CpE 645 Image Processing and Computer Vision

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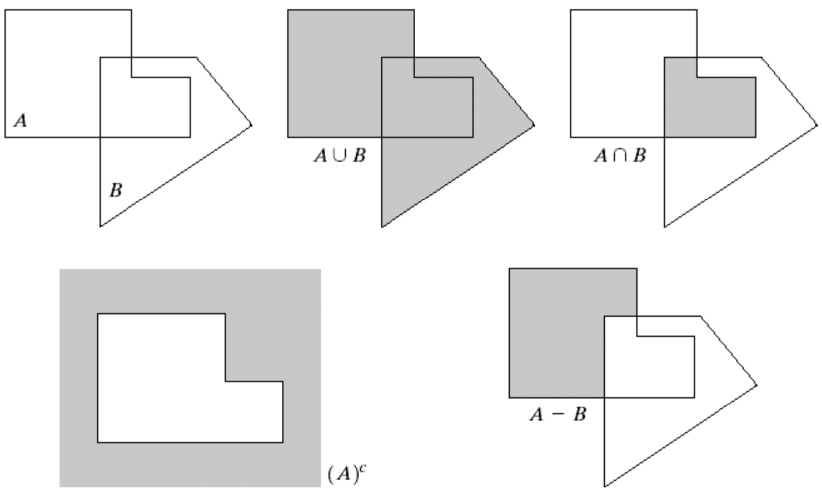
Mathematical Morphology

- Mathematical Morphology is a tool that can be used to extract useful images information such as shape, boundary, skeleton and convex hull.
- Preliminaries
  - A 2-D space is denoted Z^2, a point in a Z^2 space requires a two element vector (x, y)
  - If A is a set in Z^2 then if the point a=(a_1, a_2) is an element of A, it is denoted as $a \in A$. On the other hand, $a \notin A$ indicates a is not an element of A
  - A set with no elements is denoted as the null set $A = \varnothing$
  - If all of the elements in A are also contained in B then A is a subset of B, $A \subseteq B$

Mathematical Morphology

- Set Theory
  - Union: $A \cup B$
  - Intersection: $A \cap B$
  - Complement: $A^c = \{w | w \notin A\}$
  - Difference: $A - B = \{w | w \in A, w \notin B\} = A \cap B^c$

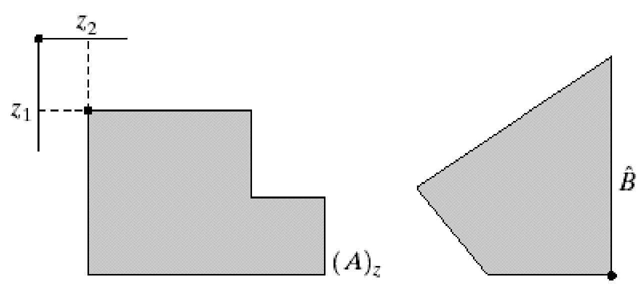


a b c
d e

FIGURE 9.1 (a) Two sets A and B. (b) The union of A and B. (c) The intersection of A and B. (d) The complement of A. (e) The difference between A and B.

Mathematical Morphology

- Two additional set operators
  - Reflection: $\hat{A}=\{w \mid w=-a,\ for\ a\in A\}$
  - Translation: $(A)_z=\{w \mid w=a+z,\ for\ a\in A, z=(z_1,z_2)\}$



a b

FIGURE 9.2 (a) Translation of A by z. (b) Reflection of B. The sets A and B are from Fig. 9.1.

Mathematical Morphology

- Logic operators

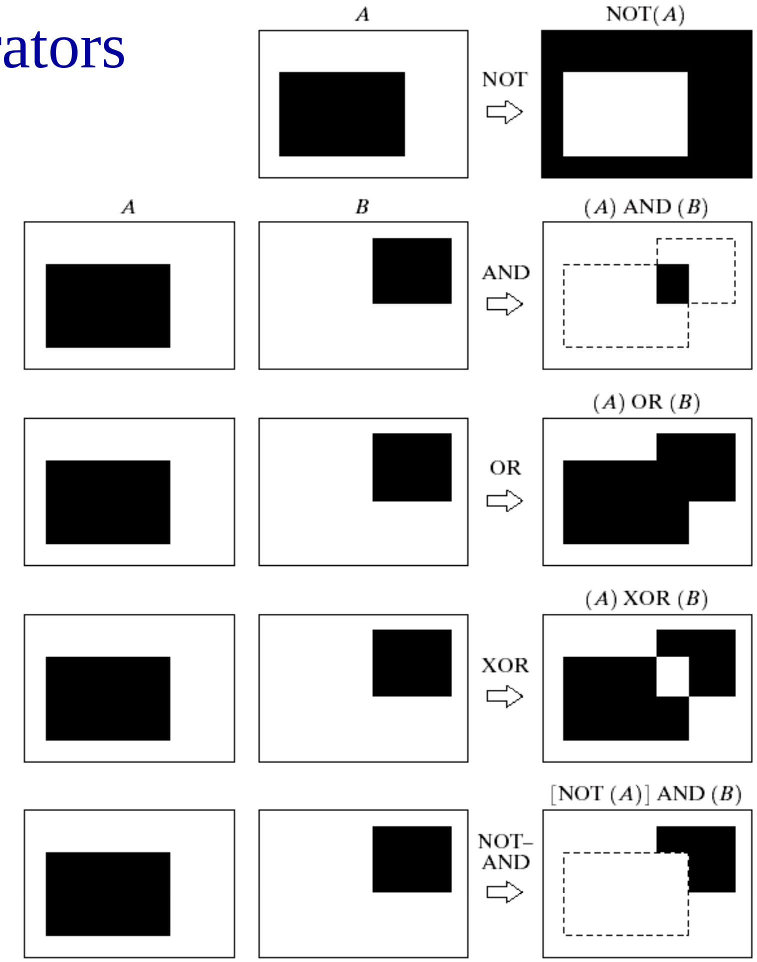
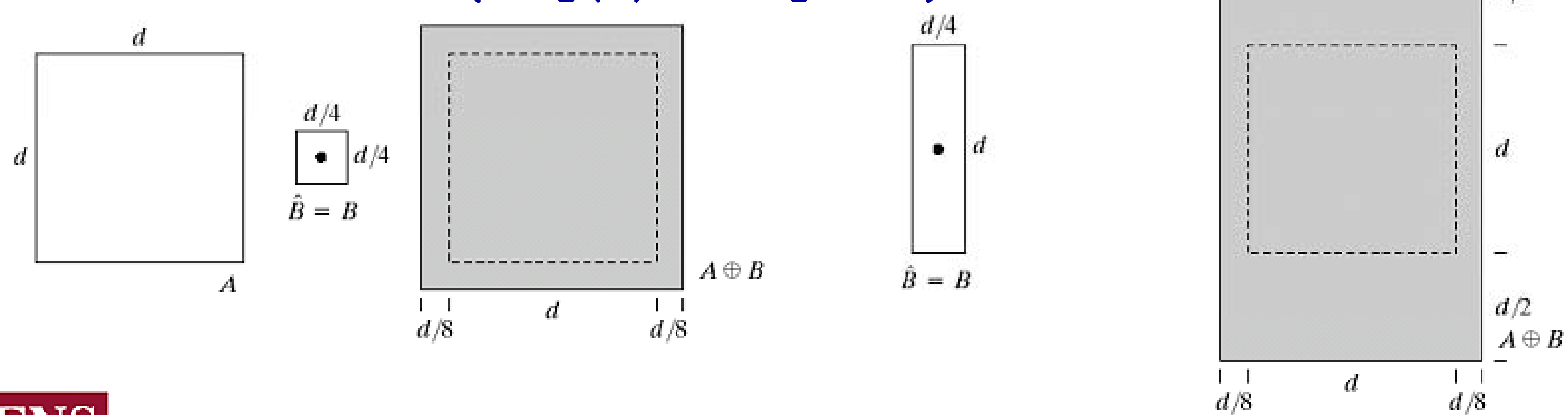


FIGURE 9.3 Some logic operations between binary images. Black represents binary 1s and white binary 0s in this example.

Mathematical Morphology

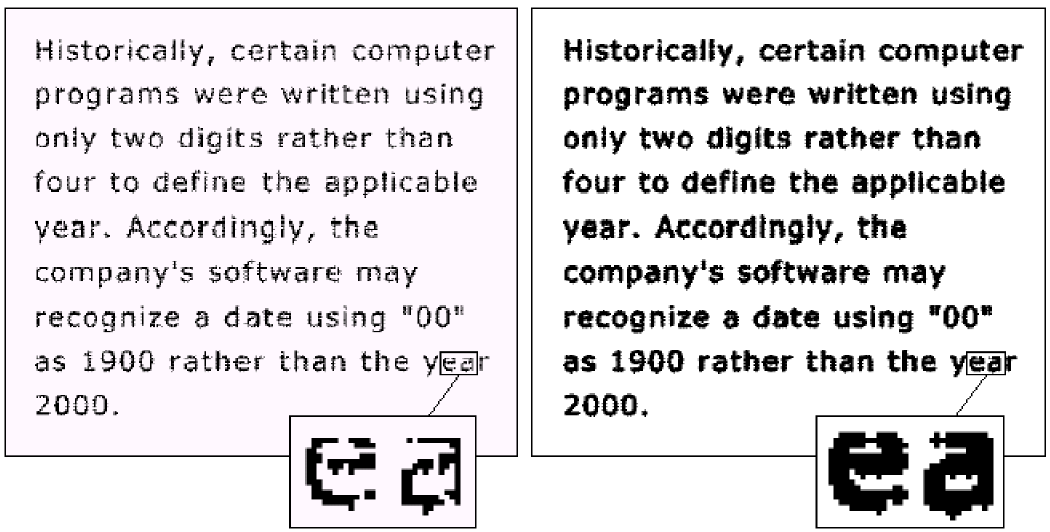
- Dilation:
  - A and B are sets in Z^2, the dilation of A by B is defined as $A \oplus B = \left\{ z \mid \left(\hat{B}\right)_z \cap A \neq \varnothing \right\}$
    B is called the structure element
  - It is equivalent to

$$A \oplus B = \left\{ z \mid \left[\left(\hat{B}\right)_z \cap A \right] \subseteq A \right\}$$



Mathematical Morphology

- Dilation can expand the object structure, and bridging any gaps in binary images.


Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.

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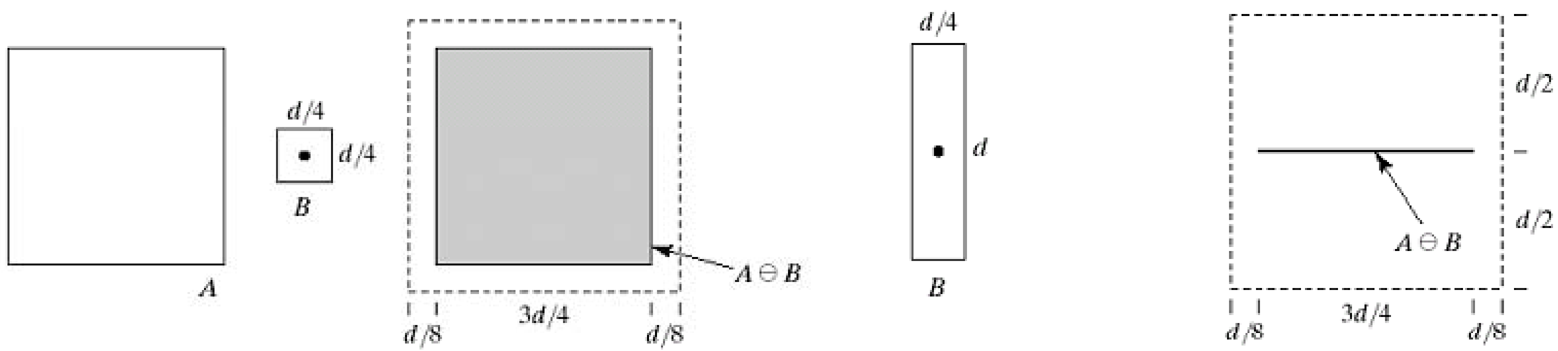
0	1	0
1	1	1
0	1	0

a c
b

FIGURE 9.5
(a) Sample text of poor resolution with broken characters (magnified view).
(b) Structuring element.
(c) Dilation of (a) by (b). Broken segments were joined.

Mathematical Morphology

- Erosion
  - A and B are sets in Z^2, the erosion of A by B is defined as $$A \ominus B = \{ z \mid (B)_z \subseteq A \}$$



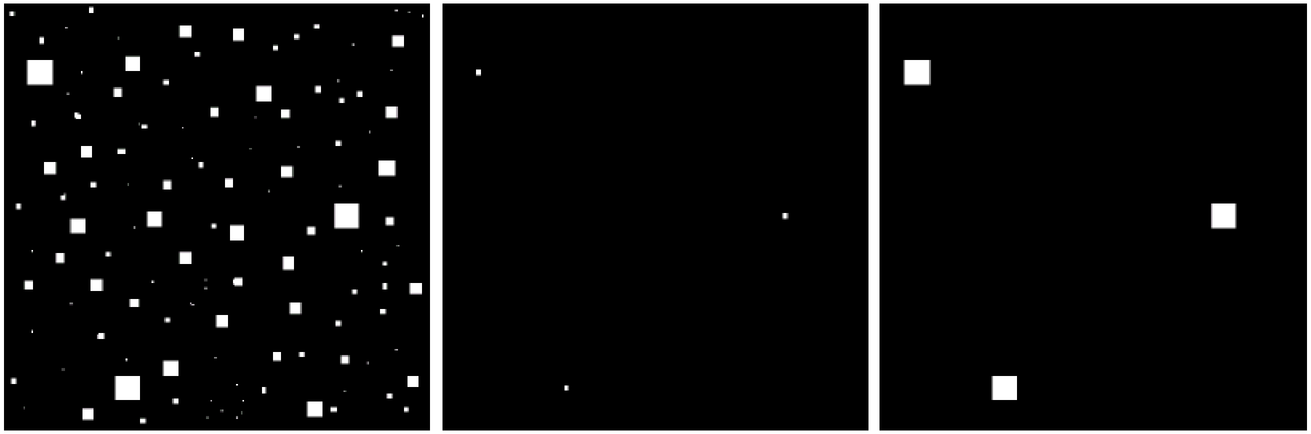
Mathematical Morphology

- Dilation and erosion are duals of each other w.r.t set complementation and reflection

Given
$$A \ominus B = \{z \mid (B)_z \subseteq A\}$$
$$(A \ominus B)^c = \{z \mid (B)_z \subseteq A\}^c$$
$$\because (B)_z \subseteq A \quad \therefore (B)_z \bigcap A^c = \varnothing$$
$$(A \ominus B)^c = \{z \mid (B)_z \bigcap A^c = \varnothing\}^c$$
$$= \{z \mid (B)_z \bigcap A^c \neq \varnothing\}$$
$$= A^c \oplus \hat{B}$$

Mathematical Morphology

- Erosion will shrink the object structure, and can remove irrelevant detail and noise in binary images



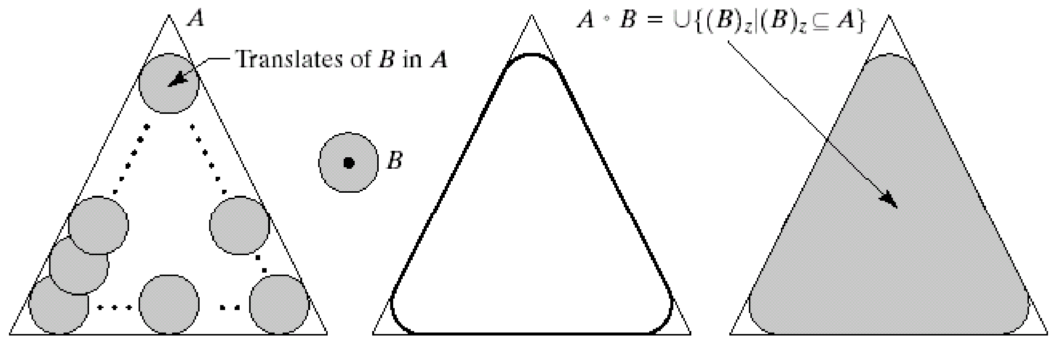
a b c

FIGURE 9.7 (a) Image of squares of size 1, 3, 5, 7, 9, and 15 pixels on the side. (b) Erosion of (a) with a square structuring element of 1's, 13 pixels on the side. (c) Dilation of (b) with the same structuring element.

Mathematical Morphology

- Opening:

$$A \circ B = (A \ominus B) \oplus B$$



a b c d

FIGURE 9.8 (a) Structuring element B "rolling" along the inner boundary of A (the dot indicates the origin of B). (c) The heavy line is the outer boundary of the opening. (d) Complete opening (shaded).

Mathematical Morphology

- Closing:

$$A \bullet B = (A \oplus B) \ominus B$$

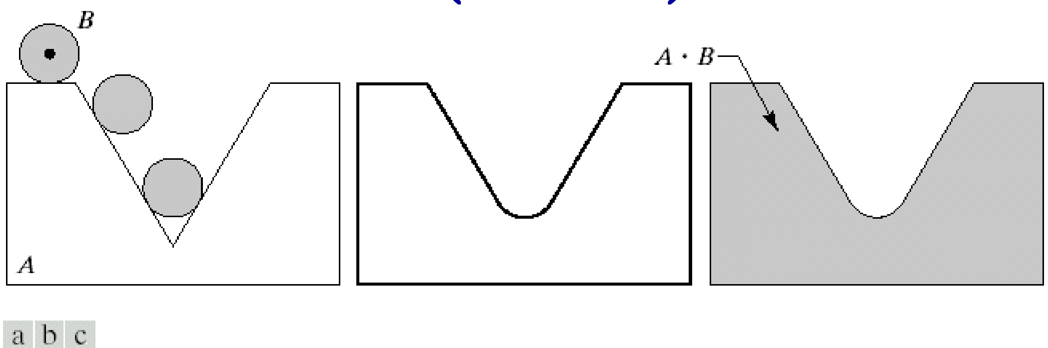


FIGURE 9.9 (a) Structuring element B "rolling" on the outer boundary of set A. (b) Heavy line is the outer boundary of the closing. (c) Complete closing (shaded).

- Opening and closing are duals of each other w.r.t. set complementation and reflection.

$$(A \bullet B)^c = (A^c \circ \hat{B})$$

Mathematical Morphology

- Properties of opening
  - $\mathbf{A} \circ \mathbf{B}$ is a subset (subimage) of $\mathbf{A}$
  - If $\mathbf{C}$ is a subset of $\mathbf{D}$, then $\mathbf{C} \circ \mathbf{B}$ is a subset of $\mathbf{D} \circ \mathbf{B}$
  - $(\mathbf{A} \circ \mathbf{B}) \circ \mathbf{B} = \mathbf{A} \circ \mathbf{B}$
- Properties of closing
  - $\mathbf{A}$ is a subset (subimage) of $\mathbf{A} \bullet \mathbf{B}$
  - If $\mathbf{C}$ is a subset of $\mathbf{D}$, then $\mathbf{C} \bullet \mathbf{B}$ is a subset of $\mathbf{D} \bullet \mathbf{B}$
  - $(\mathbf{A} \bullet \mathbf{B}) \bullet \mathbf{B} = \mathbf{A} \bullet \mathbf{B}$

Mathematical Morphology

- Opening and closing can roughly preserve the size of original object structure
- Opening can remove narrow isthmuses, capes and islands in binary images. Basically all objects that are smaller than the structure element will be removed.
- Closing can fill in gulfs and small holes, connect objects that are in close proximity, and smooth edge boundaries in binary images.

Mathematical Morphology

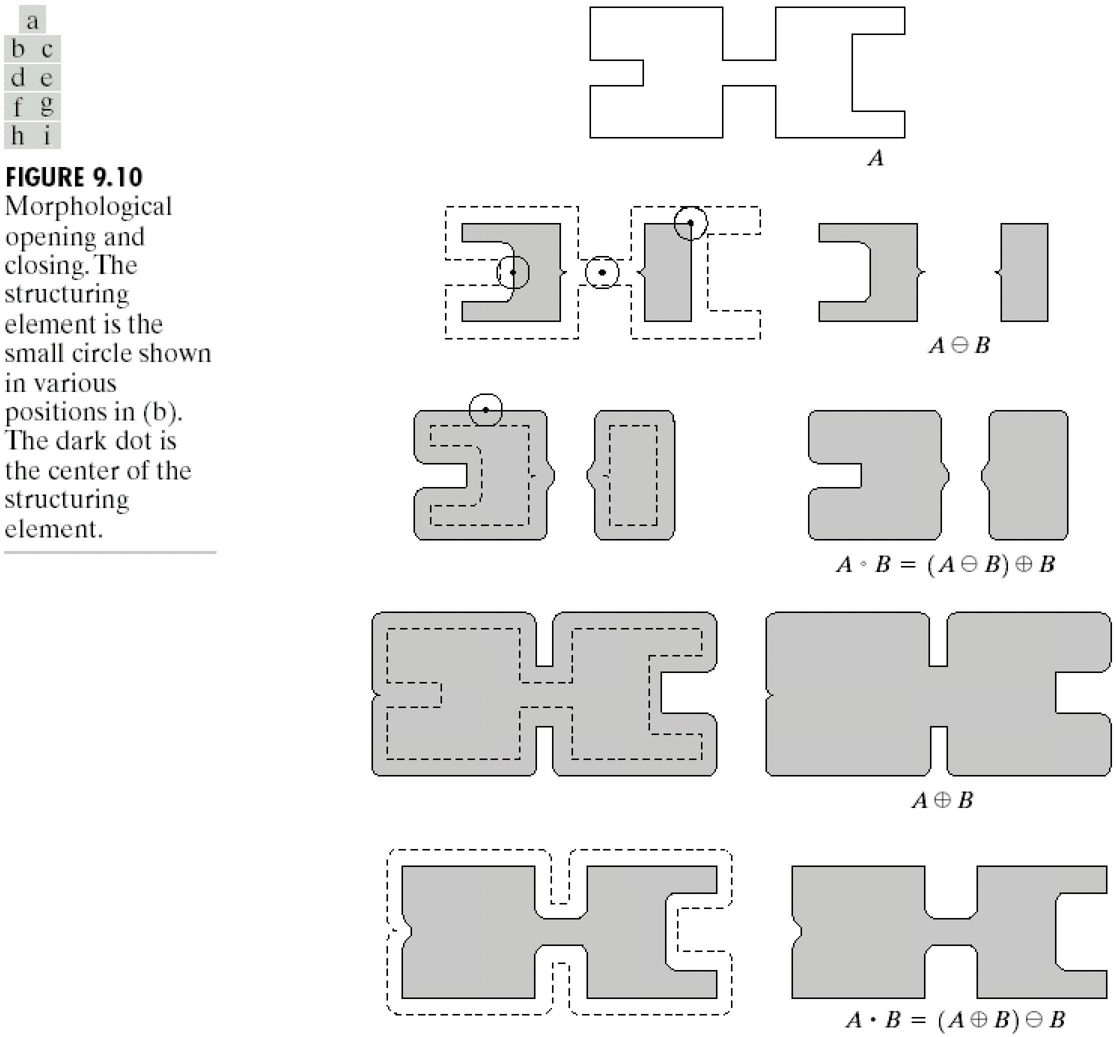


FIGURE 9.10 Morphological opening and closing. The structuring element is the small circle shown in various positions in (b). The dark dot is the center of the structuring element.

Mathematical Morphology

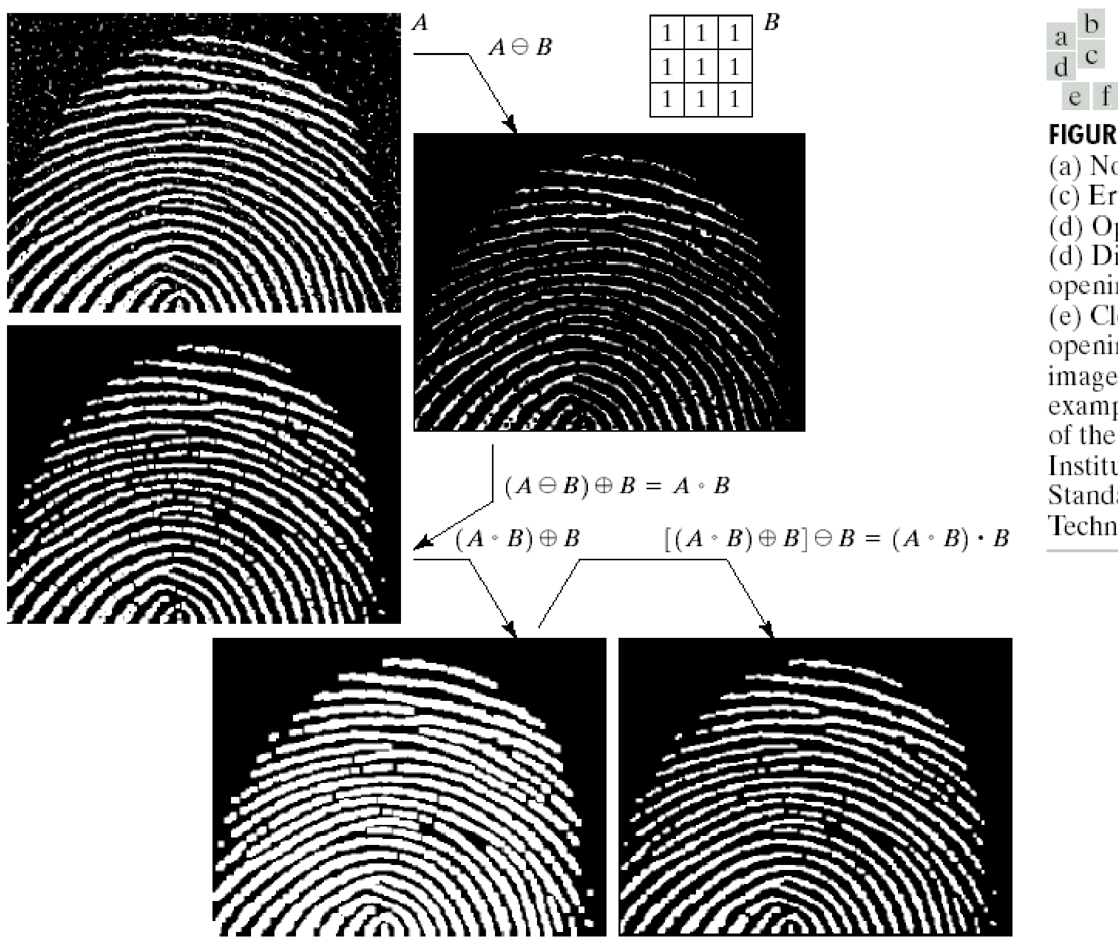


FIGURE 9.11
(a) Noisy image.
(c) Eroded image.
(d) Opening of A.
(d) Dilation of the opening.
(e) Closing of the opening. (Original image for this example courtesy of the National Institute of Standards and Technology.)

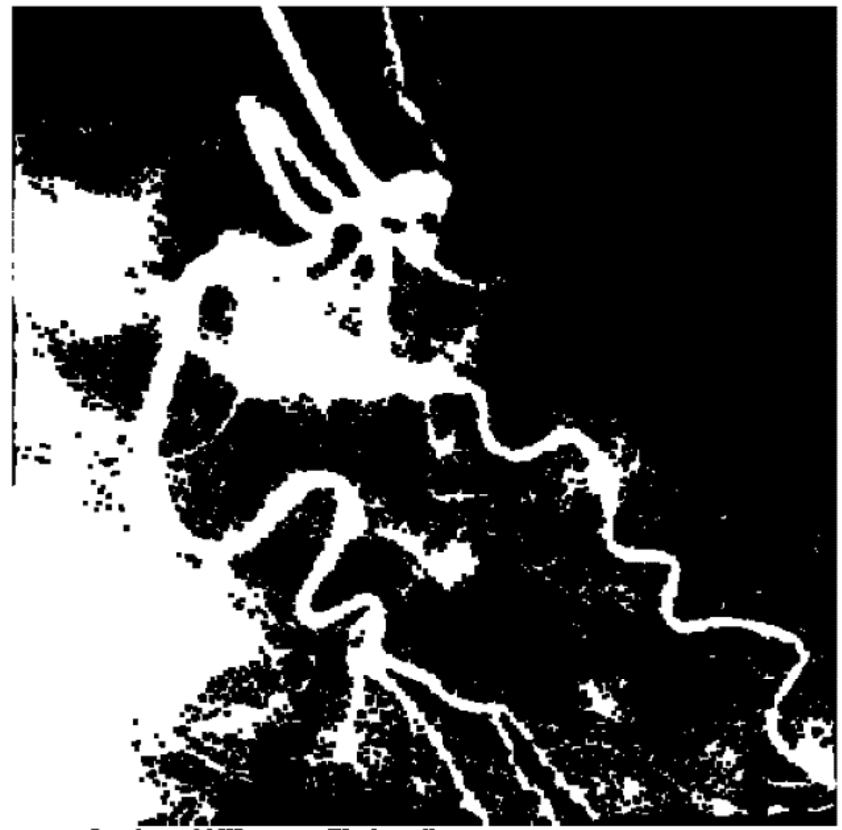
Mathematical Morphology

Original

Opening

Mathematical Morphology

Original



Closing

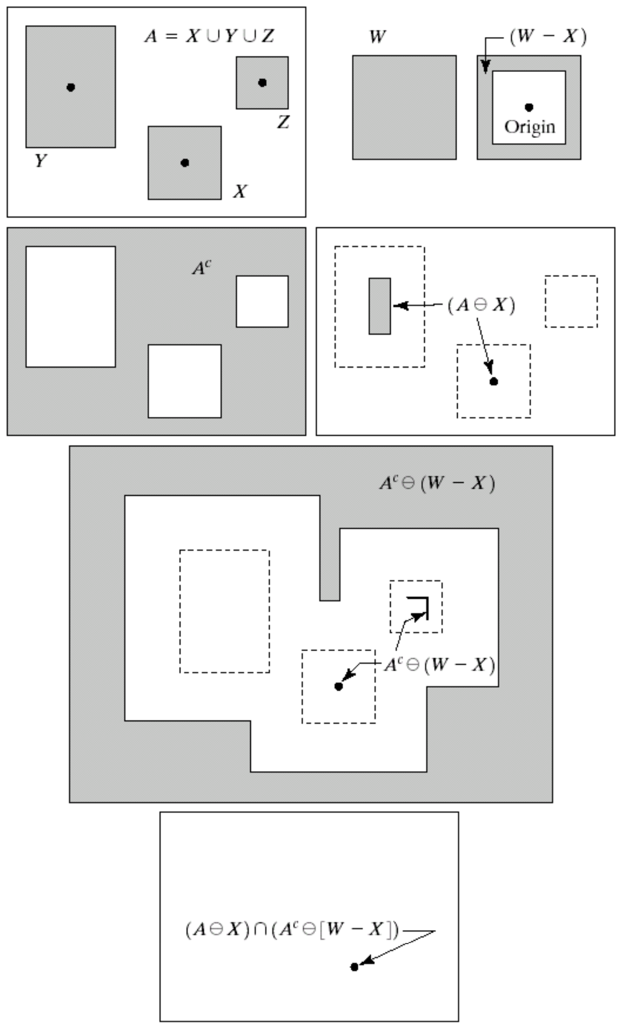
Mathematical Morphology

- The Hit-or-Miss Transform
  - Let $\mathbf{B=(B_1, B_2)}$ where $\mathbf{B_1}$ is the set formed from elements of **B** associated with an object and $\mathbf{B_2}$ is the set of elements of **B** associated with the corresponding background.
  - The match ("hit") of B in A is defined as

$$\mathbf{A \circledast B = (A \ominus B_1) \cap (A^c \ominus B_2)}$$

  - The hit-or-miss transform is basic tool for shape detection, i.e. detect the shape of $\mathbf{B_1}$ inside set **A.**
  - In the next figure (figure 9.12), we can consider $\mathbf{B_1=X}$ and $\mathbf{B_2=(W–X)}$.

Mathematical Morphology

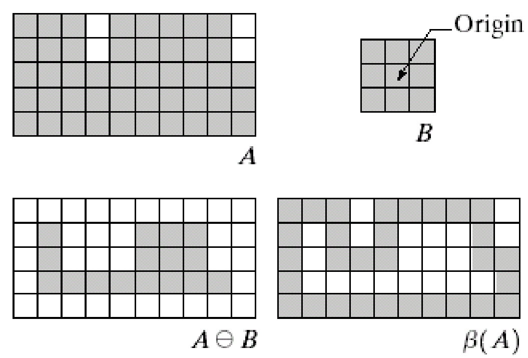


a b
c d
e
f

FIGURE 9.12
(a) Set A. (b) A window, W, and the local background of X with respect to W, $(W - X)$. (c) Complement of A. (d) Erosion of A by X. (e) Erosion of A^c by $(W - X)$. (f) Intersection of (d) and (e), showing the location of the origin of X, as desired.

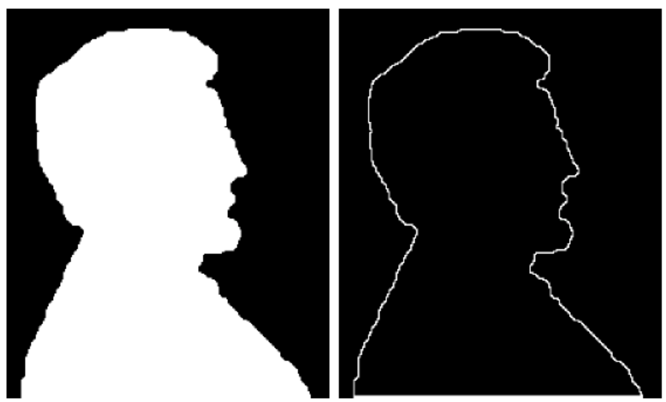
Mathematical Morphology

- Boundary extraction:
  - boundary of set A, **$\beta(A) = A - (A \ominus B)$**



a b
c d

FIGURE 9.13 (a) Set *A*. (b) Structuring element *B*. (c) *A* eroded by *B*. (d) Boundary, given by the set difference between *A* and its erosion.



a b

FIGURE 9.14 (a) A simple binary image, with 1's represented in white. (b) Result of using Eq. (9.5-1) with the structuring element in Fig. 9.13(b).

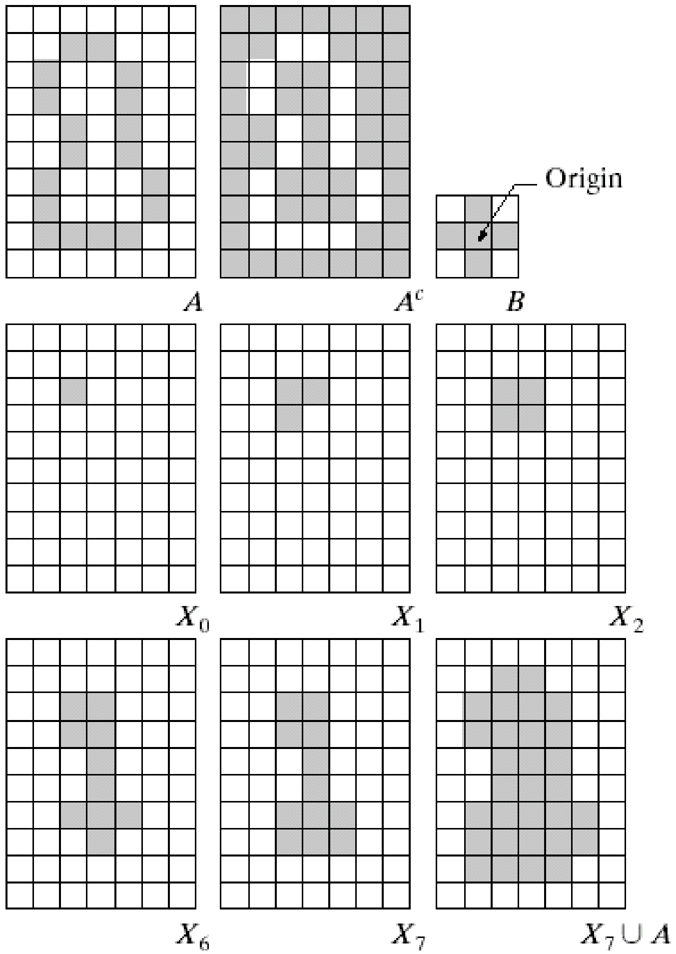
Mathematical Morphology

- Region filling:
  - **A** is a set of boundary points with value of 1's, and all non-boundary points are 0's.
  - To fill the region inside the boundary with 1's, let $\mathbf{X_0 = p}$, where $\mathbf{p \in A^c}$ inside the boundary

$$\mathbf{X_k = (X_{k-1} \oplus B) \cap A^c} \text{ for } \mathbf{k = 1, 2, 3\ldots}$$

  - The iteration stops when $\mathbf{X_k = X_{k-1}}$, and $\mathbf{X_k}$ becomes the set of filled region

Mathematical Morphology



a	b	c
d	e	f
g	h	i

FIGURE 9.15
Region filling.
(a) Set A.
(b) Complement of A.
(c) Structuring element B.
(d) Initial point inside the boundary.
(e)–(h) Various steps of Eq. (9.5-2).
(i) Final result [union of (a) and (h)].

Mathematical Morphology

- Convex hull
  - Set ***A*** is convex if a straight line joining any two points in ***A*** lines entirely in ***A*****.**
  - The convex hull ***H*** of an arbitrary set ***S*** is the smallest convex set containing ***S***. ***H* – *S*** is called the convex deficiency. They are good object descriptors.
  - To obtain the convex hull $\boldsymbol{C(A)}$ of set ***A***, let $\boldsymbol{B^i}$, $\boldsymbol{i = 1, 2, 3, 4}$ represent four structure elements as shown, let

    $\boldsymbol{X_0^i = A}$, calculate $\boldsymbol{X_k^i = (X_{k-1}^i \circledast B^i) \cup A}$

    where i = 1, 2, 3, 4 and k = 1, 2, 3, ….
  - When $\boldsymbol{X_k^i}$ converges, let $\boldsymbol{X^i = X_k^i = X_{k-1}^i}$, then

$$C(A) = \bigcup_{i=1}^{4} X^i$$

Mathematical Morphology

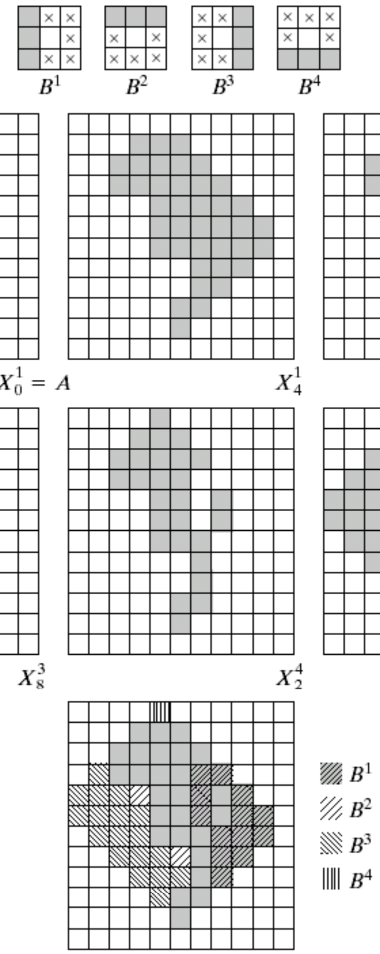


FIGURE 9.19 (a) Structuring elements. (b) Set A. (c)–(f) Results of convergence with the structuring elements shown in (a). (g) Convex hull. (h) Convex hull showing the contribution of each structuring element.

Mathematical Morphology

- Thinning
  - Thinning of a set A by a structure element B

$$\mathbf{A \otimes B = A - (A \circledast B) = A \cap (A \circledast B)^c}$$

  - Generally thinning involves a set of structure elements **B = { B^1, B^2, B^3, … B^n},** each has a spatial orientation. Thinning with the whole set **B** yields symmetric results.
- Thickening
  - Thickening of a set A by a structure element B

$$\mathbf{A \odot B = A \cap (A \circledast B)}$$

  - Generally thickening also involves a set of structure elements **B = { B^1, B^2, B^3, … B^n}.**

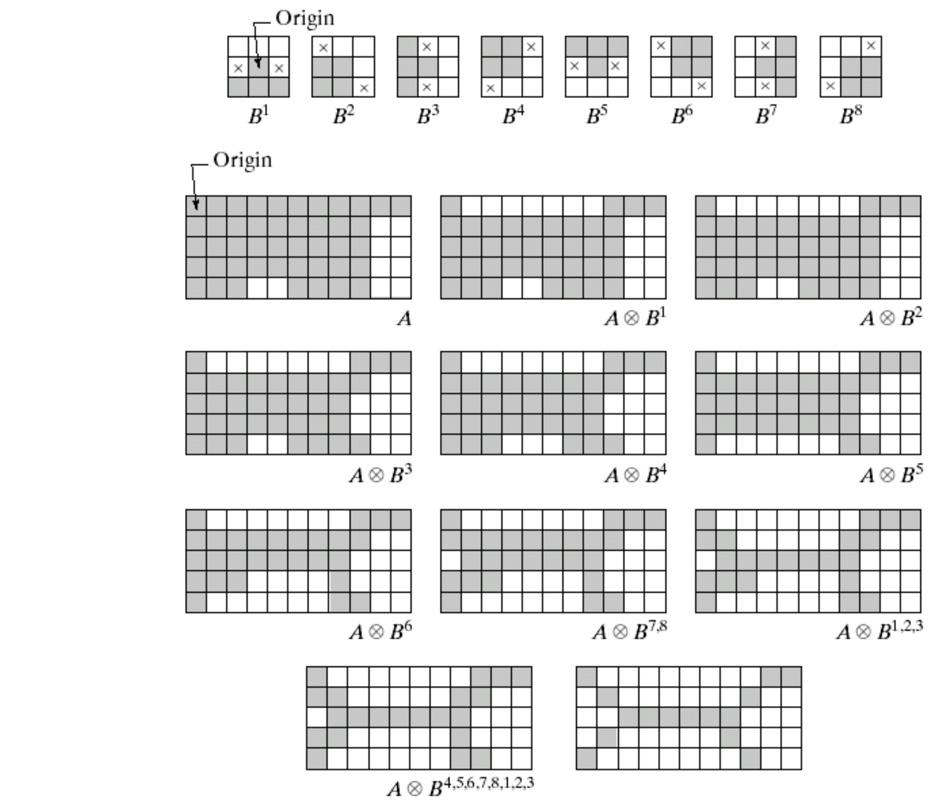


FIGURE 9.21 (a) Sequence of rotated structuring elements used for thinning. (b) Set A. (c) Result of thinning with the first element. (d)–(i) Results of thinning with the next seven elements (there was no change between the seventh and eighth elements). (j) Result of using the first element again (there were no changes for the next two elements). (k) Result after convergence. (l) Conversion to m-connectivity.

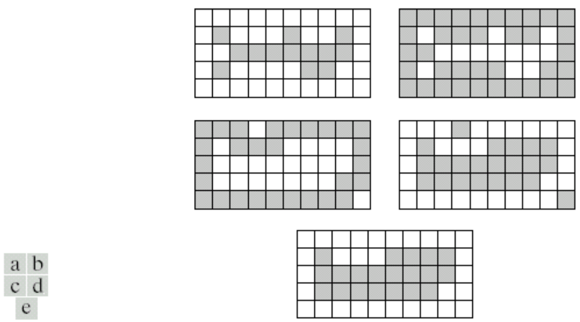


FIGURE 9.22 (a) Set A. (b) Complement of A. (c) Result of thinning the complement of A. (d) Thickened set obtained by complementing (c). (e) Final result, with no disconnected points.

Mathematical Morphology

- Skeletons
  - The skeleton $\boldsymbol{S(A)}$ of a set $\boldsymbol{A}$: $$S(A) = \bigcup_{k=0}^{K} S_k(A)$$
    where $S_k(A) = (A \ominus^k B) - (A \ominus^k B) \circ B$
    and $(A \ominus^k B) = (\ldots(\mathrm{A} \ominus \mathrm{B}) \ominus \mathrm{B})\ldots) \ominus \mathrm{B}$ for $\boldsymbol{k}$ times,
    and $\boldsymbol{K}$ is the last iteration step before $\boldsymbol{A}$ erodes to an empty set, i.e. $K=\max\{k|(A \ominus^k B) \neq \varnothing\}$
  - The set **A** can be reconstructed from its skeleton
    $$S(A) = \bigcup_{k=0}^{K} S_k(A)$$
    where $(A \oplus^k B) = (\ldots(\mathrm{A} \oplus \mathrm{B}) \oplus \mathrm{B})\ldots) \oplus \mathrm{B}$ for $\boldsymbol{k}$ times

Mathematical Morphology

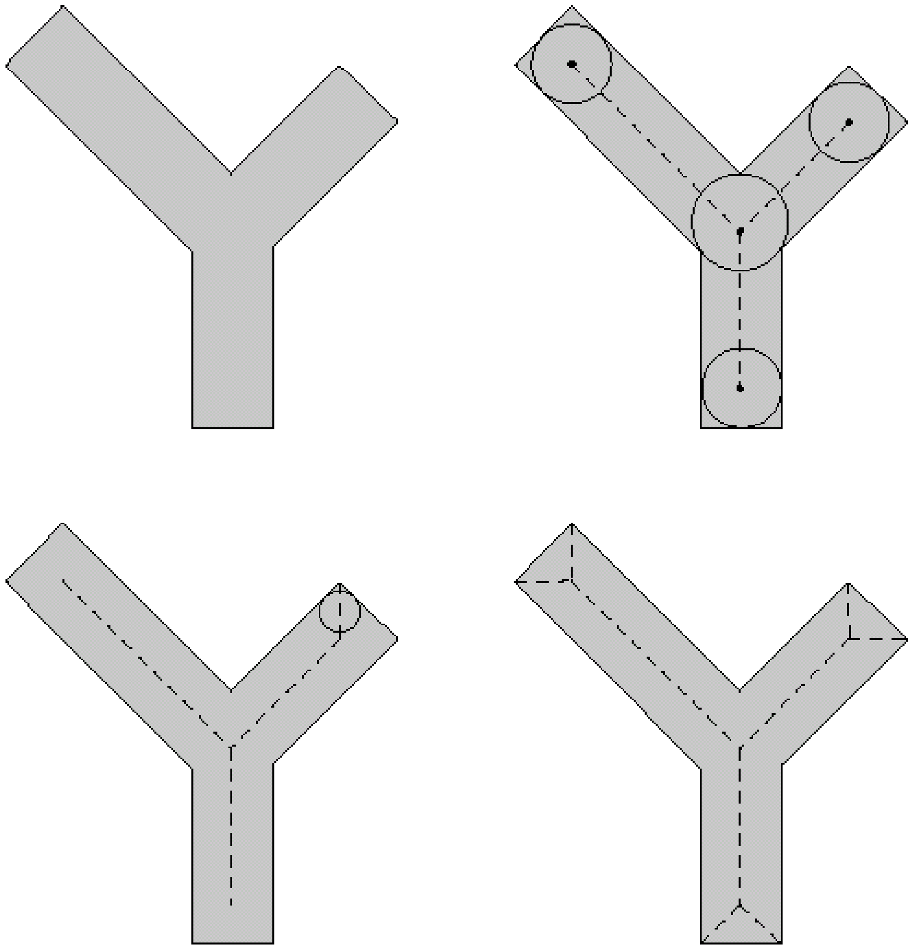


FIGURE 9.23
(a) Set A.
(b) Various positions of maximum disks with centers on the skeleton of A.
(c) Another maximum disk on a different segment of the skeleton of A.
(d) Complete skeleton.

Mathematical Morphology

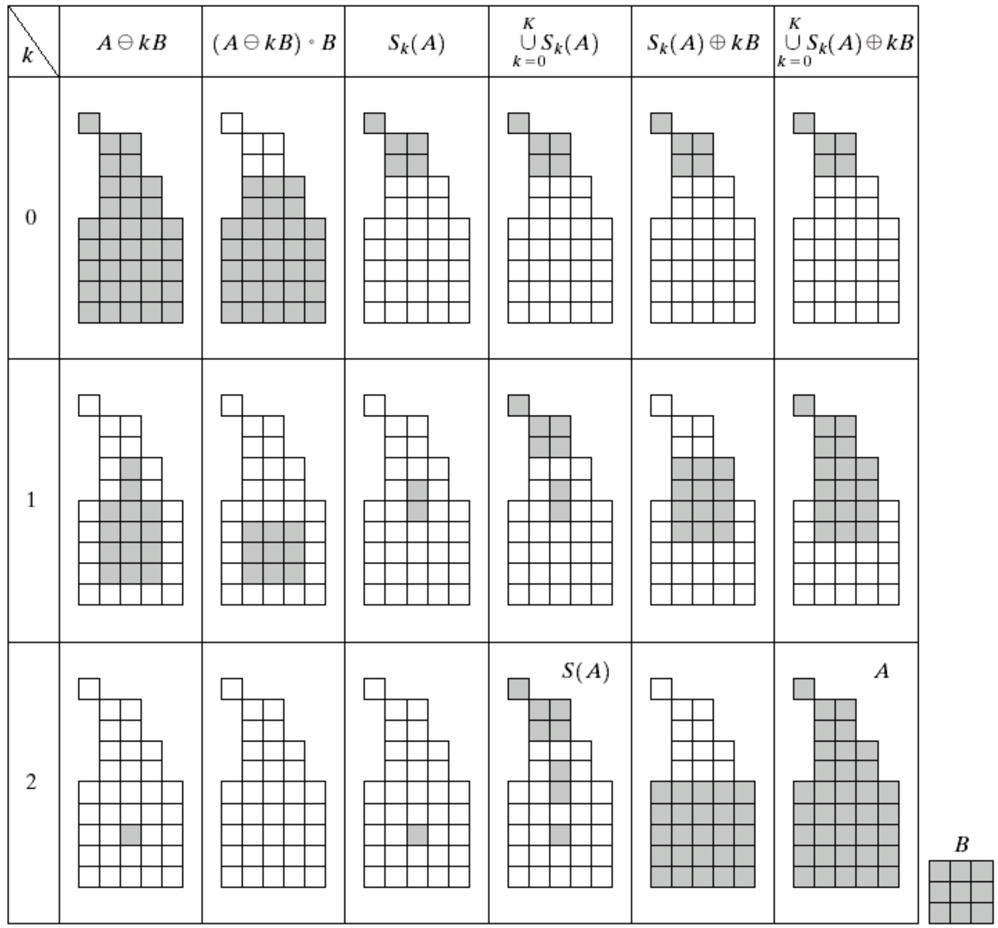


FIGURE 9.24 Implementation of Eqs. (9.5-11) through (9.5-15). The original set is at the top left, and its morphological skeleton is at the bottom of the fourth column. The reconstructed set is at the bottom of the sixth column.