PROPIEDADES DE ENTROPIA

1. Considere las siguientes fuentes de información con las probabilidades de símbolos que se muestran.

   $S = \{s_1, s_2, \ldots, s_{q-1}, s_q\}$
   $P(s_1) = P_1$, $P(s_2) = P_2, \ldots$, $P(s_{q-1}) = P_{q-1}$ y $P(s_q) = P_q$.

   $S_1 = \{s_1, s_2, \ldots, s_{q-1}, s_q, s_{q+1}\}$
   $P(s_1) = P_1$, $P(s_2) = P_2, \ldots$, $P(s_{q-1}) = P_{q-1}$, $P(s_q) = \alpha P_q$ y $P(s_{q+1}) = \overline{\alpha} P_q$.
   $\overline{\alpha} = 1 - \alpha$.

   $S_2 = \{s_1, s_2\}$
   $P(s_1) = \alpha$ y $P(s_2) = \overline{\alpha}$.

   Muestre que la función entropía satisface las siguientes propiedades:

   a) $H(S)$ es una función simétrica de $P_1, P_2, \ldots, P_q$ (grafique $H(s)$ para los casos $q = 2$ y $q = 3$).
   b) $H(S_1) = H(S) + P_q H(S_2)$
   c) $H(S_2)$ es una función contínua de α.

2. Demostrar:

$$\sum_{i=1}^{q} x_i \log\left(\frac{1}{x_i}\right) \leq \sum_{i=1}^{q} x_i \log\left(\frac{1}{y_i}\right)$$

3. Demostrar: $\log(q) - H(s) \geq 0$.

HT - 3

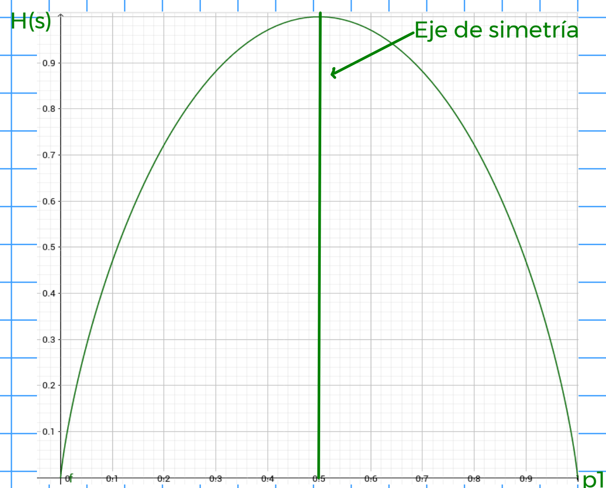
1

a Para q = 2

$$P_1 + P_2 = 1 \Rightarrow P_2 = 1 - P_1$$

$$H(S) = P_1 \log\left(\frac{1}{P_1}\right) + (1-P_1)\log\left(\frac{1}{1-P_1}\right)$$

Gráfica:

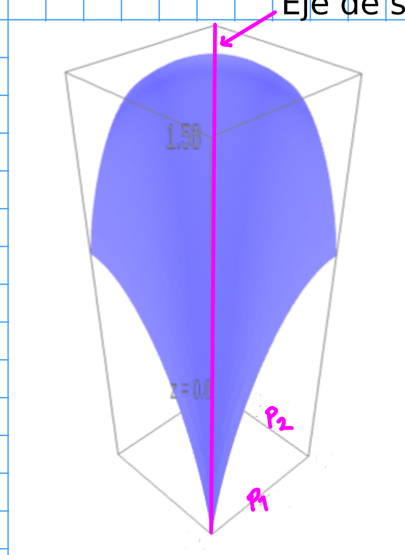


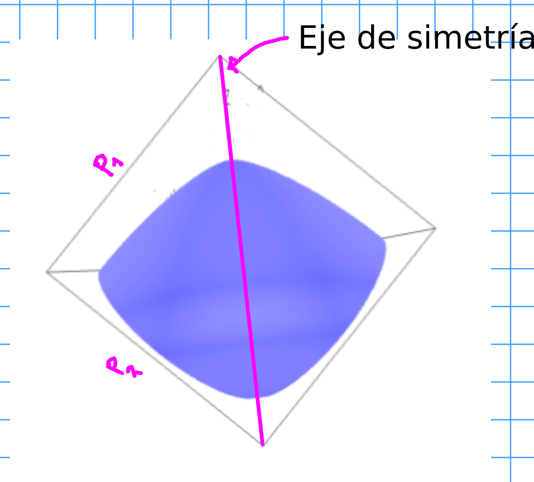
Para q = 3

$$P_1 + P_2 + P_3 = 1 \Rightarrow P_3 = 1 - (P_1 + P_2) \Rightarrow P_1 + P_2 < 1$$

$$H(S) = P_1 \log\left(\frac{1}{P_1}\right) + P_2 \log\left(\frac{1}{P_2}\right) + (1-P_1-P_2)\log\left(\frac{1}{1-P_1-P_2}\right)$$

Eje de simetría





b. $H(S_1) = H(S) + P_q H(S_2)$

$$H(S) = \sum_{i=1}^{q} p_i \log\left(\frac{1}{p_i}\right) \quad y \quad H(S2) = \alpha \log\left(\frac{1}{\alpha}\right) + \bar{\alpha} \log\left(\frac{1}{\bar{\alpha}}\right)$$

$$H(S_1) = \sum_{i=1}^{q-1} p_i \log\left(\frac{1}{p_i}\right) + \alpha P_q \log\left(\frac{1}{\alpha P_q}\right) + \bar{\alpha} P_q \log\left(\frac{1}{\bar{\alpha} P_q}\right)$$

$$H(S_1) = \sum_{i=1}^{q-1} p_i \log\left(\frac{1}{p_i}\right) + P_q\left(\alpha\left[\log\left(\frac{1}{\alpha}\right) + \log\left(\frac{1}{P_q}\right)\right] + \bar{\alpha}\left[\log\left(\frac{1}{\bar{\alpha}}\right) + \log\left(\frac{1}{P_q}\right)\right]\right)$$

$$H(S_1) = \sum_{i=1}^{q-1} p_i \log\left(\frac{1}{p_i}\right) + P_q\left[(\alpha + \bar{\alpha}) \log\left(\frac{1}{p_q}\right) + \alpha \log\left(\frac{1}{\alpha}\right) + \bar{\alpha} \log\left(\frac{1}{\bar{\alpha}}\right)\right]$$

$$H(S_1) = \sum_{i=1}^{q-1} p_i \log\left(\frac{1}{p_i}\right) + P_q \log\left(\frac{1}{P_q}\right) + P_q\left[\alpha \log\left(\frac{1}{\alpha}\right) + \bar{\alpha} \log\left(\frac{1}{\bar{\alpha}}\right)\right]$$

$$H(S_1) = \sum_{i=1}^{q} p_i \log\left(\frac{1}{p_i}\right) + P_q\left[\alpha \log\left(\frac{1}{\alpha}\right) + \bar{\alpha} \log\left(\frac{1}{\bar{\alpha}}\right)\right]$$

$$\underline{H(S_1) = H(S) + P_q H(S_2)}$$

c. $H(S2) = \alpha \log\left(\frac{1}{\alpha}\right) + \bar{\alpha} \log\left(\frac{1}{\bar{\alpha}}\right)$

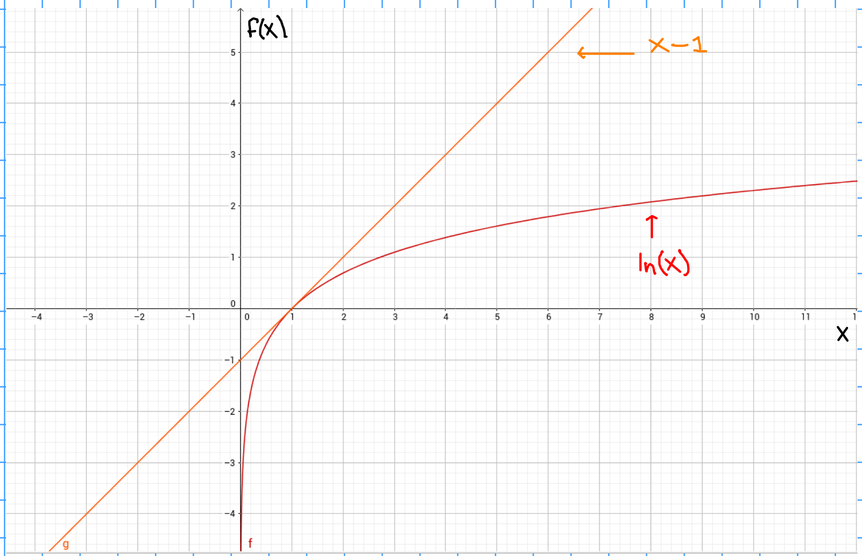
$$H(S2) = f(\alpha) = \alpha \log\left(\frac{1}{\alpha}\right) + (1-\alpha) \log\left(\frac{1}{1-\alpha}\right)$$

$$f(\alpha) = \log\left(\frac{1}{\alpha^{\alpha}}\right) + \log\left(\frac{1}{(1-\alpha)^{(1-\alpha)}}\right)$$

$$\underline{f(\alpha) = \log\left(\frac{1}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}\right)} \qquad \text{con } \alpha \in [0,1]$$

2 Demostración:

$$\sum_{i=1}^{q} x_i \log\left(\frac{1}{x_i}\right) \leq \sum_{i=1}^{q} x_i \log\left(\frac{1}{y_i}\right)$$



$$\Rightarrow \quad \ln(x) \leq x-1 \quad // (-1)$$

$$-\ln(x) \geq 1-x$$

$$\ln\left(\frac{1}{x}\right) \geq 1-x$$

Sea xi & yi dos conjuntos de probabilidades:

$$x_i \geq 0 \ , \ y_i \geq 0 \Rightarrow \sum_{i=1}^{q} x_i = \sum_{i=1}^{q} y_i = 1$$

Para limitar el logaritmo

Implica que $\sum_{i=1}^{q} x_i \log\left(\frac{1}{x_i}\right)$ podemos expresarlo de la siguiente manera $\sum_{i=1}^{q} x_i \log\left(\frac{y_i}{x_i}\right)$

$\Rightarrow$ Cambiando de base: $\frac{1}{\ln(2)} \sum_{i=1}^{q} x_i \ln\left(\frac{y_i}{x_i}\right)$

Por lo demostrado arriba sabemos que:

$$\frac{1}{\ln(2)} \sum_{i=1}^{q} x_i \ln\left(\frac{y_i}{x_i}\right) \leq \frac{1}{\ln(2)} \sum_{i=1}^{q} x_i \left[\frac{y_i}{x_i} - 1\right]$$

$$\Rightarrow \frac{1}{\ln(2)} \sum_{i=1}^{q} x_i \ln\left(\frac{y_i}{x_i}\right) \leq \frac{1}{\ln(2)} \sum_{i=1}^{q} y_i - x_i$$

$$\Rightarrow \frac{1}{\ln(2)} \sum_{i=1}^{q} x_i \ln\left(\frac{y_i}{x_i}\right) \leq \frac{1}{\ln(2)} \left[\sum_{i=1}^{q} y_i - \sum_{i=1}^{q} x_i\right]$$

$$\Rightarrow \frac{1}{\ln(2)} \sum_{i=1}^{q} x_i \ln\left(\frac{y_i}{x_i}\right) \leq 0 \quad \begin{cases} = 0 & \text{ssi} \quad y_i = x_i \\ < 0 & \text{ssi} \quad y_i \neq x_i \end{cases}$$

$$\Rightarrow \sum_{i=1}^{q} x_i \log\left(\frac{y_i}{x_i}\right) \leq 0$$

$$\Rightarrow \sum_{i=1}^{q} x_i \left[\log(y_i) + \log\left(\frac{1}{x_i}\right)\right] \leq 0$$

$$\Rightarrow \sum_{i=1}^{q} x_i \log\left(\frac{1}{x_i}\right) + \sum_{i=1}^{q} x_i \log(y_i) \leq 0$$

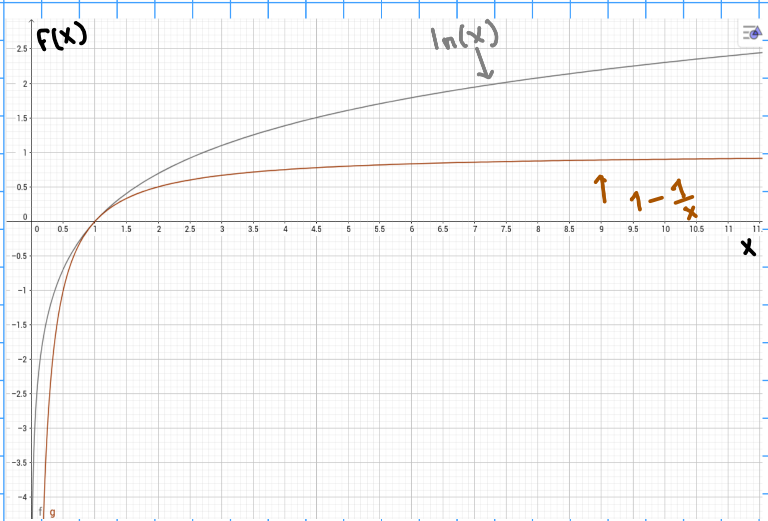
$$\Rightarrow \sum_{i=1}^{q} x_i \log\left(\frac{1}{x_i}\right) \leq -\sum_{i=1}^{q} x_i \log(y_i)$$

$$\Rightarrow \sum_{i=1}^{q} x_i \log\left(\frac{1}{x_i}\right) \leq \sum_{i=1}^{q} x_i (-\log(y_i)) \quad //$$

$$\Rightarrow \sum_{i=1}^{q} x_i \log\left(\frac{1}{x_i}\right) \leq \sum_{i=1}^{q} x_i \log\left(\frac{1}{y_i}\right)$$

3 Demostración:

$$\log(q) - H(S) \geq 0$$



$$\Rightarrow \ln(x) \geq 1 - \frac{1}{x} \quad // \cdot(-1)$$

$$-\ln(x) \leq \frac{1}{x} - 1$$

$$\ln\left(\frac{1}{x}\right) \leq \frac{1}{x} - 1$$

$$\log(q) - H(S) = \log(q) - \sum_{i=1}^{q} p_i \log\left(\frac{1}{p_i}\right)$$

Multiplicamos por un 1, que nos ayuda a asociar

$$= \left(\sum_{i=1}^{q} p_i\right) \log(q) - \sum_{i=1}^{q} p_i \log\left(\frac{1}{p_i}\right)$$

$$= \sum_{i=1}^{q} p_i \left[\log(q) - \log\left(\frac{1}{p_i}\right)\right] \quad // \quad \log\left(\frac{x}{y}\right) = \log(x) - \log(y)$$

$$\frac{q}{1/p_i} = q p_i$$

$$= \sum_{i=1}^{q} p_i \log(q p_i)$$

$$= \frac{1}{\ln(2)} \sum_{i=1}^{q} p_i \ln(q p_i)$$

Por lo demostrado arriba sabemos que:

$$\frac{1}{\ln(2)} \sum_{i=1}^{q} p_i \ln(q p_i) \geq \frac{1}{\ln(2)} \sum_{i=1}^{i} p_i \left[1 - \frac{1}{q p_i}\right]$$

$$q * \frac{1}{q} = 1$$

$$\Rightarrow \frac{1}{\ln(2)} \sum_{i=1}^{q} p_i - \frac{1}{q} = \frac{1}{\ln(2)} \left[\sum_{i=1}^{q} p_i - \sum_{i=1}^{q} \frac{1}{q}\right] = 0$$

$$\Rightarrow \frac{1}{\ln(2)} \sum_{i=1}^{q} p_i \log(q p_i) \geq 0$$

Regresando al principio:

$$\log(q) - H(S) \geq 0$$