CS601: Software Development for Scientific Computing

Autumn 2022

Week7: Motifs – Sparse Matrices (contd.), Fourier Transforms

Last week..

- Matrix Multiplication
  - ijk variants and recursive matmul
- Efficiency considerations
  - Storage (e.g. cache-oblivious data storage using Z-ordering)
  - Communication cost (data movement cost)
  - Special hardware (FMA, Vector units)
- Motif: Sparse Matrices
  - Triangular Matmul (as an e.g. that exploits structure to accelerate computation)
  - Storage scheme for sparse matrices (e.g. CSR)
  - Banded matrices (y=y+Ax with banded matrix and optimized storage)

$y=y+Ax$ with *Separable* Matrices

Refer to (Section 1 only):

https://www.math.uci.edu/~chenlong/MathPKU/FMMsimple.pdf

Faster y=Ax: Discrete Fourier Transforms (DFT)

- Very widely used
  - Image compression (jpeg)
  - Signal processing
  - Solving Poisson's Equation
- Represent A with F, a *Fourier Matrix* that has the following (remarkable) properties:
  - F^{-1} is easy to compute and consists of real numbers
  - Multiplications by F and F^{-1} is fast.
- F has complex numbers in its entries.
  - Every entry is a power of a single number w such that $w^n=1$
  - Any entry of a Fourier matrix can be written using $f_{ij} = w^{ij}$ (row and col indices start from 0)

Example: Fourier Matrix

- 4x4: $F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & 1 & w^2 \\ 1 & w^3 & w^2 & w^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}, i = \sqrt{-1}$
  - Here, w=i (also denoted as w_4=i). $w^4 = 1$ => i is a root.
- 8x8: $F_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 \\ 1 & w^2 & w^4 & w^6 & w^8 & w^{10} & w^{12} & w^{14} \\ 1 & w^3 & w^6 & w^9 & w^{12} & w^{15} & w^{18} & w^{21} \\ 1 & w^4 & w^8 & w^{12} & w^{16} & w^{20} & w^{24} & w^{28} \\ 1 & w^5 & w^{10} & w^{15} & w^{20} & w^{25} & w^{30} & w^{35} \\ 1 & w^6 & w^{12} & w^{18} & w^{24} & w^{30} & w^{36} & w^{42} \\ 1 & w^7 & w^{14} & w^{21} & w^{28} & w^{35} & w^{42} & w^{49} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 \\ 1 & w^2 & w^4 & w^6 & 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w & w^4 & w^7 & w^2 & w^5 \\ 1 & w^4 & 1 & w^4 & 1 & w^4 & 1 & w^4 \\ 1 & w^5 & w^2 & w^7 & w^4 & w^1 & w^6 & w^3 \\ 1 & w^6 & w^4 & w^2 & 1 & w^6 & w^4 & w^2 \\ 1 & w^7 & w^6 & w^5 & w^4 & w^3 & w^2 & w^1 \end{bmatrix}$

Here, $w = \frac{1+\sqrt{\mathrm{i}}}{2}$

(sqrt of i)