Advanced Studies In Mathematics Exercise

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1. Consider the quantity

$$MSE = \mathbb{E}[(\hat{\theta} - \theta)^2],$$

where $\hat{\theta}$ is an estimator of θ, and MSE denotes mean squared error. Show that

$$MSE = \mathbb{E}[(\hat{\theta} - \mathbb{E}(\hat{\theta}))^2] + \mathbb{E}[(\hat{\theta} - \theta)^2].$$

2. Let $X_1, X_2, \ldots, X_n$ be *i.i.d.* random variables. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ denotes the sample mean, and let $K^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Show that K^2 is a biased estimator for σ^2.

3. Let $X_1, X_2, \ldots, X_n$ be *i.i.d.* random variables, and let $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Compute the variance of S^2.

4. (Order Statistics)
(a) Let $X_1, X_2, \ldots, X_n$ be *i.i.d.* random variables with p.d.f. $f(x)$ and c.d.f. $F(x)$. Define $Y = \max(X_1, X_2, \ldots, X_n$. What is the distribution of Y?
(b) Define $Z = \min(X_1, X_2, \ldots, X_n$. What is the distribution of Z?

5. Consider a random sample of $x_1, x_2, \ldots, x_n$ from a uniform distribution $U(0, \theta)$ with unknown parameter θ, where $\theta > 0$. Determine the maximum likelihood estimator of θ.

6. (a) Suppose random samples are given as $(0, 0, 1, 1, 0)$ from a binomial distribution $b(1, \theta)$ where θ is unknown. Assume that $\theta \in (0, 1)$. What is the maximum likelihood estimator for θ?
(b) Suppose we impose the restriction that $\theta \in 0.2, 0.5, 0.7$. What is the maximum likelihood estimator for θ?
(c) Assume $\theta \in 0.2, 0.5, 0.7$ and we have a prior distribution $\pi_\theta(0.2) = 0.1, \pi_\theta(0.5) = 0.01, \pi_\theta(0.7) = 0.89$. What is the maximum a posteriori estimator for θ?