Advanced Studies In Mathematics Exercise

Hwijae Son

November 28, 2024

1. Show that the graph Laplacian $L = D - W$ is a positive semidefinite matrix.

2. Show that for a graph Laplacian L,
$$\min_{x \cdot \vec{1}} \frac{x^T L X}{\|x\|^2} = \lambda_2,$$
where λ_2 is the second smallest eigenvalue and the minimum is achieved when $x = x_2$, where $L x_2 = \lambda_2 x_2$

3. Let $G = (V, E)$ be a graph and assume $x_i \in \{1, -1\}$ is assigned to each node. Prove that

$$\sum_{(i,j) \in E} (x_i - x_j)^2 = x^T L x,$$

where $x = (x_1, x_2, \ldots, x_n)^T$, and L denotes the graph Laplacian matrix.

4. (Python) Implement k-means, the mixture of Gaussian, and the spectral clustering algorithms, and compare them by computing the average silhouette scores.

5. (Python) Implement your own spectral clustering algorithm and compare its results with those presented in the sample code.