

Assignment #1 (February 6, 2025)

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1. Review : Quantum Harmonic Oscillator in the Heisenberg Picture

Consider the Hamiltonian for a unit mass harmonic oscillator with frequency ω

$$H = \frac{1}{2} (\hat{p}^2 + \omega^2 \hat{x}^2) \quad (1)$$

In the Heisenberg picture $\hat{p}(t)$ and $\hat{x}(t)$ are dynamical variables which evolve with time. They obey the equal-time commutation relation

$$[\hat{x}(t), \hat{p}(t)] = i \quad (2)$$

Here and below we set $\hbar = 1$.

- (a) Obtain the Heisenberg evolution equation for $\hat{x}(t)$ and $\hat{p}(t)$.

From Heisenberg equation of motion, we can know that an operator in Heisenberg picture follows below:

$$\frac{d}{dt} \hat{A}(t) = i[H, \hat{A}(t)]$$

So, $\hat{x}(t)$ is :

$$\frac{d}{dt} \hat{x}(t) = i[H, \hat{x}(t)] = i \left(\frac{1}{2} [\hat{p}^2 + \omega^2 \hat{x}^2, \hat{x}] \right) \quad (3)$$

$$= \frac{i}{2} [\hat{p}^2, \hat{x}] \quad (4)$$

$$= \frac{i}{2} (\hat{p}[\hat{p}, \hat{x}] + [\hat{p}, \hat{x}]\hat{p}) \quad (5)$$

$$= \frac{i}{2} \times (-2i\hat{p}) \quad (6)$$

$$= \hat{p}(t) \quad (7)$$

And $\hat{p}(t)$ is:

$$\frac{d}{dt} \hat{p}(t) = i[H, \hat{p}(t)] = i \left(\frac{1}{2} [\hat{p}^2 + \omega^2 \hat{x}^2, \hat{p}] \right) \quad (8)$$

$$= \frac{i}{2} [\omega^2 \hat{x}^2, \hat{p}] \quad (9)$$

$$= \frac{i}{2} \omega^2 (\hat{x}[\hat{x}, \hat{p}] + [\hat{x}, \hat{p}]\hat{x}) \quad (10)$$

$$= \frac{i}{2} \omega^2 \times (2i\hat{x}) \quad (11)$$

$$= -\omega^2 \hat{x}(t) \quad (12)$$

(b) Suppose the initial condition at $t = 0$ are given by

$$\hat{x}(0) = \hat{x}, \quad \hat{p}(0) = \hat{p} \quad (13)$$

find $\hat{x}(t)$ and $\hat{p}(t)$.

From Pb.1.(a), we solve that

$$\frac{d}{dt}\hat{x}(t) = \hat{p}(t) \quad (14)$$

$$\frac{d}{dt}\hat{p}(t) = -\omega^2\hat{x}(t) \quad (15)$$

Combining the two differential equation to solve $\hat{x}(t)$, we can summarize it:

$$\frac{d^2}{dt^2}\hat{x}(t) = -\omega^2\hat{x}(t) \quad (16)$$

Then we can guess a general solution of $\hat{x}(t)$:

$$\hat{x}(t) = Ae^{i\omega t} + Be^{-i\omega t} \quad (17)$$

such that

$$\hat{p}(t) = iA\omega e^{i\omega t} - iB\omega e^{-i\omega t} \quad (18)$$

Given initial condition (Eqn.(??))

$$A + B = \hat{x} \quad (19)$$

$$i\omega(A - B) = \hat{p} \quad (20)$$

Then,

$$A = \frac{1}{2} \left(\hat{x} + \frac{\hat{p}}{i\omega} \right) \quad (21)$$

$$A = \frac{1}{2} \left(\hat{x} - \frac{\hat{p}}{i\omega} \right) \quad (22)$$

So, $\hat{x}(t)$ and $\hat{p}(t)$ are summarized

$$\hat{x}(t) = \frac{1}{2} \left(\left(\hat{x} + \frac{\hat{p}}{i\omega} \right) e^{i\omega t} + \left(\hat{x} - \frac{\hat{p}}{i\omega} \right) e^{-i\omega t} \right) = \hat{x} \cos \omega t + \frac{\hat{p}}{\omega} \sin \omega t \quad (23)$$

$$\hat{p}(t) = \frac{i\omega}{2} \left(\left(\hat{x} + \frac{\hat{p}}{i\omega} \right) e^{i\omega t} - \left(\hat{x} - \frac{\hat{p}}{i\omega} \right) e^{-i\omega t} \right) = -\omega \hat{x} \sin \omega t + \hat{p} \cos \omega t \quad (24)$$

(c) It is convenient to introduce operators $\hat{a}(t)$, $\hat{a}^\dagger(t)$ defined by

$$\hat{x}(t) = \sqrt{\frac{1}{2\omega}} \left(\hat{a}(t) + \hat{a}^\dagger(t) \right), \quad \hat{p}(t) = -i\sqrt{\frac{\omega}{2}} \left(\hat{a}(t) - \hat{a}^\dagger(t) \right) \quad (25)$$

Show that $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ satisfying equal-time commutation relation

$$[\hat{a}(t), \hat{a}^\dagger(t)] = 1 \quad (26)$$

From the definition(Eqn.(??)), we can rewrite that:

$$\hat{a}(t) + \hat{a}^\dagger(t) = \sqrt{2\omega}\hat{x}(t) \quad (27)$$

$$\hat{a}(t) - \hat{a}^\dagger(t) = i\sqrt{\frac{2}{\omega}}\hat{p}(t) \quad (28)$$

then, we can write down $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ in terms of $\hat{x}(t)$ and $\hat{p}(t)$:

$$\hat{a}(t) = \frac{1}{\sqrt{2\omega}} (\omega\hat{x}(t) + i\hat{p}(t)) \quad (29)$$

$$\hat{a}^\dagger(t) = \frac{1}{\sqrt{2\omega}} (\omega\hat{x}(t) - i\hat{p}(t)) \quad (30)$$

Using the commutation relation in Eqn.(??):

$$[\hat{a}(t), \hat{a}^\dagger(t)] = \frac{1}{2\omega} [\omega\hat{x}(t) + i\hat{p}(t), \omega\hat{x}(t) - i\hat{p}(t)] \quad (31)$$

$$= \frac{1}{2\omega} ([\omega\hat{x}(t), -i\hat{p}(t)] + [i\hat{p}(t), \omega\hat{x}(t)]) \quad (32)$$

$$= \frac{1}{2\omega} \times 2\omega \quad (33)$$

$$= 1 \quad (34)$$

(d) Express the Hamiltonian in terms of $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.

Firstly, expand $\hat{x}^2(t)$ and $\hat{p}^2(t)$ in terms of $\hat{a}(t)$ and $\hat{a}^\dagger(t)$:

$$\hat{x}^2(t) = \frac{1}{2\omega} \left(\hat{a}^2(t) + \hat{a}(t)\hat{a}^\dagger(t) + \hat{a}^\dagger(t)\hat{a}(t) + \hat{a}^{\dagger 2}(t) \right) \quad (35)$$

$$\omega^2 \hat{x}^2(t) = \frac{\omega}{2} \left(\hat{a}^2(t) + \hat{a}(t)\hat{a}^\dagger(t) + \hat{a}^\dagger(t)\hat{a}(t) + \hat{a}^{\dagger 2}(t) \right) \quad (36)$$

and

$$\hat{p}^2(t) = -\frac{\omega}{2} \left(\hat{a}^2(t) - \hat{a}(t)\hat{a}^\dagger(t) - \hat{a}^\dagger(t)\hat{a}(t) + \hat{a}^{\dagger 2}(t) \right) \quad (37)$$

Using the commutation relation from Pb. 1.(c) Eqn.(??), we know that

$$[\hat{a}(t), \hat{a}^\dagger(t)] = 1 \Rightarrow \hat{a}(t)\hat{a}^\dagger(t) - \hat{a}^\dagger(t)\hat{a}(t) = 1 \Rightarrow \hat{a}(t)\hat{a}^\dagger(t) = \hat{a}^\dagger(t)\hat{a}(t) + 1 \quad (38)$$

Therefore, the Hamiltonian(Eqn.(??)) can be expressed:

$$H = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{x}^2) = \frac{\omega}{4} \left(2(\hat{a}(t)\hat{a}^\dagger(t) + \hat{a}^\dagger(t)\hat{a}(t)) \right) \quad (39)$$

$$= \frac{\omega}{2} \left(2\hat{a}^\dagger(t)\hat{a}(t) + 1 \right) \quad (40)$$

$$= \omega \left(\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2} \right) \quad (41)$$

(e) Obtain the Heisenberg equations for $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.

Same process as solving Pb. 1(a),

$$\frac{d}{dt}\hat{a}(t) = i[H, \hat{a}(t)] \quad (42)$$

$$\frac{d}{dt}\hat{a}^\dagger(t) = i[H, \hat{a}^\dagger(t)] \quad (43)$$

Then, using expansion of the Hamiltonian H in terms of $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ and the commutation relation from Eqn.(??)

$$\frac{d}{dt}\hat{a}(t) = i \left[\omega \left(\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2} \right), \hat{a}(t) \right] = i\omega[\hat{a}^\dagger(t)\hat{a}(t), \hat{a}(t)] \quad (44)$$

$$\frac{d}{dt}\hat{a}^\dagger(t) = i \left[\omega \left(\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2} \right), \hat{a}^\dagger(t) \right] = i\omega[\hat{a}^\dagger(t)\hat{a}(t), \hat{a}^\dagger(t)] \quad (45)$$

rewriting the commutation,

$$\frac{d}{dt}\hat{a}(t) = i\omega(\hat{a}^\dagger(t)\hat{a}(t)\hat{a}(t) - \hat{a}(t)\hat{a}^\dagger(t)\hat{a}(t)\hat{a}(t)) = i\omega[\hat{a}^\dagger(t), \hat{a}]\hat{a} = -i\omega\hat{a}(t) \quad (46)$$

$$\frac{d}{dt}\hat{a}^\dagger(t) = i\omega(\hat{a}^\dagger(t)\hat{a}(t)\hat{a}^\dagger(t) - \hat{a}^\dagger(t)\hat{a}^\dagger(t)\hat{a}(t)) = i\omega\hat{a}^\dagger(t)[\hat{a}(t), \hat{a}^\dagger(t)] = i\omega\hat{a}^\dagger(t) \quad (47)$$

(f) Suppose the initial condition at $t = 0$ are given by

$$\hat{a}(0) = \hat{a}, \quad \hat{a}^\dagger(0) = \hat{a}^\dagger \quad (48)$$

find $\hat{a}(t)$ and $\hat{a}^\dagger(t)$

Same process as solving Pb. 1(b), using the result of Pb. 1(e)

$$\frac{d}{dt}\hat{a}(t) = -i\omega\hat{a}(t) \Rightarrow \hat{a}(t) = \hat{a}e^{-i\omega t} \quad (49)$$

$$\frac{d}{dt}\hat{a}^\dagger(t) = -i\omega\hat{a}^\dagger(t) \Rightarrow \hat{a}^\dagger(t) = \hat{a}^\dagger e^{i\omega t} \quad (50)$$

(g) Express $\hat{x}(t), \hat{p}(t)$ and the Hamiltonian H in terms of \hat{a} and \hat{a}^\dagger .

We can use the definition of $\hat{x}(t), \hat{p}(t)$ (Eqn. (??)) in Prob. 1.(c) and replace the result of $\hat{a}(t), \hat{a}^\dagger(t)$ into terms of \hat{a} and \hat{a}^\dagger . Therefore,

$$\hat{x}(t) = \sqrt{\frac{1}{2\omega}} \left(\hat{a}(t) + \hat{a}^\dagger(t) \right) = \sqrt{\frac{1}{2\omega}} \left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t} \right) \quad (51)$$

$$\hat{p}(t) = -i\sqrt{\frac{\omega}{2}} \left(\hat{a}(t) - \hat{a}^\dagger(t) \right) = -i\sqrt{\frac{\omega}{2}} \left(\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t} \right) \quad (52)$$

And also using the result of Prob.1(d)

$$H = \omega \left(\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2} \right) = \omega \left(\hat{a}^\dagger e^{i\omega t} \hat{a} e^{-i\omega t} + \frac{1}{2} \right) \quad (53)$$

$$= \omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \quad (54)$$

2. Review : Lorentz Transformation

(a) Prove that the four-dimensional δ -function

$$\delta^{(4)}(p) = \delta(p^0)\delta(p^1)\delta(p^2)\delta(p^3) \quad (55)$$

is Lorentz invariant, i.e

$$\delta^{(4)}(p) = \delta^{(4)}(\tilde{p}) \quad (56)$$

where \tilde{p}^μ is a Lorentz transformation of p .

Let Λ_ν^μ be a Lorentz transformation, then

$$\tilde{p}^\mu = \Lambda_\nu^\mu p^\nu, \quad \tilde{x}^\mu = \Lambda_\nu^\mu x^\nu \quad (57)$$

And then we have to keep in mind that $\Lambda x \cdot \Lambda p = x \cdot p$ is Lorentz invariant. Then, we will make a Fourier transformation to the δ -function

$$\delta^{(4)}(p) = \frac{1}{(2\pi)^4} \int d^4x e^{ix \cdot p} = \frac{1}{(2\pi)^4} \int d^4x e^{i\Lambda x \cdot \Lambda p} \quad (58)$$

Since the property of the Lorentz transformation that:

$$\Lambda^T \eta \Lambda = \eta \Rightarrow \det \Lambda^T \det \Lambda = (\det \Lambda)^2 = 1 \quad (59)$$

$$\Rightarrow J = |\det \Lambda| = 1 \quad (60)$$

So,

$$d^4\tilde{x} = d^4x \quad (61)$$

That makes the δ -function,

$$\frac{1}{(2\pi)^4} \int d^4x e^{i\Lambda x \cdot \Lambda p} = \frac{1}{(2\pi)^4} \int d^4\tilde{x} e^{i\tilde{x} \cdot \tilde{p}} \quad (62)$$

which is the δ -function of \tilde{p} .

(b) Show that

$$\omega_1 \delta^{(3)}(\vec{k}_1 - \vec{k}_2) \quad (63)$$

is Lorentz invariant, i.e

$$\omega_1 \delta^{(3)}(\vec{k}_1 - \vec{k}_2) = \omega'_1 \delta^{(3)}(\vec{k}'_1 - \vec{k}'_2) \quad (64)$$

where \vec{k}_1 and \vec{k}_2 are respectively the spatial part of four-vectors $k_1^\mu = (w_1, \vec{k}_1)$ and $k_2^\mu = (w_2, \vec{k}_2)$ which satisfy the on-shell condition

$$k_1^2 = k_2^2 = -m^2 \quad (65)$$

$k_1'^\mu = (w'_1, \vec{k}'_1)$ and $k_2'^\mu = (w'_2, \vec{k}'_2)$ are related to k_1^μ, k_2^μ by a same Lorentz transformation.

Since for the given condition that k^μ s are in on-shell, it gives us the mass-shell condition

$$\delta(k^2 + m^2) \quad (66)$$

We will use the property of δ -function to show the Lorentz invariance.

$$\delta(f(x)) = \sum_{x_i=0 \text{ s.t. } f(x_i)=0} \frac{1}{|f'(x_i)|} \delta(x - x_i) \quad (67)$$

We modify the mass-shell condition;

$$\delta(k^2 + m^2) = \delta(-k_0^2 + \vec{k}^2 + m^2) \quad (68)$$

$$= \delta(-k_0^2 + w_k^2) \quad (69)$$

$$= \delta((-k_0 + |w_{\vec{k}}|)(k_0 + |w_{\vec{k}}|)) \quad (70)$$

$$= \frac{1}{2|w_{\vec{k}}|} [\delta(k_0 - |w_{\vec{k}}|) + \delta(k_0 + |w_{\vec{k}}|)] \quad (71)$$

where $\omega_{\vec{k}}$ comes from the energy relationship, $\omega^2 = \vec{k}^2 + m^2$. We assume that $\omega_{\vec{k}} > 0$

We can pick out the $k_1^0 = \omega_{\vec{k}_1}$ enforcing $\delta(k_1^2 + m^2)$ by multiplying both sides by $\theta(\omega_{\vec{k}_1})$.

$$\theta(\omega_{\vec{k}_1}) \delta(k_1^2 + m^2) = \frac{1}{2|w_{\vec{k}_1}|} \theta(\omega_{\vec{k}_1}) [\delta(k_1^0 - |w_{\vec{k}_1}|) + \delta(k_1^0 + |w_{\vec{k}_1}|)] \quad (72)$$

$$= \frac{1}{2\omega_{\vec{k}_1}} \theta(\omega_{\vec{k}_1}) \delta(k_1^0 - \omega_{\vec{k}_1}) \quad (73)$$

$$= \frac{1}{2\omega_{\vec{k}_1}} \delta(k_1^0 - \omega_{\vec{k}_1}) \quad (74)$$

Then, we multiply both sides by $2\omega_{\vec{k}_1} \delta^{(3)}(\vec{k}_1 - \vec{k}_2)$

$$2\omega_{\vec{k}_1} \delta^{(3)}(\vec{k}_1 - \vec{k}_2) \theta(\omega_{\vec{k}_1}) \delta(k_1^2 + m^2) = 2\omega_{\vec{k}_1} \delta^{(3)}(\vec{k}_1 - \vec{k}_2) \cdot \frac{1}{2\omega_{\vec{k}_1}} \delta(k_1^0 - \omega_{\vec{k}_1}) \quad (75)$$

$$= \delta^{(3)}(\vec{k}_1 - \vec{k}_2) \delta(k_1^0 - \omega_{\vec{k}_1}) \quad (76)$$

$$= \delta^{(3)}(\vec{k}_1 - \vec{k}_2) \delta(k_1^0 - \omega_{\vec{k}_2}) \quad (77)$$

$$= \delta^{(3)}(\vec{k}_1 - \vec{k}_2) \delta(k_1^0 - k_2^0) \quad (78)$$

In eqn.(77), we use that the $\delta^{(3)}(\vec{k}_1 - \vec{k}_2)$ allows us to replace $\omega_{\vec{k}_1}$ to $\omega_{\vec{k}_2}$. For this step, it is crucial that $\text{sign}(\omega_{\vec{k}_1}) = \text{sign}(\omega_{\vec{k}_2})$, which is true since both are positive.

Finally, from eqn.(78), the right-handed side is Lorentz invariant, since we show that in Prob.2(a). On the left-handed side, we already know that $\delta(k_1^2 + m^2)$ is a Lorentz scalar since k_1^2 is Lorentz invariant, and $\theta(\omega_{\vec{k}_1})$ is Lorentz invariant since the energy of a particle does not change under Lorentz transformation. Therefore, we can say that $\omega_{\vec{k}_1} \delta^{(3)}(\vec{k}_1 - \vec{k}_2)$ is Lorentz invariant to satisfy both sides.

(c) For any function $f(k) = f(k^0, k^1, k^2, k^3)$, prove that

$$\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} f(k), \quad \omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2} \quad (79)$$

is Lorentz invariant in the sense that

$$\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} f(k) = \int \frac{d^3\vec{\tilde{k}}}{(2\pi)^3} \frac{1}{2\omega_{\vec{\tilde{k}}}} f(\tilde{k}) \quad (80)$$

where $\tilde{k}^\mu = \Lambda_\nu^\mu k^\nu$ is a Lorentz transformation of k^μ

Since the momentum is on the mass-shell, $f(k) = f(\omega_{\vec{k}}, \vec{k})$. So, the integral over the spatial part can be written as;

$$\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} f(k) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} f(\omega_{\vec{k}}, \vec{k}) \quad (81)$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} \delta(k^0 - \omega_{\vec{k}}) f(k^0, \vec{k}) \quad (82)$$

$$= \frac{1}{(2\pi)^3} \int d^4k \theta(\omega_{\vec{k}}) \delta(k^2 + m^2) f(k^0, \vec{k}) \quad (83)$$

Then, Lorentz transformation follows the property below:

$$d^4k = d^4\tilde{k}, \quad \theta(\omega_{\vec{k}}) = \theta(\omega_{\vec{\tilde{k}}}), \quad \delta(k^2 + m^2) = \delta(\tilde{k}^2 + m^2)$$

The integral is noted as:

$$\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} f(k) = \frac{1}{(2\pi)^3} \int d^4k \theta(\omega_{\vec{k}}) \delta(k^2 + m^2) f(k^0, \vec{k}) \quad (84)$$

$$= \frac{1}{(2\pi)^3} \int d^4k' \theta(\omega_{\vec{\tilde{k}}}) \delta(k'^2 + m^2) f((\Lambda k')^\mu) \quad (85)$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} f(\Lambda k) \quad (86)$$

$$= \int \frac{d^3\vec{\tilde{k}}}{(2\pi)^3} \frac{1}{2\omega_{\vec{\tilde{k}}}} f(\tilde{k}) \quad (87)$$

So, $\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} f(k)$ is Lorentz invariant.

3. A Complex Scalar field

Consider the field theory of a complex value scalar field $\phi(x)$ with action

$$S = \int d^4x \left[-\partial_\mu \phi^* \partial^\mu \phi - V(|\phi|^2) \right], \quad |\phi|^2 = \phi \phi^* \quad (88)$$

One could either consider the real and imaginary parts of ϕ , or ϕ and ϕ^* as independent dynamical variables. The latter is more convenient in most situations.

- (a) Check action (??) is Lorentz invariant ($\phi(x) \rightarrow \phi'(x') = \phi(x)$) and find the equations of motion.

Lorentz transformation acts as $\phi \rightarrow \phi^*$, such that $\phi'(x) = \phi(\Lambda^{-1}x)$ Then the action transforms like;

$$S \rightarrow S' = \int d^4x \left[-\partial_\mu \phi'^*(x) \partial^\mu \phi'(x) - V(|\phi'(x)|^2) \right] \quad (89)$$

$$= \int d^4x \left[-\partial_\mu \phi^*(\Lambda^{-1}x) \partial^\mu \phi(\Lambda^{-1}x) - V(|\phi(\Lambda^{-1}x)|^2) \right] \quad (90)$$

From the Prob.2(a), we know that $d^4x = d^4x'$. Use the chain rule for modifying the derivative that $\partial_\mu = (\Lambda^{-1})^\nu_\mu \partial'_\nu$.

$$S' = \int d^4x' \left[-(\Lambda^{-1})^\nu_\mu \partial'_\nu \phi^*(x') (\Lambda^{-1})^\mu_\rho \partial'^\rho \phi(x') - V(|\phi(x')|^2) \right] \quad (91)$$

$$= \int d^4x' \left[-(\Lambda^{-1})^\nu_\mu (\Lambda^{-1})^\mu_\rho \partial'_\nu \phi^*(x') \partial'^\rho \phi(x') - V(|\phi(x')|^2) \right] \quad (92)$$

$$= \int d^4x' \left[-(\delta^\nu_\rho \partial'_\nu \phi^*(x') \partial'^\rho \phi(x') - V(|\phi(x')|^2) \right] \quad (93)$$

$$= \int d^4x' \left[-(\partial'_\nu \phi^*(x') \partial'^\nu \phi(x') - V(|\phi(x')|^2) \right] \quad (94)$$

$$= S \quad (95)$$

where we can use the property of the Lorentz transformation:

$$(\Lambda^{-1})^\nu_\mu (\Lambda^{-1})^\mu_\rho = ((\Lambda^{-1})^T)^\nu_\mu (\Lambda^{-1})^\mu_\rho = \Lambda^\nu_\mu (\Lambda^{-1})^\mu_\rho = \delta^\nu_\rho$$

Therefore, the action is Lorentz invariant.

For the equation of motions, we have to calculate Euler-Lagrangian equation for ϕ and ϕ^* independently

$$\begin{cases} \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi} & \text{(i)} \\ \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \frac{\partial \mathcal{L}}{\partial \phi^*} & \text{(ii)} \end{cases} \quad (96)$$

where $\mathcal{L} = -\partial_\mu \phi'^*(x) \partial^\mu \phi'(x) - V(|\phi'(x)|^2)$, for (i);

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial_\mu \frac{\partial}{\partial(\partial_\mu \phi)} [-\partial_\nu \phi^* \partial^\nu \phi - V(|\phi|^2)] \quad (97)$$

$$= \partial_\mu (-\partial_\nu \phi^* \delta_\mu^\nu) \quad (98)$$

$$= -\partial^2 \phi^* \quad (99)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial}{\partial \phi} (-V(|\phi|^2)) \quad (100)$$

$$= -V'(|\phi|^2) \phi^* \quad (101)$$

Then, by Euler-Lagrangian equation

$$\partial^2 \phi^* - V'(|\phi|^2) \phi^* = 0 \quad (102)$$

Same sequence for (ii), we can get

$$\partial^2 \phi - V'(|\phi|^2) \phi = 0 \quad (103)$$

(b) Find the canonical conjugate momenta for ϕ and ϕ^* , and the Hamiltonian H for eqn. (??)

We can expand the 4-derivative ∂_μ to time and space derivative, that makes the Lagrangian \mathcal{L} into :

$$\mathcal{L} = \partial_t \phi^* \partial^t \phi - \vec{\nabla} \phi^* \cdot \vec{\nabla} \phi - V(|\phi|^2) \quad (104)$$

Therefore, the canonical conjugate momenta becomes:

$$\pi := \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial_t \phi^*, \quad \pi^* := \frac{\partial \mathcal{L}}{\partial(\partial_t \phi^*)} = \partial_t \phi \quad (105)$$

So the Hamiltonian is:

$$H = \int d^3x [\pi \cdot \partial_t \phi + \pi^* \cdot \partial_t \phi^* - \mathcal{L}] \quad (106)$$

$$= \int d^3x [2\pi^* \pi - \pi^* \pi + \vec{\nabla} \phi^* \cdot \vec{\nabla} \phi + V(|\phi|^2)] \quad (107)$$

$$= \int d^3x [\pi^* \pi + \vec{\nabla} \phi^* \cdot \vec{\nabla} \phi + V(|\phi|^2)] \quad (108)$$

(c) The action is invariant under transformation

$$\phi \rightarrow e^{i\alpha}\phi, \quad \phi^* \rightarrow e^{-i\alpha}\phi^* \quad (109)$$

for arbitrary constant α . When α is small, i.e. for an infinitesimal transformation, eqn.(??) become

$$\delta\phi = i\alpha\phi, \quad \delta\phi^* = -i\alpha\phi^* \quad (110)$$

Use Noether's theorem to find the corresponding conserved current j^μ and conserved charge Q .

Noether's theorem states that very continuous symmetry of the action of a physical system with conservative forces has a corresponding conservation law. So that conserve current is:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)} \delta \Phi_a - \mathcal{F}^\mu \quad \text{where} \quad \delta \mathcal{L} = \partial_\mu \mathcal{F}^\mu \quad (111)$$

Then, use the infinitesimal transformation the conserved current is

$$j^\mu = -\partial^\mu \phi^* (i\alpha\phi) - \partial^\mu \phi (-i\alpha\phi^*) \quad (112)$$

$$= i\alpha(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) \quad (113)$$

we can remove that proportional constant to simplify the current

$$j^\mu = \phi^* \partial^\mu \phi - \phi \partial^\mu \phi^* \quad (114)$$

Then the corresponding charge is

$$Q = \int d^3x \, j^0 = \int d^3x (\phi^* \partial_t \phi - \phi \partial_t \phi^*) \quad (115)$$

(d) Use the equations of motion of part (a) to verify directly that j^μ is indeed conserved.

Using the equation of motion, the conservation of the current can be checked with calculating the derivative:

$$\partial_\mu j^\mu = \partial_\mu (\phi^* \partial_t \phi - \phi \partial_t \phi^*) \quad (116)$$

$$= \phi^* \partial^2 \phi - \phi \partial^2 \phi^* \quad (117)$$

$$= V'(|\phi|^2) \phi^* \phi - V'(|\phi|^2) \phi^* \phi \quad (118)$$

$$= 0 \quad (119)$$

4. The energy-momentum tensor for the complex scalar field theory

In this problem, we work out the energy-momentum tensor of the complex scalar theory (??).

- (a) Under a spacetime translation

$$x^\mu \rightarrow x'^\mu = x^\mu + a^\mu \quad (120)$$

a scalar field transform as

$$\phi'(x') = \phi(x) \quad (121)$$

Show that the action (??) is invariant under transformation $\phi(x) \rightarrow \phi'(x)$.

Under the transformation, the scalar field satisfies $\phi'(x) = \phi(x - a)$. Then the action transforms:

$$S \rightarrow S' = \int d^4x (-\partial_\mu \phi'^*(x) \partial^\mu \phi'(x) - V(|\phi'(x)|^2)) \quad (122)$$

$$= \int d^4x (-\partial_\mu \phi^*(x - a) \partial^\mu \phi(x - a) - V(|\phi(x - a)|^2)) \quad (123)$$

$$= \int d^4x (-\partial_\mu \phi^*(x) \partial^\mu \phi(x) - V(|\phi(x)|^2)) = S \quad (124)$$

- (b) Write down the transformation of the scalar fields ϕ and ϕ^* for an infinitesimal translation, and use Noether's theorem to find the corresponding conserved currents $T^{\mu\nu}$.

An infinitesimal transformation makes the field:

$$\delta\phi = \phi'(x) - \phi(x) = \phi(x - a) - \phi(x) = -a^\mu \partial_\mu \phi(x) \quad (125)$$

$$\delta\phi^* = \phi'^*(x) - \phi^*(x) = \phi^*(x - a) - \phi^*(x) = -a^\mu \partial_\mu \phi^*(x) \quad (126)$$

Then, the Lagrangian density under translation gives the infinitesimal variance:

$$\delta\mathcal{L} = \mathcal{L}' - \mathcal{L} \quad (127)$$

$$= -\partial_\nu (\phi^*(x) - a^\mu \partial_\mu \phi^*(x)) \partial^\nu (\phi(x) - a^\mu \partial_\mu \phi(x)) \quad (128)$$

$$- V((\phi^*(x) - a^\mu \partial_\mu \phi^*(x))(\phi(x) - a^\mu \partial_\mu \phi(x))) - \mathcal{L} \quad (129)$$

$$= a^\mu (\partial_\nu \partial_\mu \phi^*(x) \partial^\nu \phi(x) + \partial_\nu \phi^* \partial^\nu \partial_\mu \phi(x)) + a^\mu V'(|\phi|^2) (\partial_\mu \phi^* \phi + \phi^* \partial_\mu \phi) + \mathcal{O}(a^\mu a^\nu) \quad (130)$$

$$= -a^\mu \partial_\mu \mathcal{L} \quad (131)$$

$$= a_\mu \partial_\nu (-\eta^{\mu\nu} \mathcal{L}) := a_\mu \partial_\nu \mathcal{F}^{\mu\nu} \quad (132)$$

Translations are parameterize with 4-vector a^μ , and we have a Noether current itself a 4-vector. Therefore, the conserved currents are encoded in the 2-rank tensor, $\mathcal{T}^{\mu\nu}$.

$$T^{\mu\nu} := (j^\mu)^\nu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} (\delta\phi)^\mu + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^*)} (\delta\phi^*)^\mu - \mathcal{F}^{\mu\nu} \quad (133)$$

$$= -\partial^\nu \phi^* (-\partial^\mu \phi) - \partial^\nu \phi (\partial^\mu \phi^*) + \eta^{\mu\nu} \mathcal{L} \quad (134)$$

$$= \partial^\nu \phi^* \partial^\mu \phi + \partial^\nu \phi \partial^\mu \phi^* - \eta^{\mu\nu} [\partial_\rho \phi^* \partial^\rho \phi + V(|\phi|^2)] \quad (135)$$

In the following, we let the first index μ of $T^{\mu\nu}$ pick out the direction of translation a^μ

(c) The conserved charge for a time translation

$$H = \int d^3x T^{00} \quad (136)$$

should be identified with the total energy of the system, while that for a spatial translation

$$P^i = \int d^3x T^{0i} \quad (137)$$

should be identified with the total momentum. Thus, $T^{\mu\nu}$ is referred to as the energy-momentum tensor. Write down the explicit expression for H and P^i . Compare H obtained here with the Hamiltonian of prob.3(b).

The Hamiltonian is:

$$H = \int d^3x T^{00} \quad (138)$$

$$= \int d^3x [2\partial^t\phi^*\partial^t\phi - \eta^{00}(\partial_\mu\phi^*\partial^\mu\phi + V(|\phi|^2))] \quad (139)$$

$$= \int d^3x [2\partial^t\phi^*\partial^t\phi + (-\partial_t\phi^*\partial_t\phi + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + V(|\phi|^2))] \quad (140)$$

$$= \int d^3x [\partial^t\phi^*\partial^t\phi + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + V(|\phi|^2)] \quad (141)$$

And the total momentum is:

$$P^i = \int d^3x T^{0i} \quad (142)$$

$$= \int d^3x [\partial^t\phi^*\partial^i\phi + \partial^i\phi^*\partial^t\phi - \eta^{0i}(\partial_\rho\phi^*\partial^\rho\phi + V(|\phi|^2))] \quad (143)$$

$$= \int d^3x [\partial^t\phi^*\partial^i\phi + \partial^i\phi^*\partial^t\phi] \quad (144)$$

$$= - \int d^3x [\partial_t\phi^*\partial_i\phi + \partial_i\phi^*\partial_t\phi] \quad (145)$$

The expression of Hamiltonian is same as prob.3(b).

(d) Use equations of motion of prob.3(a) to verify directly that $T^{\mu\nu}$ is indeed conserved.

The equations of motion are:

$$\begin{cases} \partial^2\phi^* - V'(|\phi|^2)\phi^* = 0 \\ \partial^2\phi - V'(|\phi|^2)\phi = 0 \end{cases} \quad (146)$$

Recall that μ indicates the direction of translation a^μ . Therefore, conservation from the Noether's theorem means that $\partial_\nu T^{\mu\nu}$. Caring the results from prob.4(c), $T^{\mu\nu}$ is symmetric so that we can contract the derivative with respect to either index.

$$\partial_\mu T^{\mu\nu} = \partial_\mu (\partial^\nu\phi^*\partial^\mu\phi + \partial^\mu\phi^*\partial^\nu\phi - \eta^{\mu\nu} [\partial_\rho\phi^*\partial^\rho\phi + V(|\phi|^2)]) \quad (147)$$

$$= \partial_\mu (\partial^\nu\phi^*\partial^\mu\phi + \partial^\mu\phi^*\partial^\nu\phi) - \partial^\nu (\partial_\rho\phi^*\partial^\rho\phi) - \partial^\nu V(|\phi|^2) \quad (148)$$

$$= \partial^2\phi^*\partial^\nu\phi + \partial^\nu\phi^*\partial^2\phi - \partial^\nu V(|\phi|^2) \quad (149)$$

$$= V'(|\phi|^2)\phi^*\partial^\nu\phi + \partial^\nu\phi^*V'(|\phi|^2)\phi - V'(|\phi|^2)\partial^\nu\phi^*\phi - V'(|\phi|^2)\phi^*\partial^\nu\phi \quad (150)$$

$$= 0 \quad (151)$$