CpE 645 Image Processing and Computer Vision

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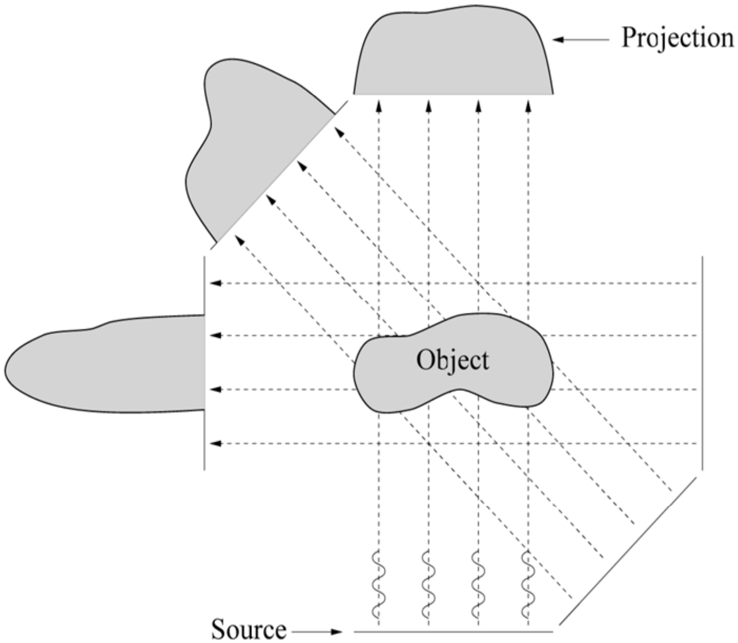
Reconstruction From Projections

- In 1895, Wilhelm Conrad Rontgen discovered that x-rays can pass through substantial amount of matter, which forms the foundation of medical X-ray imaging
- Limitations of normal X-ray images:
  - overlapping structures in the body
  - difficulties to distinguish between similar tissues
- In 1971, the first computed tomography (CT) scanner was introduced, in which X-rays from many directions are passed through one section of the patient's body.
- In 1979, two of the CT's principle contributors, G. N. Hounsfield and A. M. Cormack shared the Nobel Prize in Medicine.

Reconstruction From Projections

- The main objective is to reconstruct cross-section images of internal organs from a set of projections.
- For X-rays, this cross-section image $\boldsymbol{x(t_1, t_2)}$ reveals the density of the object which varies spatially,.



- The starting point is a set of pictures (e.g. X-rays) taken from various angles around the body. These projections measure the attenuation of the propagating wave from the source to the detector.

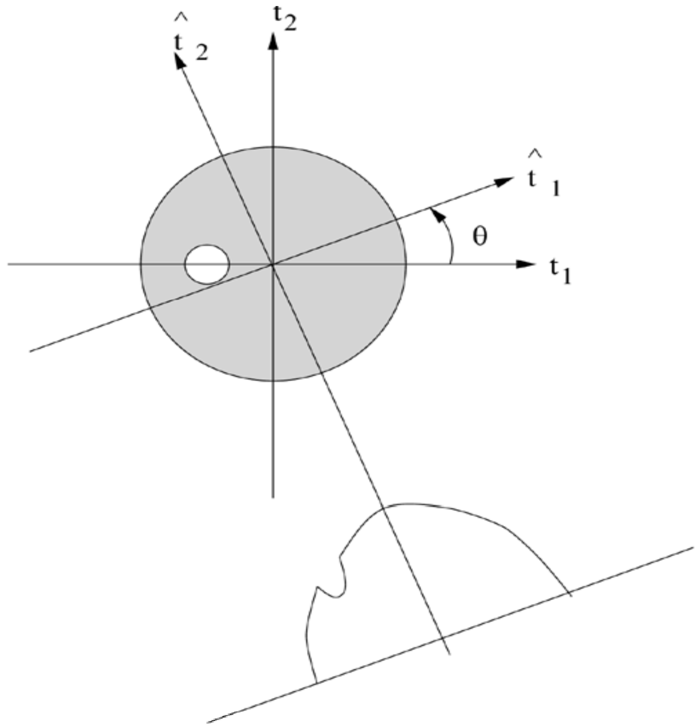
Reconstruction From Projections

- For each projection angle θ, define a rotated coordinate system (t_1', t_2') w.r.t. the original coordinate system (t_1, t_2) :

$$\begin{bmatrix} t_1' \\ t_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

and

$$\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} t_1' \\ t_2' \end{bmatrix}$$



Reconstruction From Projections

- The effects of scattering and absorption result in an exponential attenuation of a beam as it passes through a material
- In case of a homogeneous object of density ρ, and length l, the output intensity is generally modeled as

$$I = I_0 e^{-\rho l}$$

where I_0 is the source intensity.

Reconstruction From Projections

- If the material is heterogeneous, the product ρl is replaced by a line integral along t_2' direction. Then the output intensity becomes

$$I(t_1') = I_0(t_1')\exp[-\int_{-\infty}^{\infty} x(t_1, t_2)\,dt_2']$$

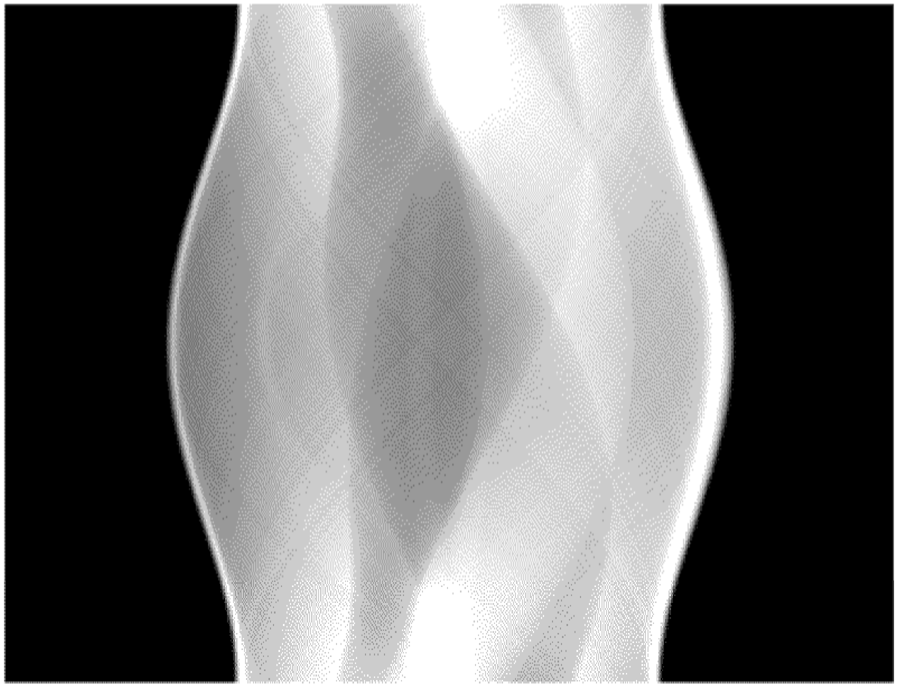
- Define the *projection* of the object at angle θ as $p_\theta(t_1')$

$$p_\theta(t_1') = -\log\frac{I(t_1')}{I_0(t_1')} = \int_{-\infty}^{\infty} x(t_1, t_2)\,dt_2' \qquad \text{for } 0 \le \theta < \pi$$

- $p_\theta(t_1')$ represents a set of line integrals of $x(t_1, t_2)$ computed along the line perpendicular to the axis t_1' at different displacement along t_1' direction.

Projection Example

Phantom (256x256)



180 projections

Reconstruction From Projections

- Consider the coordinate system conversion, we have

$$p_\theta(t_1') = \int_{-\infty}^{\infty} x\left(t_1'\cos\theta - t_2'\sin\theta,\ t_1'\sin\theta + t_2'\cos\theta\right)dt_2' \quad \text{for } \mathbf{0} \le \boldsymbol{\theta} < \boldsymbol{\pi}$$

- The relation between a 2D object, i.e. a 2D spatial function $\boldsymbol{x(t_1, t_2)}$, and its projections is called the *Radon Transform*
- Take a 1-D Fourier Transform of $\boldsymbol{p_\theta(t_1')}$, we have

$$P_\theta(\Omega) = \int_{-\infty}^{\infty} p_\theta(t_1')e^{-j\Omega t_1'}\,dt_1'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1'\cos\theta - t_2'\sin\theta, t_1'\cos\theta + t_2'\sin\theta)\, e^{-j\Omega t_1'}\,dt_2'\,dt_1'$$

Reconstruction From Projections

- Convert $P_\theta(\Omega)$ back to the original coordinate system (t_1, t_2), we have

$$P_\theta(\Omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1, t_2) e^{-j\Omega(t_1\cos\theta + t_2\sin\theta)} \frac{\partial(t_1', t_2')}{\partial(t_1, t_2)} dt_2 dt_1$$

where the Jacobian is given by

$$\frac{\partial(t_1', t_2')}{\partial(t_1, t_2)} = \begin{vmatrix} \frac{\partial t_1'}{\partial t_1} & \frac{\partial t_1'}{\partial t_1} \\ \frac{\partial t_1'}{\partial t_1} & \frac{\partial t_1'}{\partial t_1} \end{vmatrix} = \cos\theta \cdot \cos\theta + \sin\theta \cdot \sin\theta = 1$$

and then $P_\theta(\Omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1, t_2) e^{-j\Omega(t_1\cos\theta + t_2\sin\theta)} dt_2 dt_1$

Reconstruction From Projections

- Compare $\boldsymbol{P_\theta(\Omega)}$ with the 2D Fourier Transform of the object $\boldsymbol{x(t_1, t_2)}$,

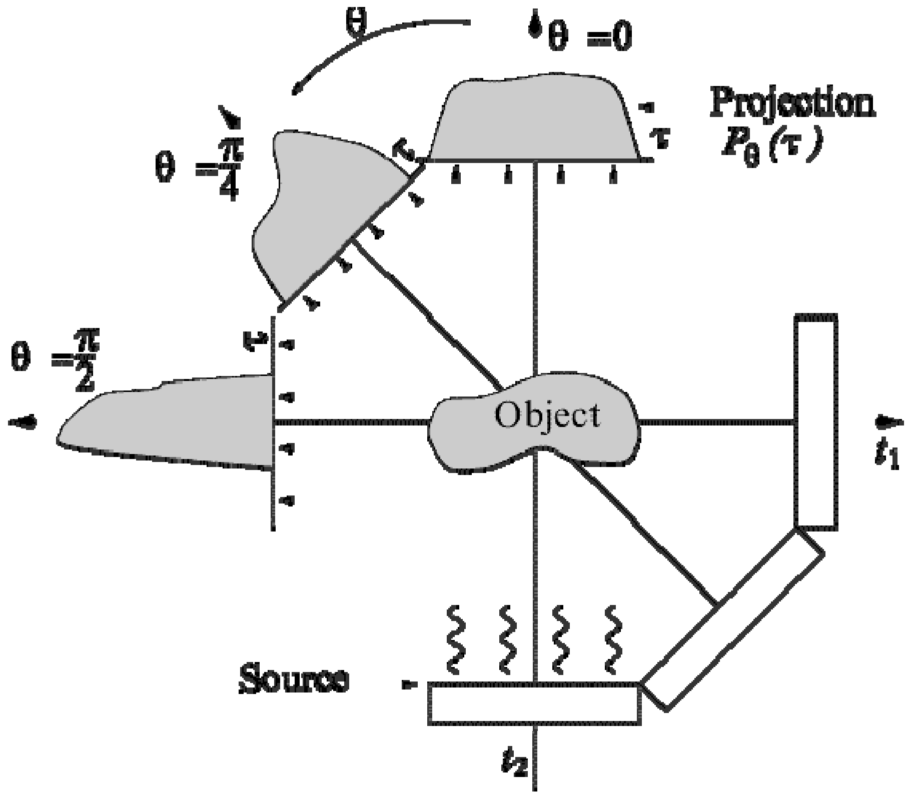
$$X(\Omega_1, \Omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1, t_2) e^{-j\Omega_1 t_1} e^{-j\Omega_2 t_2} dt_1 dt_2$$

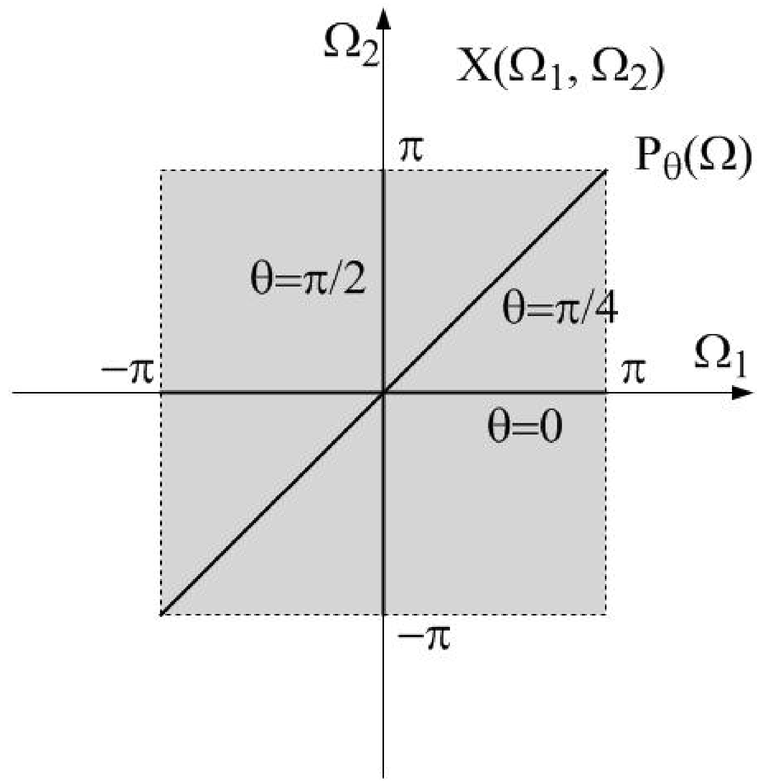
we can observe that $\underline{\boldsymbol{P_\theta(\Omega)} \text{ is identical to } \boldsymbol{X(\Omega_1, \Omega_2)}}$

<u>evaluated at</u>

$$\begin{cases} \Omega_1 = \Omega\cos\theta \\ \Omega_2 = \Omega\sin\theta \end{cases}$$

Reconstruction From Projections





$$P_\theta(\Omega) = X(\Omega_1, \Omega_2)\Big|_{\substack{\Omega_1 = \Omega\cos\theta \\ \Omega_2 = \Omega\sin\theta}}$$

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Projection Slice Theorem

- The 1-D Fourier Transform of the projection at angle θ is the 2-D Fourier Transform of object $x(t_1, t_2)$ evaluated along a line in the (Ω_1, Ω_2) plane passing through the origin at the same angle θ.
- This cross-section through the 2-D Fourier Transform is called the slice of $X(\Omega_1, \Omega_2)$ at angle θ.
- So, knowledge of several projections of an object provides knowledge of the Fourier Transform along radial lines in the Fourier plane.
- Once we have enough knowledge of the object in its Fourier domain, i.e. $X(\Omega_1, \Omega_2)$, we can reconstruct the object $x(t_1, t_2)$ in spatial domain.

Sampling of Projections

- Sampling of projections determines: number of projections, and number of samples from each projection.
- Let us assume that $\boldsymbol{x(t_1, t_2)}$ is band-limited with its maximum spatial frequency $\boldsymbol{\Omega_{max}}$, each of the projections can be sampled with no loss of information provided that the samples are taken no more than $\boldsymbol{\pi/\Omega_{max}}$ apart.
- Given that we can only obtain a finite number of projections $\boldsymbol{N}$, this number depends on some *apriori* knowledge about the object being reconstructed and how much detail we expect to preserve.
- Example:

  Consider a circular and homogeneous object, it can be characterized by only one projection at $\theta = 0$.

Sampling of Projections

- If nothing is known about $\boldsymbol{x(t_1, t_2)}$, it follows from the projection-slice theorem that an infinite number of projections will be required to reconstruct the object.
- Practically, if $\boldsymbol{x(t_1, t_2)}$ has an effective diameter $\boldsymbol{d}$ and we need to resolve details as small as $\boldsymbol{r}$, then the rule of thumb is

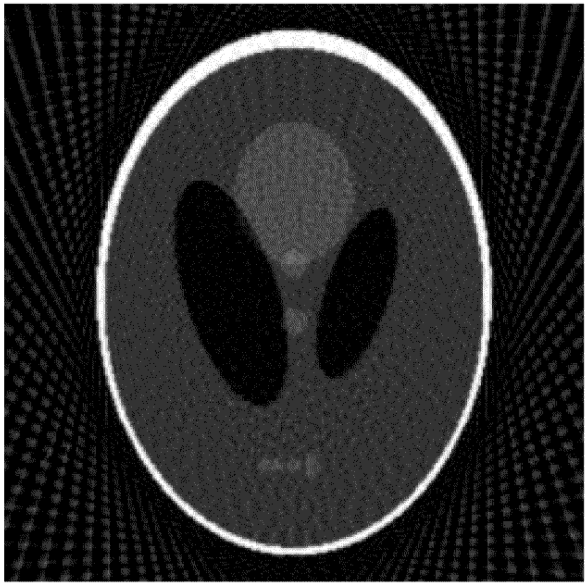
$$N > \frac{\pi d}{r}$$

Note that the number of projections increase as the desired resolution improves.

Reconstruction Examples

180 projections

90 projections

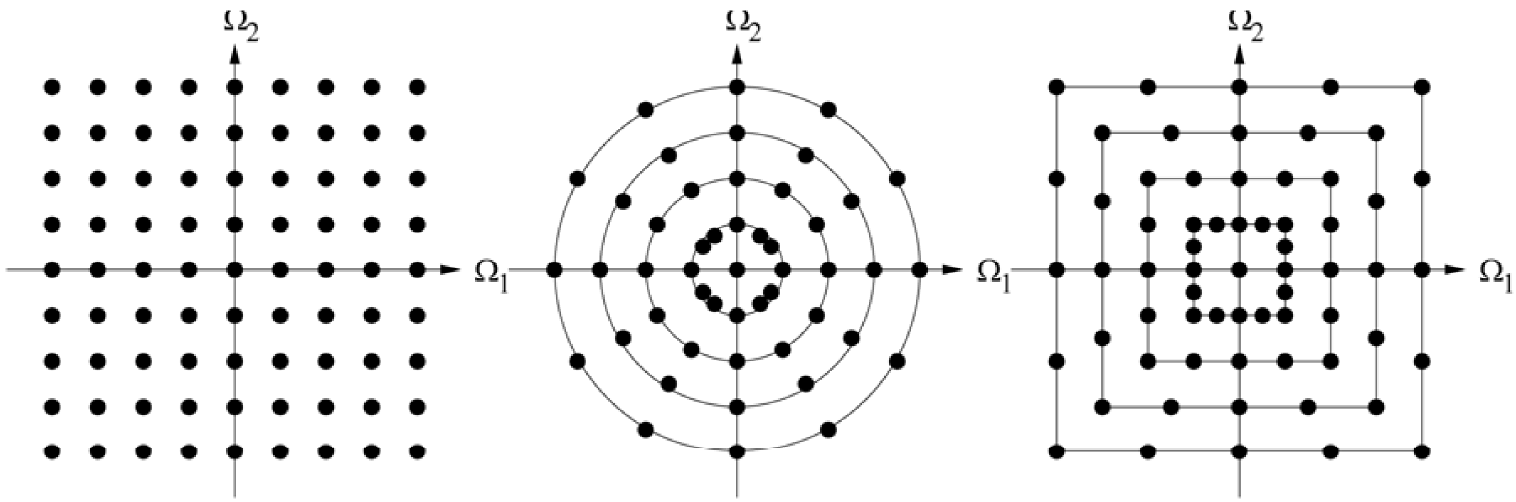


45 projections

Fourier Domain Reconstruction

- The problem of reconstructing $\boldsymbol{x(t_1, t_2)}$ is equivalent to the problem of interpolating the DFT from all the slices obtained through projections.
- The 2-D DFT represents samples of the 2-D Fourier Transform evaluated on a rectangular grid of points. The 1-D DFT slices however only provide samples located on a polar grid.
- We can interpolate from the known transform values on the polar raster to the unknown ones on the DFT raster,
- Once sufficient number of samples in the 2D DFT plane is obtained, we can perform the inverse DFT and obtain a reconstruction of $\boldsymbol{x(t_1, t_2)}$.

Fourier Domain Reconstruction



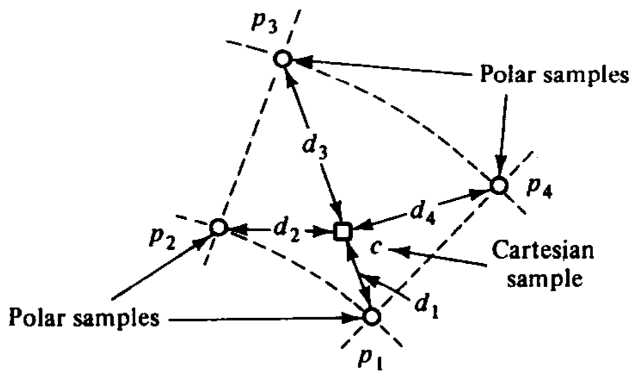
(a) (b) (c)

(a) Normal 2D DFT sample grid.

(b) Evenly sampling on 1D DFT of every projection

(c) Unevenly sampling on 1D DFT of projections to achieve a grid that is closer to rectangular grid

Fourier Domain Reconstruction



- Most points on the DFT grid are surrounded by four polar samples.
- Zero-order interpolation: the DFT sample is assigned the value of the nearest polar sample
- Linear interpolation: the DFT sample is assigned a weighted average of the four polar samples. The weighting varies inversely with the Euclidean distance between the points. This is generally superior to zero-order one.

Fourier Domain Reconstruction

- One way of simplifying the 2-D interpolation process is to transform it into a 1-D interpolation problem.
- This can be done if we have the capability to vary the sampling period for projections at different angles. We can then choose these sampling periods such that the available samples in the (Ω_1, Ω_2) plane are located on concentric squares instead of concentric circles.
- Then the samples on the rectangular grid can be interpolated from available values located on the same side of one of the concentric squares.
- The sampling period at angle θ must be chosen as
- $$T(\theta) = T(0)\max\left(|\cos\theta|, |\sin\theta|\right)$$

Convolution / Back Projection

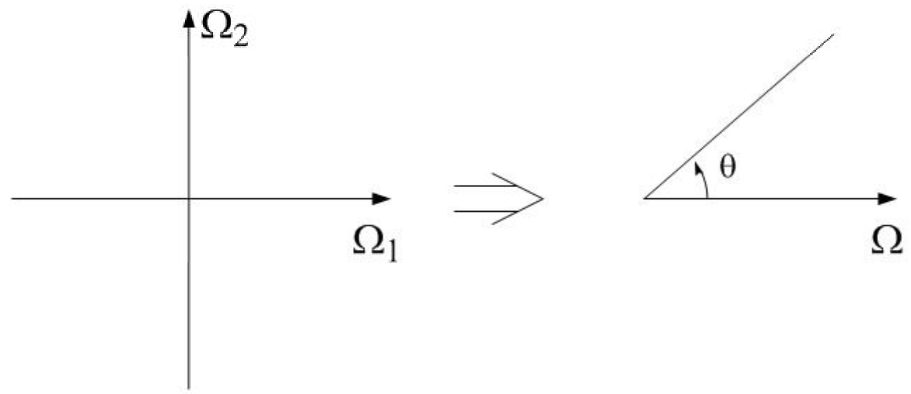
- Using the projection-slice theorem, we can derive the inverse Radon transform for the special case where all projections are known for all θ over the interval **[0, π]**

$$x(t_1, t_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(\Omega_1, \Omega_2) e^{j(\Omega_1 t_1 + \Omega_2 t_2)} d\Omega_1 d\Omega_2$$

- Describing the 2-D Fourier Transform in polar coordinates (Ω, θ):

$$\begin{cases} \Omega_1 = \Omega \cos\theta \\ \Omega_2 = \Omega \sin\theta \end{cases}$$

Convolution / Back Projection



$$d\Omega_1 d\Omega_2 = \begin{vmatrix} \dfrac{\partial\Omega_1}{\partial\Omega} & \dfrac{\partial\Omega_1}{\partial\theta} \\ \dfrac{\partial\Omega_2}{\partial\Omega} & \dfrac{\partial\Omega_2}{\partial\theta} \end{vmatrix} = |\Omega|\, d\Omega d\theta$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\Omega_1 d\Omega_2 = \int_0^{2\pi}\int_0^{\infty} \Omega d\Omega d\theta = \int_0^{\pi}\int_{-\infty}^{\infty} |\Omega|\, d\Omega d\theta$$

Convolution / Back Projection

- We have

$$x(t_1, t_2) = \frac{1}{4\pi^2} \int_0^{\pi} \int_{-\infty}^{\infty} X(\Omega\cos\theta, \Omega\sin\theta) e^{j\Omega(t_1\cos\theta + t_2\sin\theta)} |\Omega| d\Omega d\theta$$

- Notice that $P_\theta(\Omega) = X(\Omega\cos\theta, \Omega\sin\theta)$, then

$$x(t_1, t_2) = \frac{1}{4\pi^2} \int_0^{\pi} \int_{-\infty}^{\infty} P_\theta(\Omega) e^{j\Omega(t_1\cos\theta + t_2\sin\theta)} |\Omega| d\Omega d\theta$$

$$= \frac{1}{2\pi} \int_0^{\pi} \underline{\frac{1}{2\pi} \int_{-\infty}^{\infty} P_\theta(\Omega) |\Omega| e^{j\Omega(t_1\cos\theta + t_2\sin\theta)} d\Omega} d\theta$$

- Let $\tau = t_1\cos\theta + t_2\sin\theta$, we find that $g_\theta(\tau)$ is the inverse Fourier Transform of $P_\theta(\Omega)|\Omega|$

Convolution / Back Projection

- Since

$$\frac{-1}{j\tau} \leftrightarrow sign(\Omega)$$

$$\frac{d}{d\tau} \leftrightarrow j\Omega$$

$$p_\theta(\tau) \leftrightarrow P_\theta(\Omega)$$

$$\frac{d}{d\tau}\left(p_\theta(\tau) * \frac{1}{\tau}\right) \leftrightarrow P_\theta(\Omega) \cdot j\Omega \cdot \frac{sign(\Omega)}{j}$$

- we have

$$g_\theta(\tau) = \frac{d}{d\tau}\int_{-\infty}^{\infty} \frac{p_\theta(t)}{\tau - t}\,dt$$

and

$$x(t_1, t_2) = \frac{1}{2\pi}\int_0^{\pi}\left(\frac{d}{d\tau}\int_{-\infty}^{\infty}\frac{p_\theta(t)}{\tau - t}\,dt\right)d\theta$$

Convolution / Back Projection

- Practical implementation: since only a finite number of projections is available at angles θ_i , we can replace the integral over θ with the summation:

$$x(t_1,t_2)=\frac{1}{2\pi}\sum_{i=0}^{N-1} g_i(\tau)(\theta_i-\theta_{i-1})$$

- The image $x(t_1, t_2)$ can be reconstructed through
  - Take the projections $p_\theta(\tau)$ for $0 \leq \theta < \pi$
  - Compute $\mathcal{F}\{p_\theta(\tau)\} = P_\theta(\Omega)$
  - Filter with $|\Omega|$ in frequency domain
  - Compute $\mathcal{F}^{-1}\{P_\theta(\Omega)\,|\Omega|\}$
  - Back projection: accumulate rays of $\mathcal{F}^{-1}\{P_\theta(\Omega)\,|\Omega|\}$ at all $0 \leq \theta < \pi$ that pass through all spatial coordinate (t_1, t_2)

Convolution / Back Projection

- |Ω| is not a good filter since it amplifies high frequency noise. Some alternative filters exist: