§8.3 三重积分

一、填空

1．设$\Omega: x^2+y^2+z^2 \le R^2$，则$\iiint\limits_{\Omega}\left[\left(x^2+y^2\right)z+3\right]\mathrm{d}v =$ $\underline{0+\iiint\limits_{\Omega} 3\,\mathrm{d}v = 4\pi R^3}$

2．设Ω为$a \le x \le b, c \le y \le d, l \le z \le m$，则$\iiint\limits_{\Omega} xy^2z^3\mathrm{d}x\mathrm{d}y\mathrm{d}z =$ $\underline{\frac{1}{24}(b^2-a^2)(d^3-c^3)(m^4-l^4)}$

3．设Ω由曲面$z=2x^2+3y^2$及$z=3-x^2$所围成的闭区域，化三重积分

$I=\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$ 为三次积分是 $\underline{\int_{-1}^{1}\mathrm{d}x\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\mathrm{d}y}\int_{2x^2+3y^2}^{3-x^2} f(x,y,z)\mathrm{d}z$

二、设Ω是由平面$x+y+z=1$及三坐标面所围成的区域，计算

1．$I=\iiint\limits_{\Omega} z\mathrm{d}v$；

法一、投影法 $\Omega=\{(x,y,z) \mid 0 \le z \le 1-x-y,\ 0 \le y \le 1-x,\ 0 \le x \le 1\}$

$\therefore I=\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int_0^{1-x-y} z\,\mathrm{d}z = \int_0^1\mathrm{d}x\int_0^{1-x}\left.\frac{z^2}{2}\right|_0^{1-x-y}\mathrm{d}y$

$= \frac{1}{2}\int_0^1\mathrm{d}x\int_0^{1-x}(1-x-y)^2\mathrm{d}y = -\frac{1}{2}\int_0^1\mathrm{d}x\int_0^{1-x}(1-x-y)^2\mathrm{d}(1-x-y)$

$= -\frac{1}{6}\int_0^1\left.(1-x-y)^3\right|_0^{1-x}\mathrm{d}x = \frac{1}{6}\int_0^1(1-x)^3\mathrm{d}x = \frac{1}{24}$

法二、截面法

$I=\int_0^1 z\,\mathrm{d}z\iint\limits_{D_z}\mathrm{d}x\mathrm{d}y$

$=\int_0^1 z\cdot\frac{1}{2}(1-z)^2\mathrm{d}z$

$=\frac{1}{2}\int_0^1(1-z)^2\mathrm{d}z-\frac{1}{2}\int_0^1(1-z)^3\mathrm{d}z$

$=\frac{1}{6}-\frac{1}{8}=\frac{1}{24}$

2．$I=\iiint\limits_{\Omega}(x+2y+3z)\mathrm{d}v$．

Ω关于x,y,z具有轮换对称性，故

$\iiint\limits_{\Omega} x\,\mathrm{d}v=\iiint\limits_{\Omega} y\,\mathrm{d}v=\iiint\limits_{\Omega} z\,\mathrm{d}v$

$\therefore I=(1+2+3)\iiint\limits_{\Omega} z\,\mathrm{d}v = 6\times\frac{1}{24}=\frac{1}{4}$

法三、$I=\bar{z}\cdot V(\Omega)$

$=\underbrace{\frac{0+0+0+1}{4}}_{\text{形心坐标}}\times\underbrace{\frac{1}{6}}_{\text{体积}}=$

五 ~~三~~、计算 $I=\iiint\limits_{\Omega} z\sqrt{x^2+y^2}\,dxdydz$，其中 Ω 是由 $z=\sqrt{x^2+y^2}$ 与平面 $z=1$ 所围成的形体.

$D=\{(\rho,\theta)\mid 0\le\rho\le1,\ 0\le\theta\le2\pi\}$ ， $\rho\le z\le1$

$\therefore\ I=\iiint\limits_{\Omega} z\cdot\rho\cdot\rho\, d\rho d\theta dz$

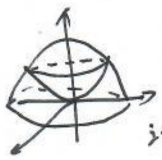
$=\int_0^{2\pi}d\theta\int_0^1 d\rho\int_\rho^1 z\cdot\rho^2 dz$

$=\int_0^{2\pi}d\theta\int_0^1 \rho^2\cdot\frac{z^2}{2}\Big|_\rho^1 d\rho = 2\pi\cdot\frac12\int_0^1(\rho^2-\rho^4)d\rho$

$=\pi\cdot(\frac13-\frac15)=\frac{2}{15}\pi$

六 ~~四~~、计算 $I=\iiint\limits_{\Omega} z\,dxdydz$，其中 Ω 是由上半球面 $z=\sqrt{4-x^2-y^2}$ 及抛物面 $x^2+y^2=3z$ 所围成的形体.

法一. 投影法: $I=\int_0^{2\pi}d\theta\int_0^{\sqrt3}\rho d\rho\int_{\frac{\rho^2}{3}}^{\sqrt{4-\rho^2}} z dz = 2\pi\int_0^{\sqrt3}\rho\cdot\frac{z^2}{2}\Big|_{\frac{\rho^2}{3}}^{\sqrt{4-\rho^2}} d\rho$



$=\cancel{4}\pi\int_0^{\sqrt3}(4-\rho^2-\frac{\rho^4}{9})\rho d\rho = \pi(2\rho^2-\frac{\rho^4}{4}-\frac{\rho^6}{54})\Big|_0^{\sqrt3}=\frac{13}{4}\pi$

法二. 截面法

$I=\int_0^1 z dz\iint\limits_{D_{z下}}dxdy+\int_1^2 z dz\iint\limits_{D_{z上}}dxdy$

$=\int_0^1 z dz\cdot(3z\cdot\pi)+\int_1^2 z dz\cdot(4-z^2)\cdot\pi = \pi\int_0^1 3z^2 dz+\pi\int_1^2 z(4-z^2)dz$

$=\pi z^3\Big|_0^1+\pi(2z^2-\frac{z^4}{4})\Big|_1^2=\pi+\pi(6-\frac{15}{4})=\frac{13}{4}\pi$

七 五、设 $\Omega: x^2+y^2+z^2\le1$，计算

1．$I=\iiint\limits_{\Omega} z\,dxdydz$；

Ω 关于 xOy 面对称，z 关于 z 是奇函数，$\therefore I=0$

（或 $I=\bar{z}\cdot V(\Omega)=0\times\frac43\pi=0$）

2．$I=\iiint\limits_{\Omega} z^2\mathrm{d}x\mathrm{d}y\mathrm{d}z$；

解法一. 截面法.

$$I = 2\iiint\limits_{\Omega_上} z^2 dxdydz = 2\int_0^1 z^2 dz \iint\limits_{D_z} dxdy = 2\int_0^1 z^2 dz \cdot \pi(1-z^2)$$

$$= 2\pi\int_0^1 (z^2-z^4)dz = \frac{4}{15}\pi$$

解法二. 球坐标法.

$$I = \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^1 (r\cos\varphi)^2 \cdot r^2 \sin\varphi \, dr$$

$$= 2\pi \int_0^{\pi} \cos^2\varphi \cdot \sin\varphi \, d\varphi \int_0^1 r^4 dr$$

$$= 2\pi \times \frac{2}{3} \times \frac{1}{5} = \frac{4\pi}{15}$$

3．$I=\iiint\limits_{\Omega}\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$．

由轮换对称性，$\iiint\limits_{\Omega} x^2 dxdydz = \iiint\limits_{\Omega} y^2 dxdydz = \iiint\limits_{\Omega} z^2 dxdydz$

$$\therefore I = \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\cdot\frac{4\pi}{15}$$

三 ~~六~~. 设Ω是平面$z=0, z=y, y=1$以及抛物柱面$y=x^2$所围成的几何体，计算.

1. $\iiint\limits_{\Omega} z\mathrm{d}x\mathrm{d}y\mathrm{d}z$；

$$\Omega = \{(x,y,z) \mid 0\le z\le y,\ x^2\le y\le 1,\ -1\le x\le 1\}$$

$$\therefore \iiint\limits_{\Omega} z\,dxdydz = \int_{-1}^1 dx \int_{x^2}^1 dy \int_0^y z\,dz$$

$$= \int_{-1}^1 dx \int_{x^2}^1 \frac{y^2}{2} dy = \int_{-1}^1 \frac{y^3}{6}\Big|_{x^2}^1 dx = \frac{1}{6}\int_{-1}^1 (1-x^6)dx$$

$$= \frac{1}{6}\times 2\times\frac{6}{7} = \frac{2}{7}$$

2. $\iiint\limits_{\Omega} xz\mathrm{d}x\mathrm{d}y\mathrm{d}z$. $\boxed{\text{对称性}}$

$$I = \int_{-1}^{1} x \cdot \left(\frac{y^3}{6}\right)\Big|_{x^2}^{1} dx = \frac{1}{6}\int_{-1}^{1} x(1-x^6)\,dx = 0$$

四 ~~七~~．设 f 在[0,1]上连续，证明：$\iiint\limits_{x^2+y^2+z^2\le 1} f(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \pi\int_{-1}^{1} f(u)(1-u^2)\mathrm{d}u$.

由截面法知，$\iiint\limits_{x^2+y^2+z^2\le 1} f(z)\,dx\,dy\,dz = \int_{-1}^{1} f(z)\,dz \iint\limits_{D_z} dx\,dy$

$$= \int_{-1}^{1} f(z)\cdot \pi(1-z^2)\,dz = \pi\int_{-1}^{1} f(u)(1-u^2)\,du.$$