MIT OCW 8.323: Quantum Field Theory I

Assignment #3

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1. Lorentz transformations for operators and states

(a) From the commutation relation of creation and annihilation operators $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = (2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}') \tag{1}$$

and the result of Pb.2(b) of P-set 1 argue that $a_{\vec{k}}$, $a^\dagger_{\vec{k}}$ should transform under a Lorentz transformation as

$$a_{\vec{k}} \to \tilde{a}_{\vec{k}} = \sqrt{\frac{\omega_{\Lambda\vec{k}}}{\omega_{\vec{k}}}}a_{\Lambda\vec{k}} \tag{2}$$

$$a^\dagger_{\vec{k}} \to \tilde{a}^\dagger_{\vec{k}} = \sqrt{\frac{\omega_{\Lambda\vec{k}}}{\omega_{\vec{k}}}}a^\dagger_{\Lambda\vec{k}} \tag{3}$$

Starting with the commutator

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = (2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}') \tag{4}$$

The LHS transforms to $[\tilde{a}_{\vec{k}}, \tilde{a}^\dagger_{\vec{k}'}]$. Furthermore, since $\omega_{\vec{k}}\delta^{(3)}(\vec{k}-\vec{k}')$ is Lorentz-invariant from Pb 2 of P-set 1, the RHS transforms to

$$\frac{\omega_{\Lambda\vec{k}}}{\omega_{\vec{k}}}(2\pi)^3\delta^{(3)}(\Lambda\vec{k}-\Lambda\vec{k}') = \frac{\omega_{\Lambda\vec{k}}}{\omega_{\vec{k}}}[a_{\Lambda\vec{k}}, a_{\Lambda\vec{k}'}] \tag{5}$$

Since the LHS and RHS must be equal,

$$a_{\vec{k}} \to \tilde{a}_{\vec{k}} = \sqrt{\frac{\omega_{\Lambda\vec{k}}}{\omega_{\vec{k}}}}a_{\Lambda\vec{k}} \tag{6}$$

$$a^\dagger_{\vec{k}} \to \tilde{a}^\dagger_{\vec{k}} = \sqrt{\frac{\omega_{\Lambda\vec{k}}}{\omega_{\vec{k}}}}a^\dagger_{\Lambda\vec{k}} \tag{7}$$

(b) Using the expression of $M_{\mu\nu}$ of Pb.4(c) of P-set 2 to compute

$$\frac{1}{2}[\omega_{\mu\nu}M^{\mu\nu}, a_{\vec{k}}], \quad \frac{1}{2}[\omega_{\mu\nu}M^{\mu\nu}, a^\dagger_{\vec{k}}] \tag{8}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are infinitesimal constant. Show that this indeed generates as infinitesimal Lorentz transformation in $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ which are consistent with (2) - (3)

$$M^{\mu\nu} = -\frac{i}{2}\int \frac{d^3k}{(2\pi)^3} k^\mu (a^\dagger_{\vec{k}}(\partial_{k_\nu} a_{\vec{k}}) - (\partial_{k_\nu} a^\dagger_{\vec{k}})a_{\vec{k}}) - (\mu \leftrightarrow \nu), \quad \partial_{k_0} a_{\vec{k}} = 0, \quad k^0 = \omega_{\vec{k}} \tag{9}$$

By antisymmetry,

$$\omega_{\mu\nu}M^{\mu\nu} = -i\omega_{\mu\nu}\int \frac{d^3k}{(2\pi)^3} k^\mu \left(a^\dagger_{\vec{k}}(\partial_{k_\nu} a_{\vec{k}}) - (\partial_{k_\nu} a^\dagger_{\vec{k}})a_{\vec{k}}\right) \tag{10}$$

Computing the commutation,

$$[\omega_{\mu\nu}M^{\mu\nu}, a_{\vec{k}}] = -i\omega_{\mu\nu}\int \frac{d^3k'}{(2\pi)^3} k'^\mu \left([a^\dagger_{\vec{k}'}, a_{\vec{k}}](\partial_{k'_\nu} a_{\vec{k}'}) - [\partial_{k'_\nu} a^\dagger_{\vec{k}'}, a_{\vec{k}}]a_{\vec{k}'}\right)\delta_{\nu\neq 0} \tag{11}$$

$$= i\omega_{\mu\nu}\int \frac{d^3k'}{(2\pi)^3} k'^\mu \left([a_{\vec{k}}, a^\dagger_{\vec{k}'}](\partial_{k'_\nu} a_{\vec{k}'}) - [a_{\vec{k}}, \partial_{k'_\nu} a^\dagger_{\vec{k}'}]a_{\vec{k}'}\right)\delta_{\nu\neq 0} \tag{12}$$

We append $\delta_{\nu\neq 0}$ for convenience because the integrand is zero when $\nu = 0$. Then divide the equation into two parts.

$$\begin{aligned}[\omega_{\mu\nu}M^{\mu\nu}, a_{\vec{k}}] = {} & i\omega_{0i}\int d^3k'\omega_{\vec{k}} \left(\delta^{(3)}(\vec{k}-\vec{k}')(\partial_{k'_i} a_{\vec{k}'}) - (\partial_{k'_i}\delta^{(3)}(\vec{k}-\vec{k}'))a_{\vec{k}'}\right) \\ & + i\omega_{ij}\int d^3k' k'^i \left(\delta^{(3)}(\vec{k}-\vec{k}')(\partial_{k'_j} a_{\vec{k}'}) - (\partial_{k'_j}\delta^{(3)}(\vec{k}-\vec{k}'))a_{\vec{k}'}\right)\end{aligned} \tag{13}$$

The first term is computed:

$$i\omega_{0i}\int d^3k'\omega_{\vec{k}} \left(\delta^{(3)}(\vec{k}-\vec{k}')(\partial_{k'_i} a_{\vec{k}'}) - (\partial_{k'_i}\delta^{(3)}(\vec{k}-\vec{k}'))a_{\vec{k}'}\right) \tag{14}$$

$$= i\omega_{0i}\int d^3k' \left(\omega_{\vec{k}}\delta^{(3)}(\vec{k}-\vec{k}')(\partial_{k'_i} a_{\vec{k}'}) + \partial_{k'_i}(\omega_{\vec{k}} a_{\vec{k}'})\delta^{(3)}(\vec{k}-\vec{k}')\right) \tag{15}$$

$$= i\omega_{0i}(\omega_{\vec{k}}(\partial_{k_i} a_{\vec{k}}) + \partial_{k_i}(\omega_{\vec{k}} a_{\vec{k}}) \tag{16}$$

$$= 2i\omega_{0i}\omega_{\vec{k}}\partial_{k_i} a_{\vec{k}} + i\omega_{0i}\frac{k^i}{\omega_{\vec{k}}}a_{\vec{k}} \tag{17}$$

The second term is computed:

$$i\omega_{ij}\int d^3k' k'^i \left(\delta^{(3)}(\vec{k}-\vec{k}')(\partial_{k'_j} a_{\vec{k}'}) - (\partial_{k'_j}\delta^{(3)}(\vec{k}-\vec{k}'))a_{\vec{k}'}\right) \tag{18}$$

$$= i\omega_{ij}\int d^3k' \left(k'^i\delta^{(3)}(\vec{k}-\vec{k}')(\partial_{k'_j} k^a_{\vec{k}'}) + (\partial_{k'_j} k'^i a_{\vec{k}'})\delta^{(3)}(\vec{k}-\vec{k}')\right) \tag{19}$$

$$= i\omega_{ij}(k^i\partial_{k_j} a_{\vec{k}} + \partial_{k_j}(k^i a_{\vec{k}})) = 2i\omega_{ij}k^i\partial_{k_j} a_{\vec{k}} + i\omega_{ij}\delta^{ij}a_{\vec{k}} \tag{20}$$

$$= 2i\omega_{ij}k^i\partial_{k_j} a_{\vec{k}} \tag{21}$$

Adding this altogether, the commutation becomes

$$[\omega_{\mu\nu}M^{\mu\nu}, a_{\vec{k}}] = 2i\omega_{\mu\nu}k^{\mu}\partial_{k_\nu}a_{\vec{k}} + i\omega_{0i}\frac{k^i}{\omega_{\vec{k}}}a_{\vec{k}} \tag{22}$$

Similarly, for the $a^\dagger_{\vec{k}}$ case,

$$[\omega_{\mu\nu}M^{\mu\nu}, a^\dagger_{\vec{k}}] = 2i\omega_{\mu\nu}k^{\mu}\partial_{k_\nu}a^\dagger_{\vec{k}} + i\omega_{0i}\frac{k^i}{\omega_{\vec{k}}}a^\dagger_{\vec{k}} \tag{23}$$

Finally, we want to show that this generates an infinitesimal Lorentz transformation consistent with Pb.1(a).

$$U_\Lambda a_{\vec{k}} U^\dagger_\Lambda = \sqrt{\frac{\omega_{\Lambda\vec{k}}}{\omega_{\vec{k}}}}a_{\Lambda\vec{k}}, \quad U_\Lambda a^\dagger_{\vec{k}} U^\dagger_\Lambda = \sqrt{\frac{\omega_{\Lambda\vec{k}}}{\omega_{\vec{k}}}}a^\dagger_{\Lambda\vec{k}} \tag{24}$$

Expanding the unitary operator for inifinitesimal transformations

$$U_\Lambda = e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}, \quad (\Lambda k)_i = k_i + \omega_i^{\nu}k^{\nu}, \quad \omega_{\Lambda\vec{k}} = (\Lambda k)_0|_{k^0=\omega_{\vec{k}}} = \omega_{\vec{k}} - \omega_0^{\nu}k_\nu \tag{25}$$

Using the expression with the BCH formula, the Lorentz transformation changes into

$$a_{\vec{k}} + \frac{i}{2}[\omega_{\mu\nu}M^{\mu\nu}, a_{\vec{k}}] = a_{\vec{k}} + \frac{i}{2}(2i\omega_{\mu\nu}k^{\mu}\partial_{k_\nu}a_{\vec{k}} + i\omega_{0i}\frac{k^i}{\omega_{\vec{k}}}a_{\vec{k}}) \tag{26}$$

$$= a_{\vec{k}} - \frac{1}{2\omega_{\vec{k}}}\omega_{0i}k^i a_{\vec{k}} - \omega_{\mu\nu}k^{\mu}\partial_{k_\nu}a_{\vec{k}} \tag{27}$$

$$= \left(1 - \frac{1}{2\omega_{\vec{k}}}\omega_{0i}k^i\right)a_{\vec{k}} - \omega_{\mu\nu}k^{\mu}\partial_{k_\nu}a_{\vec{k}} \tag{28}$$

Subtracting the $a_{\vec{k}}$ from both sides,

$$\frac{i}{2}[\omega_{\mu\nu}M^{\mu\nu}] = \left(-\frac{1}{2\omega_{\vec{k}}}\omega_{0i}k^i - \omega_{\mu\nu}k^{\mu}\partial_{k_\nu}\right)a_{\vec{k}} \tag{29}$$

$$= -\frac{1}{2}\left(2\omega_{\mu\nu}k^{\mu}\partial_{k_\nu} + \omega_{0i}\frac{k^i}{\omega_{\vec{k}}}\right)a_{\vec{k}} \tag{30}$$

$$\tag{31}$$

Therefore, $M^{\mu\nu}$ generates an infinitesimal Lorentz transformation acting on $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$

(c) Now consider unitary operator generating finite Lorentz transformations

$$U_\Lambda = e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} \tag{32}$$

where $\omega_{\mu\nu}$ are finite constants. Show that

$$U_\Lambda|0\rangle = |0\rangle \tag{33}$$

i.e. the vacuum is Lorentz invariant. One can derive equations (2)-(3) by explicitly evaluating $U_\Lambda a_{\vec{k}} U_\Lambda^\dagger$ and $U_\Lambda a_{\vec{k}}^\dagger U_\Lambda^\dagger$.

As $M^{\mu\nu}$ has the annihilation operator on the right side for each term, it is easy to compute how the vacuum state becomes

$$U_\Lambda|0\rangle = \sum_{n=0} \frac{1}{n!}(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})^n|0\rangle = |0\rangle \tag{34}$$

2. Spatially localized states of a scalar particle

In a quantum field theory there is no natural way to define position eigenvector $|\vec{x}\rangle$ as $\vec{x}$ is now simply a label, not an operator. Also there is a fundamental conflict: a perfectly localized state in space is not a Lorentz covariant concept as it picks out a reference frame (we cannot perfectly localize in time at the same time).

In this problem we will make these abstract statements concrete.

Consider states of the form

$$|\vec{r},t\rangle_f \equiv \int d^3\vec{k}\ f(\vec{k})|\vec{k}\rangle e^{-i\vec{k}\cdot\vec{r}+i\omega_{\vec{k}}t} \tag{35}$$

where $f(\vec{k})$ is a function to be determined and $|\vec{k}\rangle$ is the one-particle state $|\vec{k}\rangle = \sqrt{2\omega_{\vec{k}}}a^\dagger_{\vec{k}}|0\rangle$ of momentum $\vec{k}$ and energy $\omega_{\vec{k}}$.

(a) Determine $f(\vec{k})$ by the condition of perfect localization

$$_f\langle\vec{r}_1,t|\vec{r}_2,t\rangle_f = \delta^{(3)}(\vec{r}_1-\vec{r}_2) \tag{36}$$

$$_f\langle\vec{r}_1,t|\vec{r}_2,t\rangle_f = \int d^3\ \vec{k}\ d^3\vec{k}'\ f(\vec{k})f^*(\vec{k}')\langle k'|k\rangle e^{-i(\vec{k}\cdot\vec{r}_1-\vec{k}'\cdot\vec{r}_2)+i(\omega_{\vec{k}}-\omega_{\vec{k}'})t} \tag{37}$$

$$= \int d^3\ \vec{k}\ d^3\vec{k}'\ f(\vec{k})f^*(\vec{k}')e^{-i(\vec{k}\cdot\vec{r}_1-\vec{k}'\cdot\vec{r}_2)\cdot\vec{r}+i(\omega_{\vec{k}}-\omega_{\vec{k}'})t}2\omega_{\vec{k}}(2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}') \tag{38}$$

$$= (2\pi)^3\int d^3\vec{k}\ 2\omega_{\vec{k}}|f(\vec{k})|^2e^{-i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)} \tag{39}$$

To make the equation equal to $\delta^{(3)}(\vec{r}_1-\vec{r}_2) = \int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}$.

$$|f(\vec{k})|^2 = \frac{1}{(2\pi)^6}\frac{1}{2\omega_{\vec{k}}},\quad f(\vec{k}) = \frac{1}{(2\pi)^3\sqrt{2\omega_{\vec{k}}}} \tag{40}$$

where in going from $|f|^2$ to f the phase is not fixed, but we set it to 1 for simplicity. Therefore, using the notation $r^\mu = (t,\vec{r})$

$$|\vec{r},t\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}|k\rangle e^{-ik\cdot r} \tag{41}$$

(b) By acting the unitary operator U_Λ for a Lorentz transformation on the result in part (a) show that $|\vec{r}, t\rangle$ is *not* Lorentz covariant, i.e.

$$U_\Lambda |\vec{r}, t\rangle_f \neq |\Lambda\vec{r}, \Lambda t\rangle_f \tag{42}$$

where $\Lambda\vec{r}$ and Λt denote Lorentz transformation of $\vec{r}, t$.

Suppose that $U_\Lambda |\vec{r}, t\rangle_f = |\Lambda\vec{r}, \Lambda t\rangle_f$.
Then

$${}_f\langle \Lambda\vec{r}_1, \Lambda t | \Lambda\vec{r}_2, \Lambda t\rangle_f = {}_f \langle \vec{r}_1, t | U_\Lambda^\dagger U_\Lambda | \vec{r}_2, t\rangle_f = {}_f \langle \vec{r}_1, t | \vec{r}_2, t\rangle_f \tag{43}$$

By part (a), LHS is $\delta^{(3)}(\Lambda\vec{r}_1 - \Lambda\vec{r}_2)$, while the RHS is $\delta^{(3)}(\vec{r}_1 - \vec{r}_2)$.

$$\delta^{(3)}(\Lambda\vec{r}_1 - \Lambda\vec{r}_2) = \frac{\omega_{\vec{k}}}{\omega_{\Lambda\vec{k}}} \delta^{(3)}(\vec{r}_1 - \vec{r}_2) \tag{44}$$

This gives the contradiction.

(c) With $f(\vec{k})$ given by (a), consider the overlap $C(\vec{r}_1 - \vec{r}_2, t_1 - t_2) = {}_f \langle \vec{r}_2, t_2 | \vec{r}_1, t_1\rangle_f$. Evaluate $C(\vec{r}, t)$ for a spacelike separation: $|\vec{r}| > t$.
Suppose we interpret $|{}_f\langle \vec{r}_2, t_2 | \vec{r}_1, t_1\rangle_f|^2$ as the probability for the particle originally at $\vec{r}_1$ at time t_1 to transition to $\vec{r}_2$ at time t_2. Would the propagation be casual(*i.e.* confined to the forward light-cone)?
Note: *The above parts suggest that it is not sensible to intepret the state determined from part (a) describing a particle localized at $\vec{r}$ at time t.*

Using the result of part (a),

$$C(\vec{r}_1 - \vec{r}_2, t_1 - t_2) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{r}_2 - \vec{r}_1)} e^{-i\omega_{\vec{k}}(t_2 - t_1)} \tag{45}$$

To make the integral into a Lorentz-covariant, rewrite it as the derivative of a Lorentz-invariant integral:

$$C(\vec{r}_1 - \vec{r}_2, t_1 - t_2) = 2i\partial_{t_2 - t_1} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} e^{i\vec{k}\cdot(\vec{r}_2 - \vec{r}_1)} e^{-i\omega_{\vec{k}}(t_2 - t_1)} \tag{46}$$

Since we are interested in spacelike points $(\vec{r}_1, t_1)$, $(\vec{r}_2, t_2)$, there exists a Lorentz transformation that makes the two points simultaneous, $\Lambda t_1 = \Lambda t_2$. The integral can be calculated in spherical coordinates.

$$\begin{aligned}
\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}e^{i\vec{k}\cdot(\vec{r}_2-\vec{r}_1)}e^{-i\omega_{\vec{k}}(t_2-t_1)} &= \int \frac{d^3\Lambda k}{(2\pi)^3}\frac{1}{2\omega_{\Lambda\vec{k}}}e^{i\Lambda\vec{k}\cdot(\Lambda\vec{r}_2-\Lambda\vec{r}_1)}e^{-i\omega_{\Lambda\vec{k}}(\Lambda t_2-\Lambda t_1)} && (47)\\
&= \int \frac{d^3k'}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}'}}e^{i\vec{k}'\cdot(\Lambda\vec{r}_2-\Lambda\vec{r}_1)} && (48)\\
&= \frac{1}{2(2\pi)^3}\int_0^\infty d|\vec{k}|\frac{|\vec{k}|^2}{\sqrt{|\vec{k}|^2+m^2}}\int_0^\pi d\phi\int_0^{2\pi}d\theta\sin\theta e^{ik|\vec{r}_1-\vec{r}_2|\cos\theta} && (49)\\
&= \frac{1}{2(2\pi)^2}\int_0^\infty dk\frac{k^2}{\sqrt{k^2+m^2}}\frac{2\sin(k|\vec{r}_1-\vec{r}_2|)}{k|\vec{r}_1-\vec{r}_2|} && (50)\\
&= \frac{m}{4\pi^2|\vec{r}_1-\vec{r}_2|}K_1(m|\vec{r}_1-\vec{r}_2|) && (51)
\end{aligned}$$

where $K_1(x)$ is the modified Bessel function of the second kind.
Finally, for computing $C(\vec{r}_1-\vec{r}_2, t_1-t_2)$, we need to take the partial derivative term $\partial_{t_2-t_1}$ on the above expression. However, as we choose the frame where $t_1 = t_2$, it is unclear. To restore the time dependence, we use the fact that it is a Lorentz invariant. This means that our integral can only depend on $|r_2 - r_1| = \sqrt{-(t_2-t_1)^2 + (\vec{r}_2-\vec{r}_1)^2}$.

$$C(\vec{r}_1-\vec{r}_2, t_1-t_2) = 2i\partial_{t_2-t_1}\left[\frac{m}{4\pi^2|r_1-r_2|}K_1(m|r_1-r_2|)\right] = \frac{im^2}{2\pi^2}\frac{t_2-t_1}{(r_1-r_2)^2}K_2(m|r_1-r_2|) \quad (52)$$

Interpreting $|_f\langle\vec{r}_2, t_2|\vec{r}_1, t_1\rangle_f|^2$ as a probability, it is

$$|_f\langle\vec{r}_2, t_2|\vec{r}_1, t_1\rangle_f|^2 = \left|\frac{m^2}{2\pi^2}\frac{t_2-t_1}{(r_1-r_2)^2}K_2(m|r_1-r_2|)\right|^2 \quad (53)$$

This is non-zero, meaning that the propagator is non-zero for generic spacelike separated points.

(d) Compare your resulting state $|\vec{r}, t\rangle_f$ in part (a) with the state

$$|x\rangle \equiv \phi(x)|0\rangle \tag{54}$$

where $x = (\vec{x}, t)$. Are the sates $|x\rangle$ perfectly localized? Show that $|x\rangle$ is Lorentz covariant

$$U_\Lambda|x\rangle = |\Lambda x\rangle \tag{55}$$

$$|x\rangle = \phi(x)|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}e^{-ik\cdot x}a^\dagger_{\vec{k}}|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}e^{-ik\cdot x}|\vec{k}\rangle \tag{56}$$

For equal time,

$$\begin{aligned}
\langle x|x'\rangle &= \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}\cdot 2\omega_{\vec{k}'}}e^{ik\cdot x - ik'\cdot x'}\langle k|k'\rangle & (57)\\
&= \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}\cdot 2\omega_{\vec{k}'}}e^{ik\cdot x - ik'\cdot x'}2\omega_{\vec{k}}\delta^{(3)}(\vec{k}-\vec{k}') & (58)\\
&= \int \frac{d^3k}{(2\pi)^3}e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} & (59)\\
&= \frac{m}{2\pi^2|\vec{x}-\vec{x}'|}K_1(m|\vec{x}-\vec{x}'|) & (60)
\end{aligned}$$

So, the state $|x\rangle$ is not perfectly localized.

$$\begin{aligned}
U_\Lambda|x\rangle &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}e^{-ik\cdot x}U_\Lambda|k\rangle & (61)\\
&= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}e^{-ik\cdot x}|\Lambda k\rangle & (62)\\
&= \int \frac{d^3\Lambda k}{(2\pi)^3}\frac{1}{2\omega_{\Lambda\vec{k}}}e^{-i\Lambda k\cdot \Lambda x}|\Lambda k\rangle & (63)\\
&= |\Lambda x\rangle & (64)
\end{aligned}$$

This shows that the state is Lorentz invariance.

(e) Consider a state

$$|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3} h(\vec{k})|k\rangle \tag{65}$$

with the corresponding "wave function" defined as

$$\Psi(x) = \langle 0|\phi(x)|\Psi\rangle \tag{66}$$

Find $h(\vec{k})$ so that at $t = 0$, the single-particle wave function $\Psi(\vec{x})$ corresponding to $|\Psi\rangle$ is a Gaussian wave packet center around $\vec{x}_0$ with width a, and momentum $\vec{p}$.

$$\Psi(x) = \langle 0|\phi(x)|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3} h(k) e^{ik\cdot x} \tag{67}$$

Considering the 3D Fourier transformation,

$$h(k) = (2\sqrt{\pi}a)^{3/2} e^{-\frac{a}{2}(k-p)^2} e^{-ik\cdot x} \tag{68}$$

So,

$$\Psi(x) = \frac{1}{(\sqrt{\pi}a)^{3/2}} e^{\frac{1}{2a^2}(x-x_0)^2} e^{ip\cdot x} \tag{69}$$

3. Normal ordering and smeared field

(a) First show that

$$\langle 0|\phi^2(\vec{x},t)|0\rangle \tag{70}$$

Evaluate the vacuum expectation value

$$\sigma^2 \equiv \langle 0|\phi^2(\vec{x},t)|0\rangle \tag{71}$$

Express σ^2 as an integral over a single variable and show that the integral is divergent.

The expectation value of ϕ is

$$\langle 0|\phi(\vec{x},t)|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(\langle 0|a_k|0\rangle e^{ik\cdot x} + \langle 0|a_k^\dagger|0\rangle e^{-ik\cdot x}\right) = 0 \tag{72}$$

The variance of ϕ has 4 terms with the combination of annihilation and creation operators. The only term that does not vanish is $\langle 0|a_k a_k^\dagger|0\rangle$. Therefore,

$$\begin{aligned}
\sigma^2 &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{k'}}}\langle 0|a_k a_{k'}^\dagger|0\rangle e^{i(k-k')\cdot x} &(73)\\
&= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{k'}}}(2\pi)^3\delta^{(3)}(k-k')e^{i(k-k')\cdot x} &(74)\\
&= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k} &(75)\\
&= \frac{4\pi}{2(2\pi)^3}\int_0^\infty dk\frac{k^2}{\sqrt{k^2+m^2}} = \infty &(76)
\end{aligned}$$

This result tells us that the vacuum is not empty! While the expectation value of ϕ is zero, the fluctuation of ϕ, measured by σ^2, is nonzero, and in fact is infinitely large. This is a reflection of the fact that a QFT has an infinte number of degrees of freedom.

(b) The general philosophy of QFT is to regard operators like $\phi^2(\vec{x},t)$ as "bad" operators. One then introduces "good" operators which do not suffer divergences, a procedure often referred to as renormalization. One way to remove the divergence in part (a) is to introduce normal ordered operators. The rule of normal ordering is that whenever you see products of $a_{\vec{k}}$'s and $a^\dagger_{\vec{k}}$'s, move all the $a_{\vec{k}}$'s to the right of $a^\dagger_{\vec{k}}$'s. We denote the normal ordered version of an operator $\mathcal{O}$ by $:\mathcal{O}:$. For example,

$$: a_{\vec{k}_1} a^\dagger_{\vec{k}_2} a_{\vec{k}_3} a^\dagger_{\vec{k}_4} := a^\dagger_{\vec{k}_2} a^\dagger_{\vec{k}_4} a_{\vec{k}_1} a_{\vec{k}_3} \tag{77}$$

Express normal ordered operator $:\phi^(\vec{x},t):$ in terms of $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$. Show that $\langle 0| :\phi^2(\vec{x},t): |0\rangle = 0$ and

$$\phi^2(\vec{x},t) =: \phi^2(\vec{x},t) : +\sigma^2 \mathbf{1} \tag{78}$$

ϕ^2 is expressed with $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$:

$$\phi^2 = \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\frac{1}{\sqrt{2\omega_{k'}}}(a_{\vec{k}}a_{\vec{k}'}e^{i(k+k')\cdot x}+a^\dagger_{\vec{k}}a^\dagger_{\vec{k}'}e^{-i(k+k')\cdot x}+a_{\vec{k}}a^\dagger_{\vec{k}'}e^{i(k-k')\cdot x}+a^\dagger_{\vec{k}}a_{\vec{k}'}e^{-i(k-k')\cdot x}) \tag{79}$$

By the way, the normal ordered operator is then,

$$:\phi^2 := \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\frac{1}{\sqrt{2\omega_{k'}}}(a_{\vec{k}}a_{\vec{k}'}e^{i(k+k')\cdot x}+a^\dagger_{\vec{k}}a^\dagger_{\vec{k}'}e^{-i(k+k')\cdot x}+a^\dagger_{\vec{k}'}a_{\vec{k}}e^{i(k-k')\cdot x}+a^\dagger_{\vec{k}}a_{\vec{k}'}e^{-i(k-k')\cdot x}) \tag{80}$$

Then acting on the vacuum,

$$\langle 0| :\phi^2(\vec{x},t): |0\rangle = 0 \tag{81}$$

Now considering the difference between the two operators;

$$\phi^2 - :\phi^2: = \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\frac{1}{\sqrt{2\omega_{k'}}}\left[a_{\vec{k}}, a^\dagger_{\vec{k}'}\right] e^{i(k-k')\cdot x} \tag{82}$$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\frac{1}{\sqrt{2\omega_{k'}}}(2\pi)^3\delta^{(3)}(k-k')e^{i(k-k')\cdot x} \tag{83}$$

$$= \sigma^2 \mathbf{1} \tag{84}$$

(c) Another way to introduce "good" operators is to consider the "smeared" field

$$\tilde{\phi}(\vec{x},t) \equiv N \int d^3y\ \phi(\vec{y},t) e^{-\frac{(\vec{x}-\vec{y})^2}{a^2}}, \quad N = \frac{1}{(\sqrt{\pi}a)^3} \tag{85}$$

The definition is motivated from the fact that the divergence in part (a) comes from having two ϕ's at the same spacetime point. In (85) we thus "smear" (or average) ϕ in a region of radius a. N is a normalization factor, chosen so that

$$\lim_{a\to 0} \tilde{\phi}(\vec{x},t) = \phi(\vec{x},t) \tag{86}$$

Show that

$$\langle 0|\tilde{\phi}(\vec{x},t)|0\rangle = 0 \tag{87}$$

Then consider the fluctuation of $\tilde{\phi}$

$$\tilde{\sigma}^2 \equiv \langle 0|\tilde{\phi}^2(\vec{x},t)|0\rangle \tag{88}$$

Express $\tilde{\sigma}^2$ as an integral over a single variable and show that it is finite.

The expectation value of $\phi(\vec{y},t)$ is $\langle 0|\phi(\vec{y},t)|0\rangle = 0$ So, from the definition of $\tilde{\phi}(\vec{x},t)$:

$$\begin{aligned}\langle 0|\tilde{\phi}(\vec{x},t)|0\rangle &= N \int d^3y\ e^{-\frac{(\vec{x}-\vec{y})^2}{a^2}} \langle 0|\phi(\vec{y},t)|0\rangle &(89)\\ &= 0 &(90)\end{aligned}$$

The fluctuation of $\tilde{\phi}$ is

$$\begin{aligned}\tilde{\sigma}^2 &= \langle 0|\tilde{\phi}^2(\vec{x},t)|0\rangle &(91)\\ &= N^2 \int d^3y\ d^3z\ e^{-\frac{(\vec{x}-\vec{y})^2}{a^2}} e^{-\frac{(\vec{x}-\vec{z})^2}{a^2}} \langle 0|\phi(\vec{y},t)\phi(\vec{z},t)|0\rangle &(92)\\ &= N^2 \int d^3y\ d^3z\ \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\frac{1}{\sqrt{2\omega_{k'}}} e^{-\frac{(\vec{x}-\vec{y})^2}{a^2}} e^{-\frac{(\vec{x}-\vec{z})^2}{a^2}} \langle 0|a_{\vec{k}}a^\dagger_{\vec{k}'}|0\rangle e^{ik\cdot y - ik'\cdot z} &(93)\\ &= N^2 \int \frac{d^3k}{(2\pi)^3} d^3y\ d^3z\ \frac{1}{2\omega_k} e^{-\frac{(\vec{x}-\vec{y})^2}{a^2}} e^{-\frac{(\vec{x}-\vec{z})^2}{a^2}} e^{ik\cdot(y-z)} &(94)\\ &= N^2(\sqrt{\pi}a)^6 \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k} e^{-\frac{1}{4}a^2k^2 + ik\cdot x} e^{-\frac{1}{4}a^2k^2 - ik\cdot x} &(95)\\ &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k} e^{-\frac{1}{2}a^2k^2} = \frac{4\pi}{2(2\pi)^3}\int_0^\infty dk \frac{k^2}{\sqrt{k^2+m^2}} e^{-\frac{1}{2}a^2k^2} &(96)\end{aligned}$$

where the integral term can be evaluated in terms of hypergeometric or Bessel functions. Then, the $\tilde{\sigma}^2$ is

$$\tilde{\sigma}^2 = \frac{1}{8a^2\pi^{3/2}} U(\frac{1}{2}, 0, \frac{1}{2}a^2m^2) = \frac{m^2}{16\pi^2} e^{\frac{1}{4}a^2m^2} \left(K_1(\frac{1}{4}a^2m^2) - K_0(\frac{1}{4}a^2m^2) \right) \tag{97}$$

(d) Without evaluating the integral in part (c), show that in the limits of small a and large a, the leading term in $\tilde{\sigma}^2$ may be written as

$$\tilde{\sigma}^2 \approx \alpha a^\gamma \tag{98}$$

and calculate α and γ for each of these two limits. You should discover that at large a the average field approaches a classical variable whereas at small a it is dominated by fluctuations.

Rewritting $\tilde{\sigma}^2$ to make an approximation with a,

$$\tilde{\sigma}^2 = \frac{1}{4\pi^2}\int_0^\infty dk \frac{k^2}{\sqrt{k^2+m^2}} e^{-\frac{1}{2}a^2k^2} = \frac{1}{4\pi^2 a^2}\int_0^\infty du \frac{u^2}{\sqrt{u^2+m^2a^2}} e^{-\frac{1}{2}u^2} \tag{99}$$

where $u = ka$.
For small a, the fraction can be expanded as:

$$\frac{u^2}{\sqrt{u^2+m^2a^2}} = \frac{u^2}{u\left(1+\frac{m^2a^2}{2u^2}+\cdots\right)} \approx u \tag{100}$$

Therefore, the integral

$$\tilde{\sigma}^2 \approx \frac{1}{4\pi^2a^2}\int_0^\infty du\ u e^{-\frac{1}{2}u^2} = \frac{1}{4\pi^2a^2} \tag{101}$$

For large a, the fraction can be expanded as:

$$\frac{u^2}{\sqrt{u^2+m^2a^2}} = \frac{u^2}{ma\left(1+\frac{u^2}{2m^2a^2}+\cdots\right)} \approx \frac{u^2}{m^2a^2} \tag{102}$$

Therefore, the integral

$$\tilde{\sigma}^2 \approx \frac{1}{4\pi^2ma^3}\int_0^\infty du\ u^2 e^{-\frac{1}{2}u^2} = \frac{1}{\sqrt{32\pi^3}ma^3} \tag{103}$$

So for small a, $\tilde{\sigma} \to \infty$, meaning that the field is dominated by fluctuations. On the other hand, for large a, $\tilde{\sigma} \to 0$, meaning that the field behaves like a classical variable.

4. Correlation functions for a complex scalar

Consider a theory of a complex scalar field ϕ which is invariant under a phase rotation of ϕ. The unitary operator U_α generating a phase rotation is written as $U_\alpha = e^{-i\alpha Q}$ where $Q = i\int d^3x[\pi_{\phi^*}\phi^* - \pi_\phi\phi]$. We also assume that the vacuum of the theory is invariant under the phase rotation,

$$U_\alpha|0\rangle = |0\rangle \tag{104}$$

The theory can be interacting. That is, in this problem, you should *not* use the mode expansion for ϕ discussed in lecture which applies only to a free field theory.

(a) Show that

$$U_\alpha\phi(x)U_\alpha^\dagger = e^{i\alpha}\phi(x), \quad U_\alpha\phi^*(x)U_\alpha^\dagger = e^{-i\alpha}\phi^*(x) \tag{105}$$

First, when $Q, \phi(x)$ are evaluate at the same time t, we can use BCH formula

$$\begin{aligned} U_\alpha\phi(x)U_\alpha^\dagger = \phi(x) + \alpha\int d^3x'[-\phi(x')\pi_\phi(x'), \phi(x)] \\ + \frac{\alpha^2}{2!}\int d^3x'd^3x''[-\phi(x'')\pi_\phi(x''), [-\phi(x')\pi_\phi(x'), \phi(x)]] + \cdots \end{aligned} \tag{106}$$

Note that $[-\phi(x')\pi_\phi(x'), \phi(x)] = i\phi(x')\delta^{(3)}(x-x')$ whose $\int \frac{d^3x}{(2\pi)^3}$-integral yields $i\phi(x)$.

$$U_\alpha\phi(x)U_\alpha^\dagger = (1 + i\alpha + \cdots + \frac{(i\alpha)^n}{n!} + \cdots)\phi(x) = e^{i\alpha}\phi(x) \tag{107}$$

When Q and $\phi(x)$ are computed in different time, we can time-evolve $\phi(x)$ to the timeslice t' and use that $[H, Q] = 0$:

$$\begin{aligned} U_\alpha\phi(x,t)U_\alpha^\dagger &= e^{-i\alpha Q(t')}e^{iH(t-t')}\phi(x,t')e^{-iH(t-t')}e^{i\alpha Q(t')} && (108)\\ &= e^{iH(t-t')}e^{-i\alpha Q(t')}\phi(x,t')e^{i\alpha Q(t')}e^{-iH(t-t')} && (109)\\ &= e^{iH(t-t')}e^{i\alpha}\phi(x,t')e^{-iH(t-t')} && (110)\\ &= e^{i\alpha}\phi(x,t) && (111) \end{aligned}$$

It is the same for ϕ^*.

(b) Show that

$$\langle 0|\phi(x)\phi(x')|0\rangle = 0 \tag{112}$$

$$\begin{aligned} \langle 0|\phi(x)\phi(x')|0\rangle &= \langle 0|U_\alpha\phi(x)U_\alpha^\dagger U_\alpha\phi(x')U_\alpha^\dagger|0\rangle && (113)\\ &= e^{2i\alpha}\langle 0|\phi(x)\phi(x')|0\rangle && (114) \end{aligned}$$

If α is not a multiple of 2π, the equality is not satisfied unless $\langle 0|\phi(x)\phi(x')|0\rangle = 0$.

(c) Now consider a general n-point function of products of ϕ's and ϕ^*'s inserted at different spacetime points between the vacuum, i.e.

$$\langle 0|\phi(x_1)\cdots\phi^*(x_i)\cdots\phi_n(x_n)|0\rangle \tag{115}$$

Show that the above quantity vanishes whenever the numbers of ϕ's and ϕ^*'s are not the same.

Consider a general n-point function with M terms of ϕ and N terms of ϕ^* with $M+N$ and $N \neq M$. Similarly to (b),

$$\begin{aligned}\langle 0|\phi(x_1)\cdots\phi^*(x_i)\cdots\phi_n(x_n)|0\rangle &= \langle 0|U_\alpha\phi(x_1)U_\alpha^\dagger\cdots U_\alpha^\dagger U_\alpha\phi^*(x_i)U_\alpha^\dagger U_\alpha\cdots U_\alpha^\dagger U_\alpha\phi_n(x_n)U_\alpha^\dagger|0\rangle \\ &= e^{i\alpha(M-N)}\langle 0|\phi(x_1)\cdots\phi^*(x_i)\cdots\phi_n(x_n)|0\rangle\end{aligned}$$

For an arbitrary α, the equality is not satisfied so that $\langle 0|\phi(x_1)\cdots\phi^*(x_i)\cdots\phi_n(x_n)|0\rangle = 0$