

Inés Alarcón

18008450

Hoja de trabajo N° 9

$$1. \dot{x} = \begin{pmatrix} 0 & 100 \\ -1 & 0 \end{pmatrix}x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u, \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

a) Calcular la matriz de transición de estado.

b) Calcular $x_T(-5)$, $x_T(0)$, y $x_T(5)$.

c) Calcular $x_T(t)$ si $x(10) = \begin{pmatrix} 0.86 \\ 0.05 \end{pmatrix}$

a.

$$A = \begin{pmatrix} 0 & 100 \\ -1 & 0 \end{pmatrix}$$

valores propios

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 100 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 100$$

$$\lambda^2 + 100 = 0$$

$$\lambda^2 = -100$$

$$\lambda_1 = 10i, \quad \lambda_2 = -10i$$

ecuaciones para alfa

$$e^{10it} = \alpha_0 + 10i\alpha_1$$

$$e^{-10it} = \alpha_0 - 10i\alpha_1$$

→ euler

$$\cos(10t) + i \sin(10t) = \alpha_0 + 10i\alpha_1 \quad ①$$

$$\cos(10t) - i \sin(10t) = \alpha_0 - 10i\alpha_1 \quad ②$$

despejo α_0 :

$$\begin{aligned} e^{it} &= \cos(t) + i \sin(t) \\ e^{-it} &= \cos(t) - i \sin(t) \end{aligned}$$

$$\alpha_0 = \cos(10t) + i \sin(10t) - 10i \alpha_1 \quad (3)$$

Sustituir 3 en 2:

$$\cos(10t) - i \sin(10t) = \cos(10t) + i \sin(10t) - 10i \alpha_1 - 10i \alpha_1$$

$$\cancel{\cos(10t)} - i \sin(10t) - \cancel{\cos(10t)} - i \sin(10t) = -20i \alpha_1$$

$$-2i \sin(10t) = -20i \alpha_1$$

$$\frac{1}{10} \sin(10t) = \alpha_1 \quad (4)$$

Sustituir 4 en 3:

$$\alpha_0 = \cos(10t) + i \sin(10t) - 10i \alpha_1$$

$$= \cos(10t) + i \sin(10t) - 10i \left(\frac{1}{10} \sin(10t) \right)$$

$$\alpha_0 = \cos(10t) + \cancel{i \sin(10t)} - \cancel{i \sin(10t)}$$

$$\alpha_0 = \cos(10t)$$

$$e^{At} = \alpha_0 I + \alpha_1 A$$

$$= \cos(10t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{10} \sin(10t) \begin{pmatrix} 0 & 10 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(10t) & 0 \\ 0 & \cos(10t) \end{pmatrix} + \begin{pmatrix} 0 & 10 \sin(10t) \\ -\frac{1}{10} \sin(10t) & 0 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} \cos(10t) & 10 \sin(10t) \\ -\frac{1}{10} \sin(10t) & \cos(10t) \end{pmatrix}$$

b) Calcular $x_T(-5)$, $x_T(0)$, y $x_T(5)$. $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

c) Calcular $x_T(t)$ si $x(10) = \begin{pmatrix} 0.86 \\ 0.05 \end{pmatrix}$

b.

$$x_T(-5) = \underline{x}(-t) \underline{x}(0)$$

$$x_T(-5) = \begin{pmatrix} \cos(10 \cdot -5) & 10 \sin(10 \cdot -5) \\ -\frac{\sin(10 \cdot -5)}{10} & \cos(10 \cdot -5) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.94 & 2.42 \\ -0.03 & 0.94 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

 $x_T(-5) = \begin{pmatrix} 0.94 \\ -0.03 \end{pmatrix}$

$$x_T(0) = E(-t) \underline{x}(0)$$

$$= \begin{pmatrix} \cos(0) & 10 \sin(0) \\ -\frac{\sin(0)}{10} & \cos(0) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

 $x_T(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x_T(0) = \underline{x}(0)$

$$x_T(5) = E(t) \underline{x}(0)$$

$$= \begin{pmatrix} \cos(10 \cdot +5) & 10 \sin(10 \cdot +5) \\ -\frac{\sin(10 \cdot +5)}{10} & \cos(10 \cdot +5) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.94 & -2.42 \\ 0.03 & 0.94 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

 $x_T(5) = \begin{pmatrix} 0.94 \\ 0.03 \end{pmatrix}$

c. $\underline{x}(t) = E(t) E(-t_0) \underline{x}(t_0); \quad \text{if } \underline{x}(10) = \begin{bmatrix} 0.84 \\ 0.05 \end{bmatrix}$

$$= E(t) E(-10) \underline{x}(10)$$

$$= \begin{pmatrix} \cos(10t) & 10 \sin(10t) \\ -\frac{\sin(10t)}{10} & \cos(10t) \end{pmatrix} \begin{pmatrix} \cos(10 \cdot -10) & 10 \sin(10 \cdot -10) \\ -\frac{\sin(10 \cdot -10)}{10} & \cos(10 \cdot -10) \end{pmatrix} \begin{pmatrix} 0.84 \\ 0.05 \end{pmatrix}$$

$$\begin{aligned}
 &= \begin{pmatrix} \cos(10t) & \cos(10t) \\ -\frac{\sin(10t)}{10} & \cos(10t) \end{pmatrix} \begin{pmatrix} 0.84 \cos(-100) + 0.05 \cos(-100) \\ -\frac{0.84 \sin(-100)}{10} + 0.05 \sin(-100) \end{pmatrix} \\
 &= \begin{pmatrix} \cos(10t) & \cos(10t) \\ -\frac{\sin(10t)}{10} & \cos(10t) \end{pmatrix} \begin{pmatrix} 0.9948 \\ -0.0004 \end{pmatrix} \approx \begin{pmatrix} 0.9948 \\ -0.0004 \end{pmatrix} \\
 &= \begin{pmatrix} \cos(10t) & \cos(10t) \\ -\frac{\sin(10t)}{10} & \cos(10t) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 x(t) &= \begin{pmatrix} \cos(10t) \\ -\frac{\sin(10t)}{10} \end{pmatrix}
 \end{aligned}$$

3. Calcular el componente transitorio de la solución de las ecuaciones de estado

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad x(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$R: x_T = \begin{pmatrix} \sqrt{34}e^{-t} \cos(t + 1.0304) \\ \sqrt{68}e^{-t} \cos(t - 1.3258) \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}$$

$$\begin{aligned}
 \det(A - \lambda I) &= \begin{pmatrix} -\lambda & 1 \\ -2 & -2-\lambda \end{pmatrix} = -\lambda(-2-\lambda) - (-2 \cdot 1) \\
 &\quad 2\lambda + \lambda^2 + 2 \\
 &= \lambda^2 + 2\lambda + 2
 \end{aligned}$$

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1}$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{-4}}{2}$$

$$\lambda_1 = -1 + i, \quad \lambda_2 = -1 - i$$

vector propio λ_1

$$(A - \lambda_1 I) = \begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix} \xrightarrow{F_2 \leftarrow F_2 - (-1-i)F_1} \begin{pmatrix} 1-i & 1 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1-i & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{F_1 + \frac{1}{2}(1+i)F_2} \begin{pmatrix} 1 & \frac{1}{2}(1+i) \\ 0 & 0 \end{pmatrix}$$

entonces

$$\begin{pmatrix} 1 & \frac{1}{2}(1+i) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$1 + \frac{1}{2}(1+i)y = 0$$

$$x = -\frac{1}{2}(1+i)y$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(1+i)y \\ y \end{pmatrix} \quad \text{con } y = 2 \quad x^1 = \begin{pmatrix} -1-i \\ 2 \end{pmatrix}$$

vector propio λ_2

$$(A - \lambda_2 I) = \begin{pmatrix} 1+i & 1 \\ -2 & -1+i \end{pmatrix} \xrightarrow{F_2 \leftarrow F_2 - (-1+i)F_1} \begin{pmatrix} 1+i & 1 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1+i & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{F_1 + \frac{1}{2}(1-i)F_2} \begin{pmatrix} 1 & \frac{1}{2}(1-i) \\ 0 & 0 \end{pmatrix}$$

entonces

$$\begin{pmatrix} 1 & \frac{1}{2}(1-i) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$1 + \frac{1}{2}(1-i)y = 0$$

$$x = -\frac{1}{2}(1-i)y$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(1-i)y \\ y \end{pmatrix} \quad \text{con } y = 2 \quad x^2 = \begin{pmatrix} -1+i \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} /$$

Matrix modal

$$M = \begin{pmatrix} -1-i & -1+i \\ 2 & 2 \end{pmatrix}$$

$$M^{-1} = \frac{1}{\det(M)} \begin{pmatrix} 2 & -(-1+i) \\ -2 & -1-i \end{pmatrix}$$

$$= \frac{1}{(-2-2i) - (-2+2i)} \begin{pmatrix} 2 & 1-i \\ -2 & -1-i \end{pmatrix}$$

$$= \frac{1}{-4i} \begin{pmatrix} 2 & 1-i \\ -2 & -1-i \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{-2i} & \frac{1}{-4i}(1-i) \\ \frac{-1}{-2i} & \frac{1}{-4i}(-1-i) \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} \frac{i}{2} & \frac{1}{4}(1+i) \\ -\frac{i}{2} & \frac{1}{4}(1-i) \end{pmatrix}$$

$$\lambda_1 = -1+i, \quad \lambda_2 = -1-i$$

$$e^{\lambda t} = \begin{pmatrix} e^{(-1+i)t} & 0 \\ 0 & e^{(-1-i)t} \end{pmatrix} = \begin{pmatrix} e^{-t} e^{it} & 0 \\ 0 & e^{-t} e^{-it} \end{pmatrix}$$

aplicando enter

$$e^{\lambda t} = \begin{pmatrix} e^{-t}(\cos(t) + i \sin(t)) & 0 \\ 0 & e^{-t}(\cos(t) - i \sin(t)) \end{pmatrix}$$

$$e^{At} = M e^{\lambda t} M^{-1}$$

$$= \begin{pmatrix} -1-i & -1+i \\ 2 & 2 \end{pmatrix} \begin{pmatrix} e^{-t}(\cos(t) + i \sin(t)) & 0 \\ 0 & e^{-t}(\cos(t) - i \sin(t)) \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{4}(1+i) \\ -\frac{i}{2} & \frac{1}{4}(1-i) \end{pmatrix}$$

$$= \begin{pmatrix} (-1-i)e^{-t}(\cos(t) + i \sin(t)) & (-1+i)e^{-t}(\cos(t) - i \sin(t)) \\ 2e^{-t}\cos(t) + 2ie^{-t}\sin(t) & 2e^{-t}\cos(t) - 2ie^{-t}\sin(t) \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{4}(1+i) \\ -\frac{i}{2} & \frac{1}{4}(1-i) \end{pmatrix}$$

=

• fila 1 columna 1

$$\begin{aligned} &= \frac{i(-1-i)e^{-t}(\cos(t) + i \sin(t)) + (-1+i)e^{-t}(\cos(t) - i \sin(t)) - i}{2} \\ &= \frac{i(-1-i)e^{-t}(\cos(t) + i \sin(t)) - i(-1+i)e^{-t}(\cos(t) - i \sin(t))}{2} \\ &= \frac{ie^{-t}((-1-i)(\cos(t) + i \sin(t))i) - (-1+i)(\cos(t) - i \sin(t))}{2} \\ &= \frac{-2i^2e^{-t}(\cos(t) + i \sin(t))}{2} \\ &= e^{-t}(\cos(t) + i \sin(t)) \end{aligned}$$

= $e^{-t}(\cos(t) + i \sin(t))$

• fila 1 columna 2

$$\begin{aligned} &= (-1-i)e^{-t}(\cos(t) + i \sin(t)) \frac{1}{4}(1+i) + (-1+i)e^{-t}(\cos(t) + i \sin(t)) \frac{1}{4}(1-i) \\ &= \frac{e^{-t}(\sin(t) - i \cos(t)) + e^{-t}(i \cos(t) + \sin(t))}{2} \end{aligned}$$

$$= e^{-t} \sin(t)$$

• fila 2 columna 1

$$\begin{aligned} &= (2e^{-t}\cos(t) + 2ie^{-t}\sin(t)) \frac{i}{2} - (2e^{-t}\cos(t) - 2ie^{-t}\sin(t)) \frac{i}{2} \\ &= (ie^{-t}\cos(t) - e^{-t}\sin(t)) - (e^{-t}\sin(t) + ie^{-t}\cos(t)) \\ &= (-e^{-t}\sin(t) - e^{-t}\sin(t)) + (e^{-t}\cos(t) - \cancel{e^{-t}\cos(t)})i \\ &= -2e^{-t}\sin(t) \end{aligned}$$

. fila 2 wwmna 2.

$$\begin{aligned}
 &= (2e^{-t} \cos(t) + 2ie^{-t} \sin(t)) \frac{1}{4}(1+i) + (2e^{-t} \cos(t) - 2ie^{-t} \sin(t)) \frac{1}{4}(1-i) \\
 &= \underline{\frac{e^{-t} \cos(t) + i(e^{-t} \cos(t) + e^{-t} \sin(t)) - e^{-t} \sin(t)}{2}} + \\
 &\quad \underline{\frac{e^{-t} \cos(t) + i(e^{-t} \cos(t) - e^{-t} \sin(t)) - e^{-t} \sin(t)}{2}} \\
 &= \underline{\frac{2e^{-t} \cos(t) - 2e^{-t} \sin(t)}{2}} \\
 &= e^{-t} (\cos(t) - \sin(t))
 \end{aligned}$$

entonces

$$e^{At} = \begin{pmatrix} e^{-t}(\cos(t) + \sin(t)) & e^{-t} \sin(t) \\ -2e^{-t} \sin(t) & e^{-t} (\cos(t) - \sin(t)) \end{pmatrix}$$

resolviendo:

$$x_T(t) = e^{At} \underline{x(0)}$$

$$\begin{aligned}
 &= \begin{pmatrix} e^{-t}(\cos(t) + \sin(t)) & e^{-t} \sin(t) \\ -2e^{-t} \sin(t) & e^{-t} (\cos(t) - \sin(t)) \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\
 &= \begin{pmatrix} 3e^{-t}(\cos(t) + \sin(t)) + 2e^{-t} \sin(t) \\ -4e^{-t} \sin(t) + 2e^{-t} (\cos(t) - \sin(t)) \end{pmatrix}
 \end{aligned}$$

$$x_T = \begin{pmatrix} e^{-t} (3 \cos(t) + 5 \sin(t)) \\ e^{-t} (2 \cos(t) - 8 \sin(t)) \end{pmatrix}$$

suma de 2 sinusoides

$$A \cos(\omega t) + B \sin(\omega t) = \sqrt{A^2 + B^2} \cdot \cos(\omega t + \tan^{-1}(B/A))$$

entonces

$$x_T = \begin{pmatrix} e^{-t} \sqrt{34} \cdot \cos(t + 1.0304) \\ e^{-t} \sqrt{68} \cdot \cos(t - 1.3258) \end{pmatrix}$$