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Hoja de trabajo No. 4

1. Calcular la matriz adjunta, el determinante, y la matriz inversa:

$$A^{-1} = \frac{\text{adj } \mathbf{A}}{|\mathbf{A}|}$$

a) $A = \begin{pmatrix} 1 & 3 \\ -5 & 6 \end{pmatrix}$ b) $A = \begin{pmatrix} 1 & 3 & 2 \\ -5 & 6 & -7 \\ 8 & 5 & 4 \end{pmatrix}$

1) $A = \begin{pmatrix} 1 & 3 \\ -5 & 6 \end{pmatrix}$

$\text{Adj } A = \begin{pmatrix} 6 & -3 \\ 5 & 1 \end{pmatrix}$

$\det A = \begin{pmatrix} 1 & 3 \\ -5 & 6 \end{pmatrix} = 6 - (-15) = 6 + 15 = 21$

$A^{-1} = \frac{1}{\det A} = \begin{pmatrix} 6 & -3 \\ 5 & 1 \end{pmatrix} \frac{1}{21} = \begin{pmatrix} \frac{2}{7} & -\frac{1}{7} \\ \frac{5}{21} & \frac{1}{21} \end{pmatrix}$

2) $A = \begin{pmatrix} 1 & 3 & 2 \\ -5 & 6 & -7 \\ 8 & 5 & 4 \end{pmatrix}$

$\text{Adj } A = C^T$ matriz cofactores.

$C = \begin{bmatrix} +59 & -36 & -73 \\ -2 & -12 & +19 \\ -33 & -3 & +21 \end{bmatrix}$

$11 = \begin{matrix} 6 & -7 \\ 5 & 4 \end{matrix} = 24 + 35 = 59$

$12 = \begin{matrix} -5 & -7 \\ 8 & 4 \end{matrix} = -20 + 56 = 36$

$13 = \begin{matrix} -5 & 6 \\ 8 & 5 \end{matrix} = -25 - 48 = -73$

$21 \quad \begin{matrix}3 & 2\\ 5 & 4\end{matrix} = 12 - 10 = 2 \qquad 31 \quad \begin{matrix}3 & 2\\ 6 & -7\end{matrix} = -21 - 12 = -33$

$22 \quad \begin{matrix}1 & 2\\ 8 & 4\end{matrix} = 4 - 16 = -12 \qquad 32 \quad \begin{matrix}1 & 2\\ -5 & -7\end{matrix} = -7 + 10 = 3$

$23 \quad \begin{matrix}1 & 3\\ 8 & 5\end{matrix} = 5 - 24 = -19 \qquad 33 \quad \begin{matrix}1 & 3\\ -5 & 6\end{matrix} = 6 + 15 = 21$

$$\text{Adj } A = C^T = \begin{bmatrix} 59 & -2 & -33 \\ -36 & -12 & -3 \\ -73 & +19 & 21 \end{bmatrix}$$

$$\det A = \begin{pmatrix} 1 & 3 & 2 \\ -5 & 6 & -7 \\ 8 & 5 & 4 \end{pmatrix}$$

$$\det |A| = \begin{bmatrix} + & - & + \\ 1 & 3 & 2 \end{bmatrix}$$

$$1(6 \cdot 4 + 7 \cdot 5) - 3(-20 + 56) + 2(-25 - 48)$$

$$1(59) - 3(36) + 2(-73)$$

$$59 - 108 - 146$$

$\det A = 195$

$$A^{-1} = \frac{1}{195} \begin{pmatrix} 59 & -2 & -33 \\ -36 & -12 & -3 \\ -73 & +19 & 21 \end{pmatrix}$$

2. Calcular |A| , el determinante de la matriz A, por expansión de Laplace.

$$A = \begin{array}{c} \begin{array}{cccc} A & & & \end{array} \\ \begin{pmatrix} \boxed{1} & 3 & -1 & 4 \\ \boxed{2} & 0 & 1 & 5 \\ \cancel{-1} & \cancel{6} & \cancel{10} & \cancel{-8} \\ \boxed{0} & -2 & 7 & 1 \end{pmatrix} \end{array}$$

$$|A| = a_{11} \cdot A_{11} + a_{21} \cdot A_{21} + a_{31} \cdot A_{31} + \cancel{a_{41} \cdot A_{41}}$$

$$(1 \cdot A_{11}) - (2 \cdot A_{21}) + (1 \cdot A_{31})$$

$$= 1 \begin{vmatrix} 0 & 1 & 5 & 0 & 1 \\ 6 & 10 & -8 & 6 & 10 \\ -2 & 7 & 1 & -2 & 7 \end{vmatrix} = 0 + 16 + 210 - (6 + 0 - 100) = 1\,(226 + 94) = 320.$$

$$-2 \begin{vmatrix} 3 & -1 & 4 & 3 & -1 \\ 6 & 10 & -8 & 6 & 10 \\ -2 & 7 & 1 & -2 & 7 \end{vmatrix} = \overset{182}{(30 + -16 + 168)} - \overset{+254}{(-6 + -168 + -80)} = 436\,(-2) = -872$$

$$-1 \begin{vmatrix} 3 & -1 & 4 & 3 & -1 \\ 0 & 1 & 5 & 0 & 1 \\ -2 & 7 & 1 & -2 & 7 \end{vmatrix} = (3 + 10 + 0) - (0 + 105 + -8) = +84$$

det A = 320 − 872 + 84 = 468

3. Calcular la función de transferencia, $G(s) = Y(s)/U(s)$, para el sistema definido por

a) $\dot{\boldsymbol{x}} = \begin{pmatrix} -4 & -1.5 \\ 4 & 0 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}u, \quad y = (1.5 \quad 0.625)\boldsymbol{x}$

b) $\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}u, \quad y = (1 \quad 0 \quad 0)\boldsymbol{x}$

c) $\dot{\boldsymbol{x}} = \begin{pmatrix} 8 & -4 & 1 \\ -3 & 2 & 0 \\ 5 & 7 & -9 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} -4 \\ -3 \\ 4 \end{pmatrix}u, \quad y = (2 \quad 8 \quad -3)\boldsymbol{x}$

$$Y = GU = [C(sI - A)^{-1}B + D]\underline{U}$$

A) $$\dot{x} = \underbrace{\begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{B} u \qquad y = \underbrace{(1.5 \quad 0.625)}_{C} x$$

$$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} s+4 & 1.5 \\ -4 & s \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{Adj(sI - A)}{|sI - A|} = \frac{\begin{bmatrix} s & -1.5 \\ 4 & s+4 \end{bmatrix}}{s(s+4) + 4(1.5)} = \frac{\begin{bmatrix} s & -1.5 \\ 4 & s+4 \end{bmatrix}}{s^2 + 4s + 6}$$

$$C(sI - A)^{-1} = [1.5 \quad 0.625]\frac{\begin{bmatrix} s & -1.5 \\ 4 & s+4 \end{bmatrix}}{s^2 + 4s + 6}$$

$$= \frac{1}{s^2 + 4s + 6}\begin{bmatrix} 1.5s + 2.5 & -2.25 + 0.625s + 2.5 \end{bmatrix}$$

$$= \frac{[1.5s + 2.5 \quad 0.625s + 0.25]}{s^2 + 4s + 6}$$

$$C(SI-A)^{-1}B = \frac{1}{s^2+4s+6}\begin{bmatrix}1.5s+2.5 & 0.625s+0.25\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}$$

$$= \frac{1}{s^2+4s+6}\begin{bmatrix}3s+5\end{bmatrix}$$

$$\frac{Y(s)}{X(s)} = \frac{3s+5}{s^2+4s+6}$$

b) $\dot{x} = \underbrace{\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -3\end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix}10\\0\\0\end{pmatrix}}_{B} u, \quad y = \underbrace{(1 \quad 0 \quad 0)}_{C.} x$

$$(SI-A) = \begin{bmatrix}s & 0 & 0\\ 0 & s & 0\\ 0 & 0 & s\end{bmatrix} - \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -3\end{bmatrix} = \begin{bmatrix}s & -1 & 0\\ 0 & s & -1\\ 1 & 2 & s+3\end{bmatrix}$$

$$(SI-A)^{-1} = \frac{\begin{vmatrix} +\begin{vmatrix}s & -1\\ 2 & s+3\end{vmatrix} & -\begin{vmatrix}0 & -1\\ 1 & s+3\end{vmatrix} & +\begin{vmatrix}0 & s\\ 1 & 2\end{vmatrix} \\ -\begin{vmatrix}-1 & 0\\ 2 & s+3\end{vmatrix} & +\begin{vmatrix}s & 0\\ 1 & s+3\end{vmatrix} & -\begin{vmatrix}s & -1\\ 1 & 2\end{vmatrix} \\ +\begin{vmatrix}-1 & 0\\ s & -1\end{vmatrix} & -\begin{vmatrix}s & 0\\ 0 & -1\end{vmatrix} & +\begin{vmatrix}s & -1\\ 0 & s\end{vmatrix}\end{vmatrix}}{\begin{vmatrix}s & -1 & 0\\ 0 & s & -1\\ 1 & 2 & s+3\end{vmatrix}} = \frac{\begin{vmatrix}s^2+3s+2 & -1 & -s\\ +s+3 & s^2+3s & -2s-2\\ 1 & s & s^2\end{vmatrix}}{\begin{vmatrix}s & -1 & 0\\ 0 & s & -1\\ 1 & 2 & s+3\end{vmatrix}}$$

$$(SI-A^{-1}) = \frac{\begin{pmatrix}s^2+3s+2 & s+3 & 1\\ -1 & s^2+3s & s\\ -s & -2s-2 & s^2\end{pmatrix}}{s^3+3s^2+2s+1}$$

$$(SI-A^{-1})C = (1 \quad 0 \quad 0) \cdot \frac{\begin{pmatrix}s^2+3s+2 & s+3 & 1\\ -1 & s^2+3s & s\\ -s & -2s-2 & s^2\end{pmatrix}}{s^3+3s^2+2s+1}$$

$$= \frac{\begin{pmatrix} s^2+3s+2 & s+3 & 1 \end{pmatrix}}{s^3+3s^2+2s+1}$$

$$(sI-A)^{-1}CB = \frac{\begin{pmatrix} s^2+3s+2 & s+3 & 1 \end{pmatrix}}{s^3+3s^2+2s+1} \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{10(s^2+3s+2)}{s^3+3s^2+2s+1}$$

c) $$\dot{\boldsymbol{x}} = \underbrace{\begin{pmatrix} 8 & -4 & 1 \\ -3 & 2 & 0 \\ 5 & 7 & -9 \end{pmatrix}}_{A} \boldsymbol{x} + \underbrace{\begin{pmatrix} -4 \\ -3 \\ 4 \end{pmatrix}}_{B} u, \quad y = \underbrace{(2 \quad 8 \quad -3)}_{C.} \boldsymbol{x}$$

$$(sI-A) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 8 & -4 & 1 \\ -3 & 2 & 0 \\ 5 & 7 & -9 \end{bmatrix} = \begin{bmatrix} s-8 & 4 & -1 \\ 3 & s-2 & 0 \\ -5 & -7 & s+9 \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{\begin{pmatrix} (s-2)(s+9) & -4s+29 & s-2 \\ 3s+27 & s^2+s-77 & 3 \\ 5s-31 & +7s-76 & s^2-10s+4 \end{pmatrix}}{s^3-s^2-91s+67}$$

$$(sI-A)^{-1}C = (2 \quad 8 \quad -3) \begin{pmatrix} (s-2)(s+9) & -4s+29 & s-2 \\ 3s+27 & s^2+s-77 & 3 \\ 5s-31 & +7s-76 & s^2-10s+4 \end{pmatrix}$$

$$= \frac{\begin{pmatrix} 2s^2-25s-159 & 8s^2-21s-446 & -3s^2+32s-40 \end{pmatrix}}{s^3-s^2-91s+67}$$

$$(sI-A)^{-1}C \cdot B =$$

$$\frac{(-4)(2s^2-25s-159)+(-3)(8s^2-21s-446)+(4)(-3s^2+32s-40)}{s^3-s^2-91s+67}$$

$$= \frac{-44s^2 + 291s + 1814}{s^3 - s^2 - 91s + 67}$$

4. Calcular la matriz de transferencia H(s) para un sistema descrito por

$$\dot{x} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 0 \\ 3 & -1 & 6 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} u$$

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$T(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

$$= \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 0 \\ 3 & -1 & 0 \end{bmatrix} \begin{bmatrix} s+1 & -1 & 0 \\ 0 & s+1 & 0 \\ 0 & 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A^{-1} = \left[\begin{array}{ccc|ccc} s+1 & -1 & 0 & 1 & 0 & 0 \\ 0 & s+1 & 0 & 0 & 1 & 0 \\ 0 & 0 & s+1 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{s+1} & \frac{1}{(s+1)^2} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{s+1} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{s+1} \end{array}\right]$$

$$= \begin{pmatrix} \frac{1}{s+1} & \frac{1}{(s+1)^2} & 0 \\ 0 & \frac{1}{s+1} & 0 \\ 0 & 0 & \frac{1}{s+1} \end{pmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 0 \\ 3 & -1 & 0 \end{bmatrix} \begin{pmatrix} \frac{1}{s+1} & \frac{1}{(s+1)^2} & 0 \\ 0 & \frac{1}{s+1} & 0 \\ 0 & 0 & \frac{1}{s+1} \end{pmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{pmatrix} 2s+2 & -s-2 & 3s+3 \\ 0 & s+1 & 0 \\ 3s+3 & -s-1 & 0 \end{pmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{pmatrix} -1 & 4s+5 \\ s+1 & -s-1 \\ 5s+5 & s+1 \end{pmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{pmatrix} 0 & 4s+6 \\ s+1 & -s-1 \\ 5s+5 & s+1 \end{pmatrix} \square$$

5. Un sistema con dos entradas y dos salidas se describe con las ecuaciones diferenciales $\ddot{y}_1 + 3\dot{y}_1 + 2y_2 = u_1 + 2u_2 + 2\dot{u}_2$ y $\ddot{y}_2 + 4\dot{y}_1 + 3y_2 = \ddot{u}_2 + 3\dot{u}_2 + u_1$.

   a) Representar al sistema utilizando la forma canónica observable de las ecuaciones de estado.

   b) Calcular la matriz de transferencia.

$$\ddot{y}_1 + 3\dot{y}_1 + 2y_2 = u_1 + 2u_2 + 2\dot{u}_2$$

$$\ddot{y}_1 = 2\dot{u}_2 - 3\dot{y}_1 + (u_1 + 2u_2 - 2y_2)$$

$$\dot{y}_1 = 2u_2 - 3y_1 + \int (u_1 + 2u_2 - 2y_2)\,dt$$

$$y_1 = \int \left[2u_2 - 3y_1 + \int (u_1 + 2u_2 - 2y_2)\,dt' \right] dt''$$

$$\ddot{y}_2 + 4\dot{y}_1 + 3y_2 = \ddot{u}_2 + 3\dot{u}_2 + u_1$$

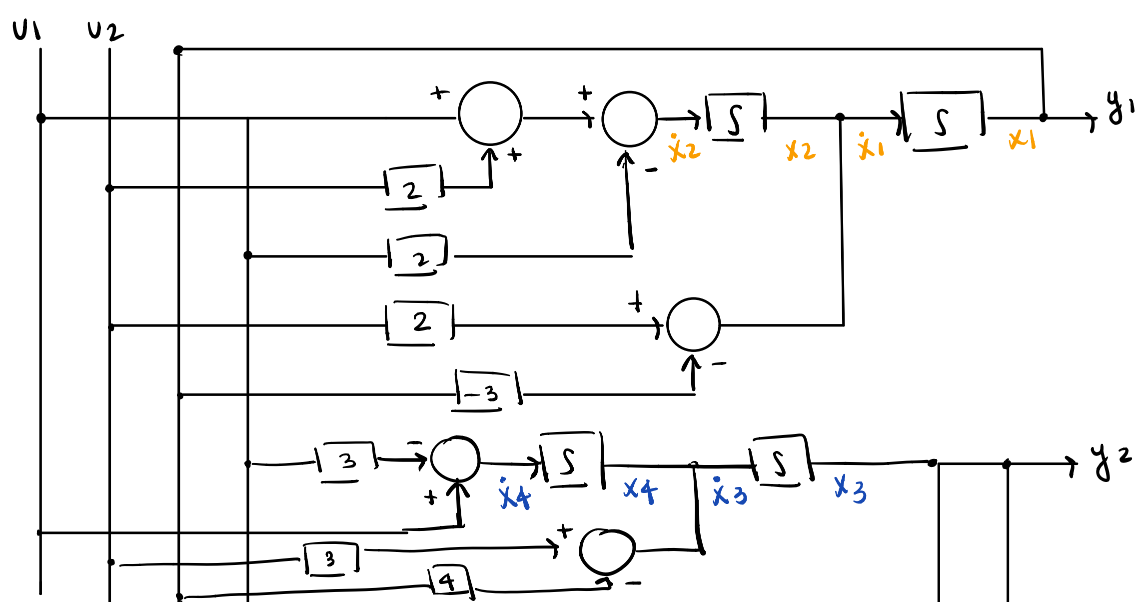
$$\ddot{y}_2 = \ddot{u}_2 + (3\dot{u}_2 - 4\dot{y}_1) + (u_1 - 3y_2)$$

$$\dot{y}_2 = \dot{u}_2 + (3u_2 - 4y_1) + \int (u_1 - 3y_2)\,dt$$

$$y_2 = u_2 + \int \left[3u_2 - 4y + \int (u_1 - 3y_2)\,dt' \right] dt''$$

$$y_1 = \int \left[2u_2 - 3y_1 + \int (u_1 + 2u_2 - 2y_2)\,dt' \right] dt''$$

$$y_2 = u_2 + \int \left[3u_2 - 4y + \int (u_1 - 3y_2)\,dt' \right] dt'$$



$\dot{x}_1 = x_2 + 2u_2 - 3x_1$

$\dot{x}_2 = (\omega_1 + 2\mu_2 - 2y_2)$

$\quad = u_1 + 2u_2 - 2(x_3 + u_2)$

$\dot{x}_2 = u_1 - 2x_3$

$y_1 = x_1$

$\dot{x}_3 = x_4 + 3u_2 - 4x_1$

$\dot{x}_4 = -3y_2 + u_1$

$\quad = -3(x_3 + u_2) + u_1$

$\dot{x}_4 = -3x_3 - 3u_2 + u_1$

$y_2 = x_3 + u_2.$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ -4 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

b). $$\dot{x} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ -4 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 3 \\ 1 & -3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} u$$

$T(s) = C(sI - A)^{-1}B + D$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s+3 & -1 & 0 & 0 \\ 0 & s & +2 & 0 \\ 4 & 0 & s & -1 \\ 0 & 0 & 3 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 3 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} \dfrac{s^2+3}{(s+1)^3} & \dfrac{s^2+3}{s(s+1)^3} & \dfrac{-2}{(s+1)^3} & \dfrac{-2}{s(s+1)^3} \\[2ex] \dfrac{-4s}{(s+1)^3} & \dfrac{-4}{(s+1)^3} & \dfrac{s(s+3)}{(s+1)^3} & \dfrac{s+3}{(s+1)^3} \end{pmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 3 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} \dfrac{s^2+1}{s(s+1)^3} & \dfrac{2s^3+6}{s(s+1)^3} \\[2ex] \dfrac{s-1}{(s+1)^3} & \dfrac{3s^2-2s-9}{(s+1)^3} \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} \dfrac{s^2+1}{s(s+1)^3} & \dfrac{2s^3+6}{s(s+1)^3} \\[2ex] \dfrac{s-1}{(s+1)^3} & \dfrac{3s^2-2s-8}{(s+1)^3} \end{pmatrix}$$