

CS 355/555

Probability and Statistics for CS

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Lecture 25, 04/07/2025

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Conditional PDF Given an Event

Suppose A and B are events.

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Conditional PDF Given an Event

The **conditional PDF** of a continuous random variable X , given an event A with $P(A) > 0$, is defined as a non-negative function $f_{X|\{X \in A\}}$, or simply $f_{X|A}$, that satisfies

$$P(X \in B|X \in A) = \int_B f_{X|A}(x) dx \text{ for any subset of } B.$$

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$$\begin{aligned} P(X \in B|X \in A) &= \frac{P(X \in B \text{ and } X \in A)}{P(X \in A)} = \frac{\int_{A \cap B} f_X(x) dx}{P(X \in A)} \\ &= \frac{\int_{A \cap B} f_X(x) dx}{\int_A f_X(x) dx} \end{aligned}$$

Conditional PDF Given an Event

$$P(X \in B | X \in A) = \int_{B \cap A} f_{X|A}(x) dx = \int_B f_{X|A}(x) dx - \int_{B \setminus A} f_{X|A}(x) dx$$

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$$P(X \in B | X \in A) = \frac{\int_{A \cap B} f_X(x) dx}{P(X \in A)}$$

$$f_{X|A}(x) = \begin{cases} ?, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}$$

Conditional PDF Given an Event

$$P(X \in B|A) = \int_B f_{X|A}(x)dx = \int_{B \cap A} f_{X|A}(x)dx + \int_{B \setminus A} f_{X|A}(x)dx$$

$$P(X \in B|X \in A) = \frac{\int_{A \cap B} f_X(x)dx}{P(X \in A)}$$

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{P(X \in A)}, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}$$

Conditional PDF Given an Event

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{P(X \in A)}, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}$$

The Conditional PDF is zero outside the conditioning set.

Within the conditioning set, the conditional PDF has exactly the same shape as the unconditional one, except that it is scaled by the constant factor $1/P(X \in A)$.

Conditional PDF Given an Event

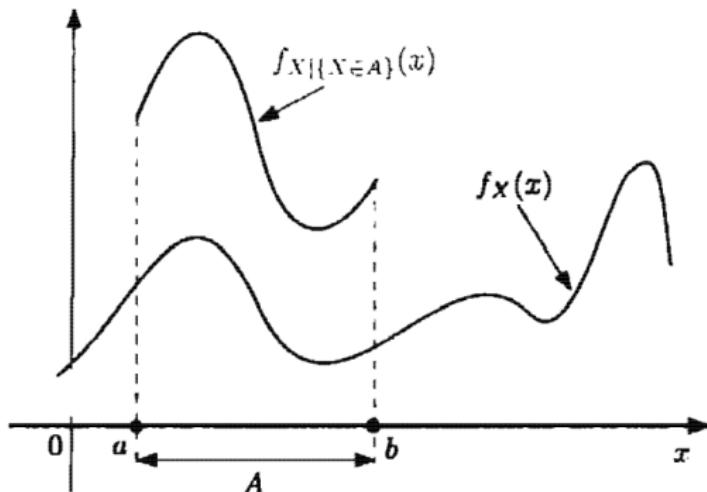


Figure 3.14: The unconditional PDF f_X and the conditional PDF $f_{X|\{X \in A\}}$, where A is the interval $[a, b]$. Note that within the conditioning event A , $f_{X|\{X \in A\}}$ retains the same shape as f_X , except that it is scaled along the vertical axis.

Example 3.13, pg 165

Example 3.13. The Exponential Random Variable is Memoryless. The time T until a new light bulb burns out is an exponential random variable with parameter λ . Ariadne turns the light on, leaves the room, and when she returns, t time units later, finds that the light bulb is still on, which corresponds to the event $A = \{T > t\}$. Let X be the additional time until the light bulb burns out. What is the conditional CDF of X , given the event A ?

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$$P(X > x|A) = P(T > t + x|T > t)$$

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$$\begin{aligned} P(X > x|A) &= P(T > t + x|T > t) \\ &= \frac{P(T > t + x \text{ and } T > t)}{P(T > t)} \end{aligned}$$

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The conditional CDF of X given A is: $P(X \leq x|A) = 1 - e^{-\lambda x}$

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The conditional CDF of X given A is: $P(X \leq x|A) = 1 - e^{-\lambda x}$

The conditional CDF of X is exponential with parameter λ , regardless of the time t that the bulb has been turned on.

Example 3.13, pg 165

The time between customers at a bank is modeled as an exponential random variable X . On the average, a customer walks in every ten minutes. How long will it be before the next customer if the teller has been waiting

- 1 minute?
- 5 minutes?
- 9 minutes?
- 20 minutes?

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Conditioning one Random Variable on Another

In the discrete case,

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

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Let X and Y be continuous random variables with joint PDF $f_{X,Y}$.
For any y with $f_Y(y) > 0$, the **conditional PDF** of X given $Y = y$, is

$$f_{X|Y}(x|Y = y) = f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

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$$\int_{-\infty}^{\infty} f_{X|Y}(x|y) dx = \frac{\int_{-\infty}^{\infty} f_{X,Y}(x,y) dx}{f_Y(y)} = \frac{f_Y(y)}{f_Y(y)} = 1$$

Example 3.15, pg 169

Circular Uniform PDF. X and Y are uniformly distributed random variables in a circular region with radius r , so that the joint PDF is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi r^2}, & \text{if } x^2 + y^2 \leq r^2, \\ 0, & \text{otherwise} \end{cases}$$

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To compute the conditional PDF $f_{X|Y}(x|y)$, we first find $f_Y(y)$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx = \frac{1}{\pi r^2} \int_{x^2+y^2 \leq r^2} dx \\ &= \frac{1}{\pi r^2} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} dx = \frac{2}{\pi r^2} \sqrt{r^2 - y^2}, \text{ if } |y| \leq r \end{aligned}$$

Example 3.15

The conditional PDF is given by

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1/\pi r^2}{(2/\pi r^2)\sqrt{r^2 - y^2}} = \frac{1}{2\sqrt{r^2 - y^2}}$$

if $x^2 + y^2 \leq r^2$.

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The marginal PDF f_Y is not uniform.

Example 3.15

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if $x^2 + y^2 \leq r^2$.

The marginal PDF f_Y is not uniform.

For a fixed value of y , the conditional PDF $f_{X|Y}$ is uniform.

Conditional PDF Given a Random Variable, pg 172

Let X and Y be jointly continuous random variables with joint PDF $f_{X,Y}$.

- The joint, marginal, and conditional PDFs are related to each other by the formulas

$$f_{X,Y}(x,y) = f_Y(y)f_{X|Y}(x|y),$$

$$f_X(x) = \int_{-\infty}^{\infty} f_Y(y)f_{X|Y}(x|y) dy.$$

The conditional PDF $f_{X|Y}(x|y)$ is defined only for those y for which $f_Y(y) > 0$.

- We have

$$\mathbf{P}(X \in A | Y = y) = \int_A f_{X|Y}(x|y) dx.$$

Summary: Conditional Expectation, pg 173

Summary of Facts About Conditional Expectations

Let X and Y be jointly continuous random variables, and let A be an event with $\mathbf{P}(A) > 0$.

- **Definitions:** The conditional expectation of X given the event A is defined by

$$\mathbf{E}[X | A] = \int_{-\infty}^{\infty} xf_{X|A}(x) dx.$$

The conditional expectation of X given that $Y = y$ is defined by

$$\mathbf{E}[X | Y = y] = \int_{-\infty}^{\infty} xf_{X|Y}(x | y) dx.$$

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Conditional Expectation, pg 173

- **The expected value rule:** For a function $g(X)$, we have

$$\mathbf{E}[g(X) | A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x) dx,$$

and

$$\mathbf{E}[g(X) | Y = y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x | y) dx.$$

Conditional Expectation, pg 173

- **Total expectation theorem:** Let A_1, A_2, \dots, A_n be disjoint events that form a partition of the sample space, and assume that $\mathbf{P}(A_i) > 0$ for all i . Then,

$$\mathbf{E}[X] = \sum_{i=1}^n \mathbf{P}(A_i) \mathbf{E}[X | A_i].$$

Similarly,

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} \mathbf{E}[X | Y = y] f_Y(y) dy.$$

Example 3.17, pg 174

Example 3.17. Mean and Variance of a Piecewise Constant PDF. Suppose that the random variable X has the piecewise constant PDF

$$f_X(x) = \begin{cases} 1/3, & \text{if } 0 \leq x \leq 1, \\ 2/3, & \text{if } 1 < x \leq 2, \\ 0, & \text{otherwise,} \end{cases}$$

(see Fig. 3.18). Consider the events

$$A_1 = \{X \text{ lies in the first interval } [0, 1]\},$$

$$A_2 = \{X \text{ lies in the second interval } (1, 2]\}.$$

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(see Fig. 3.18). Consider the events

$$A_1 = \{X \text{ lies in the first interval } [0, 1]\},$$

$$A_2 = \{X \text{ lies in the second interval } (1, 2]\}.$$

We have from the given PDF,

$$\mathbf{P}(A_1) = \int_0^1 f_X(x) dx = \frac{1}{3}, \quad \mathbf{P}(A_2) = \int_1^2 f_X(x) dx = \frac{2}{3}.$$

Example 3.17, pg 174

The conditional PDFs $f_{X|A_1}$ and $f_{X|A_2}$ are uniform. Recall that the mean of a uniform random variable over an interval $[a, b]$ is $\frac{a+b}{2}$ and its second moment is $\frac{a^2 + ab + b^2}{3}$.

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Thus,

$$\mathbf{E}[X | A_1] = \frac{1}{2}, \quad \mathbf{E}[X | A_2] = \frac{3}{2}.$$

$$\mathbf{E}[X^2 | A_1] = \frac{1}{3}, \quad \mathbf{E}[X^2 | A_2] = \frac{7}{3}.$$

Example 3.17, pg 174

The conditional PDFs $f_{X|A_1}$ and $f_{X|A_2}$ are uniform. Recall that the mean of a uniform random variable over an interval $[a, b]$ is $(a + b)/2$ and its second moment is $(a^2 + ab + b^2)/3$.

We now use the total expectation theorem to obtain

$$\mathbf{E}[X] = \mathbf{P}(A_1)\mathbf{E}[X | A_1] + \mathbf{P}(A_2)\mathbf{E}[X | A_2] = \frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{3}{2} = \frac{7}{6},$$

$$\mathbf{E}[X^2] = \mathbf{P}(A_1)\mathbf{E}[X^2 | A_1] + \mathbf{P}(A_2)\mathbf{E}[X^2 | A_2] = \frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{7}{3} = \frac{15}{9}.$$

The variance is given by

$$\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \frac{15}{9} - \frac{49}{36} = \frac{11}{36}.$$

Independence of Random Variables

- Just like the discrete case, two continuous random variables X and Y are **independent** if their joint PDF is the product of the marginal PDFs:

$$f_{X,Y}(x,y) = f_X(x)f_Y(y), \text{ for all } x, y$$

- Comparing with the formula $f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y)$, independence condition is the same as

$$f_{X|Y}(x|y) = f_X(x), \text{ for all } y \text{ with } f_Y(y) > 0$$

- Symmetrically,

$$f_{Y|X}(y|x) = f_Y(y), \text{ for all } x \text{ with } f_X(x) > 0$$

Example

Let $f_{X,Y} = cxy$ on $[0, 1] \times [0, 1]$.

Then

$$\begin{aligned} 1 &= \int_0^1 \int_0^1 cxy \, dx \, dy \\ &= \int_0^1 cy \int_0^1 x \, dx \, dy \end{aligned}$$

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Example

Let $f_{X,Y} = 4xy$ on $[0, 1] \times [0, 1]$.

$$f_X(x) = \int_0^1 4xy \, dy = 1$$

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Let $f_{X,Y} = 4xy$ on $[0, 1] \times [0, 1]$.

$$\begin{aligned}f_X(x) &= \int_0^1 4xy \, dy = 1 \\&= 4x \int_0^1 y \, dy \\&= 2x\end{aligned}$$

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Let $f_{X,Y} = 4xy$ on $[0, 1] \times [0, 1]$.

$$f_X(x) = \int_0^1 4xy \, dy = 1$$

$$= 4x \int_0^1 y \, dy$$

$$= 2x$$

$$f_Y(y) = \int_0^1 4xy \, dx = 1$$

$$= 4y \int_0^1 x \, dx$$

$$= 2y$$

$$f_{X,Y}(x, y) = 4xy = 2x \times 2y = f_X(x)f_Y(y)$$

So, X and Y are independent.

Generalization

- There is a natural generalization to the case of more than two random variables. X , Y and Z are independent if

$$f_{X,Y,Z} = f_X(x)f_Y(y)f_Z(z), \text{ for all } x, y, z.$$

Properties of Independence

- If X and Y are independent, then any two events of the form $\{X \in A\}$ and $\{Y \in B\}$ are independent

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$$\begin{aligned} P(X \in A \text{ and } Y \in B) &= \int_{x \in A} \int_{y \in B} f_{X,Y}(x, y) dy dx \\ &= \int_{x \in A} \int_{y \in B} f_X(x) f_Y(y) dy dx \\ &= \int_{x \in A} f_X(x) dx \int_{y \in B} f_Y(y) dy \\ &= P(X \in A)P(Y \in B) \end{aligned}$$

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- If X and Y are independent, then any two events of the form $\{X \in A\}$ and $\{Y \in B\}$ are independent

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- In particular, independence implies that

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- The converse is also true.

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Example 2

The joint PDF of two continuous RVs X and Y is given by

$$f_{X,Y}(x,y) = 12xy(1-y), \text{ for } 0 \leq x \leq 1, 0 \leq y \leq 1.$$

Are X and Y independent?

