自测题二（多元函数的微分学）

一、选择题（每题 3 分，共 15 分）

1、$\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{3xy}{\sqrt{xy+1}-1}=$（ B ）

A、3，　　B、6　　C、不存在但不是无穷大，　　D、∞

2、若 $\dfrac{\partial f}{\partial x}|_{(x_0,y_0)}=0$，$\dfrac{\partial f}{\partial y}|_{(x_0,y_0)}=0$，则 $f(x,y)$ 在 (x_0,y_0)（ D ）

A、连续且可微， B、连续但不一定可微 C、可微但不一定连续

D、不一定可微也不一定连续。( )

3、$z=$ ~~$f(x,y)$ 在 (x_0,y_0) 处可微且 $f_x(x_0,y_0)=f_y(x_0,y_0)=0$~~，则 $z=f(x,y)$ 在 ~~(x_0,y_0)~~

~~处(　　)~~　设 $f(x,y)=\begin{cases}\dfrac{\ln(1+x^2y)}{xy}, & xy\neq 0\\ 0, & xy=0\end{cases}$，则 $f_x(0,1)=$（ B ）

A、~~无极值 B、可能有极值，也可能没有~~ C、~~有极大值 D、有极小值~~（　　）A. 0，B. 1，C. 2，D. 不存在

4、二元函数 $f(x,y)$ 在 $(0,0)$ 可微的一个必要条件是（ A ）　$\lim\limits_{x\to 0}\dfrac{f(x,1)-f(0,1)}{x}=\lim\limits_{x\to 0}\dfrac{\ln(1+x^2)-0}{x^2}=1$.

A、$\lim\limits_{(x,y)\to(0,0)}(f(x,y)-f(0,0))=0$　B、$\lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x}=\lim\limits_{y\to 0}\dfrac{f(0,y)-f(0,0)}{y}=0$

C、$\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}}=0$　D、$f(x,y)$ 在 $(0,0)$ 偏导数存在且连续

5.曲线 $\begin{cases} y=1-2x, \\ z=\dfrac{1}{2}-\dfrac{5}{2}x^2 \end{cases}$ $(1,-2,-5)$ 在点 $(1,-1,-2)$ 处 切线与直线 $\begin{cases} 5x-3y+3z-9=0, \\ 3x-2y+z-1=0 \end{cases}$ $(3,4,-1)$ 的夹角 $\varphi=$（ D ）

A.0　　B. $\dfrac{\pi}{4}$　　C. $\dfrac{\pi}{3}$　　D. $\dfrac{\pi}{2}$

二、填空题（每题 3 分，共 15 分）

1、$\lim\limits_{(x,y)\to(0,0)}(x+y)\sin\dfrac{1}{x}\sin\dfrac{1}{y}=$ __0__

2、设 f 有一阶连续偏导数，$z=f(x^2-y^2,\mathrm{e}^{xy})$，则 $\mathrm{d}z=$ $(2xf_1'+ye^{xy}f_2')dx+(-2yf_1'+xe^{xy}f_2')dy$

3、设连续函数 $z=f(x,y)$ 满足 $\lim\limits_{(x,y)\to(0,1)}\dfrac{f(x,y)-2x-3y}{\sqrt{x^2+(y-1)^2}}=0$ 则 $\mathrm{d}z|_{(0,1)}=$ __2__ $dx+3dy$

可知 $f(0,1)=3$，

$0=\lim\limits_{(x,y)\to(0,1)}\dfrac{[f(x,y)-3]-2x-3(y-1)}{\sqrt{x^2+(y-1)^2}}$

4、$F(x,y)=\int_0^{xy}\dfrac{\sin t}{1+t^2}\mathrm{d}t$，则 $\dfrac{\partial^2 F}{\partial x^2}|_{(0,2)}=$ __4__

$F_x=y\dfrac{\sin xy}{1+(xy)^2}$

5、$u=(\dfrac{x}{y})^{\frac{1}{z}}$ 在 $(1,1,1)$ 的梯度为 __$(1,-1,0)$__

$F_{xx}=y\cdot\dfrac{y\cos(xy)(1+x^2y^2)-\sin(xy)\cdot 2xy^2}{(1+x^2y^2)^2}$,

二、解下列各题（每题 10 分，共 40 分）

1、设 $z=x^{x^y}$，求 $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$

$$\ln z = x^y \ln x$$

$$\therefore \frac{1}{z}\cdot\frac{\partial z}{\partial x} = \frac{\partial x^y}{\partial x}\cdot\ln x + \frac{1}{x}\cdot x^y = (1+y\ln x)x^{y-1},$$

$$\therefore \frac{\partial z}{\partial x} = (1+y\ln x)x^{x^y+y-1}$$

类似可得 $\frac{\partial z}{\partial y} = x^{x^y+y} + \ln^2 x$

2、设 $x^2+z^2=y\varphi(\frac{z}{y})$，其中 φ 为可微函数，求 $\frac{\partial z}{\partial y}$

令 $F(x,y,z)=x^2+z^2-y\varphi(\frac{z}{y})$

$F_y=-\varphi+\frac{z}{y}\varphi'$，$F_z=2z-\varphi'$

$$\therefore \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{\varphi-\frac{z}{y}\varphi'}{2z-\varphi'}$$

3、求由方程组$\begin{cases} u+v=x \\ u^2+v^2=y \end{cases}$所确定的函数$u=u(x,y)$的二阶偏导数$\dfrac{\partial^2 u}{\partial x \partial y}$.

$u_x = -\dfrac{v}{u-v} \quad u_y = \dfrac{+1}{2(u-v)}, \quad v_x = \dfrac{u}{u-v}, \quad v_y = \dfrac{-1}{2(u-v)}$

$\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{v}{v-u}\right) = \dfrac{v_y(v-u)-v(v_y-u_y)}{(u-v)^2} = \dfrac{u+v}{2(u-v)^3}$

4、求由方程$x^2+y^2+z^2-2x+2y-4z-10=0$所确定的函数$z=z(x,y)$的极值.

令 $F = x^2+y^2+z^2-2x+2y-4z-10=0$

则 $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{x-1}{z-2}, \quad \dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{y+1}{z-2}$

令 $\begin{cases} \dfrac{\partial z}{\partial x}=0 \\ \dfrac{\partial z}{\partial y}=0 \end{cases}$ 得 $x=1$，$y=-1$，又当 $x=1$，$y=-1$ 时，原方程为 $z^2-4z-12=$

故得两点 $(1,-1,-2)$ 及 $(1,-1,6)$，计算得

$A=8$，$B=0$，$C=-\dfrac{1}{z-2}$，故 $AC-B^2=\dfrac{1}{(z-2)^2}>0$.

故 $(1,-1)$ 为极小值点，极值-2，$(1,-1)$为极大值

极大值 6.

三、解下列各题（每题 10 分，共 30 分）

1、试证光滑曲面$F(z-x, y-z)=0$所有切平面都与一固定的非零向量平行.

令 $G(x,y,z) = F(z-x, y-z)$.

法向量 为 $\vec{n} = (G_x, G_y, G_z) = (-F_1', F_2', F_1'-F_2')$

因此 $\vec{n} \cdot (1,1,1) = 0$.

这说明该光滑曲面所有法向量都与向量$(1,1,1)$垂直，

（即所有切平面都与$(1,1,1)$平行）. 柱面?

2、设 $u=x+2y+2, v=x-y-1$，$z=z(x,y)$ 有二阶连续偏导数，变换方程 $2z_{xx}+z_{xy}-z_{yy}+z_x+z_y=0$．

$z_x = z_u \cdot u_x + z_v \cdot v_x = z_u + z_v$.　　$z_y = z_u \cdot u_y + z_v \cdot v_y = 2z_u - z_v$

$z_{xx} = \frac{\partial}{\partial x} z_u + \frac{\partial}{\partial x} z_v = (z_{uu} \cdot u_x + z_{uv} \cdot v_x) + (z_{vu} \cdot u_x + z_{vv} \cdot v_x) = z_{uu} + 2z_{uv} + z_{vv}$

$z_{xy} = (z_{uu} \cdot u_y + z_{uv} \cdot v_y) + (z_{vu} \cdot u_y + z_{vv} \cdot v_y) = 2z_{uu} + z_{uv} - z_{vv}$

$z_{yy} = 2(z_{uu} \cdot u_y + z_{uv} \cdot v_y) - (z_{vu} \cdot u_y + z_{vv} \cdot v_y) = 4z_{uu} - 4z_{uv} + z_{vv}$

$2z_{xx} + z_{xy} - z_{yy} + z_x + z_y = 9z_{uv} + 3z_u = 0$

$\therefore 3z_{uv} + z_u = 0$

3、设 $u=f(x,y,z)$ 有连续偏导数，且 $x=r\sin\theta\cos\varphi$，$y=r\sin\theta\sin\varphi$，$z=r\cos\theta$，证明：若 $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$，则 u 与 r 无关

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial r}$

$= \frac{\partial u}{\partial x} \cdot \sin\theta\cos\varphi + \frac{\partial u}{\partial y} \cdot \sin\theta\sin\varphi + \frac{\partial u}{\partial z} \cos\theta$

$= \frac{1}{r}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right) = \frac{1}{r} \cdot 0 = 0$

$\therefore u$ 与 r 无关.

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