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## Hoja de trabajo No. 7

### Problema 1

Escribir las matrices A, B, C, D para el sistema dado por la función de transferencia

$$G(p) = \frac{6 + 4p + p^2}{6 + 11p + 6p^2 + p^3} = \frac{p^3 + 7p^2 + 15p + 12}{p^3 + 4p^2 + 11p + 4}$$

1.1 En forma normal de control. canónica controlable

1.2 En forma A diagonal. normal diagonal

1.3 En forma normal de Jordan.

1.  $b_1(p) = \frac{1}{p^3 + 4p^2 + 11p + 4}$

$$b_2(p) = p^3 + 7p^2 + 15p + 12$$

$$b(p) = b_1(p)b_2(p)$$

•  $b_1(p) = \frac{1}{p^3 + 4p^2 + 11p + 4}$

$$\frac{c(s)}{v(s)} = \frac{1}{p^3 + 4p^2 + 11p + 4}$$

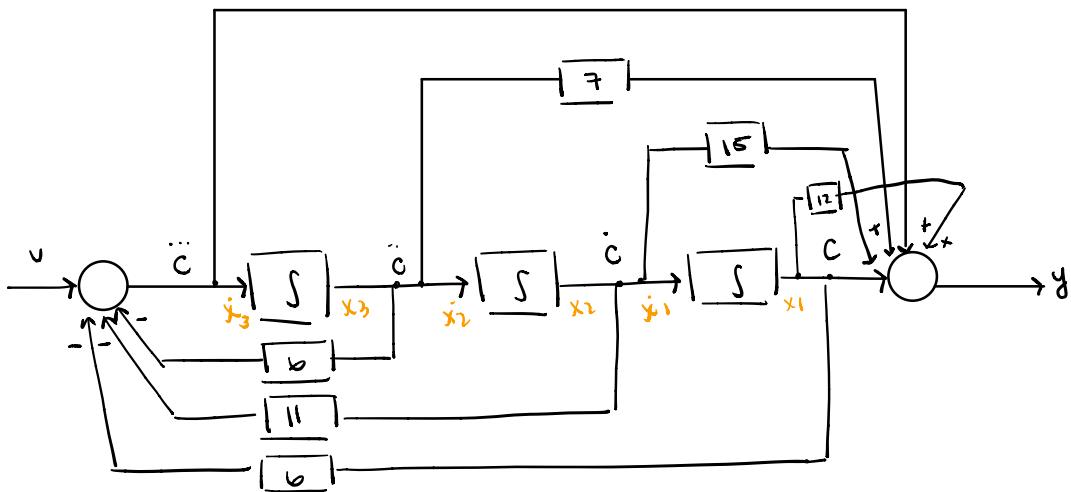
$$c(s)(p^3 + 4p^2 + 11p + 4) = v(s)$$

$$\ddot{c} + 4\dot{c} + 11c + 4v = v$$

$$\ddot{c} = v - 4\dot{c} - 11c - 4v$$

•  $\frac{Y(s)}{c(s)} = p^3 + 7p^2 + 15p + 12$

$$y = \ddot{c} + 7\dot{c} + 15c + 12v$$



FORMA MATEMATICA →

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = u - 6x_3 - 11x_2 - 4x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -11 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = 12x_1 + 15x_2 + 7x_3 + \dot{x}_3$$

~~$$y = 12x_1 + 15x_2 + 7x_3 + u - 6x_3 - 11x_2 - 4x_1$$~~

$$y = x_3 + 4x_2 + 6x_1 + u$$

$$y(t) = \begin{bmatrix} 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [1]u$$

## 2. Forma normal diagonal

$$b(p) = \frac{p^3 + 7p^2 + 15p + 12}{p^3 + 4p^2 + 11p + 4}$$

$$= \frac{(p+4)(p^2 + 3p + 3)}{(p+1)(p+2)(p+3)}$$

$$v(s) = \left| \begin{array}{c} p+4 \\ p+1 \end{array} \right| \rightarrow \left| \begin{array}{c} p^2 + 3p + 3 \\ p+2 \end{array} \right| \rightarrow \left| \begin{array}{c} 1 \\ p+3 \end{array} \right| \rightarrow Y(s)$$

$$v \rightarrow \left| \begin{array}{c} p+4 \\ p+1 \end{array} \right| \rightarrow Y$$

$$\frac{Y}{v} = \frac{p+4}{p+1} \Rightarrow (p+1)Y = (p+4)v$$

$$\dot{y} + y = \dot{v} + 4v$$

$$\dot{y} = \ddot{v} + 3\dot{v} + 3v - y$$

$$y = v + \int (4v - y)dt$$

$$\frac{Y}{v} = \frac{p^2 + 3p + 3}{p+2} \Rightarrow (p+2)Y = (p^2 + 3p + 3)v$$

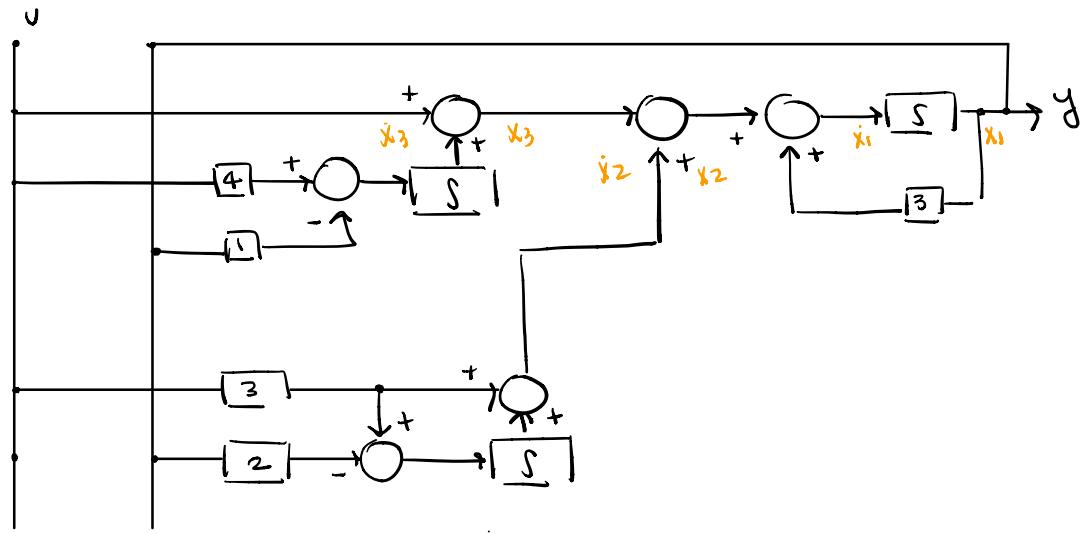
$$\dot{y} + 2y = \ddot{v} + 3\dot{v} + 3v$$

$$\dot{y} = \ddot{v} + 3\dot{v} + 3v - 2y$$

$$y = v + 3v + \int (3v - 2y)dt$$

$$y = v + \int (4v - y)dt$$

$$y = v + 3v + \int (3v - 2y)dt$$



3. Jordan.

$$b(p) = \frac{p^3 + 7p^2 + 15p + 12}{p^3 + 4p^2 + 11p + 6}$$

Fracciones parciales:

$$= 1 + \underbrace{\frac{3/2}{(p+1)}}_{T_1} - \underbrace{\frac{2}{p+2}}_{T_2} + \underbrace{\frac{3/2}{(p+3)}}_{T_3}$$

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} 3/2 & -2 & 3/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$G(p) = 1 + \frac{6 + 4p + p^2}{6 + 11p + 6p^2 + p^3} =$$

$$\frac{c}{v} = 1 + \frac{6 + 4p + p^2}{6 + 11p + 6p^2 + p^3}$$

$$\frac{(c-1)}{v} = \frac{6 + 4p + p^2}{6 + 11p + 6p^2 + p^3}$$

### Problema 2

Para el sistema  $\dot{\underline{q}} = (\underbrace{\begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}}_A) \underline{q} + (\underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_B) \underline{y}, \quad \underline{y} = (\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_C) q$

2.1 Calcular la matriz de transferencia  $G(p)$ .

2.2 Calcular  $\underline{q}_T(t)$  (estado transitorio).

2.3 Calcular  $\underline{q}_E(t)$  para  $y = \begin{pmatrix} t^2 H(t) \\ 0 \end{pmatrix}$ .

$$\underline{x}(t) = \underline{x}_T(t) + \underline{x}_E(t)$$

$$v(s) = C(sI - A)^{-1} B$$

$$\underline{x}_T(t) = e^{At} \underline{x}(0)$$

$$\underline{x}_E(t) = e^{At} \int_0^t e^{-A\tau} B v(\tau) d\tau$$

$$2.1) \quad v(s) = C(sI - A)^{-1} B$$

$$(sI - A) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} s & 0 \\ -1 & s+1 \end{pmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s(s+1)} \begin{pmatrix} s+1 & 0 \\ -1 & s \end{pmatrix}$$

⇒ reemplazando en la original

$$v(s) = C(sI - A)^{-1} B \quad \dot{q} = (\underbrace{\begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}}_A) q + (\underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_B) y, \quad x = (\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_C) q$$

$$\begin{aligned}
 &= \frac{1}{s(s+1)} \begin{pmatrix} s+1 & 0 \\ -1 & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\
 &= \frac{1}{s(s+1)} \begin{pmatrix} s+1 & 0 \\ -1+s & s \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{s} & 0 \\ \frac{s-1}{s(s+1)} & \frac{1}{s+1} \end{pmatrix}
 \end{aligned}$$

$$2.2) e^{At} = M e^{xt} \cdot M^{-1}$$

Preparar medad van eigenwaarden.

$$\lambda = 0 \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \lambda = -1 \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$M = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \quad M^{-1} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$e^{xt} = \begin{pmatrix} e^{0t} & 0 \\ 0 & e^{-t} \end{pmatrix}$$

$$\begin{aligned}
 e^{At} &= M e^{xt} \cdot M^{-1} \\
 &= \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{0t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}
 \end{aligned}$$

$$e^{At} = \begin{pmatrix} 1 & 0 \\ e^{-t}-1 & e^{-t} \end{pmatrix}$$

$$x_T(t) = e^{At} x(0)$$

$$x_T(t) = \begin{pmatrix} 1 & 0 \\ e^{-t}-1 & e^{-t} \end{pmatrix} x(0) \quad \square$$

$$2.3) \quad x_E(t) = e^{At} \int_0^t e^{-A\tau} B_U(\tau) d\tau$$

$$e^{-At} = \begin{pmatrix} 1 & 0 \\ t-1 & e^{-t} \end{pmatrix}$$

$$U(\tau) = \begin{pmatrix} \tau^2 \\ 0 \end{pmatrix}$$

$$e^{-At} B_U(\tau) = \begin{pmatrix} 1 & 0 \\ t-1 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \tau^2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \tau^2 \\ \tau^2(2e^{-t}-1) \end{pmatrix}$$

$$\int_0^t e^{-At} B_U(\tau) d\tau = \begin{pmatrix} \frac{t^3}{3} \\ -\frac{t^3}{3} + 2e^{-t}(t^2 - 2t + 2) - 4 \end{pmatrix}$$

$$e^{At} \int_0^t e^{-At} B_U(\tau) d\tau = \begin{pmatrix} 1 & 0 \\ -t-1 & e^{-t} \end{pmatrix} \begin{pmatrix} \frac{t^3}{3} \\ -\frac{t^3}{3} + 2e^{-t}(t^2 - 2t + 2) - 4 \end{pmatrix}$$

$$x_E(t) = \begin{pmatrix} \frac{t^3}{3} \\ \frac{-t^3 + 6t^2 - 12t - 12e^{-t} + 12}{3} \end{pmatrix}$$

### Problema 3

Calcular  $\mathbf{q}(t)$  para el sistema dado por  $\dot{\mathbf{q}} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{q} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{y}$  con

$$\mathbf{y} = \begin{pmatrix} 5\cos(2t) \\ 0 \end{pmatrix}, t \geq 0, \text{ y } \mathbf{q}(0) = \mathbf{0}.$$

$$\mathbf{x}_T(t) = e^{At} \mathbf{x}(0)$$

$$\mathbf{x}_E(t) = e^{At} \int_0^t e^{-A\tau} B_U(\tau) d\tau$$

$$e^{At} = \gamma_1 = -1 \quad \gamma_2 = -2$$

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad M^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e^{\gamma_1 t} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

$$e^{At} = M e^{\gamma_1 t} M^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

$$\mathbf{x}_T(t) = e^{At} \mathbf{x}(0)$$

$$\mathbf{x}_E(t) = e^{At} \int_0^t e^{-A\tau} B_U(\tau) d\tau$$

$$\mathbf{x}(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \cdot \mathbf{0} + \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \int_0^t \begin{pmatrix} e^{+\tau} & 0 \\ 0 & e^{+2\tau} \end{pmatrix} \begin{pmatrix} 5\cos(2\tau) \\ 0 \end{pmatrix} d\tau$$

$$\mathbf{x}(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \int_0^t \begin{pmatrix} e^{\tau} & 0 \\ 0 & e^{2\tau} \end{pmatrix} \begin{pmatrix} 5\cos(2\tau) \\ 0 \end{pmatrix} d\tau$$

$$= \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \int_0^t \begin{pmatrix} 5e^{\tau} \cos(2\tau) \\ 0 \end{pmatrix} d\tau$$

$$x(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} e^t (2\sin(2t) + \cos(2t)) - 1 \\ 0 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} e^{-t}(e^t (2\sin(2t) + \cos(2t)) - 1) \\ 0 \end{pmatrix}$$

Problema 4

Calcular  $\mathbf{q}(t)$  para el sistema dado por

$$\dot{\mathbf{q}} = \begin{pmatrix} -1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \mathbf{q} + \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \mathbf{y} \text{ para } \mathbf{y} = \begin{pmatrix} 2\sin(\omega t) \\ 0 \\ 0 \end{pmatrix}, t \geq 0.$$

$$e^{At} = \gamma_1 = -1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \gamma_2 = -2 \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} \quad \gamma_3 = 3 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & -5 & 0 \\ 0 & 1 & 2 \end{pmatrix} \quad \mathbf{M}^{-1} = \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{5} & 0 \\ 0 & \frac{1}{10} & \frac{1}{2} \end{pmatrix}$$

$$e^{\gamma t} = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}$$

$$e^{At} = \mathbf{M} e^{\lambda t} \mathbf{M}^{-1}$$

$$= \begin{pmatrix} 1 & 3 & 1 \\ 0 & -5 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{5} & 0 \\ 0 & \frac{1}{10} & \frac{1}{2} \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^{-t} & \frac{5e^{-t} - 4e^{-2t} + e^{3t}}{10} & \frac{e^{3t} - e^{-t}}{2} \\ 0 & e^{-2t} & 0 \\ 0 & \frac{-e^{-2t} + e^{3t}}{5} & e^{3t} \end{pmatrix}$$

$$\mathbf{x}_+(t) = e^{At} \mathbf{x}(0)$$

$$\mathbf{x}_+(t) = \begin{pmatrix} e^{-t} & \frac{5e^{-t} - 4e^{-2t} + e^{3t}}{10} & \frac{e^{3t} - e^{-t}}{2} \\ 0 & e^{-2t} & 0 \\ 0 & \frac{-e^{-2t} + e^{3t}}{5} & e^{3t} \end{pmatrix} \times 10$$

$$x_E(t) = e^{At} \int_0^t e^{-A\tau} b_U(\tau) d\tau$$

$$\Rightarrow v(t) = \begin{pmatrix} 2 \sin(w\tau) \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow b_U(\tau) = \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \sin(w\tau) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \sin(w\tau) \\ 2 \sin(w\tau) \\ 0 \end{pmatrix}$$

$$\Rightarrow e^{-At} b_U(\tau) =$$

$$\begin{pmatrix} e^{+t} & \frac{5e^{+t} - 4e^{+2t} + e^{+3t}}{10} & \frac{e^{+3t} - e^{+t}}{2} \\ 0 & e^{+2t} & 0 \\ 0 & -e^{+2t} + e^{+3t} & e^{+3t} \end{pmatrix} \begin{pmatrix} 4 \sin(w\tau) \\ 2 \sin(w\tau) \\ 0 \end{pmatrix}$$

$$e^{-At} b_U(\tau) = \begin{pmatrix} \frac{25e^\tau \sin(w\tau) + e^{-3\tau} \sin(w\tau) - 4e^{+2\tau} \sin(w\tau)}{5} \\ 2e^{+2\tau} \sin(w\tau) \\ \frac{2 \sin(w\tau) (e^{-3\tau} - e^{+2\tau})}{5} \end{pmatrix}$$

$$\Rightarrow \int_0^t e^{-A\tau} b_U(\tau) d\tau = \int_0^t \begin{pmatrix} \frac{25e^\tau \sin(w\tau) + e^{-3\tau} \sin(w\tau) - 4e^{+2\tau} \sin(w\tau)}{5} \\ 2e^{+2\tau} \sin(w\tau) \\ \frac{2 \sin(w\tau) (e^{-3\tau} - e^{+2\tau})}{5} \end{pmatrix}$$

$$x_+(t) = \begin{pmatrix} e^{-t} & \frac{5e^{-t} - 4e^{-2t} + e^{3t}}{10} & \frac{e^{3t} - e^{-t}}{2} \\ 0 & e^{-2t} & 0 \\ 0 & \frac{-e^{-2t} + e^{3t}}{5} & e^{3t} \end{pmatrix} \times 10$$

$$x_E(t) = \begin{pmatrix} e^{-t} & \frac{5e^{-t} - 4e^{-2t} + e^{3t}}{10} & \frac{e^{3t} - e^{-t}}{2} \\ 0 & e^{-2t} & 0 \\ 0 & \frac{-e^{-2t} + e^{3t}}{5} & e^{3t} \end{pmatrix} *$$

$$\int_0^t \begin{pmatrix} \frac{25e^\tau \sin(\omega\tau) + e^{-3\tau} \sin(\omega\tau) - 4e^{2\tau} \sin(\omega\tau)}{5} \\ 2e^{2\tau} \sin(\omega\tau) \\ \frac{2 \sin(\omega\tau)(e^{-3\tau} - e^{2\tau})}{5} \end{pmatrix}$$

$\dot{x} = Ax + Bu \rightarrow$  Así deben estar escritas las ec. de estado.

$$x(t) = x_T(t) + x_E(t)$$

$$x_T(t) = e^{At} x(0)$$

$$x_E(t) = e^{At} \int_0^t e^{-A\tau} Bu(\tau) d\tau$$