This assignment contains 5 bonus points.

Problem 1) In test II, we had the following problem: Two continuous uniform random variables X and Y have a joint PDF given by

$$\begin{cases} c(x+y), & \text{if } 0 \le x \le 2, 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

What is the value of c? Find the marginal PDF of X and Y, i.e., $f_X(x)$ and $f_Y(y)$, Find $E[2X+4Y]$.

(a) Solve the above questions again as similar questions will be on the final. (5 points)

(b) Find $Cov(X,Y)$ (10 points)

Solution: (a) $c = \frac{1}{8}$
$f_X(x) = \frac{1}{4}(x+1)$, if $0 \le x \le 2$, $\quad f_Y(y) = \frac{1}{4}(y+1)$, if $0 \le y \le 2$
$E[X] = \int_0^2 x f_X(x)dx = \frac{1}{4}\int_0^2 x^2 + xdx = \frac{7}{6}$
$E[Y] = \frac{7}{6}$
$E[2X+4Y] = 7$
(b)

$$\begin{aligned} E[XY] &= \int_0^2 \int_0^2 xy\frac{1}{8}(x+y)dxdy \\ &= \int_0^2 (\frac{1}{8}(\frac{1}{3}x^3y + \frac{1}{2}x^2y^2)|_0^2)dy \\ &= \int_0^2 (\frac{1}{3}y + \frac{1}{4}y^2)dy \\ &= \frac{1}{6}y^2 + \frac{1}{12}y^3|_0^2 \\ &= \frac{4}{6} + \frac{8}{12} = \frac{4}{3} \end{aligned}$$

$Cov(X,Y) = E[XY] - E[X]E[Y] = -\frac{1}{36}$

Problem 2) pg 290 #10 (15 points)

(a) Let $S_n = X_1 + \ + X_n$ be the total number of gadgets produced in n days. Note that the mean, variance, and standard deviation of S_n is 5n, 9n, and $3\sqrt{n}$, respectively. Thus,

$$
\begin{aligned}
P(S_{100} < 440) &= P(S_100 \leq 439.5) \\
&= P\left(\frac{S_{100} - 500}{30} < \frac{439.5 - 500}{30}\right) \\
&= \Phi\left(\frac{439.5 - 500}{30}\right) \\
&= \Phi(-2.02) \\
&= 1 - \Phi(2.02) \\
&= 1 - 0.9783 \\
&= 0.0217.
\end{aligned}
$$

(b) The requirement $P(S_n \geq 200 + 5n) \leq 0.05$ translates to

$$P\left(\frac{S_n - 5n}{3\sqrt{n}} \geq \frac{200}{3\sqrt{n}}\right) \geq 0.05,$$

or, using a normal approximation,

$$1 - \Phi\left(\frac{S_n - 5n}{3\sqrt{n}}\right) \leq 0.05$$

and

$$\phi\left(\frac{200}{3\sqrt{n}}\right) \geq .95$$

and

$$\frac{200}{3\sqrt{n}} \geq 1.65$$

which finally yields $n \leq 1632$.

(c) The event $N \geq 220$ (it takes at least 220 days to exceed 1000 gadgets) is the same as the event $S_{219} \leq 1000$ (no more than 1000 gadgets produced in the first 219 days). Thus,

$$
\begin{aligned}
P(N \geq 220) &= P(S_{219} \leq 1000) \\
&= P\left(\frac{S_{219} - 5 \times 219}{3\sqrt{219}} \leq \frac{1000 - 5 \times 219}{3\sqrt{219}}\right) \\
&= 1 - \Phi(2.14) \\
&= 1 - 0.9838 \\
&= 0.0162.
\end{aligned}
$$