ECE 178 W04
HOMEWORK #5 SOLUTIONS

TA: Evan Ruzanski

Problem 3.3

The transformations required to produce the individual bit planes are nothing more than mappings of the truth table for eight binary variables. In this truth table, the values of the 7^{th} bit are 0 for byte values 0 to 127, and 1 for byte values 128 to 255, thus giving the transformation mentioned in the problem statement. Note that the given transformed values of either 0 or 255 simply indicate a binary image for the 7^{th} bit plane. Any other two values would have been equally valid, though less conventional. Continuing with the truth table concept, the transformation required to produce an image of the 6^{th} bit plane outputs a 0 for byte values in the range [0, 63], a 1 for byte values in the range [64, 127], a 0 for byte values in the range [128, 191], and a 1 for byte values in the range [192, 255]. Similarly, the transformation for the 5^{th} bit plane alternates between eight ranges of byte values, the transformation for the 4^{th} bit plane alternates between 16 ranges, and so on. Finally, the output of the transformation for the 0^{th} bit plane alternates between 0 and 255 depending if the byte values are even or odd. Thus, this transformation alternates between 128 byte value ranges, which explains why an image of the 0^{th} bit plane is usually the busiest looking of all the bit plane images.

Problem 3.5

All that histogram equalization does is re-map histogram components on the intensity scale. To obtain a uniform (flat) histogram would require in general that pixel intensities be actually redistributed so that there are L groups of $n = L$ pixels with the same intensity, where L is the number of allowed discrete intensity levels and n is the total number of pixels in the input image. The histogram equalization method has no provisions for this type of (artificial) redistribution process.

Problem 3.6

Let n be the total number of pixels and let n_{rj} be the number of pixels in the input image with intensity value rj. Then, the histogram equalization transformation is

$$s_k = T(r_k) = \sum_{j=0}^{k} \frac{n_{rj}}{n} = \frac{1}{n} \sum_{j=0}^{k} n_{rj}$$

Since every pixel (and no others) with value r_k is mapped to value s_k, it follows that $n_{sk} = n_{rk}$. A second pass of histogram equalization would produce values v_k according to the transformation

$$v_k = T(s_k) = \frac{1}{n}\sum_{j=0}^{k} n_{sj}$$

But, $n_{sj} = n_{rj}$, so

$$v_k = T(s_k) = \frac{1}{n}\sum_{j=0}^{k} n_{rj} = s_k$$

which shows that a second pass of histogram equalization would yield the same result as the first pass. We have assumed negligible round-off errors.

Problem 3.10

First, we obtain the histogram equalization transformation:

$$s = T(r) = \int_{o}^{r} p_r(w)dw = \int_{0}^{r} (-2w+2)dw = -r^2 + 2r$$

Next we find,

$$v = G(z) = \int_{0}^{z} p_z(w)dw = \int_{0}^{z} 2wdw = z^2$$

Finally,

$$z = G^{-1}(v) = \pm\sqrt{v}$$

But only positive gray levels are allowed, so $z = \sqrt{v}$. Then we replace *v* with *s*, which in turn is $-r^2 + 2r$, and we have

$$z = \sqrt{-r^2 + 2r}$$

Problem 3.12

The purpose of this simple problem is to make the student think of the meaning of histograms and arrive at the conclusion that histograms carry no information about spatial properties of images. Thus, the only time that the histogram of the images formed by the operations shown in the problem statement can be determined in terms of the original histograms is when one or both of the images is (are) constant. In (d) we have the additional requirement that none of the pixels of $g(x,y)$ can be 0. Assume for convenience that the histograms are not normalized, so that, for example, $h_f(r_k)$ is the number of pixels in $f(x,y)$ having gray level r_k, assume that all the pixels in $g(x,y)$ have constant value *c*. The pixels of both images are assumed to be positive. Finally, let u_k denote the gray levels of the pixels of the images formed by any of the arithmetic operations given in the problem statement. Under the preceding set of conditions, the histograms are determined as follows:

(a) The histogram $h_{sum}(u_k)$ of the sum is obtained by letting $u_k = r_k + c$, and $h_{sum}(u_k) = h_f(r_k) \forall k$. In other words, the values (height) of the components of h_{sum} are the same as the components of h_f, but their locations on the gray axis are shifted right by an amount *c*.

(b) Similarly, the histogram $h_{diff}(u_k)$ of the difference has the same components as h_f but their locations are moved left by an amount *c* as a result of the subtraction operation.

(c) Following the same reasoning, the values (heights) of the components of histogram $h_{prod}(u_k)$ of the product are the same as h_f, but their locations are at $u_k = c \times r_k$. Note that while the spacing between components of the resulting histograms in (a) and (b) was not affected, the spacing between components of $h_{prod}(u_k)$ will be spread out by an amount *c*.

(d) Finally, assuming that $c \neq 0$, the components of $h_{div}(u_k)$ are the same as those of h_f, but their locations will be at $u_k = \frac{r_k}{c}$. Thus, the spacing between components of $h_{div}(u_k)$ will be compressed by an amount equal to $\frac{1}{c}$. The preceding solutions are applicable if image $f(x,y)$ also is constant. In this case the four histograms just discussed would each have only one component. Their location would be affected as described (a) through (c).

Problem 4.1

By direct substitution of $f(x)$ [Eq. (4.26)] into $F(u)$ [Eq. (4.25)]:

$$
\begin{aligned}
F(u) &= \frac{1}{M}\sum_{x=0}^{M-1}\left[\sum_{r=0}^{M-1} F(r)e^{j2\pi rx/M}\right]e^{-j2\pi ux/M} \\
&= \frac{1}{M}\sum_{r=0}^{M-1} F(r)\sum_{x=0}^{M-1} e^{j2\pi rx/M}e^{-j2\pi ux/M} \\
&= \frac{1}{M}F(u)(M) \\
&= F(u)
\end{aligned}
$$

where the third step follows from the orthogonality condition given in the problem statement. Substitution of $F(u)$ into $f(x)$ is handled in a similar manner.

Problem 4.2

This is a simple problem to familiarize the student with just the manipulation of the 2D Fourier transform and its inverse. The Fourier transform is linear iff:

$$\Im\left[a_1 f_1(x,y) + a_2 f_2(x,y)\right] = a_1\Im\left[f_1(x,y)\right] + a_2\Im\left[f_2(x,y)\right]$$

where a_1 and a_2 are arbitrary constants. From the definition of the 2D transform,

$$
\begin{aligned}
&\Im\left[a_1 f(x,y) + a_2 f(x,y)\right] = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\left[a_1 f_1(x,y) + a_2 f_2(x,y)\right]e^{-j2\pi(ux/M+vy/N)} \\
&= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} a_1 f_1(x,y)e^{-j2\pi(ux/M+vy/N)} + \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} a_2 f_2(x,y)e^{-j2\pi(ux/M+vy/N)} \\
&= a_1\Im\left[f_1(x,y)\right] + a_2\Im\left[f_2(x,y)\right]
\end{aligned}
$$

Problem 4.3

The inverse DFT of a constant A in the frequency domain is an impulse of strength A in the spatial domain. Convolving the impulse with the image copies (multiplies) the value of the impulse at each pixel location in the image.

<u>Problem 4.9</u>

The complex conjugate simply changes j to $-j$ in the inverse transform, so the image on the right is given by

$$\Im^{-1}\left[F^*(u,v)\right] = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} F(u,v)e^{-j2\pi(ux/M+vy/N)}$$
$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} F(u,v)e^{j2\pi(u(-x)/M+v(-y)/N)}$$
$$= f(-x,-y)$$

which simply mirrors $f(x,y)$ about the origin, thus producing the image on the right.