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Hoja de trabajo No. 7

Problema 1

Escribir las matrices A, B, C, D para el sistema dado por la función de transferencia

$$G(p) = 1 + \frac{6 + 4p + p^2}{6 + 11p + 6p^2 + p^3} = \frac{p^3 + 7p^2 + 15p + 12}{p^3 + 6p^2 + 11p + 6}$$

1.1 En forma normal de control. canonica controlable

1.2 En forma A diagonal. normal diagonal

1.3 En forma normal de Jordan.

1. $$G_1(p) = \frac{1}{p^3 + 6p + 11p + 6}$$

$$G_2(p) = p^3 + 7p^2 + 15p + 12$$

$$G(p) = G_1(p)\,G_2(p)$$

- $$G_1(p) = \frac{1}{p^3 + 6p^2 + 11p + 6}$$

$$\frac{C(S)}{U(S)} = \frac{1}{p^3 + 6p^2 + 11p + 6}$$

$$C(S)(p^3 + 6p^2 + 11p + 6) = U(p)$$

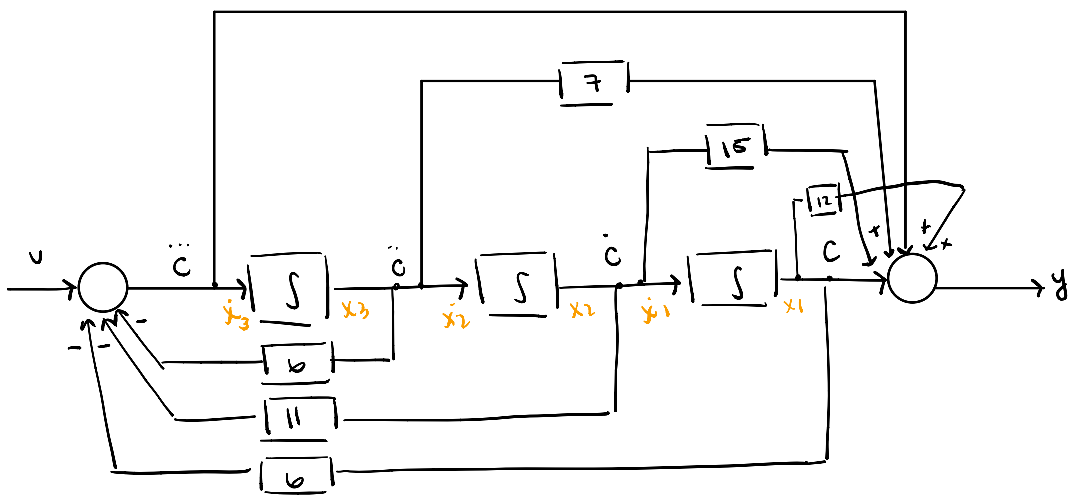
$$\dddot{c} + 6\ddot{c} + 11\dot{c} + 6c = u$$

$$\dddot{c} = u - 6\ddot{c} - 11\dot{c} - 6c$$

- $$\frac{Y(S)}{C(S)} = p^3 + 7p^2 + 15p + 12$$

$$Y(S) = (p^3 + 7p^2 + 15p + 12)\,C(S)$$

$$y = \dddot{c} + 7\ddot{c} + 15\dot{c} + 12c$$



FORMA MATRICIAL →

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = u - 6x_3 - 11x_2 - 6x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = 12x_1 + 15x_2 + 7x_3 + \dot{x}_3$$

$$y = \cancel{12x_1} + \cancel{15x_2} + \cancel{7x_3} + u - \cancel{6x_3} - \cancel{11x_2} - \cancel{6x_1}$$

$$y = x_3 + 4x_2 + 6x_1 + u$$

$$y(t) = \begin{bmatrix} 6 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [1]u$$

2. Forma normal diagonal

$$G(p) = \frac{p^3 + 7p^2 + 15p + 12}{p^3 + 6p^2 + 11p + 6}$$

$$= \frac{(p+4)(p^2+3p+3)}{(p+1)(p+2)(p+3)}$$

$$U(s) = \boxed{\frac{p+4}{p+1}} \rightarrow \boxed{\frac{p^2+3p+3}{p+2}} \rightarrow \boxed{\frac{1}{p+3}} \rightarrow Y(s)$$

$$U \rightarrow \boxed{\frac{p+4}{p+1}} \rightarrow Y$$

$$\frac{Y}{U} = \frac{p+4}{p+1} \Rightarrow (p+1)Y = (p+4)U$$

$$\dot{y} + y = \dot{u} + 4u$$

$$\dot{y} = \dot{u} + 4u - y$$

$$y = u + \int (4u - y)\,dt$$

$$\frac{Y}{U} = \frac{p^2+3p+3}{p+2} \Rightarrow (p+2)Y = (p^2+3p+3)U$$

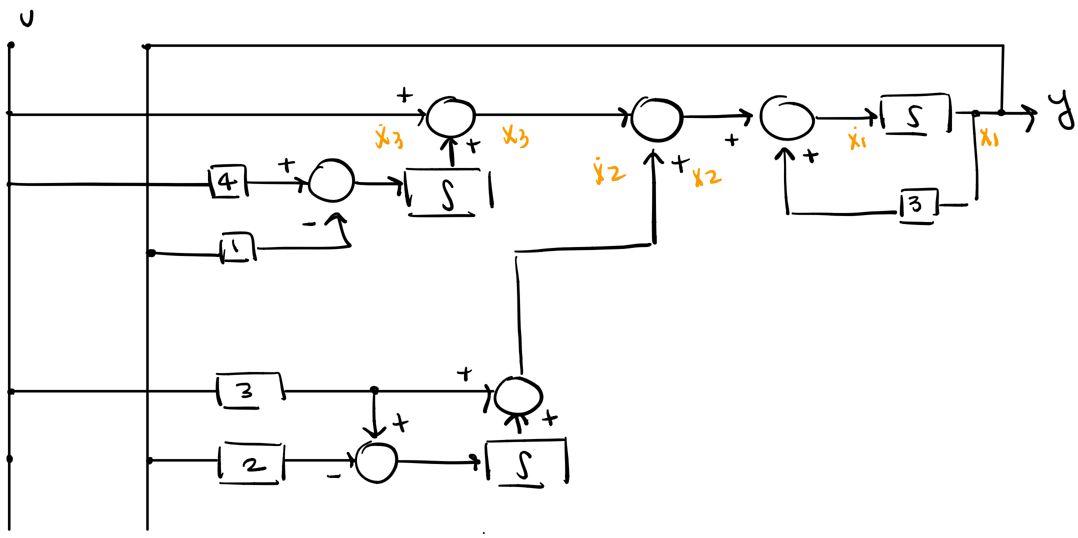
$$\dot{y} + 2y = \ddot{u} + 3\dot{u} + 3u$$

$$\dot{y} = \ddot{u} + 3\dot{u} + 3u - 2y$$

$$y = \dot{u} + 3u + \int (3u - 2y)\,dt$$

$$y = u + \int (4u - y)\,dt$$

$$y = \dot{u} + 3u + \int (3u - 2y)\,dt$$

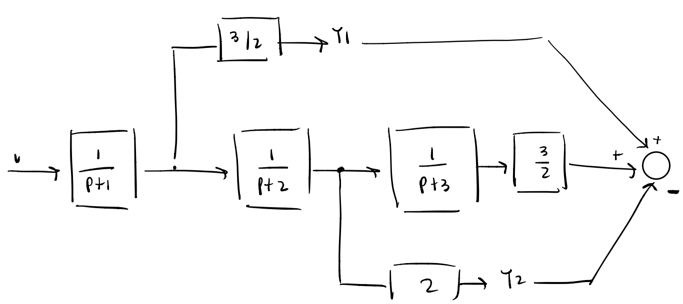


3. Jordan.

$$G(p) = \frac{p^3 + 7p^2 + 15p + 12}{p^3 + 6p^2 + 11p + 6}$$

Fracciones parciales:

$$= 1 + \underbrace{\frac{3/2}{(p+1)}}_{T_1} - \underbrace{\frac{2}{p+2}}_{T_2} + \underbrace{\frac{3/2}{(p+3)}}_{T_3}$$



$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} 3/2 & -2 & 3/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$G(p) = 1 + \frac{6 + 4p + p^2}{6 + 11p + 6p^2 + p^3} \quad =$$

$$\frac{c}{v} = 1 + \frac{6 + 4p + p^2}{6 + 11p + 6p^2 + p^3}$$

$$\frac{(c - 1)}{v} = \frac{6 + 4p + p^2}{6 + 11p + 6p^2 + p^3}$$

Problema 2

Para el sistema $\overset{\dot{x}}{\cancel{\dot{q}}} = \underbrace{\begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}}_{A} \overset{x}{\cancel{q}} + \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_{B} \overset{v}{\cancel{\boldsymbol{y}}},\quad \overset{y}{\cancel{x}} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{C} \boldsymbol{q}$

2.1 Calcular la matriz de transferencia G(p).

2.2 Calcular $\boldsymbol{q}_T(t)$ (estado transitorio).

2.3 Calcular $\boldsymbol{q}_E(t)$ para $\boldsymbol{y} = \begin{pmatrix} t^2 H(t) \\ 0 \end{pmatrix}$.

$$x(t) = \underline{x}_T(t) + \underline{x}_E(t)$$

$$G(s) = C(sI - A)^{-1} B$$

$$x_T(t) = e^{At} x(0)$$

$$x_E(t) = e^{At} \int_0^t e^{-A\tau} B v(\tau)\, d\tau$$

2.1) $G(s) = C(sI - A)^{-1} B$

$$(sI - A) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} s & 0 \\ +1 & s+1 \end{pmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s(s+1)} \begin{pmatrix} s+1 & -0 \\ -1 & s \end{pmatrix}$$

→ reemplazando en la original

$G(s) = C(sI - A)^{-1} B$

$$\dot{\boldsymbol{q}} = \underbrace{\begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}}_{A} \boldsymbol{q} + \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_{B} \boldsymbol{y},\quad \boldsymbol{x} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{C} \boldsymbol{q}$$

$$= \frac{1}{s(s+1)} \begin{pmatrix} s+1 & -0 \\ -1 & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{s(s+1)} \begin{pmatrix} s+1 & 0 \\ -1+s & s \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{s} & 0 \\ \frac{s-1}{s(s+1)} & \frac{1}{s+1} \end{pmatrix}$$

$$\begin{matrix} 0 & 0 \\ -1 & -1 \end{matrix}$$

2.2) $e^{At} = Me^{\lambda t} \cdot M^{-1}$

preparar modal con eigenvectores.

$\lambda = 0 \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \lambda = -1 \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$M = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$

$$e^{\lambda t} = \begin{pmatrix} e^{0t} & 0 \\ 0 & e^{-t} \end{pmatrix}$$

$$e^{At} = Me^{\lambda t} \cdot M^{-1}$$

$$= \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{0t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 1 & 0 \\ e^{-t}-1 & e^{-t} \end{pmatrix}$$

$$X_T(t) = e^{At} x(0)$$

$$X_T(t) = \begin{pmatrix} 1 & 0 \\ e^{-t}-1 & e^{-t} \end{pmatrix} x(0) \quad \square$$

2.3) $x_E(t) = e^{At}\int_0^t e^{-A\tau} B u(\tau)\, d\tau$

$$e^{-At} = \begin{pmatrix} 1 & 0 \\ e^{\tau}-1 & e^{\tau} \end{pmatrix}$$

$$u(\tau) = \begin{pmatrix} \tau^2 \\ 0 \end{pmatrix}$$

$$e^{-At} B u(\tau) = \begin{pmatrix} 1 & 0 \\ e^{\tau}-1 & e^{\tau} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \tau^2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \tau^2 \\ \tau^2(2e^{\tau}-1) \end{pmatrix}$$

$$\int_0^t e^{-At} B u(\tau)\, d\tau = \begin{pmatrix} \frac{t^3}{3} \\ -\frac{t^3}{3} + 2e^{t}(t^2-2t+2) - 4 \end{pmatrix}$$

$$e^{At}\int_0^t e^{-At} B u(\tau)\, d\tau = \begin{pmatrix} 1 & 0 \\ e^{-t}-1 & e^{-t} \end{pmatrix}\begin{pmatrix} \frac{t^3}{3} \\ -\frac{t^3}{3} + 2e^{t}(t^2-2t+2) - 4 \end{pmatrix}$$

$$x_E(t) = \begin{pmatrix} \frac{t^3}{3} \\ \frac{-t^3 + 6t^2 - 12t - 12e^{-t} + 12}{3} \end{pmatrix}$$

Problema 3

Calcular $\boldsymbol{q}(t)$ para el sistema dado por $\dot{\boldsymbol{q}} = \underset{A}{\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}} \boldsymbol{q} + \underset{B}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \boldsymbol{y}$ con

$\boldsymbol{y} = \underset{U}{\begin{pmatrix} 5\cos(2t) \\ 0 \end{pmatrix}}$, $t \geq 0$, y $\boldsymbol{q}(0) = \boldsymbol{0}$.

$$X_T(t) = e^{At} x(0)$$

$$X_E(t) = e^{At} \int_0^t e^{-A\tau} B U(\tau)\, d\tau$$

$$e^{At} = \quad \lambda_1 = -1 \qquad \lambda_2 = -2$$

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e^{\lambda t} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

$$e^{At} = M e^{\lambda t} M^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

$$X_T(t) = e^{At} x(0)$$

$$X_E(t) = e^{At} \int_0^t e^{-A\tau} B U(\tau)\, d\tau$$

$$x(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \cdot 0 + \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \int_0^t \begin{pmatrix} e^{+\tau} & 0 \\ 0 & e^{+2\tau} \end{pmatrix} \begin{pmatrix} 5\cos(2\tau) \\ 0 \end{pmatrix}$$

$$X(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \int_0^t \begin{pmatrix} e^{t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 5\cos(2\tau) \\ 0 \end{pmatrix} d\tau$$

$$= \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \int_0^t \begin{pmatrix} 5 e^{\tau} \cos(2\tau) \\ 0 \end{pmatrix} d\tau$$

$$x(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} e^{t} (2\sin(2t) + \cos(2t)) - 1 \\ 0 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} e^{-t}(e^{t} (2\sin(2t) + \cos(2t)) - 1) \\ 0 \end{pmatrix}$$

Problema 4

Calcular $\boldsymbol{q}(t)$ para el sistema dado por

$$\dot{\boldsymbol{q}} = \begin{pmatrix} -1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \boldsymbol{q} + \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \boldsymbol{y} \text{ para } \boldsymbol{y} = \begin{pmatrix} 2\sin(\omega t) \\ 0 \end{pmatrix}, \; t \geq 0.$$

$$e^{At} = \quad \lambda_1 = -1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \lambda_2 = -2 \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} \qquad \lambda_3 = 3 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 3 & 1 \\ 0 & -5 & 0 \\ 0 & 1 & 2 \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{5} & 0 \\ 0 & \frac{1}{10} & \frac{1}{2} \end{pmatrix}$$

$$e^{\lambda t} = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}$$

$$e^{At} = M \, e^{\lambda t} \, M^{-1}$$

$$= \begin{pmatrix} 1 & 3 & 1 \\ 0 & -5 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{5} & 0 \\ 0 & \frac{1}{10} & \frac{1}{2} \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^{-t} & \dfrac{5e^{-t} - 6e^{-2t} + e^{3t}}{10} & \dfrac{e^{3t} - e^{-t}}{2} \\ 0 & e^{-2t} & 0 \\ 0 & \dfrac{-e^{-2t} + e^{3t}}{5} & e^{3t} \end{pmatrix}$$

$$X_T(t) = e^{At} \, x(0)$$

$$X_T(t) = \begin{pmatrix} e^{-t} & \dfrac{5e^{-t} - 6e^{-2t} + e^{3t}}{10} & \dfrac{e^{3t} - e^{-t}}{2} \\ 0 & e^{-2t} & 0 \\ 0 & -e^{-2t} + e^{3t} & e^{3t} \end{pmatrix} X(0)$$

$$x_E(t) = e^{At} \int_0^t e^{-A\tau} \overline{B u(\tau)} \, d\tau \qquad \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\rightarrow \quad u(t) = \begin{pmatrix} 2 \sin(\omega\tau) \\ 0 \end{pmatrix}$$

$$\rightarrow \quad B u(\tau) = \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \sin(\omega\tau) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \sin(\omega\tau) \\ 2 \sin(\omega\tau) \\ 0 \end{pmatrix}$$

$$\rightarrow \quad e^{-A\tau} B u(\tau) =$$

$$\begin{pmatrix} e^{+t} & \dfrac{5e^{+t} - 6e^{+2t} + e^{-3t}}{10} & \dfrac{e^{-3t} - e^{+t}}{2} \\ 0 & e^{+2t} & 0 \\ 0 & \dfrac{-e^{+2t} + e^{-3t}}{5} & e^{-3t} \end{pmatrix} \begin{pmatrix} 4 \sin(\omega\tau) \\ 2 \sin(\omega\tau) \\ 0 \end{pmatrix}$$

$$e^{-A\tau} B u(\tau) = \begin{pmatrix} \dfrac{25 e^{\tau} \sin(\omega\tau) + e^{-3\tau} \sin(\omega t) - 6 e^{2t} \sin(\omega\tau)}{5} \\ 2e^{2t} \sin(\omega t) \\ \dfrac{2 \sin(\omega t)(e^{-3t} - e^{2t})}{5} \end{pmatrix}$$

$$\rightarrow \quad \int_0^t e^{-A\tau} B u(\tau) \, d\tau = \int_0^t \begin{pmatrix} \dfrac{25 e^{\tau} \sin(\omega\tau) + e^{-3\tau} \sin(\omega t) - 6 e^{2t} \sin(\omega\tau)}{5} \\ 2e^{2t} \sin(\omega t) \\ \dfrac{2 \sin(\omega t)(e^{-3t} - e^{2t})}{5} \end{pmatrix}$$

$$x_T(t) = \begin{pmatrix} e^{-t} & \dfrac{5e^{-t} - 6e^{-2t} + e^{3t}}{10} & \dfrac{e^{3t} - e^{-t}}{2} \\ 0 & e^{-2t} & 0 \\ 0 & \dfrac{-e^{-2t} + e^{3t}}{5} & e^{3t} \end{pmatrix} x(0)$$

$$x_E(\tau) = \begin{pmatrix} e^{-t} & \dfrac{5e^{-t} - 6e^{-2t} + e^{3t}}{10} & \dfrac{e^{3t} - e^{-t}}{2} \\ 0 & e^{-2t} & 0 \\ 0 & \dfrac{-e^{-2t} + e^{3t}}{5} & e^{3t} \end{pmatrix} * \int_0^t \begin{pmatrix} \dfrac{25 e^{\tau} \sin(\omega\tau) + e^{-3\tau} \sin(\omega t) - 6 e^{2t} \sin(\omega\tau)}{5} \\ 2e^{2t} \sin(\omega\tau) \\ \dfrac{2\sin(\omega t)(e^{-3t} - e^{2t})}{5} \end{pmatrix}$$

$\dot{x} = Ax + Bu$ → Así deben estar escritas las ec. de estado.

$x(t) = \underline{x}_T(t) + \underline{x}_E(t)$

$x_T(t) = e^{At}x(0)$

$x_E(t) = e^{At}\int_0^t e^{-A\tau}Bu(\tau)\,d\tau$