CS 355/555
Probability and Statistics for CS

Lecture 27, 04/14/2025

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Derived Distributions

Consider a function $Y = g(X)$ of a continuous random variable X. Given the PDF of X, how do we compute the PDF of Y?

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Calculation of the PDF of a function $Y = g(X)$ of a continuous Random Variable X

- 1. Calculate the CDF F_Y of Y using the formula

$$\begin{aligned} F_Y(y) =& P(Y \leq y) \\ &= p_Y(g(X) \leq y) \\ &= \int_{x|g(x)\leq y} f_X(x)dx \end{aligned}$$

- Differentiate to obtain the PDF of Y

Example 1

Let X be uniform on [0,1], and let $Y = \sqrt{X}$. Compute $f_Y(y)$.

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$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

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Differentiate and obtain

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \frac{d}{dy} y^2 \\ &= 2y, 0 \le y \le 1. \end{aligned}$$

Example 1

Outside the range $y \in [0, 1]$, the CDF $F_Y(y)$ is constant, i.e., $F_Y(y) = 0$ for $y \leq 0$, and $F_Y(y) = 1$ for $y \geq 1$.

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$$f_Y(y) = \begin{cases} 2y, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Example 2

John is driving from Birmingham to Atlanta, a distance of 180 miles at a constant speed, whose value is uniformly distributed between 30 and 60 miles per hour. What is the PDF of the duration of the trip?[1]

[1] The numbers are unrealistic, in order to make arithmetic easier

Example 2

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Let X be the speed and let Y be the trip duration: $Y = g(X) = \frac{180}{X}$.

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Let X be the speed and let Y be the trip duration: $Y = g(X) = \frac{180}{X}$.

The CDF of Y is given by

$$\begin{aligned} P(Y \leq y) &= P\left(\frac{180}{X} \leq y\right) \\ &= P\left(X \geq \frac{180}{y}\right) \\ &= 1 - F_X\left(\frac{180}{y}\right) \end{aligned}$$

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Example 2

The PDF of X is uniform, which is

$$f_X(x) = \begin{cases} 1/30, & \text{if } 30 \leq x \leq 60 \\ 0, & \text{otherwise} \end{cases}$$

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The corresponding CDF of X is given by

$$F_X(x) = \begin{cases} 0, & \text{if } x < 30 \\ (x-30)/30, & \text{if } 30 \le x \le 60 \\ 1, & \text{if } x > 60 \end{cases}$$

Example 2

$$
\begin{aligned}
P(Y \le y) &= 1 - F_X\left(\frac{180}{y}\right) \\
&= \begin{cases} 0, & \text{if } \frac{180}{y} > 60 \\ 1 - \left(\frac{180}{y} - 30\right)/30, & \text{if } 30 \le \frac{180}{y} \le 60 \\ 1, & \text{if } \frac{180}{y} < 30 \end{cases}
\end{aligned}
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&= \begin{cases} 0, & \text{if } y < 3 \\ 2 - \frac{6}{y}, & \text{if } 3 \le y \le 6, \\ 1, & \text{if } y > 6 \end{cases}
\end{aligned}
$$

Example 2

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 3 \\ 2 - \frac{6}{y}, & \text{if } 3 \leq y \leq 6, \\ 1, & \text{if } y > 6 \end{cases}$$

Differentiate the expression, we obtain the PDF of Y

$$f_Y(y) = \begin{cases} \frac{6}{y^2}, & \text{if } 3 \leq y \leq 6 \\ 0, & \text{otherwise} \end{cases}$$

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$$f_Y(y) = \begin{cases} \frac{6}{y^2}, & \text{if } 3 \le y \le 6 \\ 0, & \text{otherwise} \end{cases}$$

Expected value:

$$\begin{aligned} E[Y] &= \int_3^6 y \frac{6}{y^2} dy \\ &= \int_3^6 \frac{6}{y} dy \\ &= 6 \ln y |_3^6 \\ &= 6(\ln 6 - \ln 3) \\ &= 6 \ln 2 \\ &\approx 4.1 \end{aligned}$$

The Linear Case

In the special case where Y is a linear function of X

The PDF of a linear Function of a Random Variable

Let X be a continuous random variable with PDF f_X, and let

$$Y = aX + b$$

where a and b are scalars with $a \neq 0$. Then,

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

The Linear Case

For $a > 0$

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&= P\left(X \leq \frac{y-b}{a}\right) \\
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Differentiate the above equation and use the chain rule

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) \\ &= \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \end{aligned}$$

The Linear Case

Differentiate F_X and use the chain rule

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) \\ &= \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \end{aligned}$$

When $a < 0$, $f_Y(y) = -\frac{1}{a} f_X(\frac{y-b}{a})$

The Linear Case

The derivative of a composite function $f(g(x))$ can be determined by taking the product of the derivative[2] of $f(x)$ with respect to $g(x)$ and the derivative of $g(x)$ with respect to the variable x.

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

[2] The chain rule from Calculus I, again

A Linear Function of an Exponential RV

Let X be an exponential random variable with PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

where λ is a positive parameter. Let $Y = aX + b$. Then,

$$\begin{aligned} f_Y(y) &= \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) \\ &= \begin{cases} \frac{\lambda}{|a|} e^{-\lambda(y-b)/a}, & \text{if } (y-b)/a \geq 0 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

A Linear Function of a Normal Random Variable is Normal

X is a normal random variable with mean μ and variance σ^2. Let $Y = aX + b$, where a and b are scalars, with $a \neq 0$. What is $f_Y(y)$?

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$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

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$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\begin{aligned}
f_Y(y) &= \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) \\
&= \frac{1}{|a|} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(\frac{y-b}{a}-\mu)^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi}|a|\sigma} e^{-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}}
\end{aligned}$$

which is a normal PDF with mean $a\mu + b$ and variance $a^2\sigma^2$

The Monotonic Case

Monotonic Function

A function f is *monotonic* if for all x, y either

$$x \leq y \Rightarrow f(x) \leq f(y)$$

or

$$x \leq y \Rightarrow f(x) \geq f(y)$$

The Monotonic Case

Monotonic Function

A function f is *strictly monotonic* if for all x, y either

$$x < y \Rightarrow f(x) < f(y)$$

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Monotonic Function

A function f is *strictly monotonic* if for all x, y either

$$x < y \Rightarrow f(x) < f(y)$$

or

$$x < y \Rightarrow f(x) > f(y)$$

Note that for f strictly monotonic f must be one to one.

The Monotonic Case

Let $Y = g(X)$, where g is strictly monotonic over the interval I

- $g(x_1) < g(x_2)$ for all $x_1, x_2 \in I$ satisfying $x_1 < x_2$
- $g(x_1) < g(x_2)$ for all $x_1, x_2 \in I$ satisfying $x_1 > x_2$

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For some function h and all x in the range of X we have

$$y = g(x) \text{ if and only if } x = h(y)$$

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h is the **inverse function** of g

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For some function h and all x in the range of X we have

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If h is differentiable, the PDF of Y in the region where $f_Y(y) > 0$ is given by

$$f_Y(y) = f_X(h(y)) \left| \frac{d}{dy} h(y) \right|$$

Proof

Assuming that g is monotonically increasing,

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(g(X) \leq y) \\ &= P(X \leq h(y)) = F_X(h(y)) \end{aligned}$$

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$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(g(X) \leq y) \\ &= P(X \leq h(y)) = F_X(h(y)) \end{aligned}$$

Differentiate $F_Y(y)$ to obtain $f_Y(y)$

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \frac{d}{dy} F_X(h(y)) \\ &= f_X(h(y)) \frac{d}{dy} h(y) \end{aligned}$$

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Assuming that g is monotonically increasing,

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Differentiate $F_Y(y)$ to obtain $f_Y(y)$

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \frac{d}{dy} F_X(h(y)) \\ &= f_X(h(y)) \frac{d}{dy} h(y) \end{aligned}$$

Since h is monotonically increasing, its derivative is non-negative $\frac{d}{dy} h(y) = \left| \frac{d}{dy} h(y) \right|$

Proof

Assuming that g is monotonically decreasing, ...

Problem 2

Find the PDF of $Y = e^X$ in terms of the PDF of X.

Specialize the answer to the case where X is uniformly distributed in $[0, 1]$.

Find the expression for $F_Y(y)$

$$
\begin{aligned}
F_Y(y) &= P(Y = e^X \le y) \\
&= P(X \le \ln y) \\
&= F_X(\ln y), y \ge 0.
\end{aligned}
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Find the expression for $F_Y(y)$

$$\begin{aligned} F_Y(y) &= P(Y = e^X \leq y) \\ &= P(X \leq \ln y) \\ &= F_X(\ln y), y \geq 0. \end{aligned}$$

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Find the PDF of $Y = e^X$ in terms of the PDF of X. Specialize the answer to the case where X is uniformly distributed in $[0, 1]$.

Find the expression for $F_Y(y)$

$$\begin{aligned} F_Y(y) &= P(Y = e^X \leq y) \\ &= P(X \leq \ln y) \\ &= F_X(\ln y), y \geq 0. \end{aligned}$$

Differentiate to get $f_Y(y)$

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_X(\ln y) \\ &= f_X(\ln y) \cdot (\ln y)' \\ &= \frac{1}{y} f_X(\ln y), y \geq 0. \end{aligned}$$

Problem 2

Specialize the answer to the case where X is uniformly distributed in $[0, 1]$.

$f_X(x) = 1$, since X is uniform, and thus

$$f_Y(y) = \frac{1}{y} f_X(\ln y) = \frac{1}{y}, \quad y \geq 0.$$

Next Time

Read section 4.2

Example 4.2, continued

John is driving from Birmingham to Atlanta, a distance of 180 miles at a constant speed, whose value of uniformly distributed between 30 and 60 miles per hour. What is the PDF of the duration of the trip?

X denotes the speed, $X \sim \text{uniform}(30, 60)$, $f_X(x) = 1/30$

$y = g(x) = \frac{180}{x}$, and $x = h(y) = \frac{180}{y}$