Hoja 2

viernes, 12 de febrero de 2021 08:37

1. Evalúe la función compleja f dada en los puntos indicados:

   a) $f(z) = \log_e |z| + iArg(z)$

      1) 1
      2) $4i$
      3) $1 + i$.

$\log_e = \ln$ 😒

$f(z) = \ln(z) + i\,Arg(z)$

1) $\ln(1) + i(0) = 0$

2) $\ln(\sqrt{4^2}) + i\frac{\pi}{2} = \ln(4) + i\frac{\pi}{2}$

3) $\sqrt{1^2+1^2} = \sqrt{2}$, $Arg(z) = \frac{\pi}{4}$

$\ln(\sqrt{2}) + i\frac{\pi}{4}$

   b) $f(z) = (xy - x^2) + i(3x + y)$

      1) $3i$
      2) $4 + i$
      3) $3 - 5i$.

1) $(\cancel{0 \cdot 3 - 0^2})^{0} + i(3 \cdot 0 + 3)$

$= 3i$ //

2) $(4 \cdot 1 - 4^2) + i(3 \cdot 4 + 1)$

$= -12 + 13i$ //

3) $(3 \cdot (-5) - 3^2) + i(3 \cdot 3 + (-5))$

$= -24 + 4i$ //

2. Determine las partes real e imaginaria u y v de la función compleja dada como funciones de x y y:

   a) $f(z) = \frac{\overline{z}}{z+1}$
   b) $f(z) = z + \frac{1}{z}$
   c) $f(z) = e^{2z+i}$
   d) $f(z) = e^{z^2}$.

$z = x + iy$

a) $f(z) = \dfrac{\overline{z}}{z+1} = \dfrac{x - iy}{x + iy + 1} \cdot \dfrac{x - iy + 1}{x - iy + 1}$

$= \underline{x^2 - ixy + x - ixy + (iy)^2 - iy}$

$$x^2 - \cancel{ixy} + x + \cancel{ixy} - (iy)^2 \cancel{+ iy} + x - \cancel{iy} + 1$$

$$= \frac{x^2 + x - y^2 - 2ixy - iy}{x^2 + 2x + 1 + y^2}$$

$$\Rightarrow Re(z) = \frac{x^2 + x - y^2}{(x+1)^2 + y^2}$$

$$Im(z) = \frac{-2xy - y}{(x+1)^2 + y^2}$$

b) $f(z) = z + \frac{1}{z}$

$$= x + iy + \frac{1}{x+iy}$$

$$\Rightarrow \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2 - iyx + iyx - (iy)^2}$$

$$= \frac{x-iy}{x^2 - y^2}$$

$$\Rightarrow x + iy + \frac{x - iy}{x^2 - y^2}$$

$$= \frac{(x+iy)(x^2-y^2) + x - iy}{x^2 - y^2}$$

$$= \frac{x^3 - xy^2 + x + ix^2y - iy^3 - iy}{x^2 - y^2}$$

$$\Rightarrow Re(z) = \frac{x^3 - xy^2 + x}{x^2 - y^2}$$

$$Img(z) = \frac{x^2y - y^3 - y}{x^2 - y^2}$$

$$f(z) = e^{2z+i}$$

$$= e^{2z} e^{i} = e^{2x + i2y} \cdot e^{i}$$

$$= e^{2x} \cdot e^{i(2y+1)} \quad // \quad e^{i\theta} = \cos(\theta) + i\,sen(\theta)$$

$$= e^{2x}(\cos(2y+1) + i\,sen(2y+1))$$

$$\Rightarrow Re(z) = e^{2x} \cdot \cos(2y+1)$$

$$\operatorname{Im}(z) = e^{2x} \cdot \operatorname{sen}(2y+1)$$

d) $f(z) = e^{z^2}$

$$= e^{(x+iy)^2} = e^{x^2+2ixy-y^2}$$

$$= e^{x^2-y^2} \cdot e^{i\,2xy}$$

$$= e^{x^2-y^2}\left(\cos(2xy) + i \operatorname{sen}(2xy)\right)$$

$$\Rightarrow \operatorname{Re}(z) = e^{x^2-y^2} \cdot \cos(2xy)$$

$$\operatorname{Im}(z) = e^{x^2-y^2} \cdot \operatorname{sen}(2xy)$$

3. Demuestre que $\overline{e^{z}} = e^{\overline{z}}$ y deduzca que

$$\overline{\sin z} = \sin \overline{z}, \qquad \overline{\cos z} = \cos \overline{z}.$$

$$\Rightarrow e^{x+iy}$$

$$= e^{x} \cdot e^{iy}$$

$$= e^{x}(\cos(y) + i \operatorname{sen}(y))$$

$$\Rightarrow e^{x}(\cos(y) + i \operatorname{sen}(y))$$

$\ast$ el conjugado solo le cambia el signo a la parte imaginaria

$$\Rightarrow \overline{e^{z}} = e^{x}(\cos(y) - i \operatorname{sen}(y)) \Rightarrow \overline{e^{z}} = e^{\bar{z}}$$
$$e^{\bar{z}} = e^{x}(\cos(y) - i \operatorname{sen}(y))$$

QED

$$\overline{\sin(z)} = \sin(\bar{z})$$

$$\sin(x+iy) = \operatorname{sen}(x)\cos(iy) + \cos(x)\operatorname{sen}(iy)$$

$$= \operatorname{sen}(x)\cosh(y) - i\cos(x)\operatorname{senh}(y)$$

$$\Rightarrow \overline{\sin(z)} = \operatorname{sen}(x)\cosh(y) + i\cos(x)\operatorname{senh}(y)$$
$$\sin(\bar{z}) = \operatorname{sen}(x)\cosh(y) + i\cos(x)\operatorname{senh}(y)$$

$$\overline{\cos(z)} = \cos(\bar{z})$$

$$\cos(x+iy) = \cos(x)\cosh(y) + i\operatorname{sen}(x)\operatorname{senh}(y)$$

$$\Rightarrow \overline{\cos(z)} = \cos(x)\cosh(y) - i\operatorname{sen}(x)\operatorname{senh}(y)$$
$$\cos(\bar{z}) = \cos(x)\cosh(y) - i\operatorname{sen}(x)\operatorname{senh}(y)$$

4. Demuestre que para todo $z \in \mathbb{C}$,

$$\sin(z+w) = \sin z \; cosw + \cos z \sin w.$$

$$\sin(z)\cos(w) + \sin(w)\cos(z) = \frac{e^{iz} - e^{-iz}}{2i} \cdot \frac{e^{iw} + e^{-iw}}{2}$$

$$+ \frac{e^{iw} - e^{-iw}}{2i} \cdot \frac{e^{iz} + e^{-iz}}{2}$$

$$= \frac{e^{i(z+w)} + e^{i(z-w)} - e^{i(w-z)} - e^{-i(z+w)}}{4i}$$

$$+ \frac{e^{i(z+w)} + e^{i(w-z)} - e^{i(z-w)} - e^{-i(z+w)}}{4i}$$

$$= \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i}$$

$$= \sin(z+w)$$

5. Demuestre que para todo $z \in \mathbb{C}$,

$$\cos(iz) = \cosh z.$$

$$\cos(iz) = \cos(i(x+iy))$$

$$= \frac{e^{i(iz)} + e^{-i(iz)}}{2}$$

$$= \frac{e^{-z} + e^{z}}{2}$$

$$= \cosh(z) \quad \text{QED}$$

6. Demuestre que la función $f(z) = \cosh z$ es periódica con periodo $T = 2\pi i$.

Sabemos que

$$\cosh(z) = \frac{e^{z} + e^{-z}}{2}$$

e^{z} es periodica en $2\pi i$
al igual que e^{-z}

Por lo tanto

$$e^{z+2\pi i} = e^{z}$$
$$e^{-(z+2\pi i)} = e^{-z}$$

$$\cosh(z + 2\pi i) = \frac{e^{z+2\pi i} + e^{-(z+2\pi i)}}{2}$$

$$= \frac{e^{z} + e^{-z}}{2}$$

$$= \cosh(z)$$