Project - Tomatoes vs. Aphids vs. Ladybugs
MM5016

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Include the plots, answers to all questions including your calculations, and your final code in your report.

1 Introduction

In this project you will model the tomato plants I will have on my balcony this summer. Unfortunately, on the balcony there will also be aphids (bladlöss) eating the tomato leaves. By the end of the project, you will attempt to rescue the tomato plants by introducing an additional trophic-layer to this simple ecosystem, i.e. ladybugs (nyckelpigor), which are the natural (hyper-) predators of the aphids.

The interaction between predators, y (here number of aphids), and prey x (here number of tomato leaves), are famously described by the Lotka-Volterra equations.

$$\frac{dx}{dt} = \alpha x - \beta xy \tag{1}$$

$$\frac{dy}{dt} = \delta xy - \gamma y \tag{2}$$

where t is time (here in days). The term αx represents the growth of the prey, which for the tomatoes is determined by the amount of sunlight, access to water and nutrients in the soil. The term $-\beta xy$ models how the tomatoes are eaten by aphids. A high β means that there is a strong interaction between the tomatoes and aphids, i.e. the aphids are highly likely to find the tomato plants, and the tomatoes are sensitive to an aphid attack.

The aphid growth is modelled by the term δxy, i.e. their growth is dependent on the availability of tomato plants to eat. A high δ models that they do not need to eat much tomato leaves to thrive, and they easily find the tomato plants. The term $-\gamma y$ describes the normal death rate of the aphids, due to their natural life-cycle

For all the code here we are using the packages `numpy`, `matplotlib.pyplot` by importing as `np` and `plt` respectively. We are also using the RK45 method imported from the library `scipy`.

2 Questions

2.1 Part 1: Aphids

At the beginning of the summer, there are 10 aphids on the balcony. As a start, consider how the aphids would perform in the absence of tomatoes over a time period of two months (60 days), i.e. study equation (2) with $\delta = 0$. Each day 10 percent of the aphids die or disappear, i.e. $\gamma = 0.1$.

1. What is the analytical solution? Does it make sense?
2. Plot the analytical solution from time $t = 0$ to $t = 60$ days. You can use math. exp to get the exponential.
3. What is the the theoretical maximum time step that yields a stable solution, $\Delta t_{\max}$, given an forward Euler discretisation?
4. Solve the problem numerically with forward Euler and plot it (together with the analytical solution). You are not allowed to use an inbuilt function for forward Euler.
5. Try a time step that is slightly larger than the maximum stable time step, e.g. by setting $\Delta t = 1.1\Delta t_{\max}$. Is the numerical solution growing or decreasing, i.e. is it stable? Does it look accurate? Include a plot in the report.
6. Try a time step which is slightly smaller than the maximum stable time step, e.g. by setting $\Delta t = 0.9\Delta t_{\max}$. Is the numerical solution growing or decreasing, i.e. is it stable? Does it look accurate? Include a plot in the report.
7. Decrease the time step until the oscillations disappear and the solution looks very similar to the analytical solution. Which time step did you choose? Include a plot in the report.
8. Derive the exact maximum and minimum local truncation error (to do that, in the formula for the local truncation error, evaluate the second derivative at $t = 0$ and $t = 0 + \Delta t$).
9. Calculate the local truncation error numerically, by comparing the analytical and numerical solution after one time step. Does the numerical local truncation error lie within the expected range? Write down the theoretical and numerical local truncation error in the report.
10. Take a twice as small time-steps. How did the truncation error change? Is it as expected from theory?
11. How small time-step do you need to take to notice that the simulation is slow on your computer?

2.2 Part 2: Aphids and Tomatoes

You will now add tomatoes to the system, and model the interaction between aphids and tomatoes by solving the full system (1)-(2).

1. Is this a linear or nonlinear system of equations?
2. In the python script, add the first term of (2) and discretize with a forward Euler discretisation of (1). In order to check that your code is working correctly, reproduce the example with rabbits and foxes that you find in the section "A simple example" on the wikipedia page about the Lotka Volterra equations: https://en.wikipedia.org/wiki/Lotka–Volterra_equations Reproduce the plot of number of rabbits and foxes against time. The problem is quite sensitive to the time step, so if your solution looks strange, try decreasing the time step. Include a plot in your report.

3. Now you are ready to simulate the tomatoes and aphids. Simulate the system for 60 days. The aphids still die/disappear with a rate of 10% per day, while their growth factor due to eating the available tomato leaves is now increasing to $\delta = 0.05$. The tomato leaves grow with 30% ($\alpha = 0.3$) each day, while their death rate due to aphid munching is $\beta = 0.1$. In the beginning the total number of tomato leaves, as well as number of aphids, equals 10. Find a suitable time-step by decreasing it until you cannot see the solution changing. Include a plot in your report.

4. The success of the tomato plants can be measured as the amount of tomato leaves integrated over the whole summer $S = \int x(t)t$. Compute S is and include the value in your report. You are not allowed to use an inbuilt function to compute the integral.

5. How much does S decrease if you start with 11 aphids instead of 10 ? Change the initial value back to 10 after you tested.

6. What would happen if the aphids mutate to a more resistant form ("superaphids"), so that their death rate is just slighlty lower, i.e. 0.09 instead of 0.1 ? How much does S decrease? Change the death rate back to 0.1 after you tested.

7. What would happen if I get better at watering the tomato plants, so that $\alpha = 0.35$ instead of 0.3 ? How much does S increase? Change the growth rate back to 0.3 after you tested.

8. Solve the problem using the inbuilt RK45 method (it is in solve_ivp) as well and compare how many discretization points you need to get a decent result, compared to the number of discretization points you need for forward Euler.

9. Explain in maximum half a page how RK45 works.

2.3 Part 3: Adding Ladybugs

I would like to get rid of the aphids, but without the use of synthetic pesticides. One method of controlling pests (skadedjur) without using pesticides is by "biological control", which rely on introducing a hyper-predator that eats the pest. You will now simulate what happens if I introduce 10 ladybugs (nyckelpigor) on my balcony. Do this by extending the Lotka-Volterra equation with a third variable, $w(t)$, and a third equation, which describes the ladybugs and their interaction with the aphids. The ladybugs grow by feeding on the aphids, with a growth rate of 0.1. Ladybugs also disappear, either by dying or by migrating away from the balcony, at a rate of 0.5 . The aphids now do not only die naturally, but their death rate is affected additionally by the presence of the ladybugs, resulting in a a death rate of 0.3.

1. Write down the new system of the three equations that you are going to solve numerically.

2. Edit the python script so that it solves the full tomato-aphid-ladybug system and plot the solution. Include the plot in your report.

3. How much does S increase, compared to not having any ladybugs?

3 Code

See the file called `code.txt`.

4 Answers

4.1 Part 1

Question 1

We wish to solve the differential equation $\frac{dy}{dt} = -\gamma y$ with initial condition $y(0) = 10$. We make the Ansatz $y(t) = Ce^{ax}$. Differentiating and and solving for a yields $a = -0.1$. The initial condition $y(0) = 10$ gives us $C = 10$. The analytic solution is therefore $y(t) = 10e^{-0.1t}$. It is obviously not a perfect model since it gives continous values for a discrete count of bugs, but atleast it does not explode to infinity, instead it is simply monotonically decreasing which makes sense as there is no food :(

Question 2

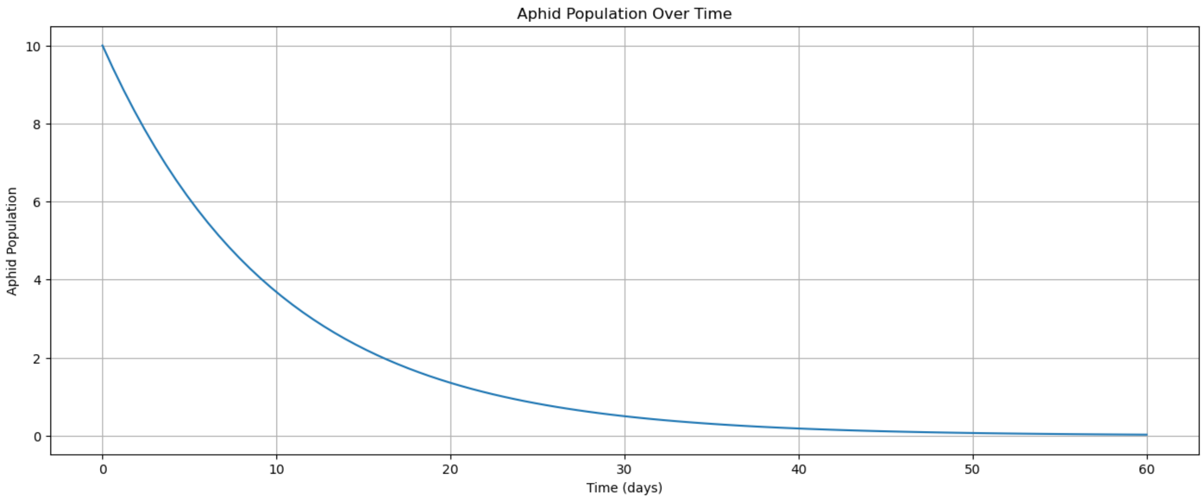

Figure 1: Analytical solution for aphid population over time without tomatoes.

Question 3

Theoretical maximum time step allowed giving a stable solution given an Euler forward discretization.

We do this by analyzing the stability criteria for the Euler method, which states that the time step must be less than or equal to a certain value to ensure convergence. The stability condition condition is

$$| 1 + h\lambda | \leq 1$$

where λ is an eigenvalue of the system. In this case, we have $\lambda = -\gamma = 0.1$ which yields

$$| 1 + h(-0.1) | \leq 1$$

Which after some algebra gives us the condition

$$-1 \leq -0.1h \leq 1$$

Simplifying further gives us the interval $[0, 20]$. This means that the maximum time step allowed for a stable solution is $h = 20$ days. Potentially worth noting here is that λ is negative which eseentially guarantees that the factor $1 - h\lambda$ is positive for all h in the interval, it does not guarantee that the solution is stable.

Question 4

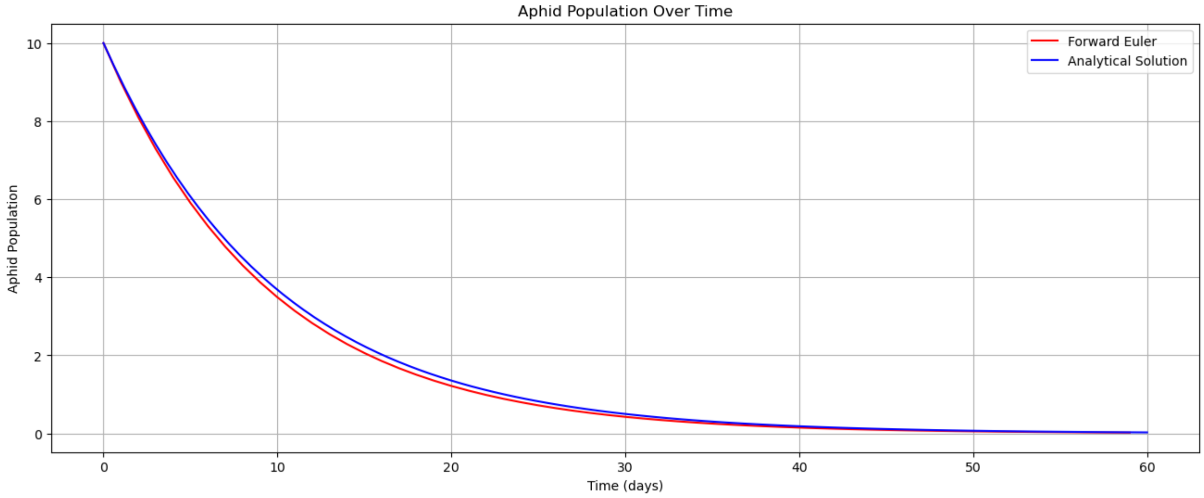

Figure 2: Numerical solution acquired via Euler forward (red) plotted together with the analytical solution from 1.

Question 5

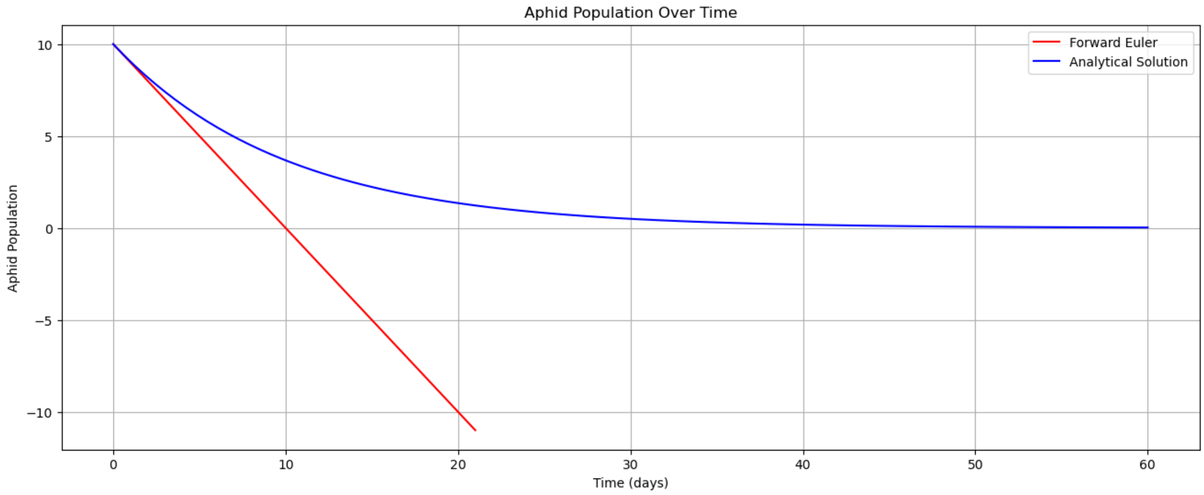

Figure 3: Numerical solution with a time step of 21, slightly larger than the calculated maximum stable time step according to 3.

Question 6

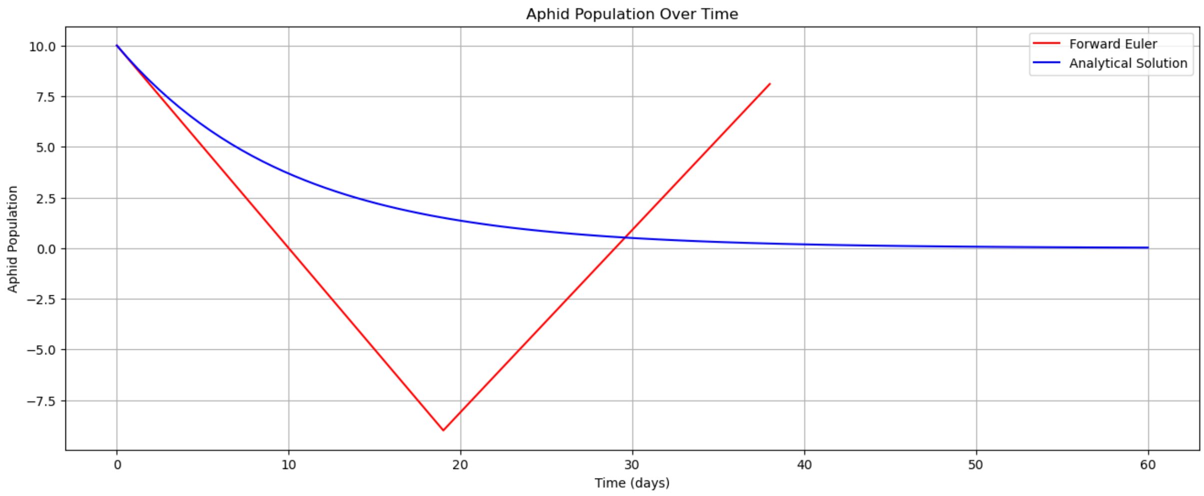

Figure 4: Numerical solution with a time step of 19, slightly smaller than the calculated maximum stable time step of 20.

Question 7

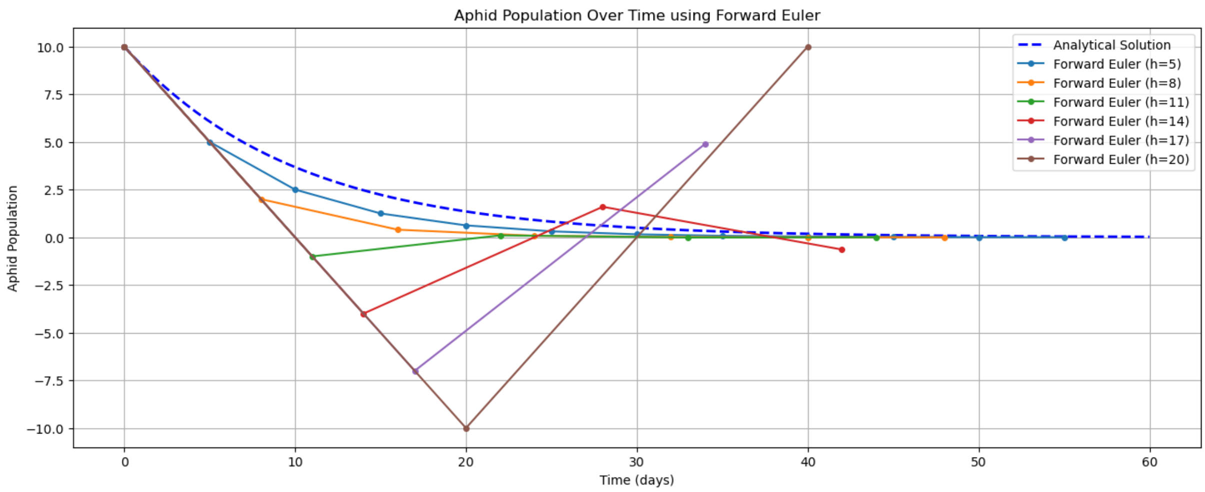

Figure 5: Numerical solutions with varying time steps: 5, 8, 11, 14, 17, 20

We see that a time step of around 8 stops the wild oscillations. This is expected as we are approaching $h = 0$.

Question 8

The local truncation error (LTE) for the forward Euler method applied to the model $y' = -\gamma y$ can be written in closed form and attains its maximum at the start of the step and its minimum at the end.

- Derivation of the local truncation error:

The LTE after one step of size h is

$$\mathrm{LTE}(t_n) \;=\; y(t_n+h) \;-\; \big(y(t_n) + h\,f(t_n, y(t_n))\big) \;=\; \frac{h^2}{2}\,y''(\xi)$$

for some $\xi \in [t_n, t_n+h]$. For the exact solution $y(t) = 10e^{-\gamma t}$ with $\gamma = 0.1$

$$y''(t) = \gamma^2 \cdot 10\,e^{-\gamma t}.$$

Since $e^{-\gamma t}$ is monotonically decreasing on $[0, h]$, the maximum of y'' occurs at $t = 0$ and the minimum at $t = h$. Hence,

$$\mathrm{LTE}_{\max} \;=\; \frac{h^2}{2}\,\gamma^2\,10, \quad \mathrm{LTE}_{\min} \;=\; \frac{h^2}{2}\,\gamma^2\,10\,e^{-\gamma h}$$

- Numerical values for $h = 1$:

Substituting $\gamma = 0.1$ and $h = 1$ gives

$$\mathrm{LTE}_{\max} = \frac{1^2}{2}(0.1^2 \cdot 10) = 0.05, \quad \mathrm{LTE}_{\min} = \frac{1^2}{2}(0.1^2 \cdot 10\,e^{-0.1}) \approx 0.04524.$$

So, a one-day Euler step on $y' = -0.1y$ gives a local truncation error between approximately 0.04524 and 0.05.

Question 9

From the code we get:

- Numerical LTE (first step,$h = 1$): 0.00483742
- Theoretical LTE (from $y''(t = 0)$): 0.00500000
- Theoretical LTE (from $y''(t = h)$): 0.00452419

Refering to the previous question we had gotten that the error is in $[0.04524, 0.05]$. We have gotten the numerical LTE to be 0.000483742 which is certainly in this interval.

Question 10

- Numerical LTE (first step, $h = 0.05$): 0.00001248
- Theoretical LTE (from $y''(t = 0)$): 0.00001250
- Theoretical LTE (from $y''(t = h)$): 0.00001244

Same thing can be said here as in 9 if we were to compute the max and min at $h = 0.5$.

Question 11

Time taken for slow simulation: 0.6829750537872314 seconds, this was done with a time step of 0.0001 on a laptop computer.

4.2 Part 2

Question 1

We have a nonlinear system of equations. The reason is that the equations are not linear due to multiplication of x, y in the first equation, which prevents us from writing it on matrix form.

Question 2

The Euler Forward step is:

$$u_{n+1} = u_n + hf(t, u)$$

in our case $u = (x, y)^T$ so we get

$$\begin{aligned} x_{n+1} &= x_n + h(\alpha x_n - \beta x_n y_n) \\ y_{n+1} &= y_n + h(\delta x_n y_n - \gamma y_n) \end{aligned}$$

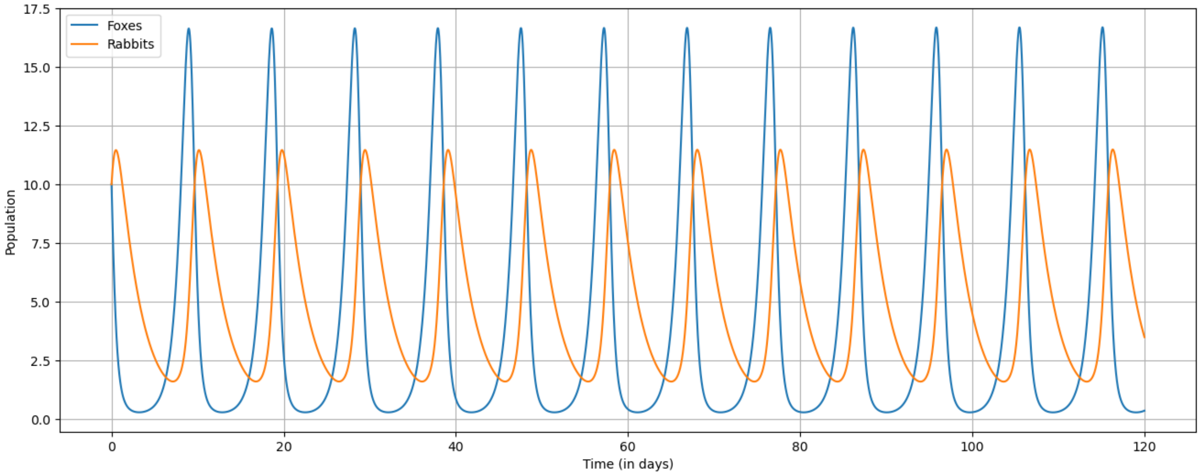

Figure 6: Reproduction of the plot of the number of foxes and rabbits from Wikipedia.

Question 3

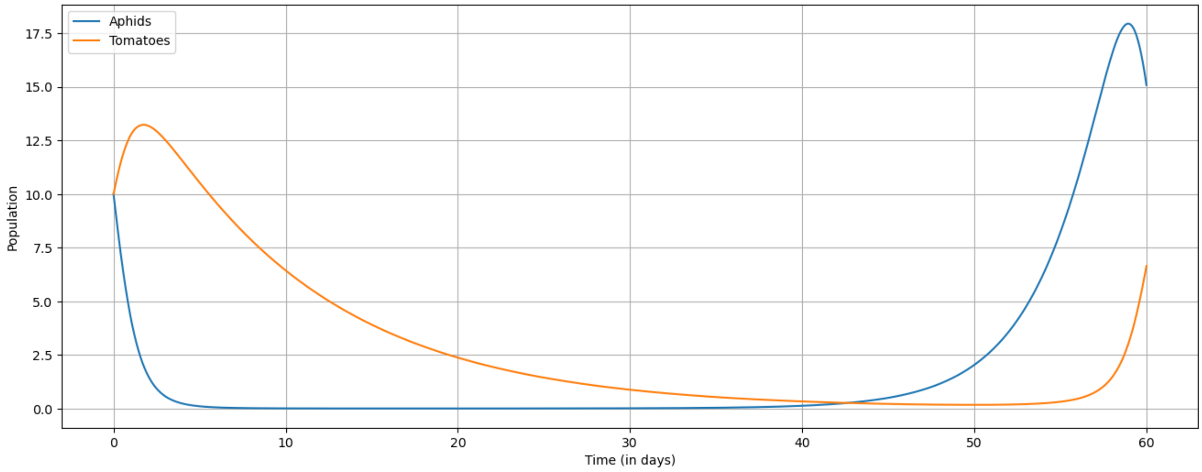

Figure 7: Simulation of how the population of aphids and tomatoes change over time (60 days, x-axis)

Question 4

We can do this by using the trapezoidal rule. We have the following:

$$S = \int_0^{60} x(t)dt = \frac{\Delta x}{2}\left(f(x_0) + 2\sum_{i=1}^{59} f(x_i) + f(x_n)\right)$$

which in our case gives us the following:

The success of the tomato plants (S), computed using the trapezoidal rule on the Forward Euler solution (`heuler=0.001`) over t=[0.0, 60.0] is: S = 111.6743

Question 5

The success of the tomato plants (S), computed using the trapezoidal rule with more aphids in the start on the Forward Euler solution (`heuler=0.001`) over t=[0.0, 60.0] is: S = 64.0344

Question 6

The success of the tomato plants (S), computed using the trapezoidal rule with super aphids on the Forward Euler solution, (`heuler=0.001`) over t=[0.0, 60.0] is: S = 32.9414.

Question 7

The success of the tomato plants (S), computed using the trapezoidal rule if watering more on the Forward Euler solution (`heuler=0.001`) over t=[0.0, 60.0] is: S = 118.1092.

Question 8

RK45 used 21 points. Forward Euler (h=0.001) needs 60001 points.

Question 9

Runge-Kutta methods improve upon simpler methods like Forward Euler by evaluating the slope (derivative) at multiple points within each step. One such method is RK45. This method has an an LTE order $O(h^5)$ which in comparison to Euler Forward's LTE order $O(h^2)$. This makes Euler Forward a first-order method since the global error scales as $O(h)$. The RK45 method (Dormand-Prince) is a 4th-order method that uses a 5th-order step to estimate the local error. This order combination is where the name comes from. RK45 uses

The RK45 method, also known as the Runge-Kutta-Fehlberg method, uses a combination of 4th and 5th order Runge-Kutta approximations to solve differential equations with adaptive step size control. The formula consists of six stages of calculations: For a differential equation $y' = f(t, y)$ with initial condition $y(t_0) = \alpha$ the RK45 method is given by:

$$\begin{aligned}
w_0 &= \alpha \\
k_1 &= hf(t_i, w_i) \\
k_2 &= hf(t_i + h/4, w_i + k_1/4) \\
&\vdots \\
k_6 &= hf(t_i + h/2, w_i - (8/27)k_1 + 2k_2 - (3544/2565)k_3 + (1859/4104)k_4 - (11/40)k_5)
\end{aligned}$$

from which we can get the 4th and 5th order approximations as

$$\begin{aligned}
w_{i+1} &= w_i + (25/216)k_1 + (1408/2565)k_3 + (2197/4104)k_4 - (1/5)k_5 \\
\tilde{w}_{i+1} &= w_i + (16/135)k_1 + (6656/12825)k_3 + (28561/56430)k_4 - (2/55)k_5
\end{aligned}$$

Due to the high order this method is very accurate, especially in comparison to Euler Forward. RK45 automatically adjusts its step size based on the estimated error, taking smaller steps where the solution changes rapidly and larger steps where it changes slowly and can as such save some computations. Due to these many positives, this method has become the most widely used method for approximating solutions to a differential equations.

4.3 Part 3

Question 1

We begin by writing down the new system of equations. We have the following:

$$\frac{dx}{dt} = \alpha x - \beta xy$$
$$\frac{dy}{dt} = \delta xy - \gamma y - \eta yw$$
$$\frac{dw}{dt} = \eta yw - \zeta w$$

where the third equation is the extension of the Lotka-Volterra equation with the ladybugs. The ladybugs grow by feeding on the aphids, with a growth rate of 0.1. Ladybugs also disappear, either by dying or by migrating away from the balcony, at a rate of 0.5 . The aphids now do not only die naturally, but their death rate is affected additionally by the presence of the ladybugs, resulting in a a death rate of 0.3. We have the following parameters:

$$\begin{cases} w(0) = 10, \\ \eta = 0.1, \\ \zeta = 0.5, \\ \gamma = 0.3, \\ \delta = 0.05, \\ \beta = 0.1, \\ \alpha = 0.3. \end{cases}$$

Question 2

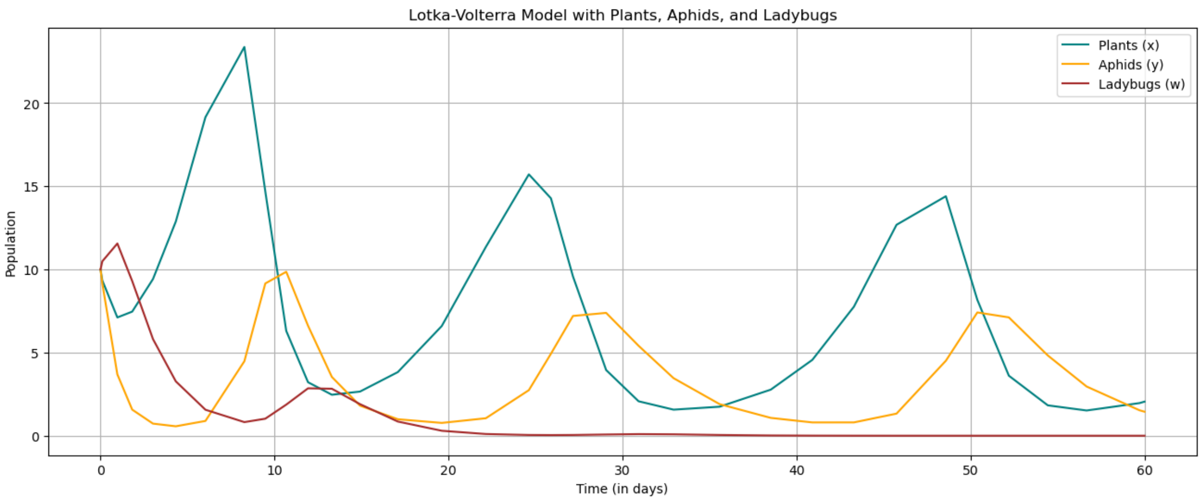

Figure 8: Solution of tomato-aphid-ladybug system using RK45

Question 3

- S with ladybugs: 442.4390

- S without ladybugs: 111.8034

- Improvement: 330.6356