, y \in \mathbb{R}$,

$$\int_0^1 f(x + \theta(y - x))d\theta \leq \frac{f(x) + f(y)}{2}.$$

4. Consider $f(x) = \frac{x^2}{2}$, for $x \in \mathbb{R}$. Show that the gradient descent

$$x_{k+1} = x_k - \eta f'(x_k)$$

with $x_0 \neq 0$ diverges if $\eta > 2$.

5. Consider $f(x) = \frac{(x-x_*)^2}{2}$ and $g(x) = \frac{(x-x_*)^2}{2} + \frac{\mu}{2}x^2$. Compare the convergence rates of gradient descent for f and g.

6. (Python, from Ernest K. Ryu's page) Consider the optimization problem

$$\underset{x \in \mathbb{R}}{\text{minimize}}\, f(x)$$

where

$$f(x) = \frac{10x^2 + e^{3(x-3)}(\frac{(x-10)^2}{2} + 50)}{1 + e^{3(x-3)}}.$$

Implement gradient descent and run it with random starting points within the range $[-5, 20]$. Experimentally demonstrate that gradient descent with learning rate $\eta = 0.01$ converges to either of the two minima, with $\eta = 0.3$ converges to the wide minimum, and with $\eta = 4$ does not converge for most starting points.