CpE 645 Image Processing and Computer Vision

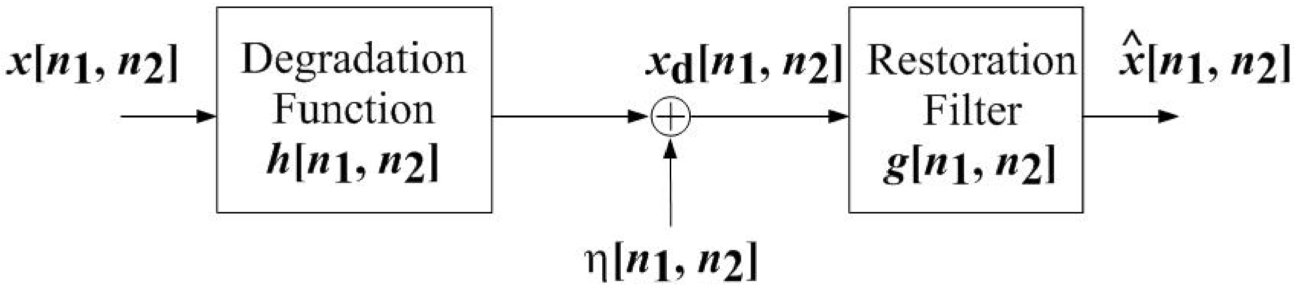
Prof. Hong Man

Department of Electrical and Computer Engineering
Stevens Institute of Technology

STEVENS Institute of Technology

Visual Information Environment Laboratory

Image Degradation Model



- A common image distortion process can be modeled as a distortion filtering followed by some additive noise.
- More specifically, let $x[n_1, n_2]$ be the original image, $h[n_1, n_2]$ be the distortion filter, and $\eta[n_1, n_2]$ be additive random noise. The distorted image $x_d[n_1, n_2]$ can be expressed as

$$x_d[n_1, n_2] = x[n_1, n_2] ** h[n_1, n_2] + \eta[n_1, n_2] .$$

Image Restoration

- Image restoration is generally a filtering process.
- Let $x_d[n_1, n_2]$ be the distorted image, the restoration filter be $g[n_1, n_2]$. The restored image

$$\hat{x}[n_1, n_2] = x_d[n_1, n_2] ** g[n_1, n_2] .$$

- The goal of image restoration is to design a restoration filter $g[n_1, n_2]$ that can minimize the difference between the restored image $\hat{x}[n_1, n_2]$ and the original image $x[n_1, n_2]$. Therefore image restoration is mostly objective.
- Image restoration assumes a prior knowledge of the distortion process, and the statistics of the random noise.

Image Restoration

- Distortions in images can be caused by:
  - Imperfections in the imaging system (aberrations, diffraction, etc…)
  - Atmospheric turbulence due to random variations in the refractive index
  - motion of objects or the camera during the exposure time
  - …
- Examples of noise sources:
  - Electronic noise in detection and recording devices
  - Film grain noise
  - …

Noise Models

- *White noise*: the Fourier spectrum of noise is nearly constant.
- Generally we assume noise is *independent* of spatial coordinates, and that it is *uncorrelated* to image pixel values.
- These assumptions could be invalid in some cases, but the solutions derived can still be effective to a wide range of applications, although not in the optimal sense.
- Statistically noise is modeled by one of several common probability density functions (PDF).

Noise Models

- *Gaussian* model, with random variable z

$$p(z) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(z-\mu)^2 / 2\sigma^2}$$

  where μ is mean, and σ^2 is variance.
- Gaussian noise model is symmetric around the mean.
- It is most frequently used model, even in situations that are not quite applicable.
- The probability for ***z*** being inside the range $[(\mu-\sigma), (\mu+\sigma)]$ is about 70%. The probability for ***z*** being inside the range $[(\mu-2\sigma), (\mu+2\sigma)]$ is about 95%.

Noise Models

- *Rayleigh* model

$$p(z) = \begin{cases} \frac{2}{b}(z-a)e^{-(z-a)^2/b} & \text{for } z \ge a \\ 0 & \text{for } z < a \end{cases}$$

where the mean $\mu = a + \sqrt{\pi b/4}$

and the variance $\sigma^2 = \frac{b(4-\pi)}{4}$.

- Rayleigh model has a skewed appearance. It is useful to approximate skewed histograms.

Noise Models

- Erlang (Gamma) model:

$$p(z) = \begin{cases} \dfrac{a^b z^{b-1}}{(b-1)!} e^{-az} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases}$$

where the mean $\mu = \dfrac{b}{a}$

and the variance $\sigma^2 = \dfrac{b}{a^2}$.

- Also a>**0**, b is a positive integer, and **!** denotes factorial.
- If the denominator **(*b*-1)!** is replaced by **Γ(*b*),** this becomes the general Gamma PDF.

Noise Models

- Exponential model:

$$p(z) = \begin{cases} ae^{-az} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases}$$

$$\text{where mean}: \mu = \frac{1}{a},$$

$$\text{and variance}: \sigma^2 = \frac{1}{a^2}.$$

- Exponential model is a special case of the Erlang model where $b=1$.

Noise Models

- Uniform model:

$$p(z) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \le z \le b \\ 0 & \text{otherwise} \end{cases}$$

where the mean $\mu = \dfrac{a+b}{2}$

and the variance $\sigma^2 = \dfrac{(b-a)^2}{12}$.

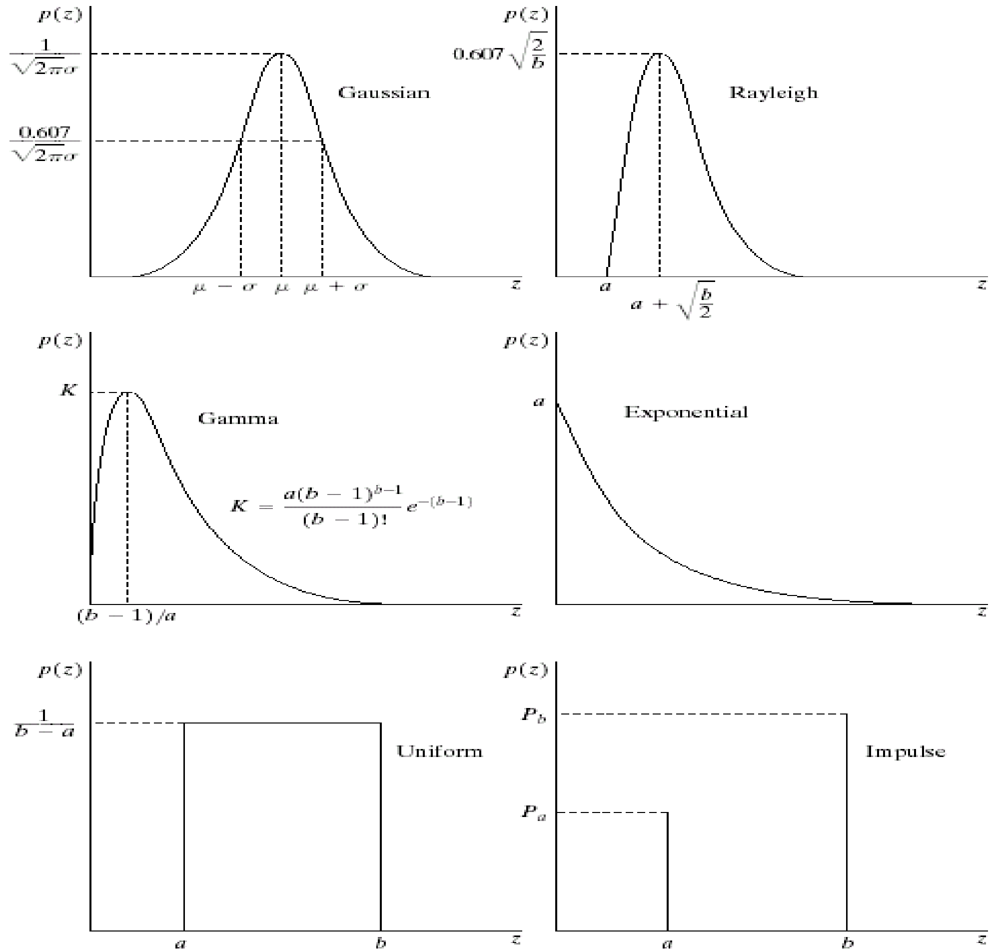
Noise Models

- Impulse (salt and pepper) model:

$$p(z) = \begin{cases} P_a & \text{for } z = a \\ P_b & \text{for } z = b \\ 0 & \text{otherwise} \end{cases}$$

- Usually we assume both ***a*** and ***b*** are "saturated", i.e. one pure black (minimum value) and the other is pure white (maximum value).

Noise Models



Noise Models

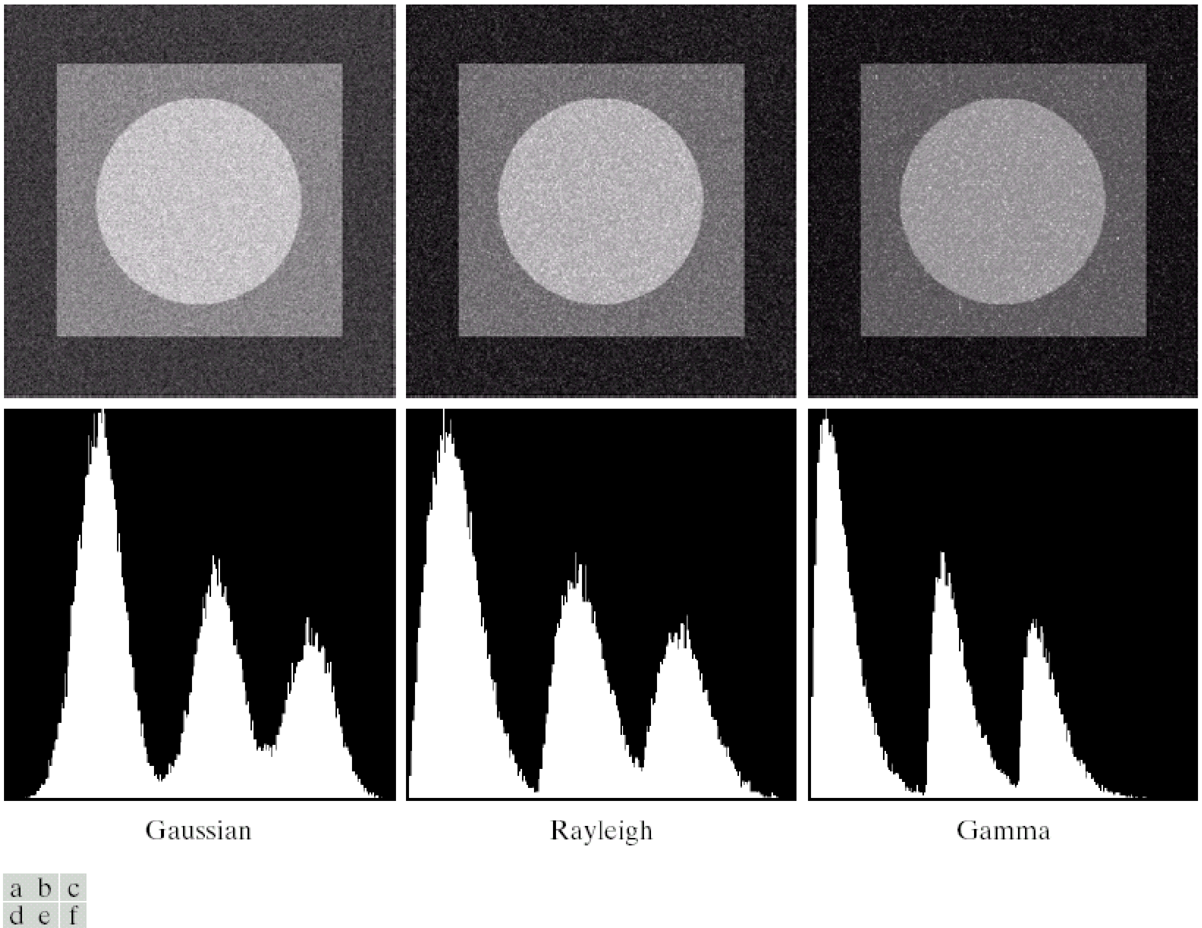


FIGURE 5.4 Images and histograms resulting from adding Gaussian, Rayleigh, and gamma noise to the image in Fig. 5.3.

Noise Models

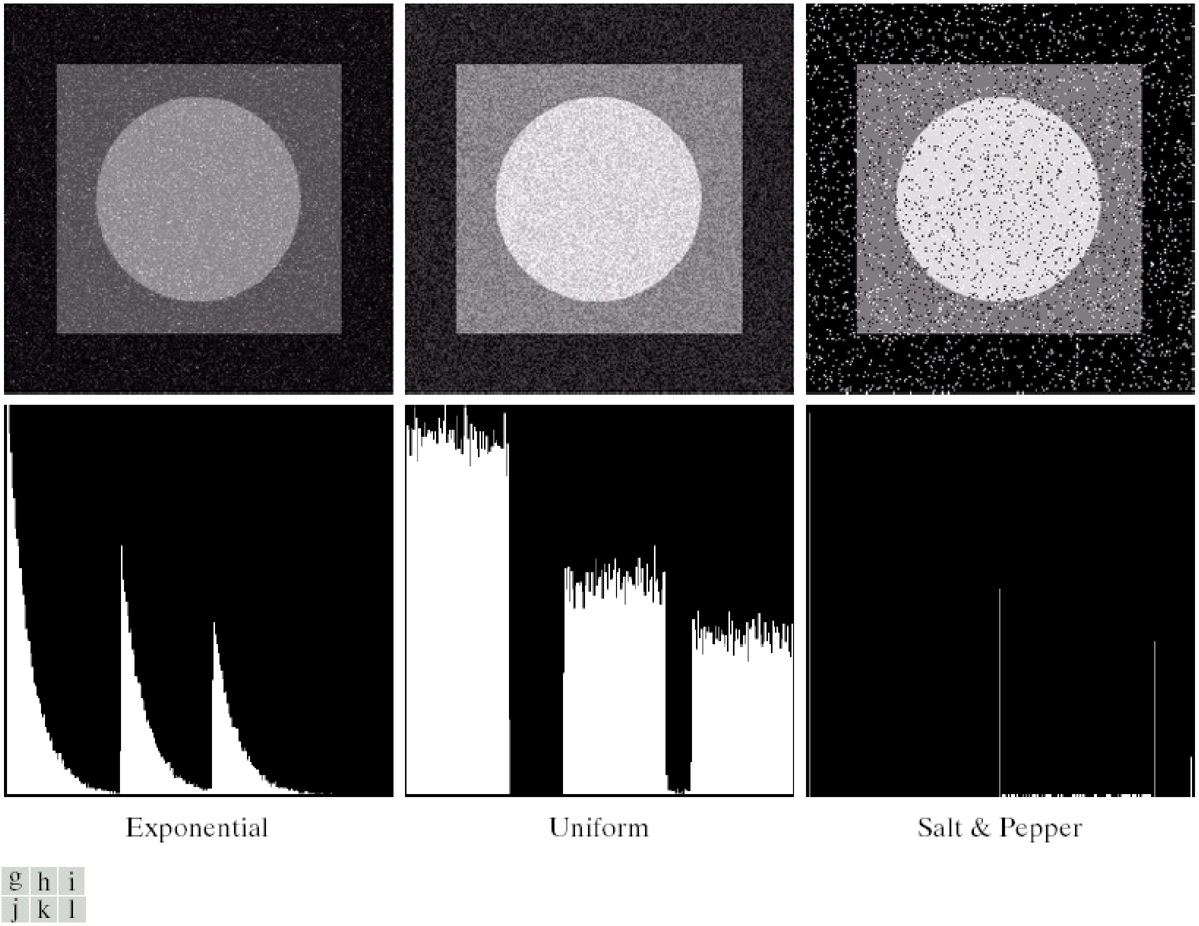


FIGURE 5.4 *(Continued)* Images and histograms resulting from adding exponential, uniform, and impulse noise to the image in Fig. 5.3.

Estimation of Noise Parameters

- If imaging device is available, parameters of noise PDF can be estimated by taking images of flat surface with uniform illumination and calculate its statistics and histogram.
- If only digital image is available, parameters of noise PDF are estimated from small regions of reasonably constant gray level.
- The shape of noise histogram $p(z_i)$ reveals the possible type of noise. The mean and variance are calculated as

$$\text{mean}: \mu = \sum_{z_i \in S} z_i p(z_i), \text{ and variance}: \sigma^2 = \sum_{z_i \in S} (z_i - \mu)^2 p(z_i).$$

Image Restoration

- Assume the degradation process is linear and position-invariant which can be modeled as a filter, the noise is wide sense stationary (WSS), white and zero-mean.
- Consider the continuous representations

$$x_d(t_1, t_2) = x(t_1, t_2) ** h(t_1, t_2) + \eta(t_1, t_2) .$$

$$\hat{x}(t_1, t_2) = x_d(t_1, t_2) ** g(t_1, t_2) .$$

- In frequency domain

$$X_d(\Omega_1, \Omega_2) = X(\Omega_1, \Omega_2) H(\Omega_1, \Omega_2) + N(\Omega_1, \Omega_2)$$

$$\hat{X}(\Omega_1, \Omega_2) = X_d(\Omega_1, \Omega_2) G(\Omega_1, \Omega_2)$$

Estimating the Degradation Function

- The degradation function, a.s.a. point spread function (PSF), can be estimated through observation, experience or mathematical modeling.
- Observation from a degraded image
  - We locate a small image region that contains a simple structure, such as a part of an flat object and some flat background. Select region with strong signal content to reduce noise effect.
  - We restore this image region to a state that we expect the original image should appear, e.g. an edge with a sharp gray level difference.

Estimating the Degradation Function

– Denote the degraded image region as $\boldsymbol{x_{sd}(t_1, t_2)}$ and the restored image region is $\boldsymbol{\hat{x}_{sd}(t_1, t_2)}$, assume noise effect is negligible, the local degradation function can be calculated as

$$H_s(\Omega_1, \Omega_2) = \frac{X_{sd}(\Omega_1, \Omega_2)}{\hat{X}_{sd}(\Omega_1, \Omega_2)}$$

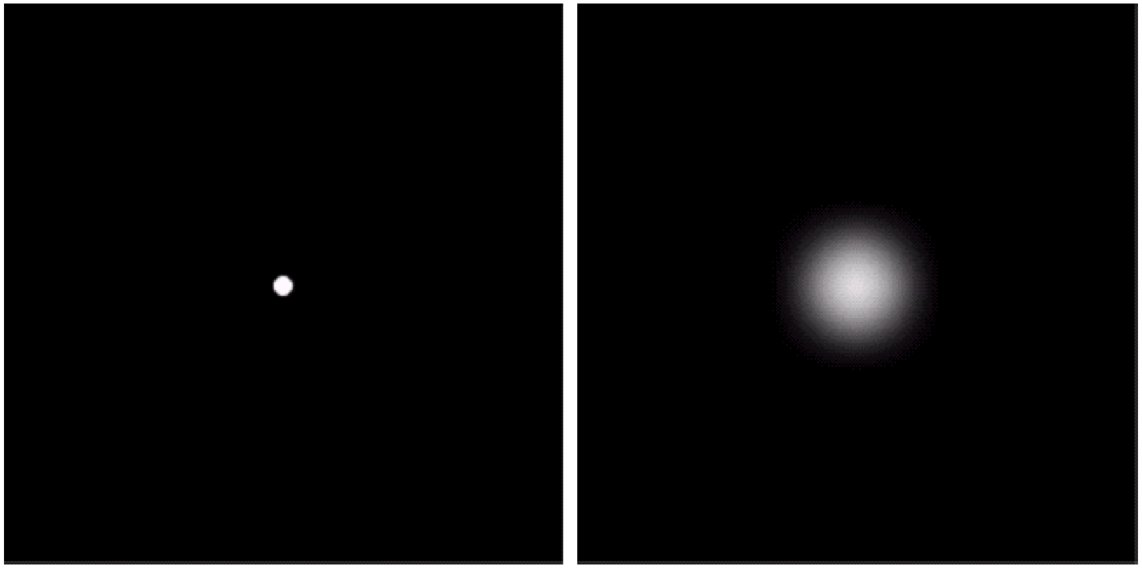
– Design a global degradation function $\boldsymbol{H(\Omega_1, \Omega_2)}$ based on this local degradation function $\boldsymbol{H_s(\Omega_1, \Omega_2)}$.

Estimating the Degradation Function

- Experiment with similar imaging equipment
  - If the original or a similar imaging equipment is available, we can adjust the system settings to duplicate the distortion effect we saw in the distorted image.
  - At this system setting, measure the system impulse response by imaging an impulse (small dot of light).
  - Denote $\boldsymbol{A}$ as the constant specifying the strength of the impulse, $\boldsymbol{H_A(\Omega_1, \Omega_2)}$ is the Fourier transform of the observed (degraded) image of this impulse, the system response is

$$H(\Omega_1, \Omega_2) = \frac{H_A(\Omega_1, \Omega_2)}{A}$$

Estimating the Degradation Function



a b

FIGURE 5.24 Degradation estimation by impulse characterization. (a) An impulse of light (shown magnified). (b) Imaged (degraded) impulse.

Estimating the Degradation Function

- Mathematical modeling
  - Mathematical models are available for a few common distortion processes.
  - Example 1: atmospheric turbulence

$$H(\Omega_1, \Omega_2) = e^{-k(\Omega_1^2 + \Omega_2^2)^{5/6}}$$

Estimating the Degradation Function

a b
c d

FIGURE 5.25 Illustration of the atmospheric turbulence model. (a) Negligible turbulence. (b) Severe turbulence, $k = 0.0025$. (c) Mild turbulence, $k = 0.001$. (d) Low turbulence, $k = 0.00025$. (Original image courtesy of NASA.)

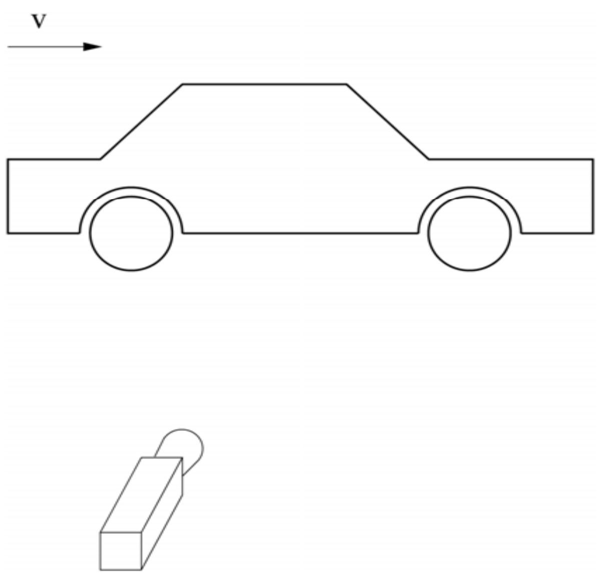
Estimating the Degradation Function

– Example 2: linear motion blur

- Motion blur occurs when there is relative motion between the camera and the object during exposure.
- The signal captured on the film is:

$$x_d(t_1, t_2) = \int_{-T/2}^{T/2} x(t_1 - v\tau, t_2)\, d\tau$$

where T is the shutter opening time, and v is object velocity.



Estimating the Degradation Function

- The Fourier transform of the motion blur PSF is

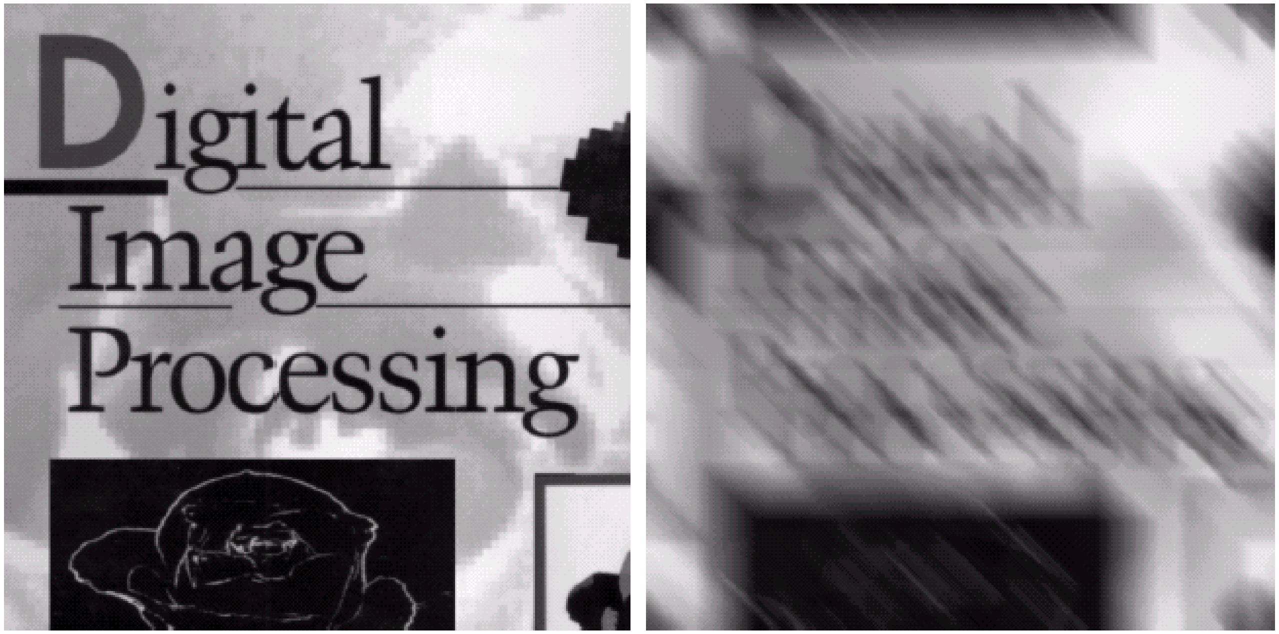
$$X_d(\Omega_1,\Omega_2) = \int_{-T/2}^{T/2} X(\Omega_1,\Omega_2) e^{-jv\tau\Omega_1} d\tau$$

$$= X(\Omega_1,\Omega_2)\frac{2\sin(\Omega_1 vT/2)}{v\Omega_1}$$

- So the PSF in frequency domain is

$$H(\Omega_1,\Omega_2) = \frac{2\sin(\Omega_1 vT/2)}{v\Omega_1}$$

Estimating the Degradation Function



a b

FIGURE 5.26 (a) Original image. (b) Result of blurring using the function in Eq. (5.6-11) with $a = b = 0.1$ and $T = 1$.

Inverse Filtering

- If assume no noise in the distortion process, i.e. $N(\Omega_1, \Omega_2)=0$, then

$$\mathbf{X_d}(\Omega_1,\Omega_2) = \mathbf{X}(\Omega_1,\Omega_2)\mathbf{H}(\Omega_1,\Omega_2)$$

$$\hat{\mathbf{X}}(\Omega_1,\Omega_2) = \mathbf{X_d}(\Omega_1,\Omega_2)\mathbf{G}(\Omega_1,\Omega_2)$$

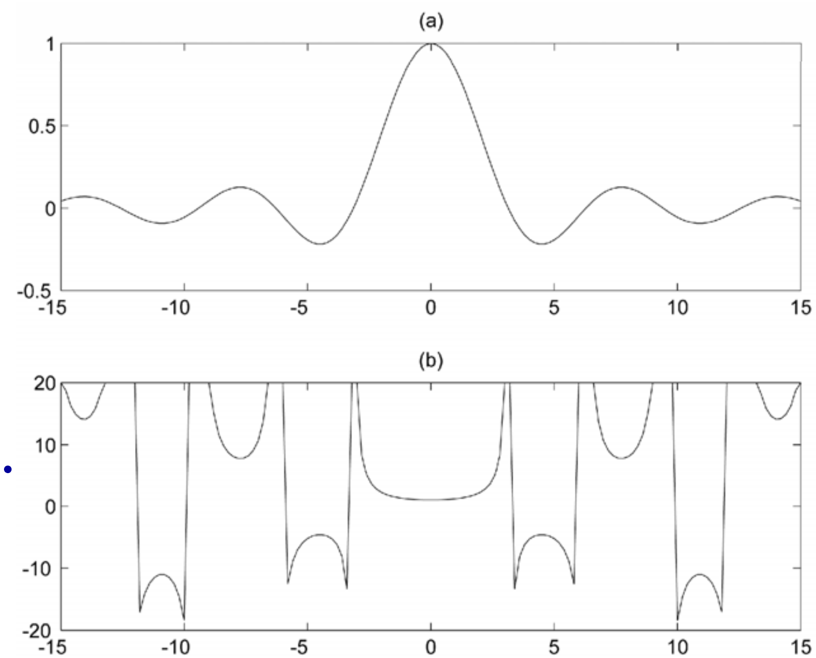
we can let $\mathbf{G}(\Omega_1,\Omega_2) = \dfrac{1}{\mathbf{H}(\Omega_1,\Omega_2)}$

and we have $\hat{\mathbf{X}}(\Omega_1,\Omega_2) = \mathbf{X}(\Omega_1,\Omega_2)$

- This produces the perfect restoration of the original image.

Inverse Filtering

- Problem of inverse filtering:
  - The degradation functions (PSFs) may have zeros, and then the inverse filters will have poles, which generate unstable situation. (example PSF of motion blur in spatial and freq.)



  - Usually PSFs are low-pass in nature, their inverse filters will become high frequency emphasis filters. This may amplify noise, which is dominant at high frequency.

Inverse Filtering

- A simple solution to solve the problem of inverse filter is restrict the magnitude of all frequency components be within a certain range, and value beyond this range will be cropped (as shown in previous figure).
- Another choice is to limit the frequency bandwidth of the PSF to be less than the first singular frequency.
- Both approaches are called pseudo-inverse filter, and they will produce imperfect restoration.
- These can solve stability problem, but still do not solve the noise sensitivity problem.

Inverse Filtering

a b
c d

FIGURE 5.27 Restoring Fig. 5.25(b) with Eq. (5.7-1). (a) Result of using the full filter. (b) Result with H cut off outside a radius of 40; (c) outside a radius of 70; and (d) outside a radius of 85.

Least MSE Restoration

- In order to avoid amplifying noise, the restoration filter should consider both PSF induced errors and noise.
- The restoration error can be calculated as

$$\varepsilon = x[n_1, n_2] - \hat{x}[n_1, n_2]$$

the mean - square - error (MSE) is

$$E\{|\varepsilon|^2\} = E\{(x[n_1, n_2] - \hat{x}[n_1, n_2])^2\}$$

- Assume the solution that can minimize the MSE is a LSI system $\boldsymbol{g[n_1, n_2]}$, then

$$\hat{X}(\omega_1, \omega_2) = X_d(\omega_1, \omega_2) G(\omega_1, \omega_2)$$

Least MSE Restoration

- Then the MSE becomes

$$MSE = E\{(X - G \cdot X_d)(X^* - G^* \cdot X_d^*)\}$$
$$= E\{|X|^2 - GX_dX^* - G^*X_d^*X + |G|^2|X_d|^2\}$$

- To minimize the MSE, the solution is

$$\frac{\partial\{MSE\}}{\partial G_R} = 0 \text{ and } \frac{\partial\{MSE\}}{\partial G_I} = 0 \quad \Rightarrow \quad G = \frac{E\{XX_d^*\}}{E\{|X_d|^2\}}$$

where G_R, and G_I are the real and imaginary terms of G.

- Given that

$$X_d(\omega_1, \omega_2) = X(\omega_1, \omega_2)H(\omega_1, \omega_2) + N(\omega_1, \omega_2)$$

- Assume $\eta[n_1, n_2]$ is zero-mean, $\eta[n_1, n_2]$ and $x[n_1, n_2]$ are uncorrelated, and $\eta[n_1, n_2]$ and $x[n_1, n_2]$ are wide sense stationary, we have $E\{X \cdot N^*\} = 0$

Least MSE Restoration

- Then we have

$$G = \frac{E\{XX^* H\}}{E\{XX^* \cdot HH^* + NN^*\}}$$

- Notice that $H(\omega_1, \omega_2)$ is deterministic, and let

$S_{xx}(\omega_1, \omega_2) = E\{XX^*\}$ and $S_{NN}(\omega_1, \omega_2) = E\{NN^*\}$

we have

$$G(\omega_1, \omega_2) = \frac{S_{xx}(\omega_1, \omega_2) H^*(\omega_1, \omega_2)}{S_{xx}(\omega_1, \omega_2) |H(\omega_1, \omega_2)|^2 + S_{NN}(\omega_1, \omega_2)}$$

- $G(\omega_1, \omega_2)$ is called the least MSE filter, or Wiener filter.

Least MSE Restoration

- Define spectral SNR as: $S_{SNR} = \dfrac{S_{xx}}{S_{NN}}$
- CASE 1: $H(\omega_1, \omega_2) = 1$ (noise only):

$$G = \frac{S_{SNR}}{1 + S_{SNR}}$$

  - if SNR is high, (occurs at low spatial frequencies)

    $G \approx 1$
  - if SNR is low, (occurs at high spatial frequencies)

    $G \approx S_{SNR}$
  - Therefore G will behave as a lowpass (smoothing) filter when only noise-induced distortion is present.

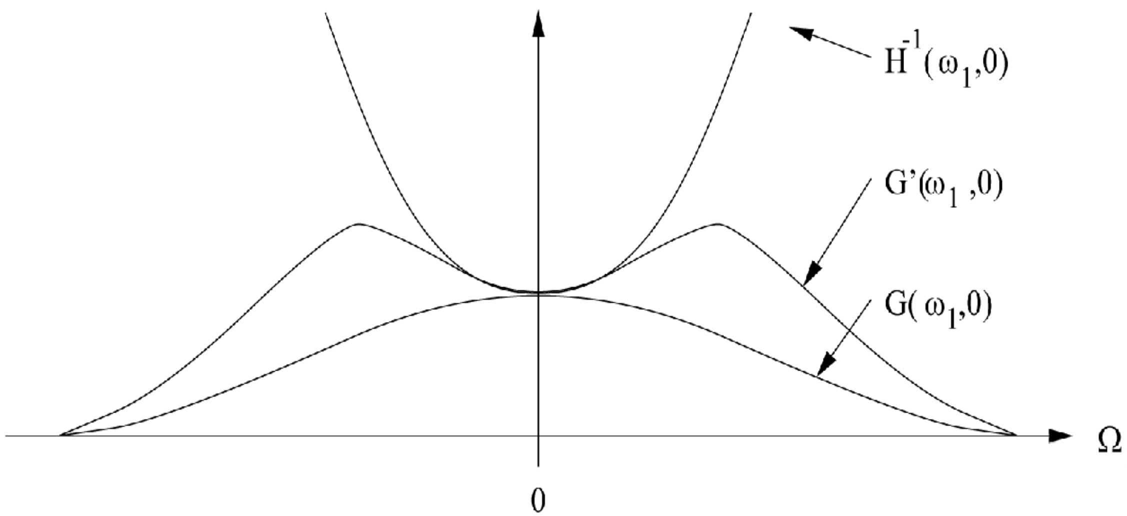
Least MSE Restoration

- CASE 2: $S_{NN}(\omega_1, \omega_2) = 0$ (no noise):

$$G(\omega_1, \omega_2) = \frac{1}{H(\omega_1, \omega_2)}$$

  – Wiener filter becomes the inverse filter.

- CASE 3: in general, we are somewhere in the middle:



Least MSE Restoration

a b c

FIGURE 5.28 Comparison of inverse- and Wiener filtering. (a) Result of full inverse filtering of Fig. 5.25(b). (b) Radially limited inverse filter result. (c) Wiener filter result.

Constrained Least Squares Restoration

- The formulation (Hunt):

  to minimize: $J = \left| Q(\omega_1, \omega_2) \hat{X}(\omega_1, \omega_2) \right|^2$

  subject to the constraint:

  $$\left| X_d(\omega_1, \omega_2) - H(\omega_1, \omega_2) \hat{X}(\omega_1, \omega_2) \right|^2 < \varepsilon$$

  where ε is a small positive number to control the restoration error, and $Q(\omega_1, \omega_2)$ is a parameter that controls the characteristics of the restoration.

Constrained Least Squares Restoration

- To solve this problem, we use Lagrangian technique, i.e. to minimize

$$J(\hat{X}) = \left|Q\hat{X}\right| + \lambda\left(\left|X_d - H\hat{X}\right|^2 - \varepsilon\right)$$

Then solve $\dfrac{\partial\{J(\hat{X})\}}{\partial\{\hat{X}\}} = 0$ and $\dfrac{\partial\{J(\hat{X})\}}{\partial\{\lambda\}} = 0$

we have $\hat{X} = \dfrac{\lambda H^*}{|Q|^2 + \lambda |H|^2} X_d$

therefore $G = \dfrac{\lambda H^*}{|Q|^2 + \lambda |H|^2}$

Constrained Least Squares Restoration

- Both $\boldsymbol{Q(\omega_1 , \omega_2)}$ and λ can control the effect of the restoration filtering.
  - $\boldsymbol{Q(\omega_1 , \omega_2)}$ can be used to control the noise. If $\boldsymbol{X}$ is a natural image with low frequency spatial energy. $\boldsymbol{Q(\omega_1 , \omega_2)}$ can be chosen as a high frequency emphasis filter so that $\boldsymbol{Q(\omega_1 , \omega_2)\hat{X}(\omega_1 , \omega_2)}$ acts as a penalty function for having high-frequency noise.
  - λ controls the mixture between the smoothing function due to $\boldsymbol{Q(\omega_1 , \omega_2)\hat{X}(\omega_1 , \omega_2)}$ and the inverse filtering generated by the constraint term. A large λ leads to the inverse filtering solution.

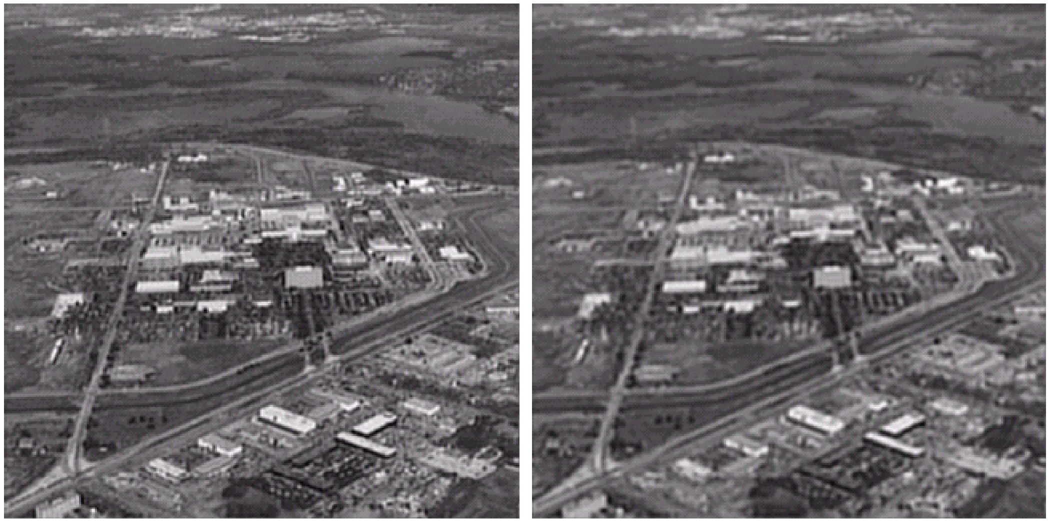
Constrained Least Squares Restoration

- It is possible to adjust the λ iteratively until acceptable results are achieved.
  - Define the restoration residual as $\boldsymbol{R} = \boldsymbol{X}_d - \boldsymbol{H}\hat{\boldsymbol{X}}$,
  - Specify an initial λ.
  - Calculate the norm of $\boldsymbol{R}$ as $\|\boldsymbol{r}\|^2 = \sum_{n_1=0}^{M_1-1} \sum_{n_2=0}^{M_2-1} r^2[n_1, n_2]$, where $\mathbf{M_1} \times \mathbf{M_2}$ is the size of the image, and $\boldsymbol{r}[n_1, n_2]$ is a sample in $\boldsymbol{R}$.
  - Check if $\|\boldsymbol{r}\|^2 = \|\eta\|^2 \pm \boldsymbol{a}$, where a is a small accuracy factor, and $\|\eta\|^2 = \mathbf{M_1 M_2}\,[\sigma_\eta^{\,2} - \mu_\eta]$.

    If $\|\boldsymbol{r}\|^2 > \|\eta\|^2 + \boldsymbol{a}$, increase λ,

    If $\|\boldsymbol{r}\|^2 < \|\eta\|^2 - \boldsymbol{a}$, decrease λ,

Constrained Least Squares Restoration



a b

FIGURE 5.31 (a) Iteratively determined constrained least squares restoration of Fig. 5.16(b), using correct noise parameters. (b) Result obtained with wrong noise parameters.

Constrained Least Squares Restoration

- There are no optimal functions for ***Q*** that can be derived. The general notion is that ***Q*** should provide some form of high-frequency emphasis.
- In practice, an approximation to the Laplacian operator (2nd order derivative) is used as ***Q***

$$q[n_1, n_2] = -\delta[n_1, n_2] + \frac{1}{4}(\delta[n_1 - 1, n_2] + \delta[n_1 + 1, n_2] + \delta[n_1, n_2 - 1] + \delta[n_1, n_2 + 1])$$

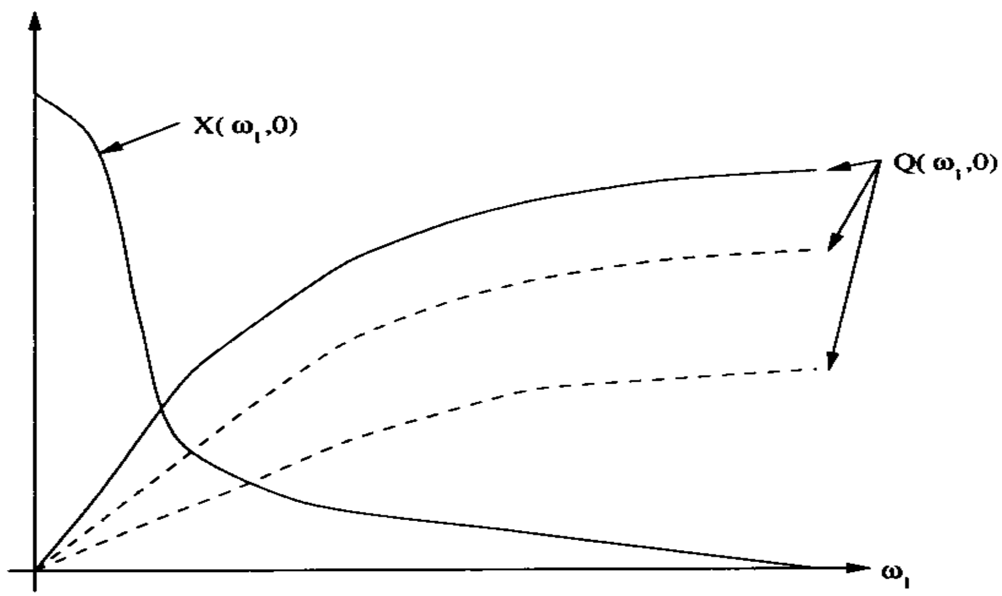
- In the frequency domain, the approximation often used is:

$$Q = -1 + \frac{1}{2}\cos\omega_1 + \frac{1}{2}\cos\omega_2$$

- Special case: if λ=1 and $|Q|^2 = S_{NN} / S_{xx}$ we obtain the wiener filter solution

Constrained Least Squares Restoration

- Typical $\boldsymbol{X(\omega_1, \omega_2)}$ and $\boldsymbol{Q(\omega_1, \omega_2)}$ relationship.



Iterative Constrained Restoration

- Iterative constrained restoration (Van Cittert) makes use of as much information as we have about the original.
- Express this information in the form of an operator $\boldsymbol{C}(\cdot)$
  For example:
  Positivity constraint: $C(x) = \begin{cases} x, & 0 \le x \le 255 \\ 0, & \text{otherwise} \end{cases}$
- If we denote the degraded signal as $x_d = H(x) + \eta$
  we want to satisfy both $x_d = H(\hat{x})$ and $\hat{x} = C(\hat{x})$
- Combining the model and the constraint equation, we obtain:

$$\hat{x} = C(\hat{x}) + \lambda\left(x_d - H\left(C(\hat{x})\right)\right)$$

Iterative Constrained Restoration

- Formulation of the iteration process:

$$\hat{x}_0 = \lambda x_d$$

$$\hat{x}_{k+1} = \lambda x_d + C(\hat{x}_k) - \lambda H(C(\hat{x}_k))$$

- If we assume the degradation function is LSI, we can examine the problem analytically and determine whether iterations will converge. This leads to:

$$\hat{x}_{k+1} = \lambda x_d + C(\hat{x}_k) - \lambda h ** (C(\hat{x}_k))$$

- If we assume that the constrain ***C*** is an identity operator. Then we will have a 1-D difference equation in the index ***k*,** and the solution becomes:

$$\hat{X}_k(\omega_1, \omega_2) = \frac{X_d(\omega_1, \omega_2)\left(1 - (1 - \lambda H(\omega_1, \omega_2))^{k+1}\right)}{H(\omega_1, \omega_2)}$$

Iterative Constrained Restoration

- Observe that

$$\lim_{k\to\infty} \hat{X}_k(\omega_1,\omega_2) = \frac{X_d(\omega_1,\omega_2)}{H(\omega_1,\omega_2)} \quad \text{if} \quad |1-\lambda H(\omega_1,\omega_2)| < 1$$

  Assume that λ is real and positive, the process converges if **Real($H(\omega_1, \omega_2)$) > 0.**

- Advantages:
  - Can incorporate constraints.
  - Can stop iterations early before converging to the inverse solution.
  - Leads to subjective improvement (inverse solution may not be desirable due to noise).
  - ***H*** can be more general, i.e. nonlinear.