Lab 5 - Part 1

Consider the BVP

$$y''(x) + y(x) = 0$$
$$y(0) = 1$$
$$y(\pi/2) = 2$$

Pen & Paper

> What is the analytical solution? We studied a very similar problem during the lecture.
>
> Write down, with pen an paper, a discretization of the problem. Write down how the matrix would look for a stepsize $h = (\pi/2)/4$
>
> Derive the local truncation error. Include your derivation in the hand in.
>
> Is your method consistent?

We get the analytical solution through the characteristic polynomial:

$$y''(x) + y(x) = 0$$
$$y(x) = c_1 \cos(x) + c_2 \sin(x)$$

We can find the constants c_1 and c_2 by using the boundary conditions:

1. $y(0) = 1 \implies c_1 = 1$
2. $y(\pi/2) = 2 \implies c_2 = 2$ Thus, the analytical solution is:

$$y(x) = \cos(x) + 2 * \sin(x)$$

We can discretize the problem using a finite difference method. We can use a central difference approximation for the second derivative:

$$y''(x) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

We can rewrite this as

$$y_{i-1} + (-2 + h^2)y_i + y_{i+1} = 0$$

by dividing by h^2. We have the boundary points, ie. $x_0 = 0, x_4 = \pi/2$ meaning that the matrix system $AU = F$ should be of dimension 3×3, ie . for the three unknowns with $N = 3$.

For $i = 1, 2, 3,$ we have

$$
\begin{aligned}
1 + (-2 + h^2)y_1 + y_2 &= 0 \\
y_1 + (-2 + h^2)y_2 + y_3 &= 0 \\
y_2 + (-2 + h^2)y_3 + y_4
\end{aligned}
$$

respectively which through plugging in values we get the values $-1, 0, -2$ respectively as $h \to 0$. So, we have

$$
A = \begin{pmatrix} -2 + h^2 & 1 & 0 \\ 1 & -2 + h^2 & 1 \\ 0 & 1 & -2 + h^2 \end{pmatrix}
$$

and

$$
F = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}
$$

Finding the local truncation error requires us to use Taylor series. The Taylor series expansion of $y(x)$ around x_i is:

$$
y(x_i + h) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(x_i) + O(h^4)
$$

$$
y(x_i - h) = y(x_i) - hy'(x_i) + \frac{h^2}{2}y''(x_i) - \frac{h^3}{6}y'''(x_i) + O(h^4)
$$

We can then write the Taylor series expansion of $y(x_i + h)$ and $y(x_i - h)$ as:

$$
y(x_i + h) - 2y(x_i) + y(x_i - h) = \frac{h^2}{2}y''(x_i) + O(h^4)
$$

Thus, the local truncation error is:

$$
\text{LTE} = \frac{h^2}{2}y''(x_i) + O(h^4)
$$

ie the local truncation error is $O(h^2)$.

We can see that the method is consistent, since the local truncation error goes to zero as h goes to zero, ie we approach the analytical solution as we let $h \to 0$. The method is also convergent, since the local truncation error is $O(h^2)$ and the method is consistent. The method is also stable, since the matrix is diagonally dominant and the eigenvalues are all positive. Thus, the method is stable.

Code

> Implement your dicretization and plot it next to the analytical solution for the stepsize $h = (\pi/2)/4(N = 3)$. You can do this by filling in the missing pieces of the script below (you are of course also allowed to

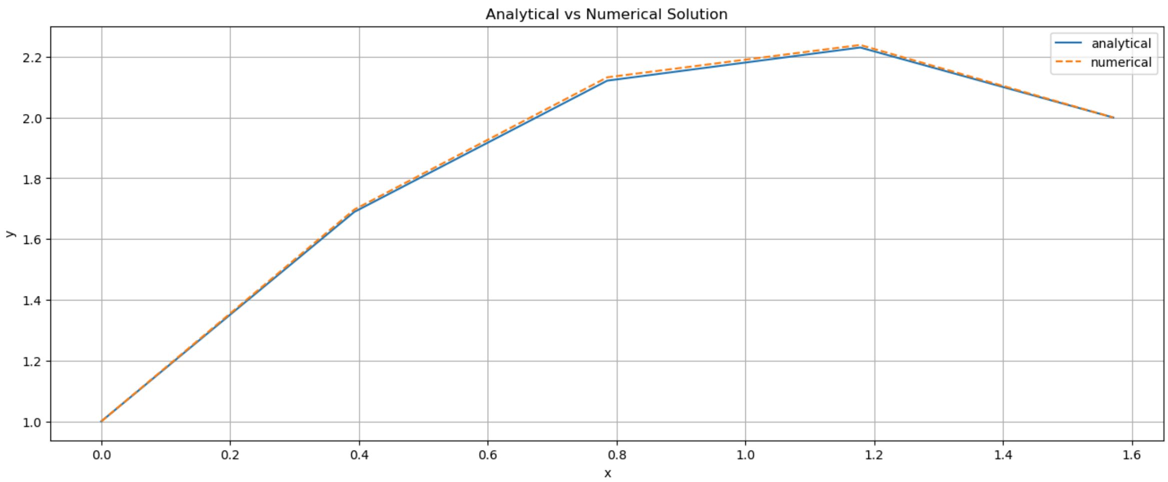
> write your own script from scratch). Include the code and the plot in the hand in.
>
> Print the matrix for $N = 3$ and compare it to the matrix that you derived by pen and paper. They should be the same. Include the printout in the hand in.
>
> Where is the error the biggest? In the beginning of the interval, the end, or in the middle? Compare to how it is for IVP:s.
>
> Produce a plot or a table which shows how the error decreases with h. You can look at your error-study from Lab 1 part 2 for inspiration. For measuring the error, choose any of the norms from the lecture. You can use `numpy.linalg.norm` to code the norm, but compensate for the fact that the 1-norm and 2-norm in numpy does not include the step size h, as in the lecture. Include the table/plot in your hand in. The result should be in accordance to the theoretical truncation error.
>
> What command is used to solve the matrix-system in the code? In the next lectures we will look closer at what is actually happening in this step.

In [15]:

```python
import numpy as np
import matplotlib.pyplot as plt
from math import pi

def ComputeAnalyticalSolution(N, leftbc, rightbc):
    x = np.linspace(0, (pi / 2), N + 2)
    y = np.cos(x) + 2* np.sin(x)
    return x, y

def ComputeNumericalSolution(N, leftbc, rightbc):
    h = (pi / 2) / (N + 1)
    x = np.linspace(0, (pi / 2), N + 2)
    A = np.zeros((N, N)) # Coefficient matrix of nxn dimension
    F = np.zeros(N) # The result vector such that AU = F of size n

    # Assembly of the system A * y_h_int = F
    # Discretized equation: y_{i-1} + (-2 + h^2)y_i + y_{i+1} = 0

    # Equation for i=1 (first interior point)
    # y_0 + (-2 + h^2)y_1 + y_2 = 0
    # leftbc + (-2 + h^2)y_1 + y_2 = 0
    # (-2 + h^2)y_1 + y_2 = -leftbc
    A[0, 0] = -2 + h**2
    A[0, 1] = 1
    F[0] = -leftbc

    for i in range(1, N - 1):
        A[i, i - 1] = 1
        A[i, i] = -2 + h**2
        A[i, i + 1] = 1
        F[i] = 0
```

```
    A[N - 1, N - 2] = 1
    A[N - 1, N - 1] = -2 + h**2
    F[N - 1] = -rightbc

    print("Matrix A for N =", N)
    print(A)
    print("\nVector F for N =", N)
    print(F)

    y_h_int = np.linalg.solve(A, F)
    y_h = np.zeros(N + 2)
    y_h[0] = leftbc
    y_h[1:N + 1] = y_h_int
    y_h[-1] = rightbc

    return x, y_h, h

leftbc = 1
rightbc = 2
N = 3
x, y = ComputeAnalyticalSolution(N, leftbc, rightbc)
x_h, y_h, h = ComputeNumericalSolution(N, leftbc, rightbc)

plt.figure(figsize=(16, 6))
plt.plot(x, y, label='analytical')
plt.plot(x_h, y_h, label='numerical', linestyle='--')
plt.title('Analytical vs Numerical Solution')
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.grid()
plt.show()
```

```
Matrix A for N = 3
[[-1.84578743  1.          0.        ]
 [ 1.         -1.84578743  1.        ]
 [ 0.          1.         -1.84578743]]

Vector F for N = 3
[-1.  0. -2.]
```



We see that the matrix is correct according to the discretization above. We also see that the error is the largest at some of the interior points since error must be 0 at the boundary points. This is different to IVP solutions since the right boundary isn't known during calculation as it is here. Ie. for IVP:s we will get highest error at the final point.

For finding the error we will use the L_2 norm. Ie.

$$\left(h\sum_{i=1}^{N} \mid e_i \mid^2\right)^{1/2}$$

implementing this requires us to use the `numpy.linalg.norm` function.

In [16]:
```
dxvec = []
errvec = []

for N in range(3, 20):
    x, y = ComputeAnalyticalSolution(N, leftbc, rightbc)
    x_h, y_h, h = ComputeNumericalSolution(N, leftbc, rightbc)

    error_vec = y - y_h
    error_interior = error_vec[1:N+1]
    dx = (np.pi / 2) / (N + 1)
    dxvec.append(dx)

    #scale according to lecture notes
    l2_error = np.sqrt(h * np.sum(error_vec**2))
    errvec.append(l2_error)


plt.figure(figsize=(16, 6))
# make log log
plt.loglog(dxvec, errvec, '-')
plt.title('Convergence of the numerical solution')
plt.xlabel('dx')
plt.ylabel('error')
plt.grid()
plt.show()
```

```
Matrix A for N = 3
[[-1.84578743  1.          0.        ]
 [ 1.         -1.84578743  1.        ]
 [ 0.          1.         -1.84578743]]

Vector F for N = 3
[-1.  0. -2.]
Matrix A for N = 4
[[-1.90130396  1.          0.          0.        ]
 [ 1.         -1.90130396  1.          0.        ]
 [ 0.          1.         -1.90130396  1.        ]
 [ 0.          0.          1.         -1.90130396]]

Vector F for N = 4
[-1.  0.  0. -2.]
Matrix A for N = 5
[[-1.93146108  1.          0.          0.          0.        ]
 [ 1.         -1.93146108  1.          0.          0.        ]
 [ 0.          1.         -1.93146108  1.          0.        ]
 [ 0.          0.          1.         -1.93146108  1.        ]
 [ 0.          0.          0.          1.         -1.93146108]]

Vector F for N = 5
[-1.  0.  0.  0. -2.]
Matrix A for N = 6
[[-1.94964488  1.          0.          0.          0.          0.        ]
 [ 1.         -1.94964488  1.          0.          0.          0.        ]
 [ 0.          1.         -1.94964488  1.          0.          0.        ]
 [ 0.          0.          1.         -1.94964488  1.          0.        ]
 [ 0.          0.          0.          1.         -1.94964488  1.        ]
 [ 0.          0.          0.          0.          1.         -1.9496448
8]]

Vector F for N = 6
[-1.  0.  0.  0.  0. -2.]
Matrix A for N = 7
[[-1.96144686  1.          0.          0.          0.          0.
   0.        ]
 [ 1.         -1.96144686  1.          0.          0.          0.
   0.        ]
 [ 0.          1.         -1.96144686  1.          0.          0.
   0.        ]
 [ 0.          0.          1.         -1.96144686  1.          0.
   0.        ]
 [ 0.          0.          0.          1.         -1.96144686  1.
   0.        ]
 [ 0.          0.          0.          0.          1.         -1.96144686
   1.        ]
 [ 0.          0.          0.          0.          0.          1.
  -1.96144686]]

Vector F for N = 7
[-1.  0.  0.  0.  0.  0. -2.]
Matrix A for N = 8
[[-1.96953826  1.          0.          0.          0.          0.
   0.          0.        ]
 [ 1.         -1.96953826  1.          0.          0.          0.
   0.          0.        ]
 [ 0.          1.         -1.96953826  1.          0.          0.
   0.          0.        ]
```

```
 [ 0.          0.          1.         -1.96953826  1.          0.
   0.          0.        ]
 [ 0.          0.          0.          1.         -1.96953826  1.
   0.          0.        ]
 [ 0.          0.          0.          0.          1.         -1.96953826
   1.          0.        ]
 [ 0.          0.          0.          0.          0.          1.
  -1.96953826  1.        ]
 [ 0.          0.          0.          0.          0.          0.
   1.         -1.96953826]]

Vector F for N = 8
[-1.  0.  0.  0.  0.  0.  0. -2.]
Matrix A for N = 9
[[-1.97532599  1.          0.          0.          0.          0.
   0.          0.          0.        ]
 [ 1.         -1.97532599  1.          0.          0.          0.
   0.          0.          0.        ]
 [ 0.          1.         -1.97532599  1.          0.          0.
   0.          0.          0.        ]
 [ 0.          0.          1.         -1.97532599  1.          0.
   0.          0.          0.        ]
 [ 0.          0.          0.          1.         -1.97532599  1.
   0.          0.          0.        ]
 [ 0.          0.          0.          0.          1.         -1.97532599
   1.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          1.
  -1.97532599  1.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   1.         -1.97532599  1.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          1.         -1.97532599]]

Vector F for N = 9
[-1.  0.  0.  0.  0.  0.  0.  0. -2.]
Matrix A for N = 10
[[-1.97960826  1.          0.          0.          0.          0.
   0.          0.          0.          0.        ]
 [ 1.         -1.97960826  1.          0.          0.          0.
   0.          0.          0.          0.        ]
 [ 0.          1.         -1.97960826  1.          0.          0.
   0.          0.          0.          0.        ]
 [ 0.          0.          1.         -1.97960826  1.          0.
   0.          0.          0.          0.        ]
 [ 0.          0.          0.          1.         -1.97960826  1.
   0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          1.         -1.97960826
   1.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          1.
  -1.97960826  1.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   1.         -1.97960826  1.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          1.         -1.97960826  1.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          1.         -1.97960826]]

Vector F for N = 10
[-1.  0.  0.  0.  0.  0.  0.  0.  0. -2.]
Matrix A for N = 11
```

```
[[-1.98286527  1.          0.          0.          0.          0.
   0.          0.          0.          0.          0.        ]
 [ 1.         -1.98286527  1.          0.          0.          0.
   0.          0.          0.          0.          0.        ]
 [ 0.          1.         -1.98286527  1.          0.          0.
   0.          0.          0.          0.          0.        ]
 [ 0.          0.          1.         -1.98286527  1.          0.
   0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          1.         -1.98286527  1.
   0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          1.         -1.98286527
   1.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          1.
  -1.98286527  1.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   1.         -1.98286527  1.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          1.         -1.98286527  1.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          1.         -1.98286527  1.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          1.         -1.98286527]]

Vector F for N = 11
[-1.  0.  0.  0.  0.  0.  0.  0.  0.  0. -2.]
Matrix A for N = 12
[[-1.98539999  1.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.        ]
 [ 1.         -1.98539999  1.          0.          0.          0.
   0.          0.          0.          0.          0.          0.        ]
 [ 0.          1.         -1.98539999  1.          0.          0.
   0.          0.          0.          0.          0.          0.        ]
 [ 0.          0.          1.         -1.98539999  1.          0.
   0.          0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          1.         -1.98539999  1.
   0.          0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          1.         -1.98539999
   1.          0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          1.
  -1.98539999  1.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   1.         -1.98539999  1.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          1.         -1.98539999  1.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          1.         -1.98539999  1.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          1.         -1.98539999  1.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          1.         -1.9853999
9]]

Vector F for N = 12
[-1.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0. -2.]
Matrix A for N = 13
[[-1.98741122  1.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.        ]
 [ 1.         -1.98741122  1.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
```

```
  0.        ]
 [ 0.          1.         -1.98741122  1.          0.          0.
  0.          0.          0.          0.          0.          0.
  0.        ]
 [ 0.          0.          1.         -1.98741122  1.          0.
  0.          0.          0.          0.          0.          0.
  0.        ]
 [ 0.          0.          0.          1.         -1.98741122  1.
  0.          0.          0.          0.          0.          0.
  0.        ]
 [ 0.          0.          0.          0.          1.         -1.98741122
  1.          0.          0.          0.          0.          0.
  0.        ]
 [ 0.          0.          0.          0.          0.          1.
 -1.98741122  1.          0.          0.          0.          0.
  0.        ]
 [ 0.          0.          0.          0.          0.          0.
  1.         -1.98741122  1.          0.          0.          0.
  0.        ]
 [ 0.          0.          0.          0.          0.          0.
  0.          1.         -1.98741122  1.          0.          0.
  0.        ]
 [ 0.          0.          0.          0.          0.          0.
  0.          0.          1.         -1.98741122  1.          0.
  0.        ]
 [ 0.          0.          0.          0.          0.          0.
  0.          0.          0.          1.         -1.98741122  1.
  0.        ]
 [ 0.          0.          0.          0.          0.          0.
  0.          0.          0.          0.          1.         -1.98741122
  1.        ]
 [ 0.          0.          0.          0.          0.          0.
  0.          0.          0.          0.          0.          1.
 -1.98741122]]

Vector F for N = 13
[-1.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0. -2.]
Matrix A for N = 14
[[-1.98903377  1.          0.          0.          0.          0.
  0.          0.          0.          0.          0.          0.
  0.          0.        ]
 [ 1.         -1.98903377  1.          0.          0.          0.
  0.          0.          0.          0.          0.          0.
  0.          0.        ]
 [ 0.          1.         -1.98903377  1.          0.          0.
  0.          0.          0.          0.          0.          0.
  0.          0.        ]
 [ 0.          0.          1.         -1.98903377  1.          0.
  0.          0.          0.          0.          0.          0.
  0.          0.        ]
 [ 0.          0.          0.          1.         -1.98903377  1.
  0.          0.          0.          0.          0.          0.
  0.          0.        ]
 [ 0.          0.          0.          0.          1.         -1.98903377
  1.          0.          0.          0.          0.          0.
  0.          0.        ]
 [ 0.          0.          0.          0.          0.          1.
 -1.98903377  1.          0.          0.          0.          0.
  0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
```

```
  1.         -1.98903377  1.          0.          0.          0.
  0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
  0.          1.         -1.98903377  1.          0.          0.
  0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
  0.          0.          1.         -1.98903377  1.          0.
  0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
  0.          0.          0.          1.         -1.98903377  1.
  0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
  0.          0.          0.          0.          1.         -1.98903377
  1.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
  0.          0.          0.          0.          0.          1.
 -1.98903377  1.        ]
 [ 0.          0.          0.          0.          0.          0.
  0.          0.          0.          0.          0.          0.
  1.         -1.98903377]]

Vector F for N = 14
[-1.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0. -2.]
Matrix A for N = 15
[[-1.99036171  1.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.        ]
 [ 1.         -1.99036171  1.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.        ]
 [ 0.          1.         -1.99036171  1.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.        ]
 [ 0.          0.          1.         -1.99036171  1.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.        ]
 [ 0.          0.          0.          1.         -1.99036171  1.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.        ]
 [ 0.          0.          0.          0.          1.         -1.99036171
   1.          0.          0.          0.          0.          0.
   0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          1.
  -1.99036171  1.          0.          0.          0.          0.
   0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   1.         -1.99036171  1.          0.          0.          0.
   0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          1.         -1.99036171  1.          0.          0.
   0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          1.         -1.99036171  1.          0.
   0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          1.         -1.99036171  1.
   0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          1.         -1.99036171
   1.          0.          0.        ]
```

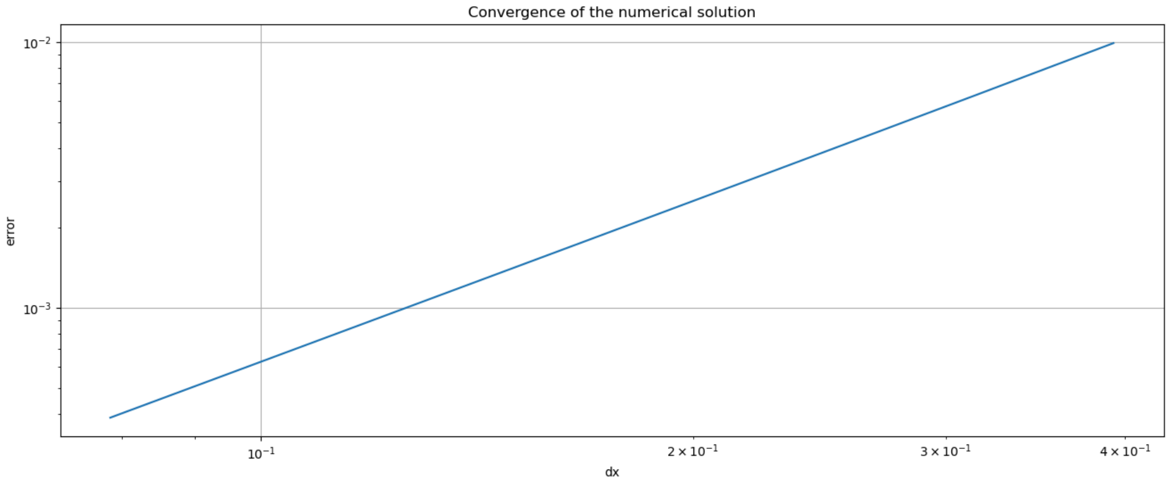
```
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          1.
  -1.99036171  1.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   1.         -1.99036171  1.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          1.         -1.99036171]]

Vector F for N = 15
[-1.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0. -2.]
Matrix A for N = 16
[[-1.99146228  1.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.        ]
 [ 1.         -1.99146228  1.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.        ]
 [ 0.          1.         -1.99146228  1.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.        ]
 [ 0.          0.          1.         -1.99146228  1.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.        ]
 [ 0.          0.          0.          1.         -1.99146228  1.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          1.         -1.99146228
   1.          0.          0.          0.          0.          0.
   0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          1.
  -1.99146228  1.          0.          0.          0.          0.
   0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   1.         -1.99146228  1.          0.          0.          0.
   0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          1.         -1.99146228  1.          0.          0.
   0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          1.         -1.99146228  1.          0.
   0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          1.         -1.99146228  1.
   0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          1.         -1.99146228
   1.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          1.
  -1.99146228  1.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   1.         -1.99146228  1.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          1.         -1.99146228  1.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
```

```
  0.          0.          1.         -1.99146228]]

Vector F for N = 16
[-1.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0. -2.]
Matrix A for N = 17
[[-1.99238456  1.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.        ]
 [ 1.         -1.99238456  1.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.        ]
 [ 0.          1.         -1.99238456  1.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.        ]
 [ 0.          0.          1.         -1.99238456  1.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          1.         -1.99238456  1.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          1.         -1.99238456
   1.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          1.
  -1.99238456  1.          0.          0.          0.          0.
   0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   1.         -1.99238456  1.          0.          0.          0.
   0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          1.         -1.99238456  1.          0.          0.
   0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          1.         -1.99238456  1.          0.
   0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          1.         -1.99238456  1.
   0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          1.         -1.99238456
   1.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          1.
  -1.99238456  1.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   1.         -1.99238456  1.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          1.         -1.99238456  1.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          1.         -1.99238456  1.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.          1.         -1.99238456]]

Vector F for N = 17
[-1.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0. -2.]
Matrix A for N = 18
```

```
[[-1.99316509  1.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.        ]
 [ 1.         -1.99316509  1.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.        ]
 [ 0.          1.         -1.99316509  1.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.        ]
 [ 0.          0.          1.         -1.99316509  1.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          1.         -1.99316509  1.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          1.         -1.99316509
   1.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          1.
  -1.99316509  1.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   1.         -1.99316509  1.          0.          0.          0.
   0.          0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          1.         -1.99316509  1.          0.          0.
   0.          0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          1.         -1.99316509  1.          0.
   0.          0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          1.         -1.99316509  1.
   0.          0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          1.         -1.99316509
   1.          0.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          1.
  -1.99316509  1.          0.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   1.         -1.99316509  1.          0.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          1.         -1.99316509  1.          0.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          1.         -1.99316509  1.          0.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.          1.         -1.99316509  1.        ]
 [ 0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          0.          0.
   0.          0.          0.          0.          1.         -1.9931650
9]]

Vector F for N = 18
[-1.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0. -2.]
Matrix A for N = 19
[[-1.9938315  1.         0.         0.         0.         0.
```

```
  0.         0.         0.         0.         0.         0.
  0.         0.         0.         0.         0.         0.
  0.       ]
 [ 1.        -1.9938315  1.         0.         0.         0.
  0.         0.         0.         0.         0.         0.
  0.         0.         0.         0.         0.         0.
  0.       ]
 [ 0.         1.        -1.9938315  1.         0.         0.
  0.         0.         0.         0.         0.         0.
  0.         0.         0.         0.         0.         0.
  0.       ]
 [ 0.         0.         1.        -1.9938315  1.         0.
  0.         0.         0.         0.         0.         0.
  0.         0.         0.         0.         0.         0.
  0.       ]
 [ 0.         0.         0.         1.        -1.9938315  1.
  0.         0.         0.         0.         0.         0.
  0.         0.         0.         0.         0.         0.
  0.       ]
 [ 0.         0.         0.         0.         1.        -1.9938315
  1.         0.         0.         0.         0.         0.
  0.         0.         0.         0.         0.         0.
  0.       ]
 [ 0.         0.         0.         0.         0.         1.
 -1.9938315  1.         0.         0.         0.         0.
  0.         0.         0.         0.         0.         0.
  0.       ]
 [ 0.         0.         0.         0.         0.         0.
  1.        -1.9938315  1.         0.         0.         0.
  0.         0.         0.         0.         0.         0.
  0.       ]
 [ 0.         0.         0.         0.         0.         0.
  0.         1.        -1.9938315  1.         0.         0.
  0.         0.         0.         0.         0.         0.
  0.       ]
 [ 0.         0.         0.         0.         0.         0.
  0.         0.         1.        -1.9938315  1.         0.
  0.         0.         0.         0.         0.         0.
  0.       ]
 [ 0.         0.         0.         0.         0.         0.
  0.         0.         0.         1.        -1.9938315  1.
  0.         0.         0.         0.         0.         0.
  0.       ]
 [ 0.         0.         0.         0.         0.         0.
  0.         0.         0.         0.         1.        -1.9938315
  1.         0.         0.         0.         0.         0.
  0.       ]
 [ 0.         0.         0.         0.         0.         0.
  0.         0.         0.         0.         0.         1.
 -1.9938315  1.         0.         0.         0.         0.
  0.       ]
 [ 0.         0.         0.         0.         0.         0.
  0.         0.         0.         0.         0.         0.
  1.        -1.9938315  1.         0.         0.         0.
  0.       ]
 [ 0.         0.         0.         0.         0.         0.
  0.         0.         0.         0.         0.         0.
  0.         1.        -1.9938315  1.         0.         0.
  0.       ]
 [ 0.         0.         0.         0.         0.         0.
```

```
  0.         0.         0.         0.         0.         0.
  0.         0.         1.        -1.9938315  1.         0.
  0.        ]
 [ 0.         0.         0.         0.         0.         0.
  0.         0.         0.         0.         0.         0.
  0.         0.         0.         1.        -1.9938315  1.
  0.        ]
 [ 0.         0.         0.         0.         0.         0.
  0.         0.         0.         0.         0.         0.
  0.         0.         0.         0.         1.        -1.9938315
  1.        ]
 [ 0.         0.         0.         0.         0.         0.
  0.         0.         0.         0.         0.         0.
  0.         0.         0.         0.         0.         1.
 -1.9938315]]

Vector F for N = 19
[-1.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.
 -2.]
```



In [17]:
```
p = np.polyfit(np.log(dxvec), np.log(errvec), 1)
print(p[0])
```

```
2.0117672459294886
```

Regarding what command is used to solve the matrix system, we use the `numpy.linalg.solve` function. This function uses LU decomposition.