Advanced Studies In Mathematics Exercise

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December 12, 2024

1. A one-dimensional rod of length L is heated at one end while the other end is kept at a constant temperature. The temperature distribution $u(x,t)$ along the rod is governed by the heat equation:

$$u_t = Du_{xx},$$

where D is the thermal diffusivity. Suppose that we want to use Physics-Informed Neural Networks(PINNs) to solve the heat equation with the collocation points $\{(x_i,t_i)\}_{i=1}^N$ drawn from $UNIF([0,L]\times[0,T])$, $\{(x_j,0)\}_{j=1}^M$ from $[0,L]\times\{0\}$, and $\{(x_k,t_k)\}_{k=1}^L$ from $\{0,L\}\times[0,T]$. Let $u_{NN}(x,t)$ denote the neural network.

(a) Derive a proper loss function for u_{NN} to approximate the solution $u(x,t)$ given the initial condition $u(x,0)=f(x)$ and the boundary conditions $u(0,t)=0, u(L,t)=T$.

(b) Suppose that the thermal diffusivity D is not known in advance, and we want to approximate it with an additional learnable parameter $\hat{D}$. Derive a proper loss function given the initial and boundary conditions, as well as additional observations $\{((x_j,t_j),u_j)\}_{j=1}^M$.