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Hoja de trabajo N°. 4

1. Calcular la matriz adjunta, el determinante, y la matriz inversa:

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

a) $A = \begin{pmatrix} 1 & 3 \\ -5 & 6 \end{pmatrix}$ b) $A = \begin{pmatrix} 1 & 3 & 2 \\ -5 & 6 & -7 \\ 8 & 5 & 4 \end{pmatrix}$

1) $A = \begin{pmatrix} 1 & 3 \\ -5 & 6 \end{pmatrix}$

$$\text{adj } A = \begin{pmatrix} 6 & -3 \\ 5 & 1 \end{pmatrix}$$

$$\det A = \begin{pmatrix} 1 & 3 \\ -5 & 6 \end{pmatrix} = 6 - (-15) = 6 + 15 = 21$$

$$A^{-1} = \frac{1}{\det A} = \begin{pmatrix} 6 & -3 \\ 5 & 1 \end{pmatrix} \frac{1}{21} = \begin{pmatrix} \frac{6}{21} & -\frac{3}{21} \\ \frac{5}{21} & \frac{1}{21} \end{pmatrix}$$

2) $A = \begin{pmatrix} 1 & 3 & 2 \\ -5 & 6 & -7 \\ 8 & 5 & 4 \end{pmatrix}$

$\text{adj } A = C^T$ matriz w factores.

$$C = \begin{bmatrix} +59 & -34 & -73 \\ -2 & -12 & +19 \\ -33 & -3 & +21 \end{bmatrix}$$

$$11 = \begin{matrix} 6 & -7 \\ 5 & 4 \end{matrix} = 24 + 35 = 59$$

$$12 = \begin{matrix} -5 & -3 \\ 8 & 4 \end{matrix} = -20 + 56 = 36$$

$$13 = \begin{matrix} -5 & 6 \\ 8 & 5 \end{matrix} = -25 - 48 = -73$$

$$21) \frac{32}{54} = 12 - 10 = 2 \quad 31) \frac{32}{6-7} = -21 - 12 = -33$$

$$22) \frac{12}{84} = 4 - 14 = -12 \quad 32) \frac{1^2}{-5-7} = -7 + 10 = 3$$

$$23) \frac{13}{85} = 5 - 24 = -19 \quad 33) \frac{13}{-54} = 6 + 15 = 21$$

$$\text{Adj } A = C^T = \begin{bmatrix} 59 & -2 & -33 \\ -34 & -12 & -3 \\ -73 & +19 & 21 \end{bmatrix}$$

$$\det A = \begin{pmatrix} 1 & 3 & 2 \\ -5 & 4 & -7 \\ 8 & 5 & 4 \end{pmatrix}$$

$$\det |A| = \underbrace{\begin{array}{c|c|c} + & - & + \\ \hline 1 & 3 & 2 \end{array}}_{\text{Cofactor expansion}}$$

$$1(6 \cdot 4 + 7 \cdot 5) - 3(-20 + 56) + 2(-25 - 48)$$

$$1(59) - 3(34) + 2(-73)$$

$$59 - 108 - 146$$

$$\det A = 195$$

$$A^{-1} = \frac{1}{195} \begin{pmatrix} 59 & -2 & -33 \\ -34 & -12 & -3 \\ -73 & +19 & 21 \end{pmatrix}$$

2. Calcular $|A|$, el determinante de la matriz A, por expansión de Laplace.

$$A = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 2 & 0 & 1 & 5 \\ -1 & 6 & 10 & -8 \\ 0 & -2 & 7 & 1 \end{pmatrix}$$

$$|A| = a_{11} \cdot A_{11} + a_{21} \cdot A_{21} + a_{31} \cdot A_{31} + \cancel{a_{41} \cdot A_{41}} \\ (1 \cdot A_{11}) - (2 \cdot A_{21}) + (1 \cdot A_{31})$$

$$= 1 \begin{vmatrix} 0 & 1 & 5 & 0 & 1 \\ 4 & 10 & -8 & 4 & 10 \\ -2 & 7 & 1 & -2 & 7 \end{vmatrix} = 0 + 16 + 210 - (4 + 0 - 100) \\ = 1 (226 + 99) = 320.$$

$$-2 \begin{vmatrix} 3 & -1 & 4 & 3 & -1 \\ 0 & 10 & -8 & 4 & 10 \\ -2 & 7 & 1 & -2 & 7 \end{vmatrix} = (30 + -10 + 100) + 254 \\ - (-4 + -160 + -80) = 434 (-2) = -872$$

$$-1 \begin{vmatrix} 3 & -1 & 4 & 3 & -1 \\ 0 & 1 & 5 & 0 & 1 \\ -2 & 7 & 1 & -2 & 7 \end{vmatrix} = (3 + 10 + 0) - (0 + 105 + -3) \\ = +84$$

$$\det A = 320 - 872 + 84 = 48$$

3. Calcular la función de transferencia, $G(s) = Y(s)/U(s)$, para el sistema definido por

a) $\dot{x} = \begin{pmatrix} -4 & -1.5 \\ 4 & 0 \end{pmatrix}x + \begin{pmatrix} 2 \\ 0 \end{pmatrix}u, \quad y = (1.5 \quad 0.625)x$

b) $\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}x + \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}u, \quad y = (1 \quad 0 \quad 0)x$

c) $\dot{x} = \begin{pmatrix} 8 & -4 & 1 \\ -3 & 2 & 0 \\ 5 & 7 & -9 \end{pmatrix}x + \begin{pmatrix} -4 \\ -3 \\ 4 \end{pmatrix}u, \quad y = (2 \quad 8 \quad -3)x$

$$y = b u = [C (sI - A)^{-1} B + D] u$$

$$A) \dot{x} = \underbrace{\begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix}}_A x + \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_B u \quad y = (1.5 \quad 0.625)x$$

$$\begin{aligned} sI - A &= s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} s+4 & 1.5 \\ -4 & s \end{bmatrix} \end{aligned}$$

$$(sI - A)^{-1} = \frac{Adj(sI - A)}{|sI - A|} = \frac{\begin{bmatrix} s & -1.5 \\ 4 & s+4 \end{bmatrix}}{s(s+4) + 4(1.5)} = \frac{\begin{bmatrix} s & -1.5 \\ 4 & s+4 \end{bmatrix}}{s^2 + 4s + 10}$$

$$C(sI - A)^{-1} = [1.5 \quad 0.625] \begin{bmatrix} s & -1.5 \\ 4 & s+4 \end{bmatrix} \frac{1}{s^2 + 4s + 10}$$

$$= \frac{1}{s^2 + 4s + 10} \left[1.5s + 2.5 \quad -2.25 + 0.625s + 2.5 \right]$$

$$= \frac{\begin{bmatrix} 1.5s + 2.5 & 0.625s + 2.5 \end{bmatrix}}{s^2 + 4s + 10}$$

$$C(sI - A)^{-1} B = \frac{1}{s^2 + 4s + 4} [1.5s + 2.5 \quad 0.625s + 0.25] \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \frac{1}{s^2 + 4s + 4} [3s + 5]$$

$$\frac{Y(s)}{X(s)} = \frac{3s + 5}{s^2 + 4s + 4}$$

b) $\dot{x} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}}_A x + \underbrace{\begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}}_B u, \quad y = \underbrace{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}}_C x$

$$(sI - A) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{\begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{vmatrix} + \begin{vmatrix} 0 & s & 0 \\ -1 & 0 & s+3 \\ 1 & 2 & s+3 \end{vmatrix} + \begin{vmatrix} 0 & s & 0 \\ 0 & 1 & 2 \\ 1 & 2 & s+3 \end{vmatrix}}{\begin{vmatrix} s^2 + 3s + 2 & -1 & -s \\ s+3 & s^2 + 3s & -2s - 2 \\ 1 & s & s^2 \end{vmatrix}}$$

$$\begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{vmatrix} = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{vmatrix}$$

$$(sI - A)^{-1} = \frac{\begin{vmatrix} s^2 + 3s + 2 & s+3 & 1 \\ -1 & s^2 + 3s & s \\ -s & -2s - 2 & s^2 \end{vmatrix}}{s^3 + 3s^2 + 2s + 1}$$

$$(sI - A^{-1}) C = (1 \quad 0 \quad 0) \cdot \frac{\begin{vmatrix} s^2 + 3s + 2 & s+3 & 1 \\ -1 & s^2 + 3s & s \\ -s & -2s - 2 & s^2 \end{vmatrix}}{s^3 + 3s^2 + 2s + 1}$$

$$= \frac{\begin{pmatrix} s^2 + 3s + 2 & s+3 & 1 \end{pmatrix}}{s^3 + 3s^2 + 2s + 1}$$

$$(sI - A)^{-1} C \cdot B = \frac{\begin{pmatrix} s^2 + 3s + 2 & s+3 & 1 \end{pmatrix}}{s^3 + 3s^2 + 2s + 1} \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1}$$

c) $\dot{x} = \underbrace{\begin{pmatrix} 8 & -4 & 1 \\ -3 & 2 & 0 \\ 5 & 7 & -9 \end{pmatrix}}_A x + \underbrace{\begin{pmatrix} -4 \\ -3 \\ 4 \end{pmatrix}}_B u, \quad y = \underbrace{\begin{pmatrix} 2 & 8 & -3 \end{pmatrix}}_C x$

$$(sI - A) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 8 & -4 & 1 \\ -3 & 2 & 0 \\ 5 & 7 & -9 \end{bmatrix} = \begin{bmatrix} s-8 & 4 & -1 \\ 3 & s-2 & 0 \\ -5 & -7 & s+9 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{\begin{pmatrix} (s-2)(s+9) & -4s+29 & s-2 \\ 3s+27 & s^2+s-77 & 3 \\ 5s-81 & +7s-76 & s^2-10s+4 \end{pmatrix}}{s^3 - s^2 - 91s + 47}$$

$$(sI - A)^{-1} C = (2 \quad 8 \quad -3) \begin{pmatrix} (s-2)(s+9) & -4s+29 & s-2 \\ 3s+27 & s^2+s-77 & 3 \\ 5s-81 & +7s-76 & s^2-10s+4 \end{pmatrix}$$

$$= \frac{\begin{pmatrix} 2s^2 - 25s - 159 & 8s^2 - 21s - 444 & -3s^2 + 32s - 40 \end{pmatrix}}{s^3 - s^2 - 91s + 47}$$

$$(sI - A)^{-1} C \cdot B = \frac{(-4)(2s^2 - 25s - 159) + (-3)(8s^2 - 21s - 444) + 10(-3s^2 + 32s - 40)}{s^3 - s^2 - 91s + 47}$$

$$= \frac{-44s^2 + 291s + 1814}{s^3 - s^2 - 91s + 17}$$

4. Calcular la matriz de transferencia $H(s)$ para un sistema descrito por

$$\dot{x} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}x + \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}u$$

$$y = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 0 \\ 3 & -1 & 6 \end{pmatrix}x + \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}u$$

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$T(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

$$= \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 0 \\ 3 & -1 & 0 \end{bmatrix} \begin{bmatrix} s+1 & -1 & 0 \\ 0 & s+1 & 0 \\ 0 & 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} s+1 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & s+1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & s+1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & \frac{1}{s+1} & \frac{1}{(s+1)^2} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{s+1} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{s+1} & 0 \end{bmatrix}$$

$$= \begin{pmatrix} \frac{1}{s+1} & \frac{1}{(s+1)^2} & 0 \\ 0 & \frac{1}{s+1} & 0 \\ 0 & 0 & \frac{1}{s+1} \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 0 \\ 3 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)^2} & 0 \\ 0 & \frac{1}{s+1} & 0 \\ 0 & 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{pmatrix} 2s+2 & -s-2 & 3s+3 \\ 0 & s+1 & 0 \\ 3s+3 & -s-1 & 0 \end{pmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{pmatrix} -1 & 4s+4 \\ s+1 & -s-1 \\ ss+s & s+1 \end{pmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

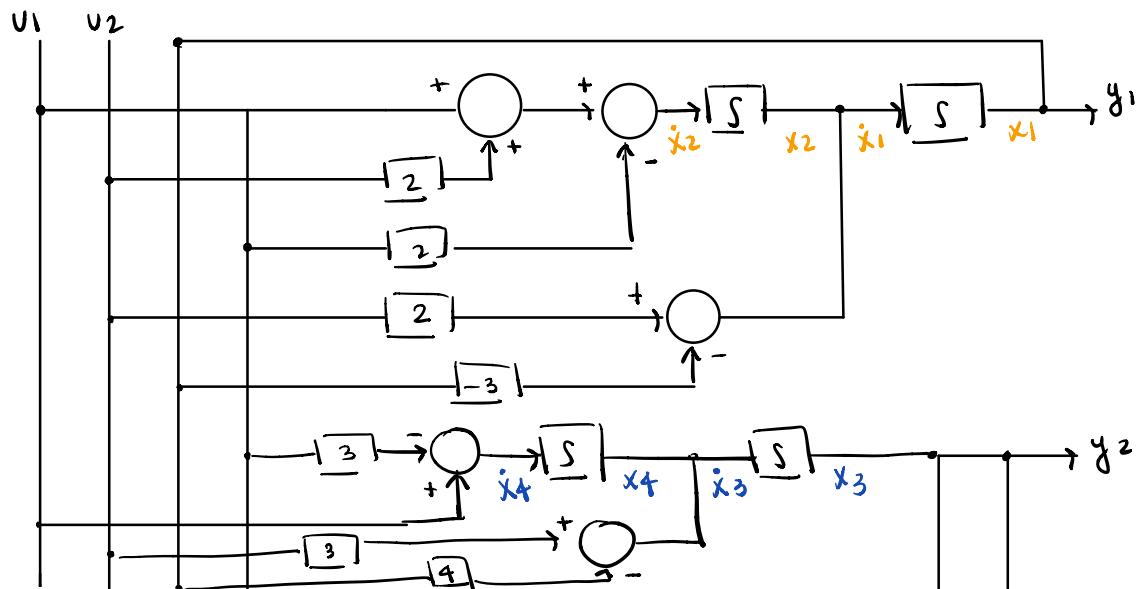
$$= \begin{pmatrix} 0 & 4s+4 \\ s+1 & -s-1 \\ ss+s & s+1 \end{pmatrix} D$$

5. Un sistema con dos entradas y dos salidas se describe con las ecuaciones diferenciales $\ddot{y}_1 + 3\dot{y}_1 + 2y_2 = u_1 + 2u_2 + 2\dot{u}_2$ y $\ddot{y}_2 + 4\dot{y}_1 + 3y_2 = \ddot{u}_2 + 3\dot{u}_2 + u_1$.

- Representar al sistema utilizando la forma canónica observable de las ecuaciones de estado.
- Calcular la matriz de transferencia.

$$\begin{aligned}
 \ddot{y}_1 + 3\dot{y}_1 + 2y_2 &= u_1 + 2u_2 + 2\dot{u}_2 \\
 \ddot{y}_1 &= 2\dot{u}_2 - 3\dot{y}_1 + (u_1 + 2u_2 - 2y_2) \\
 \dot{y}_1 &= 2u_2 - 3y_1 + \int (u_1 + 2u_2 - 2y_2) dt \\
 y_1 &= \int [2u_2 - 3y_1 + \int (u_1 + 2u_2 - 2y_2) dt'] dt'' \\
 \ddot{y}_2 + 4\dot{y}_1 + 3y_2 &= \ddot{u}_2 + 3\dot{u}_2 + u_1 \\
 \ddot{y}_2 &= \ddot{u}_2 + (3\dot{u}_2 - 4\dot{y}_1) + (u_1 - 3y_2) \\
 \dot{y}_2 &= \dot{u}_2 + (3u_2 - 4y_1) + \int (u_1 - 3y_2) dt \\
 y_2 &= u_2 + \int [3u_2 - 4y_1 + \int (u_1 - 3y_2) dt'] dt'' \\
 \end{aligned}$$

$$\begin{aligned}
 y_1 &= \int [2u_2 - 3y_1 + \int (u_1 + 2u_2 - 2y_2) dt'] dt'' \\
 y_2 &= u_2 + \int [3u_2 - 4y_1 + \int (u_1 - 3y_2) dt'] dt'
 \end{aligned}$$



$$\begin{array}{ll}
\dot{x}_1 = x_2 + 2u_2 - 3x_1 & \dot{x}_3 = x_4 + 3u_2 - 4x_1 \\
\dot{x}_2 = (w_1 + 2u_2 - 2y_2) & \dot{x}_4 = -3y_2 + u_1 \\
= u_1 + 2u_2 - 2(x_3 + u_2) & = -3(x_3 + u_2) + u_1 \\
\dot{x}_2 = u_1 - 2x_3 & \dot{x}_4 = -3x_3 - 3u_2 + u_1 \\
y_1 = x_1 & y_2 = x_3 + u_2 .
\end{array}$$

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_2 \\ \vdots \\ \dot{x}_3 \\ \vdots \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ -4 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

b). $\dot{x} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ -4 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 3 \\ 1 & -3 \end{bmatrix} u$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} v$$

$$t(s) = C(sI - A)^{-1}B + D$$

$$\begin{aligned}
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s+3 & -1 & 0 & 0 \\ 0 & s & +2 & 0 \\ 4 & 0 & s & -1 \\ 0 & 0 & 3 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 3 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \left(\begin{array}{cccc} \frac{s^2+3}{(s+1)^3} & \frac{s^2+3}{s(s+1)^3} & \frac{-2}{(s+1)^3} & \frac{-2}{s(s+1)^3} \\ \frac{-4s}{(s+1)^3} & \frac{-4}{(s+1)^3} & \frac{s(s+3)}{(s+1)^3} & \frac{s+3}{(s+1)^3} \end{array} \right) \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 3 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \left(\begin{array}{cc} \frac{s^2+1}{s(s+1)^3} & \frac{2s^3+4}{s(s+1)^3} \\ \frac{s-1}{(s+1)^3} & \frac{3s^2-2s-9}{(s+1)^3} \end{array} \right) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \left(\begin{array}{cc} \frac{s^2+1}{s(s+1)^3} & \frac{2s^3+4}{s(s+1)^3} \\ \frac{s-1}{(s+1)^3} & \frac{3s^2-2s-8}{(s+1)^3} \end{array} \right)
\end{aligned}$$