Advanced Studies In Mathematics Exercise

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October 15, 2024

1. Prove that

$$P(A^c \cap B^c) = 1 + P(A \cap B) - P(A) - P(B)$$

2. In a certain region of the country it is known from past experience that the probability of selecting an adult over 40 years of age with cancer is 0.05. If the probability of a doctor correctly diagnosing a person with cancer as having the disease is 0.78 and the probability of incorrectly diagnosing a person without cancer as having the disease is 0.06, what is the probability that an adult over 40 years of age is diagnosed as having cancer?

3. Consider the density function

$$f(x) = \begin{cases} k\sqrt{x}, & 0 < x < 1, \\ 0, & elsewhere. \end{cases}$$

(a) Evaluate k.
(b) Find $P(0.3 < X < 0.6)$.

4. Let X and Y denote the lengths of life, in years, of two components in an electronic system. If the joint density function of these variables is

$$f(x,y) = \begin{cases} e^{-(x+y)}, & x > 0, y > 0, \\ 0, & elsewhere, \end{cases}$$

find $P(0 < X < 1 | Y = 2)$.

5. The joint density function of the random variables X and Y is

$$f(x,y) = \begin{cases} 6x, & 0 < x < 1, 0 < y < 1 - x, \\ 0, & elsewhere. \end{cases}$$

(a) Show that X and Y are not independent. (b) Find $P(X > 0.3 | Y = 0.5)$

6. A chemical system that results from a chemical reaction has two important components among others in a blend. The joint distribution describing the proportions X_1 and X_2 of these two components is given by

$$f(x_1,x_2) = \begin{cases} 2, & 0 < x_1 < x_2 < 1, \\ 0, & elsewhere. \end{cases}$$

(a) Give the marginal distribution of X_1, and X_2

(b) What is the probability that component proportions produce the results $X_1 < 0.2$ and $X_2 > 0.5$?

(c) Give the conditional distribution $f(x_1|x_2)$.

7. (Python)

(a) Use Python to approximate the mean and variance of $Y = 2X + 1$ and $Z = X^2$ where $X \sim \mathcal{N}(0, 1)$.

(b) Plot the histograms of Y and Z with 10,000 samples.

(c) Plot the probability density functions of $\mathcal{N}(1, 4)$ and $\chi(1)$ (chi-squared distribution) and compare them with the histograms of Y and Z, respectively.