Sea V e.v. con $\langle \cdot | \cdot \rangle$ un producto interno.

Definimos la norma $\| f \| = \sqrt{\langle f | f \rangle}$

Ej: Sea $\mathcal{C}^0_{[-\pi,\pi]} = \{ f : [-\pi, \pi] \to \mathbb{R} \mid f \text{ es continua} \}$

def. $\langle f | g \rangle = \int_{-\pi}^{\pi} f(x) \cdot g(x)\, dx$.

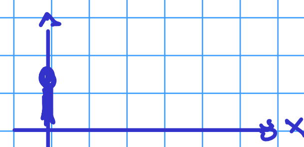
$$\langle \alpha f + \beta g | h \rangle = \int_{-\pi}^{\pi} (\alpha f + \beta g) \cdot h \, dx = \int_{-\pi}^{\pi} \alpha f \cdot h + \beta g \cdot h dx$$

$$= \alpha \int_{-\pi}^{\pi} f \cdot h \, dx + \beta \int_{-\pi}^{\pi} g \cdot h \, dx = \alpha^* \cdot \langle f | h \rangle + \beta^* \cdot \langle g | h \rangle.$$

...

$$\langle f | f \rangle = \int_{-\pi}^{\pi} f(x) \cdot f(x)\, dx = \int_{-\pi}^{\pi} f^2(x)\, dx \geq 0.$$

e.g. $f(x) = \begin{cases} 0 & \text{si } x \neq 0 \\ 1 & \text{si } x = 0 \end{cases}$



$f \neq \hat{0}$. $\int_{-\pi}^{\pi} f^2(x)\, dx = 0 \Rightarrow \langle f | f \rangle = 0.$

Calcule $\| \sin x \|$

sol: $\| \sin x \|^2 = \langle \sin x | \sin x \rangle = \int_{-\pi}^{\pi} \sin^2 x \, dx = \int_{-\pi}^{\pi} \frac{1}{2}(1 - \cos 2x) dx$

$$= \frac{1}{2}\left(x - \frac{1}{2} \sin 2x \right)\Big|_{-\pi}^{\pi} = \frac{1}{2} \cdot 2\pi - 0 = \pi.$$

$$\Rightarrow \| \sin x \| = \sqrt{\pi}.$$

Def: Sea V e.v. una norma en V es una $\|\cdot\|: V \longrightarrow \mathbb{R}$.
t.q.

i. $\|f\| \geq 0$ y $\|f\| = 0$ ssi $f = \bar{0}$.

ii. $\|\alpha f\| = |\alpha| \, \|f\|$

iii. $\|f + g\| \leq \|f\| + \|g\|$.

Propiedad: Si $\langle \cdot | \cdot \rangle$ es prod. interno $\|f\| = \sqrt{\langle f|f \rangle}$ es una norma.

dem:

i. $\|f\| = \sqrt{\langle f|f\rangle} \geq 0$ porque $\langle f|f\rangle \geq 0$ prop. pos. def.

ii. Tarea.

iii.

$z = a + ib$, $\quad z + z^* = 2a = 2\,\mathrm{Re}\, z \quad |z|^2 = a^2 + b^2 = z \cdot z^*$

$z^* = a - ib \quad z - z^* = 2ib = 2i\,\mathrm{Im}\, z$

$|z|^2 \geq (\mathrm{Re}\, z)^2 \Rightarrow |z| \geq |\mathrm{Re}\, z|$.

$\|f + g\|^2 = \langle f+g | f+g \rangle = \langle f | f+g \rangle + \langle g | f+g \rangle$ / Lineal. izq.

$= \langle f|f\rangle + \langle f|g\rangle + \langle g|f\rangle + \langle g|g\rangle$

$= \|f\|^2 + \langle f|g\rangle + \langle f|g\rangle^* + \|g\|^2$

$= \|f\|^2 + 2\,\mathrm{Re}\langle f|g\rangle + \|g\|^2$.

$(a+b)^2 = a^2 + 2ab + b^2$

$\|f + g\| \leq \|f\| + \|g\|$ ssi $\|f+g\|^2 \leq \|f\|^2 + 2\|f\|\cdot\|g\| + \|g\|^2$

ssi $\|f\|^2 + 2\,\mathrm{Re}\langle f|g\rangle + \|g\|^2 \leq \|f\|^2 + 2\|f\|\|g\| + \|g\|^2$

$$\mathrm{Re}\langle f|g\rangle \leq \|f\|\|g\|$$

Cauchy - Schwartz - Bouniakovsky. $\quad |\langle f|g\rangle| \leq \|f\| \cdot \|g\|$

Ej: En $\mathbb{R}^2$ definimos $\|(x_1, x_2)\|_\infty = \max\{|x_1|, |x_2|\}$ es una norma.

sol: 1. $\|\bar{x}\|_\infty \geq 0$ ✓ 

$$\|\bar{x}\|_\infty = \max\{|x_1|, |x_2|\} \geq \min\{|x_1|, |x_2|\}$$

como $|x_1|, |x_2| \geq 0 \Rightarrow$

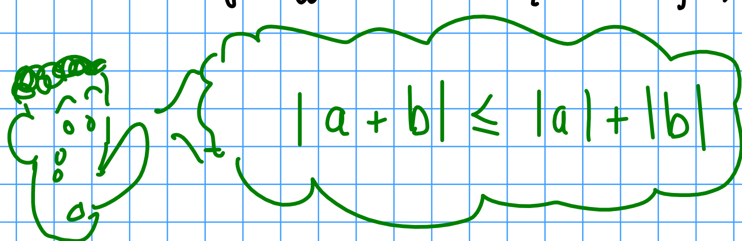
...

2. $\|\alpha\bar{x}\|_\infty = \|\alpha(x_1, x_2)\|_\infty = \|(\alpha x_1, \alpha x_2)\|_\infty =$ sup. $\max\{|x_1|, |x_2|\} = |x_2|$.

$$\max\{|\alpha x_1|, |\alpha x_2|\} = \max\{|\alpha||x_1|, |\alpha||x_2|\} =$$

$$= |\alpha| \max\{|x_1|, |x_2|\} = |\alpha| \|\bar{x}\|_\infty.$$

3. $\|\bar{x} + \bar{y}\|_\infty = \max\{|x_1 + y_1|, |x_2 + y_2|\}$



sabemos que

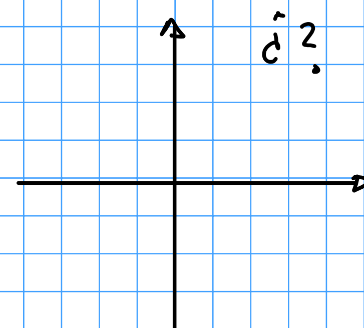
$|x_1 + y_1| \leq |x_1| + |y_1| \leq \max\{|x_1|, |x_2|\}$

$|x_2 + y_2| \leq |x_2| + |y_2| \quad + \max\{|y_1|, |y_2|\}$

$$\Rightarrow \|\bar{x} + \bar{y}\|_\infty \leq \|\bar{x}\|_\infty + \|\bar{y}\|_\infty.$$

Distancia: $d(f, g) = \|g - f\|^2$.

$C: d((x,y), (0,0)) \leq 1$. Usando $\|\cdot\|_\infty$



¿?

$\|\cdot\|$ usal