Departamento de Matemática Aplicada
Matemática VI – FISICC
Series Trigonométricas

Hoja de Trabajo No. 2- Solución

Instrucciones:

- Resuelva cada una de las cuestiones que se le presentan a continuación dejando constancia de todo procedimiento y razonamiento hecho.
- Favor de entregar su trabajo en hojas debidamente identificadas.
- Entregue su solución a través del GES, en un archivo en formato PDF.

Problema 1
Considere $m, n \in \mathbb{Z}$ demuestre las siguientes:

1. $\displaystyle\int_0^L \sin\left(\frac{n\,\pi\,x}{L}\right)\,\sin\left(\frac{m\,\pi\,x}{L}\right)\,dx = \begin{cases} \frac{L}{2} & m = n \\ 0 & m \neq n \end{cases}.$

   Para $m = n$,
   $$\int_0^L \sin\left(\frac{n\,\pi\,x}{L}\right)\,\sin\left(\frac{n\,\pi\,x}{L}\right)\,dx = \int_0^L \sin^2\left(\frac{n\,\pi\,x}{L}\right)\,dx$$
   Utilizando la siguiente identidad,
   $$\sin^2 A = \frac{1-\cos(2A)}{2}$$
   $$\int_0^L \sin\left(\frac{n\,\pi\,x}{L}\right)^2 dx = \int_0^L \frac{1-\cos(2\frac{n\,\pi\,x}{L})}{2}\,dx = \frac{1}{2}\int_0^L \left[1-\cos\left(2\frac{n\,\pi\,x}{L}\right)\right]\,dx$$
   $$= \frac{1}{2}\left[x - \frac{L}{2n\pi}\sin\left(2\frac{n\pi x}{L}\right)\right]\Bigg|_0^L = \frac{1}{2}\left[L - \frac{L}{2n\pi}\sin(2n\pi)\right]$$
   Como $n \in \mathbb{Z}$ entonces $\sin(2n\pi) = 0$, así
   $$= \frac{1}{2}\left[L - \frac{L}{2n\pi}\cancelto{0}{\sin(2n\pi)}\right] = \frac{L}{2}$$
   Para $m \neq n$, utilizando la identidad trigonométrica:
   $$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$
   Entonces,
   $$\int_0^L \sin\left(\frac{n\,\pi\,x}{L}\right)\,\sin\left(\frac{m\,\pi\,x}{L}\right)\,dx = \frac{1}{2}\int_0^L \left[\cos\left(\frac{n\pi x - m\pi x}{L}\right) - \cos\left(\frac{n\pi x + m\pi x}{L}\right)\right]\,dx$$
   $$= \frac{1}{2}\left[\frac{L}{n\pi - m\pi}\sin\left(\frac{n\pi - m\pi}{L}x\right) - \frac{L}{n\pi + m\pi}\sin\left(\frac{n\pi + m\pi}{L}x\right)\right]\Bigg|_0^L$$
   $$= \frac{1}{2}\left[\frac{L}{n\pi - m\pi}\sin(n\pi - m\pi) - \frac{L}{n\pi + m\pi}\sin(n\pi + m\pi)\right]$$

Como $m, n \in \mathbb{Z}$, entonces $(n-m), (n+m) \in \mathbb{Z}$ y $\sin((n-m)\pi) = \sin((n+m)\pi) = 0$. Así,

$$= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\cancelto{0}{\sin\left((n-m)\pi\right)} - \frac{L}{(n+m)\pi}\cancelto{0}{\sin\left((n+m)\pi\right)}\right] = 0$$

2. $\displaystyle\int_0^L \sin\left(\frac{n\,\pi\,x}{L}\right)\cos\left(\frac{m\,\pi\,x}{L}\right)\,dx = 0$ si m, n son pares.

Utilizando la identidad trigonométrica:

$$\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,dx = \frac{1}{2}\int_0^L \left[\sin\left(\frac{n\pi + m\pi}{L}x\right) + \sin\left(\frac{n\pi - m\pi}{L}x\right)\right]\,dx$$

$$= \frac{1}{2}\left[-\frac{L}{n\pi + m\pi}\cos\left(\frac{n\pi + m\pi}{L}x\right) - \frac{L}{n\pi - m\pi}\cos\left(\frac{n\pi - m\pi}{L}x\right)\right]\Bigg|_0^L$$

$$= \frac{1}{2}\left[-\frac{L}{n\pi + m\pi}\cos\left((n+m)\pi\right) - \frac{L}{n\pi - m\pi}\cos\left((n-m)\pi\right) + \frac{L}{n\pi + m\pi} + \frac{L}{n\pi - m\pi}\right]$$

$$= \frac{1}{2}\left[-\frac{L}{n\pi + m\pi}(-1)^{(n+m)} - \frac{L}{n\pi - m\pi}(-1)^{(n-m)} + \frac{L}{n\pi + m\pi} + \frac{L}{n\pi - m\pi}\right]$$

Si m, n son pares entonces, $n+m$ y $n-m$ son pares también, y así $(-1)^{(n-m)} = (-1)^{(n+m)} = 1$

$$= \frac{1}{2}\left[-\frac{L}{n\pi + m\pi} - \frac{L}{n\pi - m\pi} + \frac{L}{n\pi + m\pi} + \frac{L}{n\pi - m\pi}\right] = 0$$

Problema 2

Esboce una gráfica de $f(x)$ y calcule su serie de Fourier, dado que:

$$f(x) = \begin{cases} 1 & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ -1 & \frac{\pi}{2} \leq x < \frac{3\pi}{2} \end{cases}, \qquad f(x + 2\pi) = f(x).$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\,dx - \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 1\,dx = \frac{1}{2\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] - \frac{1}{2\pi}\left[\frac{\pi}{2} - \frac{3\pi}{2}\right]$$

$$= \frac{\pi}{2\pi} - \frac{\pi}{2\pi} = \frac{1}{2} - \frac{1}{2} = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{n\pi}{\pi}x\right)\,dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(n\pi x\right)\,dx - \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\left(n\pi x\right)\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{n\pi}\sin\left(n\pi x\right)\right]\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{1}{\pi}\left[\frac{1}{n\pi}\sin\left(n\pi x\right)\right]\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \frac{1}{n\pi^2}\left[\sin\left(\frac{n\pi^2}{2}\right) - \sin\left(\frac{-n\pi^2}{2}\right)\right] - \frac{1}{n\pi^2}\left[\sin\left(\frac{3n\pi^2}{2}\right) - \sin\left(\frac{-n\pi^2}{2}\right)\right]$$

Como $\sin(x)$ es una función impar, entonces $\sin\left(\frac{-n\pi^2}{2}\right) = -\sin\left(\frac{n\pi^2}{2}\right)$

$$= \frac{1}{n\pi^2}\left[\sin\left(\frac{n\pi^2}{2}\right) + \sin\left(\frac{n\pi^2}{2}\right)\right] - \frac{1}{n\pi^2}\left[\sin\left(\frac{3n\pi^2}{2}\right) - \sin\left(\frac{n\pi^2}{2}\right)\right]$$

$$= \frac{1}{n\pi^2}\left[3\sin\left(\frac{n\pi^2}{2}\right) - \sin\left(\frac{3n\pi^2}{2}\right)\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{n\pi}{\pi}x\right)\,dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(n\pi x)\,dx - \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin(n\pi x)\,dx$$

$$= \frac{1}{\pi}\left[\frac{-1}{n\pi}\cos(n\pi x)\right]\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{1}{\pi}\left[\frac{-1}{n\pi}\cos(n\pi x)\right]\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \frac{-1}{n\pi^2}\left[\cos\left(\frac{n\pi^2}{2}\right) - \cos\left(\frac{-n\pi^2}{2}\right)\right] - \frac{1}{n\pi^2}\left[-\cos\left(\frac{3n\pi^2}{2}\right) + \cos\left(\frac{-n\pi^2}{2}\right)\right]$$

Como $\cos(x)$ es una función par, entonces $\cos\left(\frac{-n\pi^2}{2}\right) = \cos\left(\frac{n\pi^2}{2}\right)$

$$= \frac{-1}{n\pi^2}\left[\cos\left(\frac{n\pi^2}{2}\right) - \cos\left(\frac{n\pi^2}{2}\right)\right] - \frac{1}{n\pi^2}\left[-\cos\left(\frac{3n\pi^2}{2}\right) + \cos\left(\frac{n\pi^2}{2}\right)\right]$$

$=$

$$\frac{1}{n\pi^2}\left[\cos\left(\frac{3n\pi^2}{2}\right) - \cos\left(\frac{n\pi^2}{2}\right)\right]$$

Así la serie de Fourier de la función está dada por,

$$f(x) = \frac{1}{n\pi^2}\sum_{n=0}^{\infty}\left[3\sin\left(\frac{n\pi^2}{2}\right) - \sin\left(\frac{3n\pi^2}{2}\right)\right]\cos(n\pi x) + \left[\cos\left(\frac{3n\pi^2}{2}\right) - \cos\left(\frac{n\pi^2}{2}\right)\right]\sin(n\pi x)$$

Problema 3
Considere:

$$f(x)=\begin{cases}0 & -\pi<x<0\\ \sin x & 0<x<\pi\end{cases}, \qquad f(x+2\pi)=f(x).$$

Encuentre su serie de Fourier y dibuje la gráfica de la función. Utilice Python para dibujar una aproximación de la función usando 4 términos de la serie.

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\int_0^{\pi}\sin(x)\,dx=\frac{2}{2\pi}=\frac{1}{\pi}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(n\pi x)\,dx=\frac{1}{\pi}\int_0^{\pi}\sin(x)\cos(n\pi x)\,dx=\frac{1}{2\pi}\int_0^{\pi}\left[\sin(n+1)x+\sin(n-1)x\right]\,dx$$

$$=\frac{1}{2\pi}\left[\frac{-1}{(n+1)}\cos(n+1)x-\frac{1}{(n-1)}\cos(n-1)x\right]\Bigg|_0^{\pi}$$

$$=\frac{1}{2\pi}\left[\frac{-1}{(n+1)}\cos(n+1)\pi-\frac{1}{(n-1)}\cos(n-1)\pi+\frac{1}{(n+1)}\cos(n+1)(0)+\frac{1}{(n-1)}\cos(n-1)(0)\right]$$

$$=\frac{1}{2\pi}\left[\frac{-1}{(n+1)}(-1)^{(n+1)}-\frac{1}{(n-1)}(-1)^{n-1}+\frac{1}{(n+1)}+\frac{1}{(n-1)}\right]$$

$$=\frac{1}{2\pi}\left[\frac{1}{(n+1)}(-1)^{n}+\frac{1}{(n-1)}(-1)^{n}+\frac{1}{(n+1)}+\frac{1}{(n-1)}\right]=\frac{1}{2\pi}\left[\frac{1+(-1)^n}{(n+1)}+\frac{1+(-1)^n}{(n-1)}\right]$$

$$=\frac{1}{2\pi}\left[\frac{2n+2(-1)^n n}{(n-1)(n+1)}\right]=\frac{1+(-1)^n}{\pi(1-n^2)}\qquad n=2,3,4,\cdots$$

$$a_1=\frac{1}{2\pi}\int_0^{\pi}\sin(2x)\,dx=\cos(2\pi)-\cos(0)=1-1=0$$

$$b_n=\frac{1}{\pi}\int_0^{\pi}f(x)\sin(nx)\,dx=\frac{1}{\pi}\int_0^{\pi}\sin(x)\sin(nx)\,dx=\frac{1}{2\pi}\int_0^{\pi}\left[\cos(1-n)x-\cos(1+n)x\right]\,dx$$

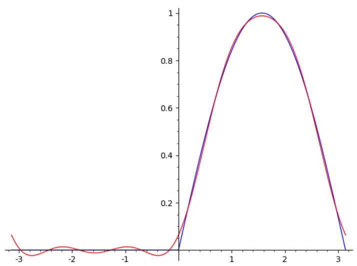
$$=\frac{1}{2\pi}\left[\frac{-1}{(n-1)}\sin(n-1)x+\frac{1}{(n+1)}\sin(n+1)x\right]\Bigg|_0^{\pi}$$

$$=\frac{1}{2\pi}\left[\frac{-\sin(n-1)\pi}{(n-1)}-\frac{\sin(n+1)\pi}{(n+1)}+\frac{\sin(n-1)(0)}{(n-1)}+\frac{\sin(n+1)(0)}{(n+1)}\right]=0\qquad n=2,3,4,\cdots$$

$$b_n=\frac{1}{2\pi}\int_0^{\pi}(1-\cos(2x))\,dx=\frac{1}{2}$$

Así la serie de Fourier de la función está dada por:

$$f(x)=\frac{1}{\pi}\frac{1}{2}\sin(x)+\sum_{n=2}^{\infty}\frac{1+(-1)^n}{\pi(1-n^2)}\cos(nx)$$



Problema 4

Considere $f(x) = x$ para $-1 < x < 1$ y $f(x+2) = f(x)$.

1. Calcule la serie de Fourier de $f(x)$.

$$a_0 = \frac{1}{2}\int_{-1}^{1} x\,dx = 0$$

$$a_n = \int_{-1}^{1} x\cos(n\pi x)\,dx = \frac{1}{n^2\pi^2}\left[n\pi\sin(n\pi x) + \cos(n\pi x)\right]\Big|_{-1}^{1}$$

$$\frac{1}{n^2\pi^2}\left[n\pi x\sin(n\pi) + \cos(n\pi) + n\pi\sin(n\pi) - \cos(n\pi)\right] = \frac{1}{n^2\pi^2}\left[0 + (-1)^n + 0 - (-1)^n\right] = 0$$

$$b_n = \int_{-1}^{1} x\sin(n\pi x)\,dx = \frac{1}{n^2\pi^2}\sin(n\pi x) - \frac{1}{n\pi}x\cos(n\pi x)\Big|_{-1}^{1}$$

$$\frac{1}{n^2\pi^2}\sin(n\pi) - \frac{1}{n\pi}\cos(n\pi) + \frac{1}{n^2\pi^2}\sin(n\pi) + \frac{1}{n\pi}(-1)\cos(n\pi) = -\frac{2\cos(n\pi)}{n\pi} = \frac{2(-1)^{n+1}}{n\pi}$$

Así la serie de Fourier de la función está dada por,

$$f(x) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x)$$

2. Utilice el teorema de Dirichlet para probar que:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

Sabemos que la función es continua en $x = \frac{1}{2}$, entonces

$$\frac{1}{2} = f\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi}{2}\right) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{2}\right)$$

$$\frac{1}{2} = \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$