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Hoja de trabajo No. 9

1. $\dot{x} = \begin{pmatrix} 0 & 100 \\ -1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

   a) Calcular la matriz de transición de estado.

   b) Calcular $x_T(-5)$, $x_T(0)$, y $x_T(5)$.

   c) Calcular $x_T(t)$ si $x(10) = \begin{pmatrix} 0.86 \\ 0.05 \end{pmatrix}$

9.

$$A = \begin{pmatrix} 0 & 100 \\ -1 & 0 \end{pmatrix}$$

valores propios

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 100 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 100$$

$$\lambda^2 + 100 = 0$$

$$\lambda^2 = -100$$

$$\lambda_1 = 10i \quad , \quad \lambda_2 = -10i$$

$$\boxed{\begin{aligned} e^{it} &= \cos(t) + i\,\text{sen}(t) \\ e^{-it} &= \cos(t) - i\,\text{sen}(t) \end{aligned}}$$

ecuaciones para alfa

$$e^{10it} = \alpha_0 + 10i\alpha_1$$

$$e^{-10it} = \alpha_0 - 10i\alpha_1$$

→ euler

$$\cos(10t) + i\,\text{sen}(10t) = \alpha_0 + 10i\alpha_1 \quad (1)$$

$$\cos(10t) - i\,\text{sen}(10t) = \alpha_0 - 10i\alpha_1 \quad (2)$$

despejo α_0:

$\alpha_0 = \cos(10t) + i\,\mathrm{sen}(10t) - 10i\,\alpha_1$ ③

Sustituir 3 en 2:

- $\cos(10t) - i\,\mathrm{sen}(10t) = \cos(10t) + i\,\mathrm{sen}(10t) - 10i\,\alpha_1 - 10i\,\alpha_1$

$$\cancel{\cos(10t)} - i\,\mathrm{sen}(10t) - \cancel{\cos(10t)} - i\,\mathrm{sen}(10t) = -20i\,\alpha_1$$

$$-2i\,\mathrm{sen}(10t) = -20i\,\alpha_1$$

$$\frac{1}{10}\,\mathrm{sen}(10t) = \alpha_1 \quad ④$$

Sustituir 4 en 3:

- $\alpha_0 = \cos(10t) + i\,\mathrm{sen}(10t) - 10i\,\alpha_1$

$$= \cos(10t) + i\,\mathrm{sen}(10t) - \cancel{10}\,i\left(\frac{1}{\cancel{10}}\,\mathrm{sen}(10t)\right)$$

$$\alpha_0 = \cos(10t) + \cancel{i\,\mathrm{sen}(10t)} - \cancel{i\,\mathrm{sen}(10t)}$$

$$\alpha_0 = \cos(10t)$$

$$e^{At} = \alpha_0 I + \alpha_1 A$$

$$= \cos(10t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{10}\,\mathrm{sen}(10t)\begin{pmatrix} 0 & 100 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(10t) & 0 \\ 0 & \cos(10t) \end{pmatrix} + \begin{pmatrix} 0 & 10\sin(10t) \\ -\frac{1}{10}\,\mathrm{sen}(10t) & 0 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} \cos(10t) & 10\sin(10t) \\ -\frac{\mathrm{sen}(10t)}{10} & \cos(10t) \end{pmatrix}$$

b) Calcular $\boldsymbol{x}_T(-5)$, $\boldsymbol{x}_T(0)$, y $\boldsymbol{x}_T(5)$. $\quad \boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

c) Calcular $\boldsymbol{x}_T(t)$ si $\boldsymbol{x}(10) = \begin{pmatrix} 0.86 \\ 0.05 \end{pmatrix}$

b.

$$x_T(-5) = \Phi(-t)\,\underline{x}(0)$$

$$x_T(-5) = \begin{pmatrix} \cos(10\cdot -5) & 10\sin(10\cdot -5) \\ -\frac{\operatorname{sen}(10\cdot -5)}{10} & \cos(10\cdot -5) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.96 & 2.62 \\ -0.03 & 0.96 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$x_T(-5) = \begin{pmatrix} 0.96 \\ -0.03 \end{pmatrix}$$

$$x_T(0) = \Phi(-t)\, \underline{x}(0)$$

$$= \begin{pmatrix} \cos(0) & 10\sin(0) \\ -\frac{\operatorname{sen}(0)}{10} & \cos(0) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$x_T(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad x_T(0) = \underline{x}(0)$$

$$x_T(5) = \Phi(t)\, \underline{x}(0)$$

$$= \begin{pmatrix} \cos(10\cdot +5) & 10\sin(10\cdot +5) \\ -\frac{\operatorname{sen}(10\cdot +5)}{10} & \cos(10\cdot +5) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.96 & -2.62 \\ 0.03 & 0.96 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$x_T(5) = \begin{pmatrix} 0.96 \\ 0.03 \end{pmatrix}$$

c. $x(t) = \Phi(t)\,\Phi(-t_0)\,\underline{x}(t_0)$; si $x(10) = \begin{bmatrix} 0.86 \\ 0.05 \end{bmatrix}$

$$= \Phi(t)\,\Phi(-10)\,\underline{x}(10)$$

$$= \begin{pmatrix} \cos(10t) & 10\sin(10t) \\ -\frac{\operatorname{sen}(10t)}{10} & \cos(10t) \end{pmatrix} \begin{pmatrix} \cos(10\cdot -10) & 10\sin(10\cdot -10) \\ -\frac{\operatorname{sen}(10\cdot -10)}{10} & \cos(10\cdot -10) \end{pmatrix} \begin{pmatrix} 0.86 \\ 0.05 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(10t) & 10\sin(10t) \\ -\frac{\operatorname{sen}(10t)}{10} & \cos(10t) \end{pmatrix} \begin{pmatrix} 0.86\cos(-100) + 0.05(10\sin(-100)) \\ -\frac{0.86\operatorname{sen}(-100)}{10} + 0.05(\cos(-100)) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(10t) & 10\sin(10t) \\ -\frac{\operatorname{sen}(10t)}{10} & \cos(10t) \end{pmatrix} \begin{pmatrix} 0.9948 \\ -0.0004 \end{pmatrix} // \begin{pmatrix} 0.9948 \\ -0.0004 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(10t) & 10\sin(10t) \\ -\frac{\operatorname{sen}(10t)}{10} & \cos(10t) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} \cos(10t) \\ -\frac{\operatorname{sen}(10t)}{10} \end{pmatrix}$$

3. Calcular el componente transitorio de la solución de las ecuaciones de estado

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad \boldsymbol{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\text{R: } \boldsymbol{x}_T = \begin{pmatrix} \sqrt{34}e^{-t}\cos(t + 1.0304) \\ \sqrt{68}e^{-t}\cos(t - 1.3258) \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{pmatrix} -\lambda & 1 \\ -2 & -2-\lambda \end{pmatrix} = -\lambda(-2-\lambda) - (-2\cdot 1)$$

$$2\lambda + \lambda^2 + 2$$

$$= \lambda^2 + 2\lambda + 2$$

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot 2}}{2\cdot 1}$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{-4}}{2}$$

$$\lambda_1 = -1 + i, \quad \lambda_2 = -1 - i$$

vector propio λ_1

$$(A - \lambda_1 I) = \begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix} \xrightarrow{F_2 \leftarrow F_2 - (-1-i)F_1} \begin{pmatrix} 1-i & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1-i & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{F_1 \leftarrow \frac{1}{2}(1+i)F_1} \begin{pmatrix} 1 & \frac{1}{2}(1+i) \\ 0 & 0 \end{pmatrix}$$

entonces

$$\begin{pmatrix} 1 & \frac{1}{2}(1+i) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x + \frac{1}{2}(1+i)y = 0$$

$$x = -\frac{1}{2}(1+i)y$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(1+i)y \\ y \end{pmatrix} \quad \text{con } y = 2 \quad x^1 = \begin{pmatrix} -1-i \\ 2 \end{pmatrix}$$

vector propio λ_2

$$(A - \lambda_1 I) = \begin{pmatrix} 1+i & 1 \\ -2 & -1+i \end{pmatrix} \xrightarrow{F_2 \leftarrow F_2 - (-1+i)F_1} \begin{pmatrix} 1+i & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1+i & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{F_1 \leftarrow \frac{1}{2}(1-i)F_1} \begin{pmatrix} 1 & \frac{1}{2}(1-i) \\ 0 & 0 \end{pmatrix}$$

entonces

$$\begin{pmatrix} 1 & \frac{1}{2}(1-i) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x + \frac{1}{2}(1-i)y = 0$$

$$x = -\frac{1}{2}(1-i)y$$

$$\begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(1-i)y \end{pmatrix} \quad \text{con } y = 2 \quad x^2 = \begin{pmatrix} -1+i \end{pmatrix}$$

$y \qquad y \qquad 2$

Matriz Modal

$$M = \begin{pmatrix} -1-i & -1+i \\ 2 & 2 \end{pmatrix}$$

$$M^{-1} = \frac{1}{\det\begin{pmatrix} -1-i & -1+i \\ 2 & 2 \end{pmatrix}} * \begin{pmatrix} 2 & -(-1+i) \\ -2 & -1-i \end{pmatrix}$$

$$= \frac{1}{(\cancel{-2}-2i)-(\cancel{-2}+2i)} * \begin{pmatrix} 2 & 1-i \\ -2 & -1-i \end{pmatrix}$$

$$= \frac{1}{-4i}\begin{pmatrix} 2 & 1-i \\ -2 & -1-i \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{-2i} & \frac{1}{-4i}(1-i) \\ \frac{-1}{-2i} & \frac{1}{-4i}(-1-i) \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} \frac{i}{2} & \frac{1}{4}(1+i) \\ -\frac{i}{2} & \frac{1}{4}(1-i) \end{pmatrix}$$

$$\lambda_1 = -1+i \quad , \quad \lambda_2 = -1-i$$

$$e^{\lambda t} = \begin{pmatrix} e^{(-1+i)t} & 0 \\ 0 & e^{(-1-i)t} \end{pmatrix} = \begin{pmatrix} e^{-t}e^{it} & 0 \\ 0 & e^{-t}e^{-it} \end{pmatrix}$$

aplicando euler

$$e^{\lambda t} = \begin{pmatrix} e^{-t}(\cos(t)+i\,\text{sen}(t)) & 0 \\ 0 & e^{-t}(\cos(t)-i\,\text{sen}(t)) \end{pmatrix}$$

$$e^{At} = M e^{\lambda t} M^{-1}$$

$$= \begin{pmatrix} -1-i & -1+i \\ 2 & 2 \end{pmatrix} \begin{pmatrix} e^{-t}(\cos(t) + i\,\mathrm{sen}(t)) & 0 \\ 0 & e^{-t}(\cos(t) - i\,\mathrm{sen}(t)) \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{4}(1+i) \\ -\frac{i}{2} & \frac{1}{4}(1-i) \end{pmatrix}$$

$$= \begin{pmatrix} (-1-i)e^{-t}(\cos(t) + i\,\mathrm{sen}(t)) & (-1+i)e^{-t}(\cos(t) - i\,\mathrm{sen}(t)) \\ 2e^{-t}\cos(t) + 2ie^{-t}\sin(t) & 2e^{-t}\cos(t) - 2ie^{-t}\sin(t) \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{4}(1+i) \\ -\frac{i}{2} & \frac{1}{4}(1-i) \end{pmatrix}$$

$=$

- fila 1 columna 1

$$= \frac{i}{2}(-1-i)e^{-t}(\cos(t) + i\,\mathrm{sen}(t)) + (-1+i)e^{-t}(\cos(t) - i\,\mathrm{sen}(t))\frac{-i}{2}$$

$$= \frac{i(-1-i)e^{-t}(\cos(t) + i\,\mathrm{sen}(t)) - i(-1+i)e^{-t}(\cos(t) - i\,\mathrm{sen}(t))}{2}$$

$$= \frac{ie^{-t}\left(((-1-i)(\cos(t) + i\,\mathrm{sen}(t))i) - (-1+i)(\cos(t) - i\,\mathrm{sen}(t)i)\right)}{2}$$

$$= \frac{-2i^2e^{-t}(\cos(t) + \sin(t))}{2}$$

$$= e^{-t}(\cos(t) + \sin(t))$$

- fila 1 columna 2

$$= (-1-i)e^{-t}(\cos(t) + i\sin(t))\frac{1}{4}(1+i) + (-1+i)e^{-t}(\cos(t) + i\sin(t))\frac{1}{4}(1-i)$$

$$= \frac{e^{-t}(\sin(t) - i\cos(t)) + e^{-t}(i\cos(t) + \sin(t))}{2}$$

$$= e^{-t}\sin(t)$$

- fila 2 columna 1

$$= (2e^{-t}\cos(t) + 2ie^{-t}\sin(t))\frac{i}{2} - (2e^{-t}\cos(t) - 2ie^{-t}\sin(t))\frac{i}{2}$$

$$= (ie^{-t}\cos(t) - e^{-t}\sin(t)) - (e^{-t}\sin(t) + ie^{-t}\cos(t))$$

$$= (-e^{-t}\sin(t) - e^{-t}\sin(t)) + (e^{-t}\cos(t) \cancel{- e^{-t}\cos(t)})i$$

$$= -2e^{-t}\sin(t)$$

- fila 2 columna 2.

$$= \left(2e^{-t}\cos(t) + 2ie^{-t}\sin(t)\right)\frac{1}{4}(1+i) + \left(2e^{-t}\cos(t) - 2ie^{-t}\sin(t)\right)\frac{1}{4}(1-i)$$

$$= \frac{e^{-t}\cos(t) + i\left(e^{-t}\cos(t) + e^{-t}\sin(t)\right) - e^{-t}\sin(t)}{2} +$$

$$\frac{e^{-t}\cos(t) + i\left(e^{-t}\cos(t) - e^{-t}\sin(t)\right) - e^{-t}\sin(t)}{2}$$

$$= \frac{2e^{-t}\cos(t) - 2e^{-t}\sin(t)}{2}$$

$$= e^{-t}\left(\cos(t) - \sin(t)\right)$$

entonces

$$e^{At} = \begin{pmatrix} e^{-t}(\cos(t) + \sin(t)) & e^{-t}\sin(t) \\ -2e^{-t}\sin(t) & e^{-t}(\cos(t) - \sin(t)) \end{pmatrix}$$

resolviendo:

$$x_T(t) = e^{At}\,\underline{x}(0)$$

$$= \begin{pmatrix} e^{-t}(\cos(t) + \sin(t)) & e^{-t}\sin(t) \\ -2e^{-t}\sin(t) & e^{-t}(\cos(t) - \sin(t)) \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3e^{-t}(\cos(t) + \sin(t)) + 2e^{-t}\sin(t) \\ -6e^{-t}\sin(t) + 2e^{-t}(\cos(t) - \sin(t)) \end{pmatrix}$$

$$x_T = \begin{pmatrix} e^{-t}(3\cos(t) + 5\sin(t)) \\ e^{-t}(2\cos(t) - 8\sin(t)) \end{pmatrix}$$

suma de 2 sinusoides

$$A\cos(\omega t) + B\sin(\omega t) = \sqrt{A^2 + B^2}\cdot\cos\left(\omega t + \tan^{-1}(B/A)\right)$$

entonces

$$x_T = \begin{pmatrix} e^{-t}\sqrt{34}\cdot\cos(t + 1.0304) \\ e^{-t}\sqrt{68}\cdot\cos(t - 1.3258) \end{pmatrix}$$