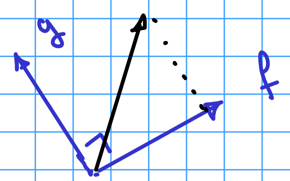
Def: $f, g \in V$ son ortogonales ssi $\langle f|g\rangle = 0$.

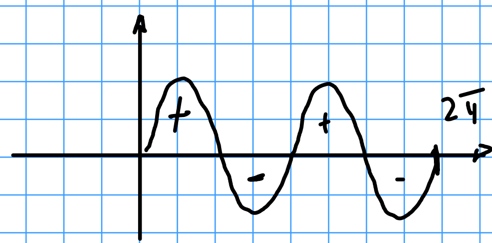


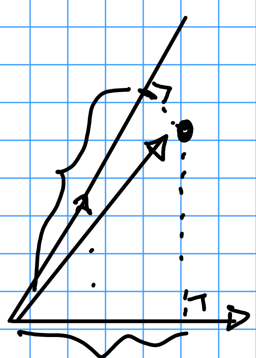
Ej: El producto $\langle f|g\rangle = \int_a^b f(x)\,g(x)\,dx$

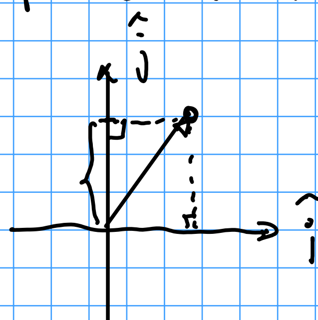
$$\langle \sin x|\cos x\rangle = \int_0^{2\pi} \sin x \cos x \, dx =$$

$$\frac{1}{2}\int_0^{2\pi} \sin 2x \, dx = 0$$

por simetría.







Def. Un conjunto $\Phi = \{f_1, \dots\}$ es ortogonal ssi

$$\langle f_i|f_j\rangle = 0 \quad \text{para } i \neq j. \qquad (f_i \neq \vec{0}).$$

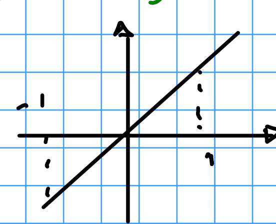
Ej: Para $[-1, 1]$ determines: $\Phi = \{1, x, \frac{3}{2}x^2 - \frac{1}{2}\}$ es ortogonal.

$f_0 \quad f_2 \quad f_3$

sol:

$$\langle f_1|f_2\rangle = \int_{-1}^{1} 1 \cdot x \, dx = 0$$

por simetría.



$$\langle f_2|f_1\rangle = \langle f_1|f_2\rangle^* = 0^* = 0.$$

$$\langle f_1|f_3\rangle = \int_{-1}^{1} \frac{3}{2}x^2 - \frac{1}{2}\,dx = \frac{1}{2}x^3 - \frac{1}{2}x \Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} - 0 = 0.$$

$$\langle f_2|f_3\rangle = \int_{-1}^{1} x \cdot \left(\frac{3}{2}x^2 - \frac{1}{2}\right) dx = \int_{-1}^{1} \frac{3}{2}x^3 - \frac{1}{2}x \, dx =$$

impares.

$$\frac{3}{8}x^4 - \frac{1}{4}x^2 \Big|_{-1}^{1} = 0 \text{ . por simetria.}$$

$$\langle f_3 | f_1 \rangle = \langle f_1 | f_3 \rangle^* = 0 \, ; \quad \langle f_3 | f_2 \rangle = \langle f_2 | f_3 \rangle^* = 0 .$$

$\therefore$ $\mathcal{O}$ es ortogonal.

Sea $\mathcal{B}$ base de $S \subset V$ subespacio y $f \in \mathcal{B}$

$$f = \lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_m b_m .$$

$\lambda_1, \dots, \lambda_m$ las coordenadas de f.

$$[f]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$ El vector de coordenadas.

$\mathcal{B}$ es base es que sea generador y l.i.

$$\langle \mathcal{B} \rangle = \left\{ \sum \lambda_i b_i \;\middle|\; \lambda_i \in \mathbb{C} \right\}$$

Teorema: $\mathcal{O} = \{b_1, \dots, b_n\}$ es ortogonal entonces es l.i.

$\langle 1, x, x^2 \rangle$ = Polinomios de orden 2 $\rightsquigarrow$ Gram-Schmidt.

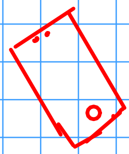
$a + bx + cx^2$

$$\left\{ 1, x, \frac{3}{2}x^2 - \frac{1}{2} \right\}.$$

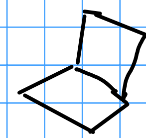
Teorema: Si $f = \sum_{k=1}^{n} \alpha_k \phi_k$ con $\mathcal{O} = \{\phi_1, \dots\}$ ortogonal entonces.

$$\langle \phi_m | f \rangle = \left\langle \phi_m \middle| \sum_{k=1}^{n} \alpha_k \phi_k \right\rangle = \sum_{k=}^{n} \alpha_k \langle \phi_m | \phi_k \rangle$$

$$\Rightarrow \langle \phi_m | f \rangle = \alpha_m \langle \phi_m | \phi_m \rangle \Rightarrow$$



$$\boxed{\alpha_m = \frac{1}{\|\phi_m\|^2}\langle \phi_m | f\rangle}$$



Coeficientes de Fourier.

Coeficientes de Fourier de $f(x) = 3 + 2x + x^2$
respecto de $\mathcal{O} = \{1,\ x,\ \frac{3}{2}x^2 - \frac{1}{2}\}$.

Op. 1. $3 + 2x + x^2 = A + Bx^2 + C\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$

$\vdots$

Op. 2. $\alpha_1 = \frac{1}{\|\phi_1\|^2}\langle \phi_1 | f\rangle$; $\quad \|\phi_1\|^2 = \langle \phi_1 | \phi_1\rangle = \int_{-1}^{1} 1\cdot 1\, dx = 2.$

$\langle \phi_1 | f\rangle = \int_{-1}^{1} 1\cdot(3 + 2x + x^2)\,dx =$

$\alpha_1 = \frac{1}{2}\cdot\frac{20}{3} = \frac{10}{3}.$ $\qquad 6 + 0 + \frac{1}{3}x^3\Big|_{-1}^{1} = 6 + \frac{2}{3} = \frac{20}{3}$

$\alpha_2 = \frac{1}{\|\phi_2\|^2}\langle \phi_2 | f\rangle \qquad \|\phi_2\|^2 = \langle \phi_2 | \phi_2\rangle = \int_{-1}^{1} x\cdot x\,dx = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{2}{3}$

$\langle \phi_2 | f\rangle = \int_{-1}^{1} x\cdot(3 + 2x + x^2)\,dx = \int_{-1}^{1} 3x + 2x^2 + x^3\,dx =$

$\frac{2}{3}x^3\Big|_{-1}^{1} = \frac{4}{3} \Rightarrow \alpha_2 = \frac{1}{2/3}\cdot\frac{4}{3} = 2.$

$\alpha_3 = \frac{1}{\|\phi_3\|^2}\langle \phi_3 | f\rangle \qquad \|\phi_3\|^2 = \langle \phi_3 | \phi_3\rangle = \int_{-1}^{1}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)\cdot\left(\frac{3}{2}x^2 - \frac{1}{2}\right)dx$

$= \int_{-1}^{1} \frac{9}{4}x^4 - \frac{3}{2}x^2 + \frac{1}{4}\,dx = \frac{9}{4}\cdot\frac{1}{5}x^5 - \frac{1}{2}x^3 + \frac{1}{4}x\Big|_{-1}^{1} = \frac{9}{10} - 1 + \frac{1}{2}$

$= \frac{2}{5}$; $\quad \langle \phi_3 | f\rangle = \int_{-1}^{1}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)\cdot(3 + 2x + x^2)\,dx = \frac{4}{15}$

$\alpha_3 = \frac{1}{2/5}\cdot\frac{4}{15} = \frac{5}{2}\cdot\frac{4}{15} = \frac{2}{3}.$

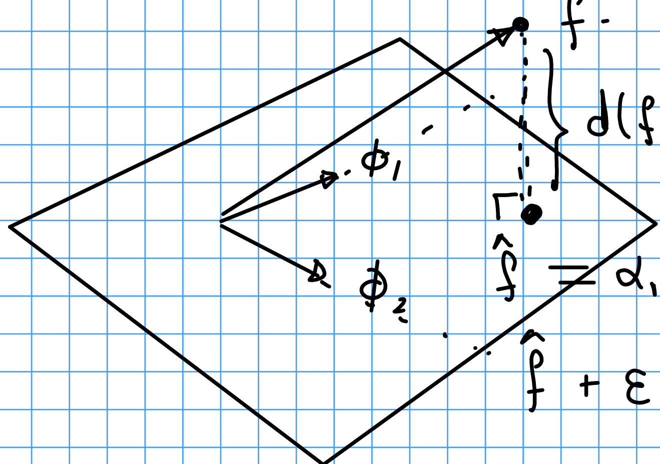
$$\boxed{f = \frac{10}{3}\phi_1 + 2\phi_2 + \frac{2}{3}\phi_3} + \ldots$$

$$\alpha_k = \frac{1}{\|\phi_k\|^2} \langle \phi_k | f \rangle$$

1. $\|\phi_k\|^2$ finitas acotadas.

2. $\sum |\alpha_k|^2 = \sum_{k=1}^{\infty} \|\alpha_k \phi_k\|^2 < \|f\|^2.$

$$\left\| f - \sum_{k=1}^{N} \frac{1}{\|\phi_k\|^2} \langle \phi_k | f \rangle \, \phi_k \right\|^2 \leq \left\| f - \sum_{k=1}^{N} a_k \, \phi_k \right\|^2 .$$



$$\hat{f} = \alpha_1 \phi_1 + \alpha_2 \phi_2$$

$$\hat{f} + \varepsilon = f$$