Lab 5 - Part 1

Consider the BVP

$$y''(x) + y(x) = 0$$

 $y(0) = 1$
 $y(\pi/2) = 2$

Pen & Paper

What is the analytical solution? We studied a very similar problem during the lecture.

Write down, with pen an paper, a discretization of the problem. Write down how the matrix would look for a stepsize $h=(\pi/2)/4$

Derive the local truncation error. Include your derivation in the hand in.

Is your method consistent?

We get the analytical solution through the characteristic polynomial:

$$y''(x) + y(x) = 0$$

 $y(x) = c_1 \cos(x) + c_2 \sin(x)$

We can find the constants c_1 and c_2 by using the boundary conditions:

1.
$$y(0) = 1 \implies c_1 = 1$$

2. $y(\pi/2) = 2 \implies c_2 = 2$ Thus, the analytical solution is:

$$y(x) = \cos(x) + 2 * \sin(x)$$

We can discretize the problem using a finite difference method. We can use a central difference approximation for the second derivative:

$$y''(x)pprox rac{y_{i+1}-2y_i+y_{i-1}}{h^2}$$

We can rewrite this as

$$y_{i-1} + (-2 + h^2)y_i + y_{i+1} = 0$$

by dividing by h^2 . We have the boundary points, ie. $x_0=0, x_4=\pi/2$ meaning that the matrix system AU=F should be of dimension 3×3 , ie . for the three unknowns with N=3.

For i = 1, 2, 3, we have

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$$1 + (-2 + h^2)y_1 + y_2 = 0$$

 $y_1 + (-2 + h^2)y_2 + y_3 = 0$
 $y_2 + (-2 + h^2)y_3 + y_4$

respectively which through plugging in values we get the values -1,0,-2 respectively as h o 0. So, we have

$$A = egin{pmatrix} -2 + h^2 & 1 & 0 \ 1 & -2 + h^2 & 1 \ 0 & 1 & -2 + h^2 \end{pmatrix}$$

and

$$F = egin{pmatrix} -1 \ 0 \ -2 \end{pmatrix}$$

Finding the local truncation error requires us to use Taylor series. The Taylor series expansion of y(x) around x_i is:

$$y(x_i+h)=y(x_i)+hy'(x_i)+rac{h^2}{2}y''(x_i)+rac{h^3}{6}y'''(x_i)+O(h^4)$$

$$y(x_i-h)=y(x_i)-hy'(x_i)+rac{h^2}{2}y''(x_i)-rac{h^3}{6}y'''(x_i)+O(h^4)$$

We can then write the Taylor series expansion of $y(x_i+h)$ and $y(x_i-h)$ as:

$$y(x_i+h)-2y(x_i)+y(x_i-h)=rac{h^2}{2}y''(x_i)+O(h^4)$$

Thus, the local truncation error is:

$$ext{LTE} = rac{h^2}{2}y''(x_i) + O(h^4)$$

ie the local truncation error is $O(h^2)$.

We can see that the method is consistent, since the local truncation error goes to zero as h goes to zero, ie we approach the analytical solution as we let $h \to 0$. The method is also convergent, since the local truncation error is $O(h^2)$ and the method is consistent. The method is also stable, since the matrix is diagonally dominant and the eigenvalues are all positive. Thus, the method is stable.

Code

Implement your dicretization and plot it next to the analytical solution for the stepsize $h=(\pi/2)/4(N=3)$. You can do this by filling in the missing pieces of the script below (you are of course also allowed to

write your own script from scratch). Include the code and the plot in the hand in.

Print the matrix for N=3 and compare it to the matrix that you derived by pen and paper. They should be the same. Include the printout in the hand in.

Where is the error the biggest? In the beginning of the interval, the end, or in the middle? Compare to how it is for IVP:s.

Produce a plot or a table which shows how the error decreases with h. You can look at your error-study from Lab 1 part 2 for inspiration. For measuring the error, choose any of the norms from the lecture. You can use $\verb|numpy.linalg.norm||$ to code the norm, but compensate for the fact that the 1-norm and 2-norm in numpy does not include the step size h, as in the lecture. Include the table/plot in your hand in. The result should be in accordance to the theoretical truncation error.

What command is used to solve the matrix-system in the code? In the next lectures we will look closer at what is actually happening in this step.

```
In [15]: import numpy as np
         import matplotlib.pyplot as plt
         from math import pi
         def ComputeAnalyticalSolution(N, leftbc, rightbc):
             x = np.linspace(0, (pi / 2), N + 2)
             y = np.cos(x) + 2* np.sin(x)
             return x, y
         def ComputeNumericalSolution(N, leftbc, rightbc):
             h = (pi / 2) / (N + 1)
             x = np.linspace(0, (pi / 2), N + 2)
             A = np.zeros((N, N)) # Coefficient matrix of nxn dimension
             F = np.zeros(N) # The result vector such that AU = F of size n
             # Assembly of the system A * y_h_{int} = F
             # Discretized equation: y_{i-1} + (-2 + h^2)y_i + y_{i+1} = 0
             # Equation for i=1 (first interior point)
             # y_0 + (-2 + h^2)y_1 + y_2 = 0
             \# leftbc + (-2 + h^2)y_1 + y_2 = 0
             \# (-2 + h^2)y_1 + y_2 = -leftbc
             A[0, 0] = -2 + h**2
             A[0, 1] = 1
             F[0] = -leftbc
             for i in range(1, N - 1):
                 A[i, i-1] = 1
                 A[i, i] = -2 + h**2
                 A[i, i + 1] = 1
                 F[i] = 0
```

```
A[N - 1, N - 2] = 1
      A[N - 1, N - 1] = -2 + h**2
      F[N-1] = -rightbc
      print("Matrix A for N =", N)
      print(A)
      print("\nVector F for N =", N)
     print(F)
     y_h_int = np.linalg.solve(A, F)
     y_h = np.zeros(N + 2)
     y_h[0] = leftbc
      y_h[1:N + 1] = y_h_int
     y_h[-1] = rightbc
      return x, y_h, h
 leftbc = 1
 rightbc = 2
 N = 3
 x, y = ComputeAnalyticalSolution(N, leftbc, rightbc)
 x_h, y_h, h = ComputeNumericalSolution(N, leftbc, rightbc)
 plt.figure(figsize=(16, 6))
 plt.plot(x, y, label='analytical')
 plt.plot(x_h, y_h, label='numerical', linestyle='--')
 plt.title('Analytical vs Numerical Solution')
 plt.xlabel('x')
 plt.ylabel('y')
 plt.legend()
 plt.grid()
 plt.show()
Matrix A for N = 3
[[-1.84578743 1.
                             0.
                                        1
               -1.84578743 1.
 [ 1.
                                        ]
                            -1.84578743]]
 [ 0.
                1.
Vector F for N = 3
[-1. 0. -2.]
                                 Analytical vs Numerical Solution
 2.2
                                                                        --- numerical
 1.8
> 1.6
 1.4
 1.2
 1.0
```

We see that the matrix is correct according to the discretization above. We also see that the error is the largest at some of the interior points since error must be 0 at the boundary points. This is different to IVP solutions since the right boundary isn't

known during calculation as it is here. le. for IVP:s we will get highest error at the final point.

For finding the error we will use the L_2 norm. le.

$$\left(h\sum_{i=1}^{N}\mid e_{i}\mid^{2}
ight)^{1/2}$$

implementing this requires us to use the numpy.linalg.norm function.

```
In [16]: dxvec = []
         errvec = []
         for N in range(3, 20):
             x, y = ComputeAnalyticalSolution(N, leftbc, rightbc)
             x_h, y_h, h = ComputeNumericalSolution(N, leftbc, rightbc)
             error_vec = y - y_h
             error_interior = error_vec[1:N+1]
             dx = (np.pi / 2) / (N + 1)
             dxvec.append(dx)
             #scale according to lecture notes
             l2_error = np.sqrt(h * np.sum(error_vec**2))
             errvec.append(l2_error)
         plt.figure(figsize=(16, 6))
         # make log log
         plt.loglog(dxvec, errvec, '-')
         plt.title('Convergence of the numerical solution')
         plt.xlabel('dx')
         plt.ylabel('error')
         plt.grid()
         plt.show()
```

```
Matrix A for N = 3
                                      ]
[[-1.84578743 1.
                           0.
 [ 1.
              -1.84578743 1.
                                      1
 [ 0.
              1. -1.84578743]]
Vector F for N = 3
[-1. 0. -2.]
Matrix A for N = 4
[[-1.90130396 1.
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 [ 1.
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 [ 0.
                          -1.90130396 1.
               1.
 [ 0.
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                                       -1.9013039611
Vector F for N = 4
[-1. 0. 0. -2.]
Matrix A for N = 5
[[-1.93146108 1.
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              -1.93146108 1.
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 [ 0.
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                                                   -1.93146108]]
 [ 0.
               0.
                           0.
                                        1.
Vector F for N = 5
[-1. 0. 0. 0. -2.]
Matrix A for N = 6
[[-1.94964488 1.
                           0.
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              -1.94964488 1.
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8]]
Vector F for N = 6
[-1. 0. 0. 0. 0. -2.]
Matrix A for N = 7
[[-1.96144686 1.
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              -1.96144686 1.
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  -1.96144686]]
Vector F for N = 7
[-1. 0. 0. 0. 0. 0. -2.]
Matrix A for N = 8
[[-1.96953826 1.
                           0.
                                        0.
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   0.
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 [ 1.
              -1.96953826 1.
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[ 0.
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                -1.96953826]]
Vector F for N = 8
[-1. 0. 0. 0. 0. 0.
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Matrix A for N = 9
[[-1.97532599 1.
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Vector F for N = 9
[-1. 0. 0. 0.
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Matrix A for N = 10
[[-1.97960826 1.
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Vector F for N = 10
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Matrix A for N = 11
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Vector F for N = 11
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Matrix A for N = 12
[-1.98539999
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Vector F for N = 12
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Matrix A for N = 13
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  -1.98741122]]
Vector F for N = 13
[-1. 0.
                 0.
            0.
                       0.
                            0.
                                 0.
                                      0.
                                           0.
                                                0.
                                                     0.
                                                           0. -2.
Matrix A for N = 14
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[-1.98903377
                                  0.
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-1.98903377
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   0.
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                  -1.98903377]]
Vector F for N = 14
      0.
            0.
                  0.
                       0.
                            0.
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                                                           0.
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Matrix A for N = 15
[[-1.99036171
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                  -1.99036171
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```

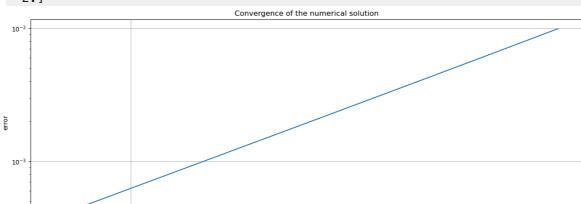
```
[ 0.
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Vector F for N = 15
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Matrix A for N = 16
[[-1.99146228
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0.
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                                                -1.99146228]]
Vector F for N = 16
[-1. 0. 0.
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Matrix A for N = 17
[[-1.99238456
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Vector F for N = 17
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Matrix A for N = 18
```

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[[-1.99316509
                    1.
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9]]
Vector F for N = 18
                   0.
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Matrix A for N = 19
[[-1.9938315
                                  0.
                                                 0.
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```

0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
]	• •	• •		-
		1	0	0	0
[1.	-1.9938315	1.	0.	0.	0.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
0.]				
[0.	1.	-1.9938315	1.	0.	0.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
]				
[0.	0.	1.	-1.9938315	1.	0.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
]				
[0.	0.	0.	1.	-1.9938315	1.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
0.]				
[0.	0.	0.	0.	1.	-1.9938315
1.	0.	0.	0.	0.	0.
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]				
[0.	0.	0.	0.	0.	1.
-1.9938315	1.	0.	0.	0.	0.
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]				
[0.	0.	0.	0.	0.	0.
1.	-1.9938315	1.	0.	0.	0.
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0.]				
[0.	0.	0.	0.	0.	0.
0.		-1.9938315	1.	0.	0.
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[0.	0.	0.	0.		0.
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[0.		0.	0.	0.	0.
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1.	0.	0.	0.	0.	0.
]				
[0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	1.
-1.9938315		0.	0.	0.	0.
]				
		0	0	0	0
[0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
1.	-1.9938315	1.	0.	0.	0.
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[0.		0.	0.	0.	0.
VI -	V) -	0.	٥.		
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0. 0. 0.	0. 1.]	0. -1.9938315	1.	0.	0.
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  -1.9938315]]
Vector F for N = 19
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[-1.
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                                                                                         0.
-2.]
```



 2×10^{-1}

 3×10^{-1}

 4×10^{-1}

```
In [17]: p = np.polyfit(np.log(dxvec), np.log(errvec), 1)
    print(p[0])
```

2.0117672459294886

Regarding what command is used to solve the matrix system, we use the numpy.linalg.solve function. This function uses LU decomposition.