HOJA DE TRABAJO No. 5 - Solución

1.

6. $\int_{-\infty}^{0} \frac{1}{2x-5}\, dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{2x-5}\, dx = \lim_{t\to-\infty} \left[\frac{1}{2} \ln |2x-5| \right]_{t}^{0} = \lim_{t\to\infty} \left[\frac{1}{2} \ln 5 - \frac{1}{2} \ln |2t-5| \right] = -\infty$

Divergent

2.

24. Integrate by parts with $u=\ln x$, $dv=dx/x^3 \Rightarrow du=dx/x$, $v=-1/\left(2x^2\right)$.

$$\int_{1}^{\infty} \frac{\ln x}{x^3}\, dx = \lim_{t\to\infty} \int_{1}^{t} \frac{\ln x}{x^3}\, dx = \lim_{t\to\infty} \left(\left[-\frac{1}{2x^2} \ln x \right]_{1}^{t} + \frac{1}{2} \int_{1}^{t} \frac{1}{x^3}\, dx \right)$$

$$= \lim_{t\to\infty} \left(-\frac{1}{2} \frac{\ln t}{t^2} + 0 - \frac{1}{4t^2} + \frac{1}{4} \right) = \frac{1}{4}$$

since $\lim_{t\to\infty} \frac{\ln t}{t^2} = \lim_{t\to\infty} \frac{1/t}{2t} = \lim_{t\to\infty} \frac{1}{2t^2} = 0$. Convergent

3.

13. $\int_{-\infty}^{\infty} xe^{-x^2}\, dx = \int_{-\infty}^{0} xe^{-x^2}\, dx + \int_{0}^{\infty} xe^{-x^2}\, dx$.

$\int_{-\infty}^{0} xe^{-x^2}\, dx = \lim_{t\to-\infty} \left(-\frac{1}{2}\right) \left[e^{-x^2} \right]_{t}^{0} = \lim_{t\to\infty} \left(-\frac{1}{2}\right) \left(e^{-t^2} - 1 \right) = -\frac{1}{2} \cdot 1 = -\frac{1}{2}$, and

$\int_{0}^{\infty} xe^{-x^2}\, dx = \lim_{t\to\infty} \left(-\frac{1}{2}\right) \left[e^{-x^2} \right]_{0}^{t} = \lim_{t\to\infty} \left(-\frac{1}{2}\right) \left(e^{-t^2} - 1 \right) = -\frac{1}{2} \cdot (-1) = \frac{1}{2}$.

Therefore, $\int_{-\infty}^{\infty} xe^{-x^2}\, dx = -\frac{1}{2} + \frac{1}{2} = 0$. Convergent

4.

$$\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-} \int_{0}^{t} \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-} \left[\sin^{-1} x \right]_{0}^{t} = \lim_{t\to 1^-} \sin^{-1} t = \frac{\pi}{2}$$. Convergent

5.

There is an infinite discontinuity at $x=0$. $\int_{-1}^{1} \frac{e^x}{e^x-1}\,dx=\int_{-1}^{0} \frac{e^x}{e^x-1}\,dx+\int_{0}^{1} \frac{e^x}{e^x-1}\,dx$.

$$\int_{-1}^{0} \frac{e^x}{e^x-1}\,dx=\lim_{t\to 0^-}\int_{-1}^{t} \frac{e^x}{e^x-1}\,dx=\lim_{t\to 0^-}\Big[\ln\left|e^x-1\right|\Big]_{-1}^{t}=\lim_{t\to 0^-}\Big[\ln\left|e^t-1\right|-\ln\left|e^{-1}-1\right|\Big]=-\infty\ ,$$

so $\int_{-1}^{1} \frac{e^x}{e^x-1}\,dx$ is divergent. The integral $\int_{0}^{1} \frac{e^x}{e^x-1}\,dx$ also diverges since

$$\int_{0}^{1} \frac{e^x}{e^x-1}\,dx=\lim_{t\to 0^+}\int_{t}^{1} \frac{e^x}{e^x-1}\,dx=\lim_{t\to 0^+}\Big[\ln\left|e^x-1\right|\Big]_{t}^{1}=\lim_{t\to 0^+}\Big[\ln\left|e-1\right|-\ln\left|e^t-1\right|\Big]=\infty\ .$$

Divergent

6.

$$I=\int_{0}^{2} z^2\ln z\,dz=\lim_{t\to 0^+}\int_{t}^{2} z^2\ln z\,dz=\lim_{t\to 0^+}\left[\frac{z^3}{3^2}(3\ln z-1)\right]_{t}^{2}$$

$$=\lim_{t\to 0^+}\left[\frac{8}{9}(3\ln 2-1)-\frac{1}{9}t^3(3\ln t-1)\right]=\frac{8}{3}\ln 2-\frac{8}{9}-\frac{1}{9}\lim_{t\to 0^+}\left[t^3(3\ln t-1)\right]=\frac{8}{3}\ln 2-\frac{8}{9}-\frac{1}{9}L\ .$$

Now $L=\lim_{t\to 0^+}\left[t^3(3\ln t-1)\right]=\lim_{t\to 0^+}\frac{3\ln t-1}{t^{-3}}=\lim_{t\to 0^+}\frac{3/t}{-3/t^4}=\lim_{t\to 0^+}\left(-t^3\right)=0$. Thus, $L=0$ and $I=\frac{8}{3}\ln 2-\frac{8}{9}$

Convergent