MIT OCW 8.323: Quantum Field Theory I

Assignment #1

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1. Review : Quantum Harmonic Oscillator in the Heisenberg Picture

Consider the Hamiltonian for a unit mass harmonic oscillator with frequency ω

$$H = \frac{1}{2}\left(\hat{p}^2 + \omega^2 \hat{x}^2\right) \tag{1}$$

In the Heisenberg picture $\hat{p}(t)$ and $\hat{x}(t)$ are dynamical variables which evolve with time. They obey the equal-time commutation relation

$$[\hat{x}(t), \hat{p}(t)] = i \tag{2}$$

Here and below we set $\hbar = 1$.

(a) Obtain the Heisenberg evolution equation for $\hat{x}(t)$ and $\hat{p}(t)$.

From Heisenberg equation of motion, we can know that an operator in Heisenberg picture follows below:

$$\frac{d}{dt}\hat{A}(t) = i[H, \hat{A}(t)]$$

So, $\hat{x}(t)$ is :

$$\begin{aligned}
\frac{d}{dt}\hat{x}(t) = i[H, \hat{x}(t)] = i\left(\frac{1}{2}[\hat{p}^2 + \omega^2 x^2, \hat{x}]\right) && (3)\\
= \frac{i}{2}[\hat{p}^2, \hat{x}] && (4)\\
= \frac{i}{2}\left(\hat{p}[\hat{p}, \hat{x}] + [\hat{p}, \hat{x}]\hat{p}\right) && (5)\\
= \frac{i}{2} \times (-2i\hat{p}) && (6)\\
= \hat{p}(t) && (7)
\end{aligned}$$

And $\hat{p}(t)$ is:

$$\begin{aligned}
\frac{d}{dt}\hat{p}(t) = i[H, \hat{p}(t)] = i\left(\frac{1}{2}[\hat{p}^2 + \omega^2 x^2, \hat{p}]\right) && (8)\\
= \frac{i}{2}[\omega^2 \hat{x}^2, \hat{p}] && (9)\\
= \frac{i}{2}\omega^2\left(\hat{x}[\hat{x}, \hat{p}] + [\hat{x}, \hat{p}]\hat{x}\right) && (10)\\
= \frac{i}{2}\omega^2 \times (2i\hat{x}) && (11)\\
= -\omega^2 \hat{x}(t) && (12)
\end{aligned}$$

(b) Suppose the initial condition at $t = 0$ are given by

$$\hat{x}(0) = \hat{x}, \quad \hat{p}(0) = \hat{p} \tag{13}$$

find $\hat{x}(t)$ and $\hat{p}(t)$.

From Pb.1.(a), we solve that

$$\frac{d}{dt}\hat{x}(t) = \hat{p}(t) \tag{14}$$

$$\frac{d}{dt}\hat{p}(t) = -\omega^2\hat{x}(t) \tag{15}$$

Combining the two differential equation to solve $\hat{x}(t)$, we can summarize it:

$$\frac{d^2}{dt^2}x(t) = -\omega^2 x(t) \tag{16}$$

Then we can guess a general solution of $\hat{x}(t)$:

$$\hat{x}(t) = Ae^{i\omega t} + Be^{-i\omega t} \tag{17}$$

such that

$$\hat{p}(t) = iA\omega e^{i\omega t} - iB\omega e^{-i\omega t} \tag{18}$$

Given initial condition (Eqn.(13))

$$A + B = \hat{x} \tag{19}$$

$$i\omega(A - B) = \hat{p} \tag{20}$$

Then,

$$A = \frac{1}{2}\left(\hat{x} + \frac{\hat{p}}{i\omega}\right) \tag{21}$$

$$A = \frac{1}{2}\left(\hat{x} - \frac{\hat{p}}{i\omega}\right) \tag{22}$$

So, $\hat{x}(t)$ and $\hat{p}(t)$ are summarized

$$\hat{x}(t) = \frac{1}{2}\left(\left(\hat{x} + \frac{\hat{p}}{i\omega}\right)e^{i\omega t} + \left(\hat{x} - \frac{\hat{p}}{i\omega}\right)e^{-i\omega t}\right) = \hat{x}\cos\omega t + \frac{\hat{p}}{\omega}\sin\omega t \tag{23}$$

$$\hat{p}(t) = \frac{i\omega}{2}\left(\left(\hat{x} + \frac{\hat{p}}{i\omega}\right)e^{i\omega t} - \left(\hat{x} - \frac{\hat{p}}{i\omega}\right)e^{-i\omega t}\right) = -\omega\hat{x}\sin\omega t + \hat{p}\cos\omega t \tag{24}$$

(c) It is convenient to introduce operators $\hat{a}(t)$, $\hat{a}^\dagger(t)$ defined by

$$\hat{x}(t) = \sqrt{\frac{1}{2\omega}}\left(\hat{a}(t) + \hat{a}^\dagger(t)\right), \quad \hat{p}(t) = -i\sqrt{\frac{\omega}{2}}\left(\hat{a}(t) - \hat{a}^\dagger(t)\right) \tag{25}$$

Show that $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ satisfying equal-time commutation relation

$$[\hat{a}(t), \hat{a}^\dagger(t)] = 1 \tag{26}$$

From the definition(Eqn.(25)), we can rewrite that:

$$\hat{a}(t) + \hat{a}^\dagger(t) = \sqrt{2\omega}\hat{x}(t) \tag{27}$$

$$\hat{a}(t) - \hat{a}^\dagger(t) = i\sqrt{\frac{2}{\omega}}\hat{p}(t) \tag{28}$$

then, we can write down $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ in terms of $\hat{x}(t)$ and $\hat{p}(t)$:

$$\hat{a}(t) = \frac{1}{\sqrt{2\omega}}\left(\omega\hat{x}(t) + i\hat{p}(t)\right) \tag{29}$$

$$\hat{a}^\dagger(t) = \frac{1}{\sqrt{2\omega}}\left(\omega\hat{x}(t) - i\hat{p}(t)\right) \tag{30}$$

Using the commutation relation in Eqn.(2):

$$\begin{aligned}
[\hat{a}(t), \hat{a}^\dagger(t)] &= \frac{1}{2\omega}[\omega\hat{x}(t) + i\hat{p}(t), \omega\hat{x}(t) - i\hat{p}(t)] &(31)\\
&= \frac{1}{2\omega}\left([\omega\hat{x}(t), -i\hat{p}(t)] + [i\hat{p}(t), \omega\hat{x}(t)]\right) &(32)\\
&= \frac{1}{2\omega} \times 2\omega &(33)\\
&= 1 &(34)
\end{aligned}$$

(d) Express the Hamiltonian in terms of $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.

Firstly, expand $\hat{x}^2(t)$ and $\hat{p}^2(t)$ in terms of $\hat{a}(t)$ and $\hat{a}^\dagger(t)$:

$$\hat{x}^2(t) = \frac{1}{2\omega}\left(\hat{a}^2(t) + \hat{a}(t)\hat{a}^\dagger(t) + \hat{a}^\dagger(t)\hat{a}(t) + \hat{a}^{\dagger^2}(t)\right) \tag{35}$$
$$\omega^2\hat{x}^2(t) = \frac{\omega}{2}\left(\hat{a}^2(t) + \hat{a}(t)\hat{a}^\dagger(t) + \hat{a}^\dagger(t)\hat{a}(t) + \hat{a}^{\dagger^2}(t)\right) \tag{36}$$

and

$$\hat{p}^2(t) = -\frac{\omega}{2}\left(\hat{a}^2(t) - \hat{a}(t)\hat{a}^\dagger(t) - \hat{a}^\dagger(t)\hat{a}(t) + \hat{a}^{\dagger^2}(t)\right) \tag{37}$$

Using the commutation relation from Pb. 1.(c) Eqn.(26), we know that

$$[\hat{a}(t), \hat{a}^\dagger(t)] = 1 \Rightarrow \hat{a}(t)\hat{a}^\dagger(t) - \hat{a}^\dagger(t)\hat{a}(t) = 1 \Rightarrow \hat{a}(t)\hat{a}^\dagger(t) = \hat{a}^\dagger(t)\hat{a}(t) + 1 \tag{38}$$

Therefore, the Hamiltonian(Eqn.(1)) can be expressed:

$$H = \frac{1}{2}(\hat{p}^2 + \omega^2\hat{x}^2) = \frac{\omega}{4}\left(2(\hat{a}(t)\hat{a}^\dagger(t) + \hat{a}^\dagger(t)\hat{a}(t))\right) \tag{39}$$
$$= \frac{\omega}{2}\left(2\hat{a}^\dagger(t)\hat{a}(t) + 1\right) \tag{40}$$
$$= \omega\left(\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2}\right) \tag{41}$$

(e) Obtain the Heisenberg equations for $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.

Same process as solving Pb. 1(a),

$$\frac{d}{dt}\hat{a}(t) = i[H, \hat{a}(t)] \tag{42}$$
$$\frac{d}{dt}\hat{a}^\dagger(t) = i[H, \hat{a}^\dagger(t)] \tag{43}$$

Then, using expasion of the Hamiltonian H in terms of $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ and the commutation relation from Eqn.(26)

$$\frac{d}{dt}\hat{a}(t) = i\left[\omega\left(\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2}\right), \hat{a}(t)\right] \qquad = i\omega[\hat{a}^\dagger(t)\hat{a}(t), \hat{a}(t)] \tag{44}$$
$$\frac{d}{dt}\hat{a}^\dagger(t) = i\left[\omega\left(\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2}\right), \hat{a}^\dagger(t)\right] \qquad = i\omega[\hat{a}^\dagger(t)\hat{a}(t), \hat{a}^\dagger(t)] \tag{45}$$

rewriting the commutation,

$$\frac{d}{dt}\hat{a}(t) = i\omega(\hat{a}^\dagger(t)\hat{a}(t)\hat{a}(t) - \hat{a}(t)\hat{a}^\dagger(t)\hat{a}(t)\hat{a}(t)) \qquad = i\omega[\hat{a}^\dagger(t), \hat{a}]\hat{a} = -i\omega\hat{a}(t) \tag{46}$$
$$\frac{d}{dt}\hat{a}^\dagger(t) = i\omega(\hat{a}^\dagger(t)\hat{a}(t)\hat{a}^\dagger(t) - \hat{a}^\dagger(t)\hat{a}^\dagger(t)\hat{a}(t)) \qquad = i\omega\hat{a}^\dagger(t)[\hat{a}(t), \hat{a}^\dagger(t)] = i\omega\hat{a}^\dagger(t) \tag{47}$$

(f) Suppose the initial condition at $t = 0$ are given by

$$\hat{a}(0) = \hat{a}, \quad \hat{a}^\dagger(0) = \hat{a}^\dagger \tag{48}$$

find $\hat{a}(t)$ and $\hat{a}^\dagger(t)$

Same process as solving Pb. 1(b), using the result of Pb. 1(e)

$$\frac{d}{dt}\hat{a}(t) = -i\omega\hat{a}(t) \Rightarrow \hat{a}(t) = \hat{a}e^{-iwt} \tag{49}$$

$$\frac{d}{dt}\hat{a}^\dagger(t) = -i\omega\hat{a}^\dagger(t) \Rightarrow \hat{a}^\dagger(t) = \hat{a}^\dagger e^{iwt} \tag{50}$$

(g) Express $\hat{x}(t), \hat{p}(t)$ and the Hamiltonian H in terms of a and $\hat{a}^\dagger$.

We can use the definition of $\hat{x}(t), \hat{p}(t)$(Eqn. (26)) in Prob. 1.(c) and replace the result of $\hat{a}(t)$, $\hat{a}^\dagger(t)$ into terms of $\hat{a}$ and $\hat{a}^\dagger$. Therefore,

$$\hat{x}(t) = \sqrt{\frac{1}{2\omega}}\left(\hat{a}(t) + \hat{a}^\dagger(t)\right) = \sqrt{\frac{1}{2\omega}}\left(\hat{a}e^{-iwt} + \hat{a}^\dagger e^{iwt}\right) \tag{51}$$

$$\hat{p}(t) = -i\sqrt{\frac{\omega}{2}}\left(\hat{a}(t) - \hat{a}^\dagger(t)\right) = -i\sqrt{\frac{\omega}{2}}\left(\hat{a}e^{-iwt} - \hat{a}^\dagger e^{iwt}\right) \tag{52}$$

And also using the result of Prob.1(d)

$$H = \omega\left(\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2}\right) = \omega\left(\hat{a}^\dagger e^{iwt}\hat{a}e^{-iwt} + \frac{1}{2}\right) \tag{53}$$

$$= \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{54}$$

2. Review : Lorentz Transformation

(a) Prove that the four-dimensional δ-function

$$\delta^{(4)}(p) = \delta(p^0)\delta(p^1)\delta(p^2)\delta(p^3) \tag{55}$$

is Lorentz invariant, i.e

$$\delta^{(4)}(p) = \delta^{(4)}(\tilde{p}) \tag{56}$$

where $\tilde{p}^\mu$ is a Lorentz transformation of p.

Let Λ^μ_ν be a Lorentz transformation, then

$$\tilde{p}^\mu = \Lambda^\mu_\nu p^\nu, \quad \tilde{x}^\mu = \Lambda^\mu_\nu x^\nu \tag{57}$$

And then we have to keep in mind that $\Lambda x \cdot \Lambda p = x \cdot p$ is Lorentz invariant. Then, we will make a Fourier transformation to the δ-function

$$\delta^{(4)}(p) = \frac{1}{(2\pi)^4} \int d^4 x e^{ix\cdot p} = \frac{1}{(2\pi)^4} \int d^4 x e^{i\Lambda x \cdot \Lambda p} \tag{58}$$

Since the property of the Lorentz transformation that:

$$\Lambda^T \eta \Lambda = \eta \Rightarrow \det \Lambda^T \det \Lambda = (\det \Lambda)^2 = 1 \tag{59}$$

$$\Rightarrow J = |\det \Lambda| = 1 \tag{60}$$

So,

$$d^4 \tilde{x} = d^4 x \tag{61}$$

That makes the δ-function,

$$\frac{1}{(2\pi)^4} \int d^4 x e^{i\Lambda x \cdot \Lambda p} = \frac{1}{(2\pi)^4} \int d^4 \tilde{x} e^{i\tilde{x}\cdot\tilde{p}} \tag{62}$$

which is the δ-function of $\tilde{p}$.

(b) Show that

$$\omega_1 \delta^{(3)}(\vec{k}_1 - \vec{k}_2) \tag{63}$$

is Lorentz invariant, i.e

$$\omega_1 \delta^{(3)}(\vec{k}_1 - \vec{k}_2) = \omega_1' \delta^{(3)}(\vec{k}_1' - \vec{k}_2') \tag{64}$$

where $\vec{k}_1$ and $\vec{k}_2$ are respectively the spatial part of four-vectors $k_1^\mu = (w_1, \vec{k}_1)$ and $k_2^\mu = (w_2, \vec{k}_2)$ which satisfy the on-shell condition

$$k_1^2 = k_2^2 = -m^2 \tag{65}$$

$k_1'^\mu = (w_1', \vec{k}_1')$ and $k_2'^\mu = (w_2', \vec{k}_2')$ are related to k_1^μ, k_2^μ by a same Lorentz transformation.

Since for the given condition that k^μs are in on-shell, it gives us the mass-shell condition

$$\delta(k^2 + m^2) \tag{66}$$

We will use the property of δ-function to show the Lorentz invariance.

$$\delta(f(x)) = \sum_{x_i=0 \ s.t \ f(x_i)=0} \frac{1}{|f'(x_i)|}\delta(x - x_i) \tag{67}$$

We modify the mass-shell condition;

$$\begin{aligned}
\delta(k^2 + m^2) &= \delta(-k_0^2 + \vec{k}^2 + m^2) && (68)\\
&= \delta(-k_0^2 + w_{\vec{k}}^2) && (69)\\
&= \delta((-k_0 + |w_{\vec{k}}|)(k_0 + |w_{\vec{k}}|)) && (70)\\
&= \frac{1}{2|\omega_{\vec{k}}|}\left[\delta(k_0 - |\omega_{\vec{k}}|) + \delta(k^0 + |\omega_{\vec{k}}|)\right] && (71)
\end{aligned}$$

where $\omega_{\vec{k}}$ comes from the energy relationship, $\omega^2 = \vec{k}^2 + m^2$. We assume that $\omega_{\vec{k}} > 0$
We can pick out the $k_1^0 = \omega_{\vec{k}_1}$ enforcing $\delta(k_1^2 + m^2)$ by multiplying both sides by $\theta(\omega_{\vec{k}_1})$.

$$\begin{aligned}
\theta(\omega_{\vec{k}_1})\delta(k_1^2 + m^2) &= \frac{1}{2|\omega_{\vec{k}_1}|}\theta(\omega_{\vec{k}_1})\left[\delta(k_1^0 - |\omega_{\vec{k}_1}|) + \delta(k_1^0 + |\omega_{\vec{k}_1}|)\right] && (72)\\
&= \frac{1}{2\omega_{\vec{k}_1}}\theta(\omega_{\vec{k}_1})\delta(k_1^0 - \omega_{\vec{k}_1}) && (73)\\
&= \frac{1}{2\omega_{\vec{k}_1}}\delta(k_1^0 - \omega_{\vec{k}_1}) && (74)
\end{aligned}$$

Then, we multiply both sides by $2\omega_{\vec{k}_1}\delta^{(3)}(\vec{k}_1 - \vec{k}_2)$

$$\begin{aligned}
2\omega_{\vec{k}_1}\delta^{(3)}(\vec{k}_1 - \vec{k}_2)\theta(\omega_{\vec{k}_1})\delta(k_1^2 + m^2) &= 2\omega_{\vec{k}_1}\delta^{(3)}(\vec{k}_1 - \vec{k}_2) \cdot \frac{1}{2\omega_{\vec{k}}}\delta(k_1^0 - \omega_{\vec{k}_1}) && (75)\\
&= \delta^{(3)}(\vec{k}_1 - \vec{k}_2)\delta(k_1^0 - \omega_{\vec{k}_1}) && (76)\\
&= \delta^{(3)}(\vec{k}_1 - \vec{k}_2)\delta(k_1^0 - \omega_{\vec{k}_2}) && (77)\\
&= \delta^{(3)}(\vec{k}_1 - \vec{k}_2)\delta(k_1^0 - k_2^0) && (78)
\end{aligned}$$

In eqn.(77), we use that the $\delta^{(3)}(\vec{k}_1 - \vec{k}_2)$ allows us to replace $\omega_{\vec{k}_1}$ to $\omega_{\vec{k}_2}$ For this step, it is crucial that $sign(\omega_{\vec{k}_1}) = sign(\omega_{\vec{k}_2})$, which is true since both are positive.
Finally, from eqn.(78), the right-handed side is Lorentz invariant, since we show that in Prob.2(a). On the left-handed side, we already know that $\delta(k_1^2 + m^2)$ is a Lorentz scalar since k_1^2 is Lorentz invariant, and $\theta(\omega_{\vec{k}_1})$ is Lorentz invariant since the energy of a particle does not change under Lorentz transformation. Therefore, we can say that $\omega_{\vec{k}_1}\delta^{(3)}(\vec{k}_1 - \vec{k}_2)$ is Lorentz invariant to satisfy both sides.

(c) For any function $f(k) = f(k^0, k^1, k^2, k^3)$, prove that

$$\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(k), \quad \omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2} \tag{79}$$

is Lorentz invariant in the sense that

$$\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(k) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(\tilde{k}) \tag{80}$$

where $\tilde{k}^\mu = \Lambda^\mu_\nu k^\nu$ is a Lorentz transformation of k^μ

Since the momentum is on the mass-shell, $f(k) = f(\omega_{\vec{k}}, \vec{k})$ So, the integral over the spatial part can be written as;

$$\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(k) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(\omega_{\vec{k}}, \vec{k}) \tag{81}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\delta(k^0 - \omega_{\vec{k}})f(k^0, \vec{k}) \tag{82}$$

$$= \frac{1}{(2\pi)^3}\int d^4k\ \theta(\omega_{\vec{k}})\delta(k^2 + m^2)f(k^0, \vec{k}) \tag{83}$$

Then, Lorentz transformation follows the property below:

$$d^4k = d^4\tilde{k}, \quad \theta(\omega_{\vec{k}}) = \theta(\omega_{\vec{k}'}), \quad \delta(k^2 + m^2) = \delta(k'^2 + m^2)$$

The integral is noted as:

$$\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(k) = \frac{1}{(2\pi)^3}\int d^4k\ \theta(\omega_{\vec{k}})\delta(k^2 + m^2)f(k^0, \vec{k}) \tag{84}$$

$$= \frac{1}{(2\pi)^3}\int d^4k'\ \theta(\omega_{\vec{k}'})\delta(k'^2 + m^2)f((\Lambda k')^\mu) \tag{85}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(\Lambda k) \tag{86}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(\tilde{k}) \tag{87}$$

So, $\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(k)$ is Lorentz invariant.

3. A Complex Scalar field

Consider the field theory of a complex value scalar field $\phi(x)$ with action

$$S = \int d^4x \ \left[-\partial_\mu \phi^* \partial^\mu \phi - V(|\phi|^2)\right], \quad |\phi|^2 = \phi\phi^* \tag{88}$$

One could either consider the real and imaginary parts of ϕ, or ϕ and ϕ^* as independent dynamical variables. The latter is more convenient in most situations.

(a) Check action (88) is Lorentz invariant ($\phi(x) \to \phi'(x') = \phi(x)$) and find the equations of motion.

Lorentz transformation acts as $\phi \to \phi^*$, such that $\phi'(x) = \phi(\Lambda^{-1}x)$ Then the action transforms like;

$$\begin{aligned} S \to S' &= \int d^4x \ \left[-\partial_\mu \phi'^*(x) \partial^\mu \phi'(x) - V(|\phi'(x)|^2)\right] &(89)\\ &= \int d^4x \ \left[-\partial_\mu \phi^*(\Lambda^{-1}x) \partial^\mu \phi(\Lambda^{-1}x) - V(|\phi(\Lambda^{-1}x)|^2)\right] &(90)\end{aligned}$$

From the Prob.2(a), we know that $d^4x = d^4x'$. Use the chain rule for modifying the derivative that $\partial_\mu = (\Lambda^{-1})^\nu_\mu \partial'_\nu$.

$$\begin{aligned} S' &= \int d^4x' \ \left[-(\Lambda^{-1})^\nu_\mu \partial'_\nu \phi^*(x')(\Lambda^{-1})^\mu_\rho \partial'^\rho \phi(x') - V(|\phi(x')|^2)\right] &(91)\\ &= \int d^4x' \ \left[-(\Lambda^{-1})^\nu_\mu (\Lambda^{-1})^\mu_\rho \partial'_\nu \phi^*(x') \partial'^\rho \phi(x') - V(|\phi(x')|^2)\right] &(92)\\ &= \int d^4x' \ \left[-(\delta^\nu_\rho \partial'_\nu \phi^*(x') \partial'^\rho \phi(x') - V(|\phi(x')|^2)\right] &(93)\\ &= \int d^4x' \ \left[-(\partial'_\nu \phi^*(x') \partial'^\nu \phi(x') - V(|\phi(x')|^2)\right] &(94)\\ &= S &(95)\end{aligned}$$

where we can use the property of the Lorentz transformation:

$$(\Lambda^{-1})^\nu_\mu (\Lambda^{-1})^\mu_\rho = ((\Lambda^{-1})^T)^\nu_\mu (\Lambda^{-1})^\mu_\rho = \Lambda^\nu_\mu (\Lambda^{-1})^\mu_\rho = \delta^\nu_\rho$$

Therefore, the action is Lorentz invariant.
For the equation of motions, we have to calculate Euler-Lagrangian equation for ϕ and ϕ^* independently

$$\begin{cases} \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi} & \text{(i)} \\ \\ \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \frac{\partial \mathcal{L}}{\partial \phi^*} & \text{(ii)} \end{cases} \tag{96}$$

where $\mathcal{L} = -\partial_\mu \phi'^*(x)\partial^\mu \phi'(x) - V(|\phi'(x)|^2$, for (i);

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial_\mu \frac{\partial}{\partial(\partial_\mu \phi)} \left[-\partial_\nu \phi^* \partial^\nu \phi - V(|\phi|^2)\right] \tag{97}$$

$$= \partial_\mu(-\partial_\nu \phi^* \delta^\nu_\mu) \tag{98}$$

$$= -\partial^2 \phi^* \tag{99}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial}{\partial \phi}(-V(|\phi|^2)) \tag{100}$$

$$= -V'(|\phi|^2)\phi^* \tag{101}$$

Then, by Euler-Lagrangian equation

$$\partial^2 \phi^* - V'(|\phi|^2)\phi^* = 0 \tag{102}$$

Same sequence for (ii), we can get

$$\partial^2 \phi - V'(|\phi|^2)\phi = 0 \tag{103}$$

(b) Find the cancnonical conjugate momenta for ϕ and ϕ^*, and the Hamiltonian H for eqn.(88)

We can expand the 4-derivative∂_μ to time and space derivative, that makes the Lagrangian $\mathcal{L}$ into :

$$\mathcal{L} = \partial_t \phi^* \partial^t \phi - \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi - V(|\phi|^2) \tag{104}$$

Therefore, the canonical conjugate momenta becomes:

$$\pi := \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial_t \phi^*, \quad \pi^* := \frac{\partial \mathcal{L}}{\partial(\partial_t \phi^*)} = \partial_t \phi \tag{105}$$

So the Hamiltonian is:

$$H = \int d^3x \left[\pi \cdot \partial_t \phi + \pi^* \cdot \partial_t \phi^* - \mathcal{L}\right] \tag{106}$$

$$= \int d^3x \left[2\pi^*\pi - \pi^*\pi + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + V(|\phi|^2)\right] \tag{107}$$

$$= \int d^3x \left[\pi^*\pi + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + V(|\phi|^2)\right] \tag{108}$$

(c) The action is invariant under transformation

$$\phi \to e^{i\alpha}\phi, \quad \phi^* \to e^{-i\alpha}\phi^* \tag{109}$$

for arbitrary constant α. When α is small, i.e. for an infinitesimal transformation, eqn.(109) become

$$\delta\phi = i\alpha\phi, \quad \delta\phi^* = -i\alpha\phi^* \tag{110}$$

Use Noether's theorem to find the corresponding conserved current j^μ and conserved change Q.

Noether's theorem states that very continuous symmetry of the action of a physical system with conservative forces has a corresponding conservation law. So that conserve current is:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)}\delta\Phi_a - \mathcal{F}^\mu \quad \text{where} \quad \delta\mathcal{L} = \partial_\mu \mathcal{F}^\mu \tag{111}$$

Then, use the infinitesimal transformation the conserved current is

$$\begin{aligned} j^\mu &= -\partial^\mu\phi^*(i\alpha\phi) - \partial^\mu\phi(-i\alpha\phi^*) &(112)\\ &= i\alpha(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) &(113)\end{aligned}$$

we can remove that proportional constant to simplify the current

$$j^\mu = \phi^*\partial^\mu\phi - \phi\partial^\mu\phi^* \tag{114}$$

Then the corresponding charge is

$$Q = \int d^3x\ j^0 = \int d^3x(\phi^*\partial_t\phi - \phi\partial_t\phi^*) \tag{115}$$

(d) Use the equations of motion of part (a) to verify directly that j^μ is indeed conserved.

Using the equation of motion, the conservation of the current can be checked with calculating the derivative:

$$\begin{aligned} \partial_\mu j^\mu &= \partial_\mu(\phi^*\partial_t\phi - \phi\partial_t\phi^*) &(116)\\ &= \phi^*\partial^2\phi - \phi\partial^2\phi^* &(117)\\ &= V'(|\phi|^2)\phi^*\phi - V'(|\phi|^2)\phi^*\phi &(118)\\ &= 0 &(119)\end{aligned}$$

4. The energy-momentum tensor for the complex scalar field theory

In this problem, we work out the energy-momentum tensor of the complex scalar theory (88).

(a) Under a spacetime translation

$$x^\mu \to x'^\mu = x^\mu + a^\mu \tag{120}$$

a scalar field transform as

$$\phi'(x') = \phi(x) \tag{121}$$

Show that the action (88) is invariant under transformation $\phi(x) \to \phi'(x)$.

Under the transformation, the scalar field satisfies $\phi'(x) = \phi(x-a)$. Then the action transforms:

$$S \to S' = \int d^4x(-\partial_\mu \phi'^*(x)\partial^\mu \phi'(x) - V(|\phi'(x)|^2) \tag{122}$$

$$= \int d^4x(-\partial_\mu \phi^*(x-a)\partial^\mu \phi(x-a) - V(|\phi(x-a)|^2) \tag{123}$$

$$= \int d^4x(-\partial_\mu \phi^*(x)\partial^\mu \phi(x) - V(|\phi(x)|^2) = S \tag{124}$$

(b) Write down the transformation of the scalar fields ϕ and ϕ^* for an infinitesimal translation, and use Noether's theorem to find the corresponding conserved currents $T^{\mu\nu}$.

An infinitesimal transformation makes the field:

$$\delta\phi = \phi'(x) - \phi(x) = \phi(x-a) - \phi(x) = -a^\mu \partial_\mu \phi(x) \tag{125}$$

$$\delta\phi^* = \phi'^*(x) - \phi^*(x) = \phi^*(x-a) - \phi^*(x) = -a^\mu \partial_\mu \phi^*(x) \tag{126}$$

Then, the Lagrangian density under translation gives the infinitesimal variance:

$$\delta\mathcal{L} = \mathcal{L}' - \mathcal{L} \tag{127}$$

$$= -\partial_\nu(\phi^*(x) - a^\mu\partial_\mu\phi^*(x))\partial^\nu(\phi(x) - a^\mu\partial_\mu\phi(x)) \tag{128}$$

$$- V\left((\phi^*(x) - a^\mu\partial_\mu\phi^*(x)(\phi(x) - a^\mu\partial_\mu\phi(x)))\right) - \mathcal{L} \tag{129}$$

$$= a^\mu(\partial_\nu\partial_\mu\phi^*(x)\partial^\nu\phi(x) + \partial_\nu\phi^*\partial^\nu\partial_\mu\phi(x)) + a^\mu V'(|\phi|^2)(\partial_\mu\phi^*\phi + \phi^*\partial_\mu\phi) + \mathcal{O}(a^\mu a^\nu) \tag{130}$$

$$= -a^\mu\partial_\mu\mathcal{L} \tag{131}$$

$$= a_\mu\partial_\nu(-\eta^{\mu\nu}\mathcal{L}) := a_\mu\partial_\nu\mathcal{F}^{\mu\nu} \tag{132}$$

Translations are parameterize with 4-vector a^μ, and we have a Noether current itself a 4-vector. Therefore, the conserved currents are encoded in the 2-rank tensor, $\mathcal{T}^{\mu\nu}$.

$$T^{\mu\nu} := (j^\mu)^\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\delta\phi)^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}(\delta\phi^*)^\mu - \mathcal{F}^{\mu\nu} \tag{133}$$

$$= -\partial^\nu\phi^*(-\partial^\mu\phi) - \partial^\nu\phi(\partial^\mu\phi^*) + \eta^{\mu\nu}\mathcal{L} \tag{134}$$

$$= \partial^\nu\phi^*\partial^\mu\phi + \partial^\nu\phi\partial^\mu\phi^* - \eta^{\mu\nu}\left[\partial_\rho\phi^*\partial^\rho\phi + V(|\phi|^2)\right] \tag{135}$$

In the following, we let the first index μ of $T^{\mu\nu}$ pick out the direction of translation a^μ

(c) The conserved charge for a time translation

$$H = \int d^3x T^{00} \tag{136}$$

should be identified with the total energy of the system, while that for a spatial translation

$$P^i = \int d^3x T^{0i} \tag{137}$$

should be identified with the total momentum. Thus, $T^{\mu\nu}$ is referred to as the energy-momentum tensor. Write down the explicit expression for H and P^i. Compare H obtained here with the Hamiltonian of prob.3(b).

The Hamiltonian is:

$$\begin{aligned}
H &= \int d^3x T^{00} && (138)\\
&= \int d^3x \left[2\partial^t\phi^*\partial^t\phi - \eta^{00}(\partial_\mu\phi^*\partial^\mu\phi + V(|\phi|^2))\right] && (139)\\
&= \int d^3x \left[2\partial^t\phi^*\partial^t\phi + (-\partial_t\phi^*\partial_t\phi + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + V(|\phi|^2))\right] && (140)\\
&= \int d^3x \left[\partial^t\phi^*\partial^t\phi + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + V(|\phi|^2))\right] && (141)
\end{aligned}$$

And the total momentum is:

$$\begin{aligned}
P^i &= \int d^3x T^{0i} && (142)\\
&= \int d^3x \left[\partial^t\phi^*\partial^i\phi + \partial^i\phi^*\partial^t\phi - \eta^{0i}(\partial_\rho\phi^*\partial^\rho\phi + V(|\phi|^2))\right] && (143)\\
&= \int d^3x \left[\partial^t\phi^*\partial^i\phi + \partial^i\phi^*\partial^t\phi\right] && (144)\\
&= -\int d^3x \left[\partial_t\phi^*\partial_i\phi + \partial_i\phi^*\partial_t\phi\right] && (145)
\end{aligned}$$

The expression of Hamiltonian is same as prob.3(b).

(d) Use equations of motion of prob.3(a) to verify directly that $T^{\mu\nu}$ is indeed conserved.

The equations of motion are:

$$\begin{cases} \partial^2\phi^* - V'(|\phi|^2)\phi^* = 0 \\ \partial^2\phi - V'(|\phi|^2)\phi = 0 \end{cases} \tag{146}$$

Recall that μ indicates the direction of translation a^μ. Therefore, conservation from the Noether's theorem means that $\partial_\nu T^{\mu\nu}$. Caring the results from prob.4(c), $T^{\mu\nu}$ is symmetric so that we can contract the derivative with respect to either index.

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \partial_\mu(\partial^\nu\phi^*\partial^\mu\phi + \partial^\mu\phi^*\partial^\nu\phi - \eta^{\mu\nu}\left[\partial_\rho\phi^*\partial^\rho\phi + V(|\phi|^2)\right]) && (147)\\
&= \partial_\mu(\partial^\nu\phi^*\partial^\mu\phi + \partial^\mu\phi^*\partial^\nu\phi) - \partial^\nu(\partial_\rho\phi^*\partial^\rho\phi) - \partial^\nu V(|\phi|^2) && (148)\\
&= \partial^2\phi^*\partial^\nu\phi + \partial^\nu\phi^*\partial^2\phi - \partial^\nu V(|\phi|^2) && (149)\\
&= V'(|\phi|^2)\phi^*\partial^\nu\phi + \partial^\nu\phi^* V'(|\phi|^2)\phi - V'(|\phi|^2)\partial^\nu\phi^*\phi - V'(|\phi|^2)\phi^*\partial^\nu\phi && (150)\\
&= 0 && (151)
\end{aligned}$$