Hoja 2

sábado, 1 de agosto de 2020 07:14 p. m.

Problema 1

Considere $m, n \in \mathbb{Z}$ demuestre las siguientes:

1. $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} \frac{L}{2} & m = n \\ 0 & m \neq n \end{cases}.$

$sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)].$

identidad

si $m \neq n$:

$$\frac{1}{2}\int_0^L \cos\left(\frac{n\pi x - m\pi x}{L}\right) - \cos\left(\frac{n\pi x + m\pi x}{L}\right) dx$$

$$\frac{1}{2}\int_0^L \cos\left(x\left(\frac{n\pi - m\pi}{L}\right)\right) - \cos\left(x\left(\frac{n\pi + m\pi}{L}\right)\right) dx$$

$$\frac{1}{2}\left(\frac{L}{n\pi - m\pi}\operatorname{sen}\left(x\left(\frac{n\pi - m\pi}{L}\right)\right) - \frac{L}{n\pi + m\pi}\operatorname{sen}\left(x\left(\frac{n\pi + m\pi}{L}\right)\right)\Bigg|_0^L\right)$$

$$\frac{1}{2}\left(\frac{L}{n\pi - m\pi}\operatorname{sen}(\pi(n-m)) - \frac{L}{n\pi + m\pi}\operatorname{sen}(\pi(n+m))\right)$$

$$\frac{L}{2}\left(\frac{\cancel{\operatorname{sen}(\pi(n-m))}^{\,0}}{n\pi - m\pi} - \frac{\cancel{\operatorname{sen}(\pi(n+m))}^{\,0}}{n\pi + m\pi}\right) \quad // \quad \operatorname{sen}(\pi \cdot K) = 0$$

$$= 0$$

si $m = n$

$$\frac{1}{2}\int_0^L \cos\left(\cancel{\frac{m\pi x - m\pi x}{L}}^{\,1}\right) - \cos\left(\frac{m\pi x + m\pi x}{L}\right) dx$$

$$= \frac{1}{2}\int_0^L 1 - \cos\left(x\left(\frac{2m\pi}{L}\right)\right) dx$$

$$= \frac{1}{2}\left(x - \frac{L}{2m\pi}\operatorname{sen}\left(x\left(\frac{2m\pi}{L}\right)\right)\Bigg|_0^L\right)$$

$$= \frac{1}{2}\left(L - \frac{L}{2m\pi}\operatorname{sen}\cancel{(2m\pi)}^{\,0}\right)$$

$$= \frac{L}{2}$$

2. $\displaystyle\int_0^L \sin\left(\frac{n\,\pi\,x}{L}\right)\cos\left(\frac{m\,\pi\,x}{L}\right)dx = 0$ si m, n son pares.

Identidad: $\cos\alpha\,\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$.

$$\frac{1}{2}\int_0^L \sin\left(\frac{n\pi x + m\pi x}{L}\right) + \sin\left(\frac{n\pi x - m\pi x}{L}\right) dx$$

$$= \frac{1}{2}\int_0^L \sin\left(\frac{n\pi x + m\pi x}{L}\right) + \sin\left(\frac{n\pi x - m\pi x}{L}\right) dx$$

$$= \frac{1}{2}\int_0^L \sin\left(x\left(\frac{n\pi + m\pi}{L}\right)\right) + \sin\left(x\left(\frac{n\pi - m\pi}{L}\right)\right) dx$$

$$= \frac{1}{2}\left(\frac{-L}{n\pi + m\pi}\cos\left(x\left(\frac{n\pi + m\pi}{L}\right)\right) - \frac{L}{n\pi - m\pi}\cos\left(x\left(\frac{n\pi - m\pi}{L}\right)\right)\Bigg|_0^L\right)$$

$$= \frac{1}{2}\left(\frac{-L}{n\pi + m\pi}\cos(n\pi + m\pi) - \frac{L}{n\pi - m\pi}\cos(n\pi - m\pi) + \frac{L}{n\pi + m\pi} + \frac{L}{n\pi - m\pi}\right)$$

Para que sea par

$$= \frac{1}{2}\left(\frac{-L}{2\pi(n+m)}\cancelto{1}{(-1)^{2(m+n)}} - \frac{L}{2\pi(n-m)}\cancelto{1}{(-1)^{2(m-n)}} + \frac{L}{2\pi(n+m)} + \frac{L}{2\pi(n-m)}\right)$$

$$= \frac{1}{2}(0)$$

$$= 0$$

Problema 2

Esboce una gráfica de $f(x)$ y calcule su serie de Fourier, dado que:

$$f(x) = \begin{cases} 1 & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ -1 & \frac{\pi}{2} \le x < \frac{3\pi}{2} \end{cases}, \qquad f(x + 2\pi) = f(x).$$

$$a_0 = \frac{1}{2\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\, dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 1\, dx\right) = \frac{1}{2\pi}(\pi - \pi)$$

$$a_0 = 0$$

$$a_n = \frac{1}{\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(nx)\, dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos(nx)\, dx\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{n}\sin(nx)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{1}{n}\sin(nx)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right)$$

$$\frac{1}{\pi}\left(\frac{\sin(\frac{\pi}{2}n)}{n} + \frac{\sin(\frac{\pi}{2}n)}{n} - \left(\frac{\sin(\frac{3\pi}{2}n)}{n} - \frac{\sin(\frac{\pi}{2}n)}{n}\right)\right)$$

$$\frac{1}{\pi}\left(\frac{2\sin(\frac{\pi}{2}n)}{n} - \frac{\sin(\frac{3\pi}{2}n)}{n} + \frac{\sin(\frac{\pi}{2}n)}{n}\right)$$

$$= \frac{1}{\pi}\left(\frac{3\sin(\frac{\pi}{2}n)}{n} - \frac{\sin(\frac{3\pi}{2}n)}{n}\right)$$

$$b_n = \frac{1}{\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(nx)\, dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin(nx)\, dx\right)$$

$$\int_{-\frac{\pi}{2}} \qquad \int_{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left(-\frac{1}{n}\cos(nx)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{n}\cos(nx)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \right) \quad // \cos(t) = \cos(-t)$$

$$= \frac{1}{\pi}\left(\overbrace{\cancel{\frac{-\cos(\frac{\pi}{2}n)}{n} + \frac{\cos(-\frac{\pi}{2}n)}{n}}}^{0} \right) + \frac{1}{n}\cos\left(\frac{3\pi}{2}n\right) - \frac{1}{n}\cos\left(\frac{\pi}{2}n\right)$$

$$= \frac{1}{\pi}\left(\frac{\cos(\frac{3\pi}{2}n)}{n} - \frac{\cos(\frac{\pi}{2}n)}{n} \right) \quad //$$

$$\hat{f}(x) = \sum_{n=1}^{\infty}\left(\frac{3\sin(\frac{\pi}{2}n) - \sin(\frac{3\pi}{2}n)}{\pi n}\cdot\cos(nx) + \frac{\cos(\frac{3\pi}{2}n) - \cos(\frac{\pi}{2}n)}{n\pi}\cdot\sin(nx) \right) \quad //$$

Problema 3
Considere:

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}, \qquad f(x+2\pi) = f(x).$$

Encuentre su serie de Fourier y dibuje la grafica de la función. Utilice Python para dibujar una aproximación de la función usando 4 términos de la serie.

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} \sin(x)\,dx = \frac{1}{2\pi}\left(-\cos(x)\Big|_0^{\pi} \right)$$

$$= \frac{1}{2\pi}(2) = \frac{1}{\pi} \quad //$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin(x+nx) + \sin(x-nx)\;dx$$

$$= \frac{1}{2\pi}\left(\frac{-1}{1+n}\cos(x(1+n)) - \frac{1}{1-n}\cos(x(1-n))\Big|_0^{\pi} \right)$$

$$= \frac{1}{2\pi}\left(\frac{-1}{1+n}(-1)^{1+n} - \frac{1}{1-n}(-1)^{1-n} - \left(\frac{-1}{1+n} - \frac{1}{1-n} \right) \right)$$

$$= \frac{1}{2\pi}\left\{ \frac{1}{1+n}\left((-1)^{2+n}+1\right) + \frac{1}{1-n}\left((-1)^{2-n}+1\right) \right\}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin(x)\,\sin(nx)\,dx$$

$$\frac{1}{2\pi}\int_0^{\pi} \cos(x(1+n)) - \cos(x(1-n))\,dx$$

$$\frac{1}{2\pi}\left(\cancel{\frac{\sin(x(1+n))}{1+n} - \frac{\sin(x(1-n))}{1-n}\Big|_0^{\pi}}\right) \quad 0$$

$$= 0$$

$$\hat{f}(x) = \frac{1}{\pi} + \sum_{n=1}^{\infty} \frac{1}{2\pi}\left(\frac{1}{1+n}\left((-1)^{2+n}+1\right) + \frac{1}{1-n}\left((-1)^{2-n}+1\right)\right)\cos(nx)$$

```
In [1]: f = piecewise([((-pi,0),0),((0,pi),sin(x))])
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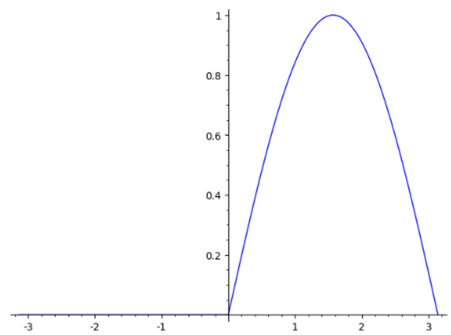
```
In [2]: plot(f,(-pi,pi))
```

Out[2]:




```
In [8]: s4 = f.fourier_series_partial_sum(4)
Out[8]: -2/15*cos(4*x)/pi - 2/3*cos(2*x)/pi + 1/pi + 1/2*sin(x)
```

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In [7]: plot(s4, (x,-pi,pi))
```
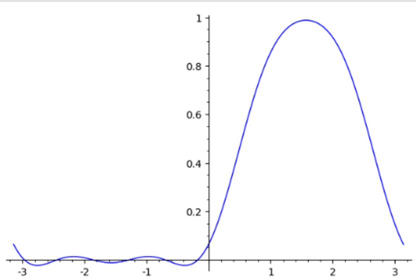

Out[7]:



Problema 4
Considere $f(x) = x$ para $-1 < x < 1$ y $f(x+2) = f(x)$.

1. Calcule la serie de Fourier de $f(x)$.
2. Utilice el teorema de Dirichlet para probar que:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

$L = 1$

$$a_0 = \frac{1}{1}\int_{-1}^{1} x\,dx = \frac{1}{4}x^2\Big|_{-1}^{1} = \frac{1}{2}$$

$$a_0 = \frac{1}{2}\int_{-1}^{1} x\,dx = \frac{1}{4}x^2\Big|_{-1}^{1} = \frac{1}{2}$$

$$a_n = \int_{-1}^{1} x\cos(n\pi x)\,dx \quad // \quad \begin{array}{l} f(-x) = -x\cos(\pi n x) \\ -f(x) = -x\cos(n\pi x) \end{array}$$

función impar

$$a_n = 0$$

$$b_n = \int_{-1}^{1} x\sin(n\pi x)\,dx$$

$$\begin{array}{ll} u = x & dv = \sin(n\pi x)\,dx \\ du = dx & v = -\frac{1}{n\pi}\cos(n\pi x) \end{array}$$

$$= \frac{-x}{n\pi}\cos(n\pi x) + \frac{1}{n\pi}\int_{-1}^{1}\cos(n\pi x)\,dx$$

$$= \frac{-1}{n\pi}\cos(n\pi) + \cancel{\frac{1}{(n\pi)^2}\sin(n\pi)}^{\,0} - \left(\frac{\cos(n\pi)}{n\pi} - \cancel{\frac{1}{(n\pi)^2}\sin(n\pi)}^{\,0}\right)$$

$$= -\frac{2}{n\pi}\cos(n\pi) = \frac{2}{n\pi}(-1)^{n+1}$$

$$\hat{f}(x) = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{2}{n\pi}(-1)^{n+1}\sin(n\pi x)$$

2) $\hat{f}(1) = \frac{1}{2}(-1+1) = 0$

0

$$\hat{f}(1) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{n\pi} \cancel{(-1)^{n+1} \sin(n\pi)}$$

$$= \underline{\underline{\frac{1}{2}}}$$