§7.3 复合函数和隐函数的偏导数

一、用链法则求下列函数的导数或偏导数：

1. $z=u^v, u=x+2y, v=x-y$，求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$；

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= v\cdot u^{v-1}\cdot 1 + u^v\cdot \ln u\cdot 1 = (x-y)(x+2y)^{x-y-1} + (x+2y)^{x-y}\cdot\ln(x+2y) = (x+2y)^{x-y}\left[\frac{x-y}{x+2y} + \ln(x+2y)\right]$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = v\cdot u^{v-1}\cdot 2 + u^v \ln u\cdot(-1) = (x+2y)^{x-y}\left[\frac{2(x-y)}{x+2y} - \ln(x+2y)\right]$$

2. $z=\dfrac{y}{x}, x=e^t, y=1-e^{2t}$，求 $\dfrac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} = \left(-\frac{y}{x^2}\right)\cdot e^t + \frac{1}{x}\cdot(-2e^{2t})$$

$$= -\frac{1-e^{2t}}{e^{2t}}\cdot e^t + \frac{1}{e^t}(-2e^{2t}) = -e^t - e^{-t}$$

二、求下列复合函数的一阶偏导数：

1. $u=f(x,xy,xyz)$，求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial z}$；

$$\frac{\partial u}{\partial x} = f_1' + y\cdot f_2' + yz f_3'$$

$$\frac{\partial u}{\partial y} = x f_2' + xz f_3'$$

$$\frac{\partial u}{\partial z} = xy f_3'$$

2. $z=f(xy,\dfrac{x}{y})+\varphi(\dfrac{y}{x})$，其中 f,φ 均可微，求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = y f_1' + \frac{1}{y} f_2' + \varphi'\cdot\left(-\frac{y}{x^2}\right)$$

$$\frac{\partial z}{\partial y} = x f_1' - \frac{x}{y^2} f_2' + \frac{1}{x}\varphi'$$

三、设函数 $z=f(x,y)$ 在点 $(1,1)$ 处可微，且

$$f(1,1)=1,\frac{\partial f}{\partial x}\Big|_{(1,1)}=2,\frac{\partial f}{\partial y}\Big|_{(1,1)}=3,\varphi(x)=f(x,f(x,x)),\text{求}\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\Big|_{x=1}.$$

$\varphi(1)=f(1,f(1,1))=f(1,1)=1$

$\frac{d}{dx}\varphi^3(x)=3\varphi^2(x)\cdot[f_1'(x,f(x,x))+f_2'(x,f(x,x))(f_1'(x,x)+f_2'(x,x))]$

将 $x=1$ 代入得

$\frac{d}{dx}\varphi^3(x)\Big|_{x=1}=3\cdot1\cdot[2+3(2+3)]=51$

四、设 $z=f(x^2+y^2)$，其中 f 具有二阶导数，求 $\frac{\partial^2 z}{\partial x^2},\frac{\partial^2 z}{\partial x\partial y},\frac{\partial^2 z}{\partial y^2}$.

$\frac{\partial z}{\partial x}=f'(u)\cdot 2x$ ， $\frac{\partial z}{\partial y}=f'(u)\cdot 2y$

$\frac{\partial^2 z}{\partial x^2}=2f'+2x\cdot f''\cdot 2x=2f'+4x^2f''$

$\frac{\partial^2 z}{\partial x\partial y}=2xf''\cdot 2y=4xyf''$

$\frac{\partial^2 z}{\partial y^2}=2f'+2yf''\cdot 2y=2f'+4y^2f''$.

五、设 $z=yf(\mathrm{e}^x,xy)$，其中 f 具有二阶连续偏导数，求 $\frac{\partial^2 z}{\partial x^2},\frac{\partial^2 z}{\partial x\partial y},\frac{\partial^2 z}{\partial y^2}$.

$\frac{\partial z}{\partial x}=y[f_1'\cdot e^x+yf_2']=ye^xf_1'+y^2f_2'$

$\frac{\partial z}{\partial y}=f+xyf_2'$

$\frac{\partial^2 z}{\partial x^2}=e^xyf_1'+ye^{2x}f_{11}''+2y^2e^xf_{12}''+y^3f_{22}''$

$\frac{\partial^2 z}{\partial x\partial y}=e^xf_1'+2yf_2'+xye^xf_{12}''+xy^2f_{22}''$

$\frac{\partial^2 z}{\partial y^2}=2xf_2'+x^2yf_{22}''$

六、求下列方程所确定的隐函数 $y=f(x)$ 的一阶导数：

（1） $x^2+xy-e^y=0$;　　　　（2） $x^y=y^x$.

法一：两边对 x 求导，

$2x+y+xy'-e^y\cdot y'=0 \Rightarrow y'=\frac{2x+y}{e^y-x}$

法二：令 $F(x,y)=x^2+xy-e^y$，则

$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{2x+y}{x-e^y}$.

令 $F(x,y)=x^y-y^x$，则

$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{yx^{y-1}-y^x\ln y}{x^y\ln x-xy^{x-1}}$

$=\frac{xy\ln y-y^2}{xy\ln x-x^2}=\frac{y\ln y^x-y^2}{x\ln x^y-x^2}=\frac{y\ln x^y-y^2}{x\ln y^x-x^2}$

$=\frac{y^2(\ln x-1)}{x^2(\ln y-1)}$

七、求方程 $z=e^{2x-3z}+2y$ 所确定的隐函数 $z=f(x,y)$ 的一阶偏导数：

两边对 x 求偏导，$\frac{\partial z}{\partial x}=e^{2x-3z}(2-3\frac{\partial z}{\partial x})$

$\therefore \frac{\partial z}{\partial x}=\frac{2e^{2x-3z}}{1+3e^{2x-3z}}$

两边对 y 求偏导，$\frac{\partial z}{\partial y}=e^{2x-3z}(-3\frac{\partial z}{\partial y})+2$

$\therefore \frac{\partial z}{\partial y}=\frac{2}{1+3e^{2x-3z}}$

八、已知 $x^2+y^2+z^2=4z$，求 $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y},\frac{\partial^2 z}{\partial x^2}$.

令 $F(x,y,z)=x^2+y^2+z^2-4z$

则 $\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{2x}{2z-4}=\frac{x}{2-z}$

$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{2y}{2z-4}=\frac{y}{2-z}$

$\frac{\partial^2 z}{\partial x^2}=\left(\frac{x}{2-z}\right)'_x=\frac{2-z+x\cdot\frac{\partial z}{\partial x}}{(2-z)^2}=\frac{2-z+\frac{x^2}{2-z}}{(2-z)^2}=\frac{(2-z)^2+x^2}{(2-z)^3}$

九、已知 $z+\ln z-\int_y^x e^{-t^2}dt=0$，求 $\frac{\partial^2 z}{\partial x\partial y}$.

令 $F(x,y,z)=z+\ln z-\int_y^x e^{-t^2}dt$

则 $\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{e^{-x^2}}{1+\frac{1}{z}}=\frac{z}{z+1}e^{-x^2}$，$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{e^{-y^2}}{1+\frac{1}{z}}=-\frac{z}{z+1}e^{-y^2}$

$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{z}{z+1}e^{-x^2}\right)=\frac{(z+1)\frac{\partial z}{\partial y}-z\frac{\partial z}{\partial y}}{(z+1)^2}e^{-x^2}=\frac{-\frac{z}{z+1}e^{-y^2}}{(z+1)^2}e^{-x^2}=\frac{z}{(z+1)^3}e^{-x^2-y^2}$.

十、设 $u=f(x,y,z)$ 有连续的偏导数，$y=y(x)$ 和 $z=z(x)$ 分别由方程 $e^{xy}-y=0$ 和 $e^z-xz=0$ 所确定，求 $\dfrac{du}{dx}$.

$$\frac{du}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}+\frac{\partial f}{\partial z}\cdot\frac{dz}{dx}$$

由 $e^{xy}-y=0$ 得 $e^{xy}(y+x\frac{dy}{dx})-\frac{dy}{dx}=0$，$\therefore \frac{dy}{dx}=\frac{ye^{xy}}{1-xe^{xy}}=\frac{y^2}{1-xy}$

由 $e^z-xz=0$ 得 $e^z\frac{dz}{dx}-z-x\frac{dz}{dx}=0$ $\quad\therefore \frac{dz}{dx}=\frac{z}{e^z-x}=\frac{z}{xz-x}$

于是 $\frac{du}{dx}=f_x+\frac{y^2}{1-xy}\cdot f_y+\frac{z}{xz-x}\cdot f_z$

十一、求由下列方程组所确定的隐函数的导数或偏导数：

1. $\begin{cases}x+y+z=0,\\ xyz=1,\end{cases}$ 求 $\dfrac{dz}{dx},\dfrac{dy}{dx}$.

方程组两边对 x 求导得 $\begin{cases}1+\frac{dy}{dx}+\frac{dz}{dx}=0\\ yz+xz\frac{dy}{dx}+xy\frac{dz}{dx}=0\end{cases}\Rightarrow\begin{cases}\frac{dy}{dx}+\frac{dz}{dx}=-1\\ xz\frac{dy}{dx}+xy\frac{dz}{dx}=-yz\end{cases}$

解得 $\frac{dy}{dx}=\frac{y(z-x)}{x(y-z)}$. $\frac{dz}{dx}=\frac{z(x-y)}{x(y-z)}$

2. $\begin{cases}u=f(ux,v+y),\\ v=g(u-x,v^2y),\end{cases}$ 其中 f,g 具有一阶连续偏导数，求 $\dfrac{\partial u}{\partial x},\dfrac{\partial v}{\partial x}$.

对 x 求偏导，得

$$\begin{cases}\frac{\partial u}{\partial x}=f_1'\cdot(u+x\frac{\partial u}{\partial x})+f_2'\cdot\frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial x}=g_1'\cdot(\frac{\partial u}{\partial x}-1)+2g_2'\cdot yv\cdot\frac{\partial v}{\partial x}\end{cases}$$

解方程组得 $\frac{\partial u}{\partial x}=\frac{-uf_1'\cdot(2yvg_2'-1)-f_2'\cdot g_1'}{(xf_1'-1)(2yvg_2'-1)-f_2'\cdot g_1'}$

$$\frac{\partial v}{\partial x}=\frac{g_1'\cdot(xf_1'+uf_1'-1)}{(xf_1'-1)(2yvg_2'-1)-f_2'\cdot g_1'}$$