§7.1 多元函数的极限与连续 §7.2 偏导数和全微分

一、填空题

1. 函数 $z=\arcsin 2x+\dfrac{\sqrt{4x-y^2}}{\ln(1-x^2-y^2)}$ 的定义域为 $\{(x,y) \mid y^2 \le 4x$. $x^2+y^2<1,\ 0<x\le\frac{1}{2}\}$

2. 设三角形区域 D 由直线 $y=1, y=x, y=-x$ 所围，则 D 可用 X 型和 Y 型区域形式分别表示为 $D=\{(x,y) \mid -x\le y\le 1, -1\le x\le 0\}\cup\{(x,y)\mid x\le y\le 1, 0\le x\le 1\}$, $D=\{(x,y)\mid -y\le x\le y,\ 0\le y\le 1\}$.

3. 函数 $z=\dfrac{1}{\sin x\cdot\sin y}$ 在 $x=k_1\pi,\ y=k_2 y,\quad k_1,k_2\in Z$ 处是间断的.

4. $\lim\limits_{(x,y)\to(1,0)}\dfrac{\ln(x+e^y)}{\sqrt{x^2+y^2}}=$ $\ln 2$;

5. $\lim\limits_{(x,y)\to(0,0)}\dfrac{2-\sqrt{xy+4}}{xy}=$ $-\frac{1}{4}$;

6. $\lim\limits_{(x,y)\to(2,0)}\dfrac{\sin xy}{y}=$ 2;

7. $\lim\limits_{(x,y)\to(0,0)}\sqrt{x^2+y^2}\sin\dfrac{1}{x^2+y^2}=$ 0;

8. $\lim\limits_{(x,y)\to(0,1)}\dfrac{1-x+xy}{x^2+y^2}=$ 1;

9. $\lim\limits_{(x,y)\to(0,0)}\dfrac{1-\cos(x^2+y^2)}{(x^2+y^2)e^{x^2y^2}}=$ 0;

10. $\lim\limits_{\substack{x\to\infty\\ y\to a}}(1+\dfrac{1}{xy})^{\frac{x^2}{x+y}}(a\ne 0)=$ $e^{\frac{1}{a}}$.

$\left[\left(1+\frac{1}{xy}\right)^{xy}\right]^{\frac{x}{y(x+y)}}$ $\nearrow\frac{1}{a}$ $\downarrow e$

二、讨论函数 $f(x,y)=\begin{cases}(x^2+y^2)\ln(x^2+y^2), & x^2+y^2\ne 0,\\ 0, & x^2+y^2=0\end{cases}$ 在 $(0,0)$ 点的连续性.

令 $x^2+y^2=t$, 则 $\lim\limits_{(x,y)\to(0,0)}f(x,y)=\lim\limits_{t\to 0}t\ln t=\lim\limits_{t\to 0}\dfrac{\ln t}{\frac{1}{t}}=\lim\limits_{t\to 0}\dfrac{\frac{1}{t}}{-\frac{1}{t^2}}=0=f(0,0)$

故 $f(x,y)$ 在 $(0,0)$ 点 连续.

$t\to 0^+$

三、选择题

1. 二元函数 $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2},(x,y)\neq(0,0),\\ 0,\ (x,y)=(0,0)\end{cases}$ 在点$(0,0)$处（ C ）

A. 连续，偏导数存在 B. 连续，偏导数不存在

C. 不连续，偏导数存在 D. 不连续，偏导数不存在

2. 已知函数 $z=x^2e^y+(x-1)\arctan\dfrac{y}{x}$，则 $z_x(1,0)=$(c)

A. 0 B. 1 C. 2 D. 不存在

$z(x,0)=x^2$

$\therefore z_x=2x$

$z_x(1,0)=2$

四、求下列函数的偏导数：

1. $z=x^2y-xy^3$;

$\frac{\partial z}{\partial x}=2xy-y^3$

$\frac{\partial z}{\partial y}=x^2-3xy^2$

2. $z=\ln\cos(2x+y)$;

$\frac{\partial z}{\partial x}=-2\tan(2x+y)$

$\frac{\partial z}{\partial y}=-\tan(2x+y)$

3. $u=(\frac{x}{y})^z$

$\frac{\partial u}{\partial x}=z\cdot(\frac{x}{y})^{z-1}\cdot\frac{1}{y}=\frac{z\cdot x^{z-1}}{y^z}$

$\frac{\partial u}{\partial y}=z\cdot(\frac{x}{y})^{z-1}\cdot(-\frac{x}{y^2})=-\frac{z\cdot x^z}{y^{z+1}}$

$\frac{\partial u}{\partial z}=(\frac{x}{y})^z\cdot\ln(\frac{x}{y})$

4. $u=\int_{xz}^{yz}e^{t^2}dt.$

$\frac{\partial u}{\partial x}=-e^{x^2z^2}\cdot z$

$\frac{\partial u}{\partial y}=e^{y^2z^2}\cdot z$

$\frac{\partial u}{\partial z}=ye^{y^2z^2}-xe^{x^2z^2}$

五、求旋转曲面 $z=\sqrt{1+x^2+y^2}$ 与平面 $x=1$ 的交线在点 $(1,1,\sqrt{3})$ 处的切线与 y 轴正向之间的夹角.

~~曲~~ 曲线 $\begin{cases} z=\sqrt{1+x^2+y^2} \\ x=1 \end{cases}$ 在点 $(1,1,\sqrt{3})$ 处切线对 y 轴的斜率即为 $\left.\frac{\partial z}{\partial y}\right|_{(1,1)}$

即 $\tan\beta = \left.\frac{d}{dy}\sqrt{2+y^2}\right|_{y=1} = \left.\frac{y}{\sqrt{2+y^2}}\right|_{y=1} = \frac{1}{\sqrt{3}}$，$\therefore \beta=\frac{\pi}{6}$

六、求下列函数的 $\frac{\partial^2 z}{\partial x^2},\frac{\partial^2 z}{\partial y^2},\frac{\partial^2 z}{\partial x\partial y}$:

1. $z=x^4+y^4-4x^2y^2$;

$\frac{\partial z}{\partial x} = 4x^3-8xy^2$，$\frac{\partial z}{\partial y} = 4y^3-8x^2y$

$\frac{\partial^2 z}{\partial x^2}=12x^2-8y^2$，$\frac{\partial^2 z}{\partial y^2} = 12y^2-8x^2$，$\frac{\partial^2 z}{\partial x\partial y} = -16xy$

2. $z=x\arcsin\sqrt{y}$;

$\frac{\partial z}{\partial x} = \arcsin\sqrt{y}$，$\frac{\partial z}{\partial y} = \frac{x}{\sqrt{1-y}\cdot 2\sqrt{y}}$

$\frac{\partial^2 z}{\partial x^2} = 0$，$\frac{\partial^2 z}{\partial y^2} = -\frac{x(1-2y)}{4\sqrt{y^3(1-y)^3}}$，$\frac{\partial^2 z}{\partial x\partial y} = \frac{1}{2\sqrt{y}\cdot\sqrt{1-y}}$

3. $z=e^{xy^2}$.

$\frac{\partial z}{\partial x} = y^2e^{xy^2}$，$\frac{\partial z}{\partial y} = 2xye^{xy^2}$

$\frac{\partial^2 z}{\partial x^2} = y^4e^{xy^2}$，$\frac{\partial^2 z}{\partial y^2} = (2x+4x^2y^2)e^{xy^2}$，

$\frac{\partial^2 z}{\partial x\partial y} = (2y+2xy^3)e^{xy^3}$

七、求函数 $z=5x^2+y^2$ 当 $x=1, y=2, \Delta x=0.005, \Delta y=0.1$ 时的全增量和全微分.

$$\Delta z = f(x+\Delta x, y+\Delta y)-f(x,y) = 5(x+\Delta x)^2+(y+\Delta y)^2-(5x^2+y^2)$$
$$= 10x\Delta x + 2y\Delta y + 5(\Delta x)^2+(\Delta y)^2$$
$$dz = 10x\Delta x + 2y\Delta y$$
$$\therefore \Delta z(1,2)\Big|_{\substack{\Delta x=0.005\\ \Delta y=0.1}} = 10\times1\times0.005+2\times2\times0.1+5\times(0.005)^2+(0.1)^2 = 0.460125$$
$$dz(1,2)\Big|_{\substack{\Delta x=0.005\\ \Delta y=0.1}} = 10\times1\times0.005+2\times2\times0.1 = 0.45$$

八、设二元函数 $z=xe^{x+y}+(x+1)\ln(1+y)$, 求 dz 和 $dz|_{(1,0)}$.

$$\frac{\partial z}{\partial x} = e^{x+y}+xe^{x+y}+\ln(1+y)$$
$$\frac{\partial z}{\partial y} = xe^{x+y}+\frac{x+1}{1+y}$$
$$\therefore dz = [(x+1)e^{x+y}+\ln(1+y)]dx + \left(xe^{x+y}+\frac{x+1}{1+y}\right)dy$$

$$dz\big|_{(1,0)} = 2e\,dx + (e+2)dy$$

九、设二元函数 $f(x,y)=\begin{cases}(x^2+y^2)\cos\dfrac{1}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0\end{cases}$,

（1）求 $f_x(0,0), f_y(0,0)$;

（2）讨论 $f(x,y)$ 在点 $(0,0)$ 是否可微.

(1) $f_x(0,0)=\lim\limits_{x\to0}\dfrac{f(x,0)-f(0,0)}{x} = \lim\limits_{x\to0}\dfrac{x^2\cos\frac{1}{|x|}}{x}=0$，同理 $f_y(0,0)=0$

(2) $$\lim_{\rho\to0}\frac{\Delta f-[f_x\Delta x+f_y\Delta y]}{\rho} = \lim_{\rho\to0}\frac{f(x,y)-0-[0+0]}{\rho} = \lim_{\rho\to0}\frac{(x^2+y^2)\cos\frac{1}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} = \lim_{\rho\to0}\rho\cos\frac{1}{\rho}=0$$

故 $f(x,y)$ 在 $(0,0)$ 点处可微.