MIT OCW 8.323: Quantum Field Theory I

Assignment #2

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1. Problem with Relativistic Quantum Mechanics

The Schrodinger equation for a free non-relativistic particle is

$$i\partial_t\psi(\vec{x},t) = -\frac{1}{2m}\nabla^2\psi(\vec{x},t) \tag{1}$$

The generalization of the above equation to a free relativistic particle is the so-called Klein-Gordon equation

$$\partial_t^2\psi(\vec{x},t) - \nabla^2\psi(\vec{x},t) + m^2\psi(\vec{x},t) = 0 \tag{2}$$

We emphasize that in both (1) and (2), $\psi(\vec{x},t)$ is interpreted ad a wavefunction for dynamical variable $\vec{x}(t)$ rather than a dynamical field.

(a) As a reminder, derive from (1) the continuity equation for the probability

$$\partial_t\rho + \nabla\cdot\vec{J} = 0 \tag{3}$$

where

$$\rho = |\psi|^2, \qquad \vec{J} = -\frac{i}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*) \tag{4}$$

.

Consider the time derivative of the ρ,

$$\partial_t\rho = \psi\partial_t\psi^* + \psi^*\partial_t\psi$$

From the Schrodinger equation, the time derivative of ρ can be computed

$$\begin{aligned}
\partial_t\rho &= \psi\partial_t\psi^* + \psi^*\partial_t\psi && (5)\\
&= \psi(-\frac{i}{2m}\nabla^2\psi^*) + \psi^*(\frac{i}{2m}\nabla^2\psi) && (6)\\
&= -\frac{i}{2m}\left(\psi\nabla^2\psi^* - \psi^*\nabla^2\psi\right) && (7)\\
&= -\frac{i}{2m}\nabla\cdot(\psi\nabla\psi^* - \psi^*\nabla\psi) && (8)\\
&= \nabla\cdot\left[-\frac{i}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi)\right] && (9)\\
&= -\nabla\cdot\left[-\frac{i}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*)\right] && (10)\\
&= -\nabla\cdot\vec{J} && (11)
\end{aligned}$$

Therefore,

$$\partial_t\rho + \nabla\cdot\vec{J} = 0$$

(b) Suppose $\psi(\vec{x},t)$ has the plane wave form, i.e.

$$\psi(\vec{x},t) \propto e^{i\vec{k}\cdot\vec{x}} \tag{12}$$

for some real vector $\vec{k}$, find the solutions to (2)

Assume that $\psi(\vec{x},t) = \phi(t)e^{i\vec{k}\cdot\vec{x}}$. Computing each terms,

$$\partial_t^2\psi = \partial_t^2\phi(t)\cdot e^{i\vec{k}\cdot\vec{x}} \tag{13}$$

$$\nabla^2\psi = -k^2\phi(t)\cdot e^{i\vec{k}\cdot\vec{x}} \tag{14}$$

Then the equation simplifies to

$$(\partial_t^2 + k^2 + m^2)\phi \cdot e^{i\vec{k}\cdot\vec{x}} = 0 \tag{15}$$

Since the plane wave term can not be zero, ϕ satisfies the differential harmonic oscillation equation. Therefore,

$$\phi(t) = C_1 e^{i\omega_{\vec{k}}t} + C_2 e^{-i\omega_{\vec{k}}t} \tag{16}$$

where $\omega_{\vec{k}}^2 = k^2 + m^2$. Then, the final wave function looks like

$$\psi(\vec{x},t) = C_1 e^{i(\omega_{\vec{k}}t + \vec{k}\cdot\vec{x})} + C_2 e^{-i(\omega_{\vec{k}}t - \vec{k}\cdot\vec{x})} \quad \text{where} \quad \omega_{\vec{k}}^2 = k^2 + m^2 \tag{17}$$

(c) Show that the Klein-Gordon equation also leads to a continuity equation (3) with now ρ and $\vec{J}$ given by

$$\rho = \frac{i}{2m}(\psi^*\partial_t\psi - \psi\partial_t\psi^*) \qquad \vec{J} = -\frac{i}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*) \tag{18}$$

Similarly, consider the time derivative of ρ

$$\begin{aligned}
\partial_t\rho &= \frac{i}{2m}(\psi^*\partial_t^2\psi - \psi\partial_t^2\psi^*) && (19)\\
&= \frac{i}{2m}\left[\psi^*(\nabla^2\psi - m^2\psi) - \psi(\nabla^2\psi^* - m^2\psi^*)\right] && (20)\\
&= \frac{i}{2m}\left[\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right] && (21)\\
&= \frac{i}{2m}\nabla\cdot(\psi^*\nabla\psi - \psi\nabla\psi^*) && (22)\\
&= -\nabla\cdot\left[-\frac{i}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*)\right] && (23)\\
&= -\nabla\cdot\vec{J} && (24)
\end{aligned}$$

Therefore,

$$\partial_t\rho + \nabla\cdot\vec{J} = 0$$

(d) Argue that ρ in (18) cannot be interpreted as probability density.

To be interpreted as probability density, ρ must be positive.
However,

$$\rho = \frac{i}{2m}(\psi^* \partial_t \psi - \psi \partial_t \psi^*) = \frac{1}{m}\mathrm{Im}(\psi^* \partial_t \psi)$$

If $\psi(\vec{x}, t) = Ce^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}}$, then

$$\rho = \frac{1}{m}\mathrm{Im}(Ce^{i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}}(-i\omega_{\vec{k}})Ce^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}}) = -\frac{C^2\omega_{\vec{k}}}{m} < 0$$

2. Commutation Relations of annihilation and creation operators

For the real scalar field theory discussed in lecture, i.e.

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{25}$$

we showed that the time evolution of quantum operator $\phi(\vec{x}, t)$ is given by

$$\phi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(a_{\vec{k}}u_{\vec{k}}(\vec{x}, t) + a^\dagger_{\vec{k}}u^*_{\vec{k}}(\vec{x}, t)\right) \tag{26}$$

where

$$\omega^2_{\vec{k}} = \vec{k}^2 + m^2 \qquad u_{\vec{k}}(\vec{x}, t) = e^{-i\omega_{\vec{k}}t + i\vec{k}\cdot\vec{x}} \tag{27}$$

We use $\pi(\vec{x}, t)$ to denote the momentum density conjugate to ϕ. The canonical commutation realtions among ϕ and π are

$$[\phi(\vec{x}, t), \phi(\vec{x}', t)] = [\pi(\vec{x}, t), \pi(\vec{x}', t)] = 0, \qquad [\phi(\vec{x}, t), \pi(\vec{x}', t)] = i\delta^{(3)}(\vec{x} - \vec{x}') \tag{28}$$

(a) Show that it is enough to impose (28) at $t = 0$. In other words, once we impose them at $t = 0$, then the relations at general t are automatically satisfied.
Note: This statement in fact applies not only to $V(\phi) = \frac{1}{2}m^2\phi^2$, but any potential $V(\phi)$

Considering the Heisenberg picture of the operators, we have:

$$[A(\vec{x}, t), B(\vec{x}, t)] = [e^{iHt}A(\vec{x}, 0)e^{-iHt}, e^{iHt}B(\vec{x}, 0)e^{-iHt}] = e^{iHt}[A(\vec{x}, 0), B(\vec{x}, 0)]e^{-iHt}$$

Therefore, for the canonical commutation relations

$$[\phi(\vec{x}, t), \phi(\vec{x}', t)] = e^{iHt}[\phi(\vec{x}, 0), \phi(\vec{x}', 0)]e^{-iHt} = e^{iHt}0e^{-iHt} = 0 \tag{29}$$

$$[\pi(\vec{x}, t), \pi(\vec{x}', t)] = e^{iHt}[\pi(\vec{x}, 0), \pi(\vec{x}', 0)]e^{-iHt} = e^{iHt}0e^{-iHt} = 0 \tag{30}$$

And also for the second relation,

$$[\phi(\vec{x}, t), \pi(\vec{x}', t)] = e^{iHt}[\phi(\vec{x}, 0), \pi(\vec{x}', 0)]e^{-iHt} \tag{31}$$

$$= e^{iHt}i\delta^{(3)}(\vec{x} - \vec{x}')e^{-iHt} \tag{32}$$

$$= i\delta^{(3)}(\vec{x} - \vec{x}') \tag{33}$$

(b) Express $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ in terms of $\phi(\vec{k})$ and $\pi(\vec{k})$ where $\phi(\vec{k})$ and $\pi(\vec{k})$ are Fourier transformation of $\phi(\vec{x}, t=0)$ and $\pi(\vec{x}, t=0)$, i.e.

$$\phi(\vec{k}) = \int d^3x e^{i\vec{k}\cdot\vec{x}} \phi(\vec{x}, t=0) \qquad \pi(\vec{k}) = \int d^3x e^{i\vec{k}\cdot\vec{x}} \pi(\vec{x}, t=0) \tag{34}$$

From Eq.(26) and considering the mode expansions

$$\phi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left(a_{\vec{k}} e^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}} e^{i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}} \right) \tag{35}$$

$$\pi(\vec{x}, t) = -i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_{\vec{k}}}{2}} \left(a_{\vec{k}} e^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}} - a^\dagger_{\vec{k}} e^{i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}} \right) \tag{36}$$

Then, the expansion is also the Fourier transformation such that we can change the variable $\vec{x}$ to $\vec{k}$:

$$\phi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} \phi(\vec{k}, t) e^{i\vec{k}\cdot\vec{x}} \quad \Longrightarrow \quad \phi(\vec{k}, t) = \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left(a_{\vec{k}} e^{-i\omega_{\vec{k}} t} + a^\dagger_{-\vec{k}} e^{i\omega_{\vec{k}} t} \right) \tag{37}$$

$$\pi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} \pi(\vec{k}, t) e^{i\vec{k}\cdot\vec{x}} \quad \Longrightarrow \quad \pi(\vec{k}, t) = -i\sqrt{\frac{\omega_{\vec{k}}}{2}} \left(a_{\vec{k}} e^{-i\omega_{\vec{k}} t} - a^\dagger_{-\vec{k}} e^{i\omega_{\vec{k}} t} \right) \tag{38}$$

Then we can take $t = 0$ and solve this as a regular system of equation of $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$:

$$a_{\vec{k}} = \sqrt{\frac{\omega_{\vec{k}}}{2}} \phi(\vec{k}) + i \frac{1}{\sqrt{2\omega_{\vec{k}}}} \pi(\vec{k}) \tag{39}$$

$$a^\dagger_{\vec{k}} = \sqrt{\frac{\omega_{\vec{k}}}{2}} \phi(-\vec{k}) - i \frac{1}{\sqrt{2\omega_{\vec{k}}}} \pi(-\vec{k}) \tag{40}$$

(c) Using the expressions you derived in part (b) to deduce the commutations relations

$$[a_{\vec{k}}, a_{\vec{k}'}] \qquad [a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}] \qquad [a_{\vec{k}}, a^\dagger_{\vec{k}'}] \tag{41}$$

from the commutation relations (28) at $t = 0$

First, consider the Fourier transform of the commutation relations at $t = 0$

$$[\phi(\vec{k}), \phi(\vec{k}')] = \mathcal{F}_{\vec{x}\to\vec{k}} \circ \mathcal{F}_{\vec{x}'\to\vec{k}'}[\phi(\vec{x}), \phi(\vec{x}')] = \mathcal{F}_{\vec{x}\to\vec{k}} \circ \mathcal{F}_{\vec{x}'\to\vec{k}'}(0) = 0 \tag{42}$$
$$[\pi(\vec{k}), \pi(\vec{k}')] = \mathcal{F}_{\vec{x}\to\vec{k}} \circ \mathcal{F}_{\vec{x}'\to\vec{k}'}[\pi(\vec{x}), \pi(\vec{x}')] = \mathcal{F}_{\vec{x}\to\vec{k}} \circ \mathcal{F}_{\vec{x}'\to\vec{k}'}(0) = 0 \tag{43}$$
$$\tag{44}$$

And for $[\phi(\vec{k}), \pi(\vec{k}')]$,

$$\begin{aligned}
[\phi(\vec{k}), \pi(\vec{k}')] &= \mathcal{F}_{\vec{x}\to\vec{k}} \circ \mathcal{F}_{\vec{x}'\to\vec{k}'}[\phi(\vec{x}), \pi(\vec{x}')] && (45)\\
&= \mathcal{F}_{\vec{x}\to\vec{k}} \circ \mathcal{F}_{\vec{x}'\to\vec{k}'}\left(i\delta^{(3)}\left(\vec{x} - \vec{x}'\right)\right) && (46)\\
&= \iint d^3x d^3x' \; i\delta^{(3)}(\vec{x} - \vec{x}')e^{-i\vec{k}\cdot\vec{x}}e^{-i\vec{k}'\cdot\vec{x}'} && (47)\\
&= i\int d^3x \; e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}} && (48)\\
&= i(2\pi)^3\delta^{(3)}(\vec{k} + \vec{k}') && (49)
\end{aligned}$$

Then, we can compute the commutator of the creation and annihilation operators using the result in (b).

$$[a_{\vec{k}}, a_{\vec{k}'}] = [a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}] = 0 \tag{50}$$
$$\tag{51}$$

For $[a_{\vec{k}}, a^\dagger_{\vec{k}'}]$,

$$\begin{aligned}
[a_{\vec{k}}, a^\dagger_{\vec{k}'}] &= -\frac{i}{2}[\phi(\vec{k}), \pi(-\vec{k}')] + \frac{i}{2}[\pi(\vec{k}), \phi(-\vec{k}')] && (52)\\
&= -\frac{i}{2}([\phi(\vec{k}), \pi(-\vec{k}')] - [\pi(\vec{k}), \phi(-\vec{k}')]) && (53)\\
&= -\frac{i}{2}\left(i(2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}') + i(2\pi)^3\delta^{(3)}(-\vec{k}' + \vec{k})\right) && (54)\\
&= (2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}') && (55)
\end{aligned}$$

3. Expressing Noether charges in terms of creation and annihilation operators

In P-Set 1, you obtained the conserved charges associated with spacetime translational symmetries for a complex scalar field theory. The results there can be easily converted to the corresponding expressions for a real scalar field theory over (25).

(a) Express the Hamiltonian H of (25) in terms of $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$.

From P-Set 1 Pb.3(b), we can induce the Hamiltonian H from the given Lagrangian density $\mathcal{L}$. The canonical conjugate momenta becomes:

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = -\partial_t \phi \tag{56}$$

Therefore, considering the Hamiltonian density function $\mathcal{H}$

$$\begin{aligned}
\mathcal{H} &= \pi\dot{\phi} - \mathcal{L}(\phi, \partial_\mu \phi) &(57)\\
&= -\pi^2 + \frac{1}{2}\left(\partial_t \phi \partial^t \phi - |\nabla\phi|^2 + m^2\phi^2\right) &(58)\\
&= -\frac{1}{2}\pi^2 - \frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}m^2\phi^2 &(59)
\end{aligned}$$

Using the general convention and considering that Hamiltonian implies the Energy, let the Hamiltonian density regard as $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}m^2\phi^2$. Since our Hamiltonian density is defined in x-space and the annihilation and creation are defined in k-space , we first consider the Fourier transformation using the below identity:

$$\begin{aligned}
\int d^3x \; f(\vec{x})g(\vec{x}) &= \frac{1}{(2\pi)^6}\int d^3x \; d^3k \; d^3k' \; f(\vec{k})g(\vec{k}')e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} &(60)\\
&= \frac{1}{(2\pi)^3}\int d^3k \; d^3k' \; \delta^{(3)}(\vec{k}+\vec{k}')f(\vec{k})g(\vec{k}') &(61)\\
&= \frac{1}{(2\pi)^3}\int d^3k f(\vec{k})g(-\vec{k}) &(62)
\end{aligned}$$

In the second line, we do the integral for x first and it yields the δ function. So, that we can reduce the integral for one momentum space.

Finally, the Hamiltonian can be written,

$$\begin{aligned}
H &= \int d^3x \; \mathcal{H} &(63)\\
&\Leftarrow \left[\nabla\phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\nabla(\phi(k)\cdot e^{i\vec{k}\cdot\vec{x}}) = \int \frac{d^3k}{(2\pi)^3} \; i\vec{k}\cdot\phi(\vec{k})\cdot e^{i\vec{k}\cdot\vec{x}}\right] &(64)\\
&= \int \frac{d^3k}{(2\pi)^3}\left(\pi(\vec{k},t)\pi(-\vec{k},t) + k^2\phi(\vec{k},t)\phi(-\vec{k},t) + m^2\phi(\vec{k},t)\phi(-\vec{k},t)\right) &(65)\\
&= \int \frac{d^3k}{(2\pi)^3}\left(\pi(\vec{k},t)\pi(-\vec{k},t) + \omega_{\vec{k}}^2\phi(\vec{k},t)\phi(-\vec{k},t)\right) &(66)
\end{aligned}$$

Then, we could substitute $\phi(\vec{k},t)$ and $\pi(\vec{k},t))$ with the results from Pb.2(b):

$$
\begin{aligned}
H &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\Big(\pi(\vec{k},t)\pi(-\vec{k},t) + \omega_{\vec{k}}^2\phi(\vec{k},t)\phi(-\vec{k},t)\Big) && (67)\\
&= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\Big(-\frac{\omega_{\vec{k}}}{2}(a_{\vec{k}}e^{-i\omega_{\vec{k}}t} - a^\dagger_{-\vec{k}}e^{i\omega_{\vec{k}}t})(a_{-\vec{k}}e^{-i\omega_{\vec{k}}t} - a^\dagger_{\vec{k}}e^{i\omega_{\vec{k}}t})\\
&\qquad\qquad + \omega_{\vec{k}}^2\frac{1}{2\omega_{\vec{k}}}(a_{\vec{k}}e^{-i\omega_{\vec{k}}t} + a^\dagger_{-\vec{k}}e^{i\omega_{\vec{k}}t})(a_{-\vec{k}}e^{-i\omega_{\vec{k}}t} + a^\dagger_{\vec{k}}e^{i\omega_{\vec{k}}t})\Big) && (68)\\
&= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{\omega_{\vec{k}}}{2}\cdot 2(a_{\vec{k}}a^\dagger_{\vec{k}} + a^\dagger_{-\vec{k}}a_{-\vec{k}}) && (69)\\
&= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\omega_{\vec{k}}\Big([a_{\vec{k}}, a^\dagger_{\vec{k}}] + 2a^\dagger_{\vec{k}}a_{\vec{k}}\Big) && (70)\\
&= \frac{1}{2}\int d^3k\ \omega_{\vec{k}}\delta(0) + \int \frac{d^3k}{(2\pi)^3}\omega_{\vec{k}}a^\dagger_{\vec{k}}a_{\vec{k}} && (71)
\end{aligned}
$$

This can be written as follows:

$$
H = \int \frac{d^3k}{(2\pi)^3}\omega_{\vec{k}}N_{\vec{k}} + E_0 \qquad (72)
$$

where $N_{\vec{k}} = a^\dagger_{\vec{k}}a_{\vec{k}}$ is the number operator, and $E_0 = \frac{1}{2}\delta(0)\int d^3k\ \omega_{\vec{k}}$ is the zero-point energy.

(b) Express the conserved charges P^i for spatial translations for (28) in terms of $a_{\vec{k}}$and $a^\dagger_{\vec{k}}$.

The conserved charge P^i can be calculated as below (referring to the P set 1 Pb.4 (c)):

$$
\begin{aligned}
P^i &= \int d^3x\pi\partial^i\phi && (73)\\
&= \int \frac{d^3k}{(2\pi)^3}\pi(\vec{k},t)(-ik^i)\phi(-\vec{k},t) && (74)\\
&= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}k^i(a_{\vec{k}}(t) - a^\dagger_{-\vec{k}}(t))(a_{-\vec{k}}(t) + a^\dagger_{\vec{k}}(t)) && (75)\\
&= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}k^i(a_{\vec{k}}a_{-\vec{k}}e^{-2i\omega_{\vec{k}}t} + a_{\vec{k}}a^\dagger_{\vec{k}} - a^\dagger_{-\vec{k}}a_{-\vec{k}} - a^\dagger_{-\vec{k}}a^\dagger_{\vec{k}}e^{2i\omega_{\vec{k}}t}) && (76)
\end{aligned}
$$

In the third line, considering the commutation relations with k^i , the first term and the fourth term are odd under the change of variable $\vec{k} \to -\vec{k}$ such that they must vanish. Then the remaining terms can be computed below:

$$
\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}k^i(a_{\vec{k}}a^\dagger_{\vec{k}} + a^\dagger_{\vec{k}}a_{\vec{k}}) = \int \frac{d^3k}{(2\pi)^3}k^i a^\dagger_{\vec{k}}a_{\vec{k}} + \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}k^i(2\pi)^3\delta(0) \qquad (77)
$$

With the result from Pb.3(a) we can regard P^i as

$$
P^i = \int \frac{d^3k}{(2\pi)^3}k^iN_{\vec{k}} \to P^\mu = \int \frac{d^3k}{(2\pi)^3}k^\mu N_{\vec{k}} + \delta^{\mu 0}E_0 \quad \text{where} \quad k^\mu = \omega_{\vec{k}} \qquad (78)
$$

(c) Starting with

$$\phi(0,0) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} (a_{\vec{k}} + a^\dagger_{\vec{k}}) \tag{79}$$

show that under the action of translation operators

$$\phi(\vec{x},t) = e^{iHt - iP^i x^i} \phi(0,0) e^{-iHt + iP^i x^i} \tag{80}$$

Note : This problem becomes trivial if you recall the following formual for a harmonic oscillator

$$e^{i\alpha N} a e^{-i\alpha N}, \quad N = a^\dagger a \text{ and } \alpha \text{ is a constant} \tag{81}$$

From BCH fromula $e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \cdots$, we could first compute the hint.

$$[i\alpha N, a] = i\alpha[a^\dagger a, a] = -i\alpha a \tag{82}$$

$$e^{i\alpha N} a e^{-i\alpha N} = a + (-i\alpha a) + \frac{1}{2!}(-i\alpha)^2 a + \cdots = e^{-i\alpha} a \tag{83}$$

Similarly for $a^\dagger$, $e^{i\alpha N} a^\dagger e^{-i\alpha N} = e^{i\alpha} a^\dagger$.

Generalization to $a_{\vec{k}}$

$$\exp\left(i \int \frac{d^3k'}{(2\pi)^3} \alpha(k') N_{\vec{k}'}\right) a_{\vec{k}} \exp\left(-i \int \frac{d^3k'}{(2\pi)^3} \alpha(k') N_{\vec{k}'}\right) \tag{84}$$

$$= \exp\left(i \int \frac{d^3k'}{(2\pi)^3} \delta^{(3)}(\vec{k}' - \vec{k}) \alpha(k') N_{\vec{k}'}\right) a_{\vec{k}} \exp\left(-i \int \frac{d^3k'}{(2\pi)^3} \delta^{(3)}(\vec{k}' - \vec{k}) \alpha(k') N_{\vec{k}'}\right) \tag{85}$$

$$= e^{i\alpha(\vec{k}) N_{\vec{k}}} a_{\vec{k}} e^{-i\alpha(\vec{k}) N_{\vec{k}}} \tag{86}$$

$$= e^{-i\alpha(\vec{k})} a_{\vec{k}} \tag{87}$$

Also the same result for $a^\dagger_{\vec{k}} \Rightarrow e^{i\alpha} a^\dagger_{\vec{k}}$.

Then, returning to the problem, we use the expression H and P^i from Pb.3(a) and Pb.3(b) such that

$$e^{iHt - iP^i x^i} a_{\vec{k}} e^{-iHt + iP^i x^i} = e^{i \int \frac{d^3k'}{(2\pi)^3} (\omega_{\vec{k}'} t - \vec{k}' \cdot \vec{x}) N_{\vec{k}'} + iE_0 t} a_{\vec{k}} e^{-i \int \frac{d^3k'}{(2\pi)^3} (\omega_{\vec{k}'} t - \vec{k}' \cdot \vec{x}) N_{\vec{k}'} - iE_0 t} \tag{88}$$

$$= e^{-i(\omega_{\vec{k}} t - \vec{k} \cdot \vec{x})} a_{\vec{k}} \tag{89}$$

$$e^{iHt - iP^i x^i} a^\dagger_{\vec{k}} e^{-iHt + iP^i x^i} = e^{i(\omega_{\vec{k}} t - \vec{k} \cdot \vec{x})} a^\dagger_{\vec{k}} \tag{90}$$

Therefore, Eq.(80) become

$$e^{iHt - iP^i x^i} \phi(0,0) e^{-iHt + iP^i x^i} = e^{iHt - iP^i x^i} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} (a_{\vec{k}} + a^\dagger_{\vec{k}}) e^{-iHt + iP^i x^i} \tag{91}$$

$$= \int \frac{d^3k}{(2\pi)^3} (e^{-i(\omega_{\vec{k}} t - \vec{k} \cdot \vec{x})} a_{\vec{k}} + e^{i(\omega_{\vec{k}} t - \vec{k} \cdot \vec{x})} a^\dagger_{\vec{k}}) \tag{92}$$

$$= \phi(\vec{x}, t) \tag{93}$$

4.Noether charges for Lorentz symmetries of the real scalar field theory

In this problem we work out the conserved current corresponding to Lorentz symmetries of (25).

(a) Consider an infinitesimal Lorentz transformation . Show that $\Lambda^{\nu}_{\mu} = \delta^{\nu}_{\mu} + \omega^{\nu}_{\mu}$ where $\omega_{\mu\nu} = -\omega_{\nu\mu}$, $\omega^{\nu}_{\mu} = \eta^{\nu\lambda}\omega_{\mu\lambda}$ satisfies $\Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\lambda}\eta^{\rho\lambda} = \eta^{\mu\nu}$ to first order in $\omega_{\mu\nu}$

$$\begin{aligned}
\Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\lambda}\eta^{\rho\lambda} &= (\delta^{\mu}_{\rho} + \omega^{\mu}_{\rho})(\delta^{\nu}_{\lambda} + \omega^{\nu}_{\lambda})\eta^{\rho\lambda} && (94)\\
&= \eta^{\mu\nu} + \omega^{\mu}_{\nu}\eta^{\lambda\nu} + \omega^{\mu}_{\nu}\eta^{\nu\lambda} + O(\omega^2) && (95)\\
&= \eta^{\mu\nu} + \omega^{\lambda\mu} + \omega^{\mu\lambda} + O(\omega^2) && (96)\\
&= \eta^{\mu\nu} + O(\omega^2) && (97)
\end{aligned}$$

(b) Write down how ϕ transforms under an infinitesimal Lorentz transformation and show that the conserved Noether current for this transformation can be written as:

$$J^{\mu\lambda\nu} = x^{\lambda}T^{\mu\nu} - x^{\nu}T^{\mu\lambda} \qquad (98)$$

where $T^{\mu\nu}$ is the conserved energy-momentum tensor which we have already derived in P-set 1.

Note: this part does not involve complicated calculations. If you find yourself in a massive calculation, pause, and try to find a simpler approach.

Under the Lorentz scalar transformation, a scalar field transforms to $\phi(x) \to \phi'(x') = \phi(x)$. So, the inifinitesimal difference under Lorentz transformation of the scalar field becomes:

$$\begin{aligned}
\delta\phi &= \phi'(x) - \phi(x) && (99)\\
&= \phi((\Lambda^{-1})^{\mu}_{\nu}x^{\nu}) - \phi(x^{\nu}) && (100)\\
&= \phi((\delta^{\mu}_{\nu} - \omega^{\mu}_{\nu})x^{\nu}) - \phi(x^{\nu}) && (101)\\
&= \phi(x^{\mu}) - \omega^{\mu}_{\nu}x^{\mu}\partial_{\mu}\phi(x^{\mu}) - \phi(x^{\nu}) && (102)\\
&= -\omega^{\mu}_{\nu}x^{\mu}\partial_{\mu}\phi(x^{\mu}) && (103)
\end{aligned}$$

Therefore, the infinitesimal Lorentz transformation of the Lagrangian is;

$$\begin{aligned}
\delta\mathcal{L} &= \mathcal{L}[\phi'] - \mathcal{L}[\phi] && (104)\\
&= \mathcal{L}[\phi - \omega^{\mu}_{\nu}x^{\nu}\partial_{\mu}\phi] - \mathcal{L}[\phi] && (105)\\
&= -\omega^{\mu}_{\nu}x^{\nu}\partial_{\mu}\phi\frac{\partial\mathcal{L}}{\partial\phi} && (106)\\
&= -\omega^{\mu}_{\nu}x^{\nu}\partial_{\mu}\mathcal{L} && (107)\\
&= -\partial_{\mu}(\omega^{\mu}_{\nu}x^{\nu}\mathcal{L}) && (108)
\end{aligned}$$

We can rewrite the difference as $\delta\mathcal{L} = \partial_{\mu}\mathcal{F}^{\mu}$ where $\mathcal{F}^{\mu} = -\omega^{\mu}_{\nu}x^{\nu}\mathcal{L}$.

The conserved Noether current for this transformation becomes:

$$j^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi - \mathcal{F}^\mu \tag{109}$$
$$= \omega_\nu^\lambda x^\nu \partial^\mu \partial_\nu \phi + \omega_\nu^\mu x^\nu \mathcal{L} \tag{110}$$
$$= \omega_{\lambda\nu} x^\nu T^{\mu\lambda} \tag{111}$$

for the energy-momentum tensor from P-set 1

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi + \eta^{\mu\nu}\mathcal{L} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}(\partial_\rho\phi\partial^\rho\phi + m^2\phi^2) \tag{112}$$

Note that $\omega_{\lambda\nu}$ is an arbitrary antisymmetric tensor which parameterizes the infinitesimal transformation. In total we have an antisymmetric tensor worth of conserved currents, which we can package in $J^{\mu\lambda\nu}$:

$$J^{\mu\lambda\nu} = x^\lambda T^{\mu\nu} - x^\nu T^{\mu\lambda}, \quad j^\mu = -\frac{1}{2}\omega_{\lambda\nu}J^{\mu\lambda\nu} \tag{113}$$

(c) Use the conservation of the energy-momentum tensor to verify that the current is indeed conserved, i.e.

$$\partial_\mu J^{\mu\lambda\nu} = 0 \tag{114}$$

$$\partial_\mu J^{\mu\lambda\nu} = \partial_\mu(x^\lambda T^{\mu\nu} - x^\nu T^{\mu\lambda}) = \delta_\mu^\lambda T^{\mu\nu} + x^\lambda \partial_\mu T^{\mu\nu} - \delta_\mu^\nu T^{\mu\lambda} - x^\nu \partial_\mu T^{\mu\lambda} \tag{115}$$
$$= T^{\lambda\nu} - T^{\nu\lambda} = 0 \tag{116}$$

Since energy-momentum tensor conserves, $\partial_\mu T^\mu \nu = 0$. And since $T^{\mu\nu}$ is symmetric, the last equation is zero.

(d) **(Bonus)** Consider the conserved charges associated with $J^{\mu\lambda\nu}$

$$M^{\lambda\nu} = \int d^3x J^{0\lambda\nu} \tag{117}$$

Express the conserved charges $M^{\mu\nu}$ for Lorentz symmetries for Eq.(25) in terms of $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$.

From Pb.4(b),

$$M^{\lambda\nu} = \int d^3x J^{0\lambda\nu} = \int d^3x(x^\lambda T^{0\nu} - x^\nu T^{0\lambda}) \tag{118}$$

Therefore, $M^{\mu\nu}$ is antisymmetric in μ and ν, so we need to compute M^{0i} and M^{ij}. First, consider the $T^{0\mu}$ in creation and annihilation operators:

$$\begin{aligned} T^{0\mu} &= -\pi\partial^\mu\phi - \frac{1}{2}\eta^{0\mu}(-\pi^2 + (\nabla\phi)^2 + m^2\phi^2) && (119)\\ &= \left(\frac{1}{2}(\pi^2 + (\nabla\phi)^2 + m^2\phi^2), -\pi\partial^i\phi\right) = (\mathcal{H}(x), \mathcal{P}^i(x)) && (120)\end{aligned}$$

Since $M^{\mu\nu}$ is conserved, we compute at $t = 0$. We need an identity

$$\begin{aligned} \int d^3x\ x^i f(x)g(x) &= \int d^3x d^3k d^3k' e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}\ x^i f(k)g(k') && (121)\\ &= \frac{1}{(2\pi)^3}\int d^3k d^3k'(-i\partial_{k^i}\delta^{(3)}(\vec{k}+\vec{k}'))f(k)g(k') && (122)\\ &= i\int \frac{d^3k}{(2\pi)^3}\partial_{k^i} f(k)g(-k) && (123)\end{aligned}$$

where we have used the integration by parts in the last equation. More generally, if there are derivatives acting on f or g, each derivative acting on f drags down a factor of $i\vec{k}$, while each derivative acting on g drags down a factor of $i\vec{k}' \to -i\vec{k}$ after performing d^3x.

$$
\begin{aligned}
M^{0i} &= \int d^3x \left(t\mathcal{P}^i(x) - x^i\mathcal{H}(x)\right)|_{t=0} && (124)\\
&= -\frac{1}{2}\int d^3x x^i(\pi^2 + (\nabla\phi)^2 + m^2\phi^2) && (125)\\
&= \frac{i}{2}\int \frac{d^3k}{(2\pi)^3}(\partial_{k^i}\pi(\vec{k})\pi(-\vec{k}) + \omega_{\vec{k}}^2\partial_{k^i}\phi(\vec{k})\phi(-\vec{k})) && (126)\\
&= -\frac{i}{4}\int \frac{d^3k}{(2\pi)^3}(-\partial_{k^i}(\sqrt{\omega_{\vec{k}}}(a_{\vec{k}} - a^\dagger_{-\vec{k}}))(\sqrt{\omega_{\vec{k}}}(a_{-\vec{k}} - a^\dagger_{\vec{k}}))\\
&\qquad + \omega_{\vec{k}}^2\partial_{k^i}(\frac{1}{\sqrt{\omega_{\vec{k}}}}(a_{\vec{k}} + a^\dagger_{-\vec{k}}))(\frac{1}{\sqrt{\omega_{\vec{k}}}}(a_{-\vec{k}} + a^\dagger_{\vec{k}}))) && (127)\\
&= \frac{i}{4}\int \frac{d^3k}{(2\pi)^3}\Big((k^i + \omega_{\vec{k}}\partial_{k^i})(a_{\vec{k}} - a^\dagger_{-\vec{k}})(a_{-\vec{k}} - a^\dagger_{\vec{k}}) + (k^i - \omega_{\vec{k}}\partial_{k^i})(a_{\vec{k}} + a^\dagger_{-\vec{k}})(a_{-\vec{k}} + a^\dagger_{\vec{k}})\Big) && (128)\\
&= -\frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\omega_{\vec{k}}\Big((\partial_{k^i}a_{\vec{k}})a^\dagger_{\vec{k}} + (\partial_{k^i}a^\dagger_{-\vec{k}})a_{-\vec{k}}\Big) && (129)\\
&= -\frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\omega_{\vec{k}}\Big((\partial_{k^i}a_{\vec{k}})a^\dagger_{\vec{k}} - (\partial_{k^i}a^\dagger_{\vec{k}})a_{\vec{k}}\Big) && (130)\\
&= -\frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\omega_{\vec{k}}\Big(a^\dagger_{\vec{k}}(\partial_{k^i}a_{\vec{k}}) + (2\pi)^3\delta(\vec{k} - \vec{k}')|_{\vec{k}=\vec{k}'} - (\partial_{k^i}a^\dagger_{\vec{k}})a_{\vec{k}}\Big) && (131)\\
&= -\frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\omega_{\vec{k}}\Big(a^\dagger_{\vec{k}}(\partial_{k^i}a_{\vec{k}}) - (\partial_{k^i}a^\dagger_{\vec{k}})a_{\vec{k}}\Big) && (132)
\end{aligned}
$$

Then similar for M^{ij},

$$
\begin{aligned}
M^{ij} &= \int d^3x \left(x^i\mathcal{P}^j(x) - x^j\mathcal{P}^i(x)\right) && (133)\\
&= \int d^3x \left(x^i\pi\partial^j\phi(x) - x^j\pi\partial^i\phi(x)\right) && (134)\\
&= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\Big(k^j\pi(\vec{k})\partial_{k^i}\phi(-\vec{k}) - (i \leftrightarrow j)\Big) && (135)\\
&= -\frac{i}{4}\int \frac{d^3k}{(2\pi)^3}\left(k^j\sqrt{\omega_{\vec{k}}}(a_{\vec{k}} - a^\dagger_{-\vec{k}})\partial_{k^i}(\frac{1}{\sqrt{\omega_{\vec{k}}}}(a_{-\vec{k}} + a_{\vec{k}})) - (i \leftrightarrow j)\right) && (136)\\
&= -\frac{i}{4}\int \frac{d^3k}{(2\pi)^3}\left(k^j\sqrt{\omega_{\vec{k}}}(a_{\vec{k}} - a^\dagger_{-\vec{k}})(-\frac{k^i}{\omega_{\vec{k}}} + \partial_{k^i})(a_{-\vec{k}} + a_{\vec{k}}) - (i \leftrightarrow j)\right) && (137)\\
&= -\frac{i}{4}\int \frac{d^3k}{(2\pi)^3}k^j(a_{\vec{k}}\partial_{k^i}a_{-\vec{k}} + a_{\vec{k}}\partial_{k^i}a^\dagger_{\vec{k}} - a^\dagger_{-\vec{k}}\partial_{k^i}a_{-\vec{k}} - a^\dagger_{-\vec{k}}\partial_{k^i}a^\dagger_{-\vec{k}}) - (i \leftrightarrow j) && (138)\\
&= -\frac{i}{4}\int \frac{d^3k}{(2\pi)^3}k^j(-(\partial_{k^i}a_{\vec{k}})a^\dagger_{\vec{k}} - a^\dagger_{\vec{k}}(\partial_{k^i}a_{\vec{k}}) - (i \leftrightarrow j) && (139)\\
&= -\frac{i}{4}\int \frac{d^3k}{(2\pi)^3}k^j((2\pi)^3\delta(\vec{k} - \vec{k}')|_{\vec{k}=\vec{k}'} - 2a^\dagger_{\vec{k}}(\partial_{k^i}a_{\vec{k}}) - (i \leftrightarrow j) && (140)\\
&= -\frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\Big(k^i a^\dagger_{\vec{k}}(\partial_{k^i}a_{\vec{k}}) - k^j(\partial_{k^i}a^\dagger_{\vec{k}})a_{\vec{k}}\Big) && (141)
\end{aligned}
$$

Altogether,

$$M^{\mu\nu} = -\frac{i}{2}\int \frac{d^3k}{(2\pi)^3} k^{\mu}(a^{\dagger}_{\vec{k}}(\partial_{k_\nu} a_{\vec{k}}) - (\partial_{k_\nu} a^{\dagger}_{\vec{k}})a_{\vec{k}}) - (\mu \leftrightarrow \nu) \tag{142}$$