A 1d stationary convection diffusion problem

TMA4212

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS BY DIFFERENCE METHODS

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1 Summary

In this project we've discussed $\mathbb{P}^1$ finite element methods (FEM) for solving elliptic one-dimensional differential equations. The idea of the $\mathbb{P}^1$-FEM is to reformulate the problem into a weak problem, to which a solution is also a solution to the original differential equation. Through some functional analysis, the solution to the weak formulation can be approximated as the solution of some linear system.

Specifically, we've looked at an elliptic differential equation modelling the stationary concentration of some diffusive, convective matter in a finite one dimensional domain. We've found and discussed its weak formulation and implemented its $\mathbb{P}^1$-FEM. The error has been studied both analytically, both in general and for our problem, and empirically, using test functions.

Lastly, some special cases has been examined. We've discussed the use of our method on non-smooth test functions (where the actual differential equation isn't well-defined), and whether the use of non-equidistant points could be beneficial on certain problems.

2 Weak formulation and it's properties

2.1 Weak formulation of ODEs

Consider the general Dirichlet problem

$$\begin{cases} -Du = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{1}$$

where Ω is the domain of u and D is some differential operator. A weak formulation of the problem can then be formulated: Find $u \in V$ such that for all $v \in V$:

$$\begin{cases} a(u,v) = F(v) & \\ u = g & \text{on } \partial\Omega \end{cases} \tag{2}$$

where V is some Hilbert function space on Ω with norm $\|\cdot\|_V$ such that the solution u of 1 is in V, and $a : V \times V \to \mathbb{R}$ is bilinear, continuous and coercive ($|a(v,v)| \geq \alpha\|v\|_V^2$) for all $v \in V$ for some $\alpha > 0$), and $F : V \to \mathbb{R}$ is a linear, bounded functional such that a solution of 2 also solves 1.

Galerkin methods approximate solutions of 2 by solving it on some finite-dimensional function space V_h, approximating V. FEM-methods are special cases of Galerkin methods.

2.2 Weak formulation of the 1d stationary convection diffusion problem

Through mass-conservation, Fick's diffusion law, and Reynolds transport theorem, we obtain the following model for the stationary concentration of some convective, diffusive substance in a finite, 1-dimensional scaled domain:

$$-\partial_x(\alpha(x)u_x) + \partial_x(b(x)u) + c(x)u = f(x) \quad \text{in } \Omega = (0,1) \tag{3}$$

where $\alpha(x) > 0$ is the diffusion coefficient, $b(x)$ is the convective velocity, $c(x)$ is the substance decay rate, and $f(x)$ is the fluid source. In the following discussion, let $\alpha(x) \geq \alpha_0 > 0$, $c(x) > 0$ and $\|\alpha\|_{L^\infty(0,1)} + \|b\|_{L^\infty(0,1)} + \|c\|_{L^\infty(0,1)} + \|f\|_{L^2(0,1)} \leq K$ where $K > 0$. We impose the zero-boundary conditions $u(0) = 0 = u(1)$.

We then know that $u \in H_0^1(0,1)$, where $H_0^1(0,1) = \{g \in H^1(0,1) : g(0) = 0 = g(1)\}$, and $H^i(0,1) = \{g : g, \nabla g, \dots, \nabla^i g \in L^2\}$. $H_0^1(0,1)$ is Hilbert with inner product

$$\langle u, v\rangle_{H_0^1} = \int_0^1 uvdx + \int (\nabla u)(\nabla v)dx$$

Here, ∇g is the weak derivative of g. For now it is enough to know that $\nabla g = g_x$ if g_x exists. We can find a weak formulation of 3 by multiplying with some arbitrary $v \in H_0^1(0,1)$, then integrating over $(0,1)$ and expanding through integrating by parts:

$$\begin{aligned}
&\int_0^1 (-\partial_x(\alpha(x)u_x)v + \partial_x(b(x)u)v + c(x)uv)dx = \int_0^1 f(x)vdx =: F(v) = \\
&= -[\alpha u_x v]_0^1 + \int_0^1 \alpha u_x v_x dx + [buv]_0^1 - \int_0^1 buv_x dx + \int_0^1 cuvdx = \\
&= \int_0^1 (\alpha u_x v_x - buv_x + cuv)dx =: a(u,v)
\end{aligned} \tag{4}$$

as $v \in H_0^1(0,1)$, and thus is 0 evaluated at $x = 0$ and $x = 1$. We then have our functions $F(v)$ and $a(u,v)$. Lets first see that $F(v)$ is linear and bounded:

$$\begin{aligned}
&F(ku + lv) = \int_0^1 (ku + lv)dx = k\int_0^1 udx + l\int_0^1 vdx = kF(u) + lF(v)\forall k, l \in \mathbb{R} \\
&|F(v)| = |\langle f, v\rangle_{L^2(0,1)}| \leq \|f\|_{L^2(0,1)}\|v\|_{L^2(0,1)} \leq K\|v\|_{H_0^1}
\end{aligned}$$

Where $\|u\|_{H_0^1} = \sqrt{\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2}$ is the induced norm on $H_0^1(0,1)$, and thus always greater or equal to $\|u\|_{L^2(0,1)}$ and $\|u_x\|_{L^2(0,1)}$. We now show biliearity and boundedness for $a(u,v)$:

$$\begin{aligned}
&a(ku_1 + lu_2, v) = \int_0^1 (\alpha(ku_1 + lu_2)_x v_x - b(ku_1 + lu_2)v_x + c(ku_1 + lu_2)v)dx = \\
=&k\int_0^1 (\alpha(u_1)xv_x - bu_1v_x + cu_1v)dx + l\int_0^1 (\alpha(u_2)_x v_x - bu_2v_x + cu_2v)dx = ka(u_1,v) + la(u_2,v) \\
&\forall k, l \in \mathbb{R}
\end{aligned}$$

and similarly for the second argument.

$$\begin{aligned}
&|a(u,v)| \leq |\int_0^1 \alpha u_x v_x| + |\int_0^1 buv_x dx| + |\int_0^1 cuvdx| \leq \max_{x\in(0,1)}(\alpha(x))\|u_x\|_{L^2}\|v_x\|_{L^2} + \\
&\max_{x\in(0,1)}(b(x))\|u\|_{L^2}\|v_x\|_{L^2} + \max_{x\in(0,1)}(c(x))\|u\|_{L^2}\|v\|_{L^2} \leq K\|u\|_{H_0^1}\|v\|_{H_0^1}
\end{aligned} \tag{5}$$

2.3 Existence of solution in H_0^1

According to the Lax-Milgram theorem, if $F(v)$ is a continuous linear functional and $a(u,v)$ is a continuous, coercive bi-linear form, i.e. there exist $M, \tilde{\alpha} \geq 0$ such that

1. (Continuous) $|a(u,v)| \leq M\|u\|_V\|v\|_V, \forall u, v \in V$ for some $M > 0$,
2. (Coercive) $|a(v,v)| \geq \tilde{\alpha}\|v\|_V^2, \forall v \in V$ for some $\tilde{\alpha} > 0$,

then 2 admits a unique solution u [1, s. 15].

In our case $V = H_0^1$. We have already shown that F is a continuous linear functional on H_0^1, and that $a(u,v)$ is a continuous bi-linear form on $H_0^1 \times H_0^1$. From now on, let $\alpha \geq 0, b \neq 0, c \geq 0$ be constants. We want to prove that $a(u,v)$ is coercive when $c > \frac{|b|^2}{2\alpha}$. To do this we first show the following relation

$$a(u,u) \geq (\alpha - \frac{\epsilon}{2}|b|)\int_1^0 u_x^2\,dx + (c - \frac{1}{2\epsilon}|b|)\int_1^0 u^2\,dx \forall\epsilon > 0$$

"Young's" inequality, $-ab \geq -\frac{1}{2\epsilon}a^2 - \frac{\epsilon}{2}b^2 \forall\epsilon > 0$, used on the part of $a(u,u)$ which involves the constant b yields

$$\begin{aligned}
&- b\int_0^1 u_x u\,dx \geq -\frac{\epsilon}{2}|b|\int_0^1 u_x^2\,dx - \frac{1}{2\epsilon}|b|\int_0^1 u^2\,dx, \forall\epsilon > 0 \\
&\implies a(u,u) \geq \alpha\int_0^1 u_x^2\,dx - \frac{\epsilon}{2}|b|\int_0^1 u_x^2\,dx + c\int_0^1 u^2\,dx - \frac{1}{2\epsilon}|b|\int_0^1 u^2\,dx \\
&= (\alpha - \frac{\epsilon}{2}|b|)\int_0^1 u_x^2\,dx + (c - \frac{1}{2\epsilon}|b|)\int_0^1 u^2\,dx, \forall\epsilon > 0
\end{aligned}$$

If ϵ is chosen to be $\frac{\alpha}{|b|}$ then

$$a(v,v) = (\frac{\alpha}{2})\int_0^1 v_x^2\,dx + (c - \frac{|b|^2}{2\alpha})\int_0^1 v^2\,dx \geq C\|v\|_{H^1}^2, C = min\{\frac{\alpha}{2}, c - \frac{|b|^2}{2\alpha}\} \tag{6}$$

$a(u,v)$ is coercive when both of the coefficients are strictly positive. This restricts c to $c > \frac{|b|^2}{2\alpha}$. Together with the boundary condition, we have proven that there exist a unique solution $u \in H_0^1$.

3 Implementation and testing of the $\mathbb{P}^1$- FEM

For some triangulation into subintervals $\mathcal{T}_h = \{K_i\}_{i=1}^M$, we can approximate the weak formulation 2 with

$$(V_h) \qquad \text{Find } u \in V_h \text{ such that } \; a(u_h, v_h) = F(v_h) \;\forall\; v_h \in V_h, \tag{7}$$

where $V_h = X_h^1 \cap H_0^1(0,1)$ is the space of continuous functions piece-wise linear on $\mathcal{T}_h$ with 0-boundary condition. This is the $\mathbb{P}_1$-FEM on a general grid on $(0,1)$, implemented below.

3.1 Implementation

From the derivation of $a(u,v)$ we have:

$$a(u,v) := \int_0^1 \alpha u_x v_x - buv_x + cuvdx = \int_0^1 f(x)vdx =: F(v).$$

We can describe any $v_h \in V_h$ by a linear combination of linearly independent basis functions $(\varphi_i(x),\;\; i = 0,1,\ldots,M)$ of V_h:

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_i}, & x \in [x_{i-1}, x_i) = K_i \\ \frac{x_{i+1} - x}{h_{i+1}}, & x \in [x_i, x_{i+1}) = K_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

Where we have M intervals $K_1, \ldots, K_M$ between the global nodes $0 = x_0 < x_1 < \ldots < x_M = 1$, together making up the triangulation $\mathcal{T}_h$. The definition is equivalent for φ_0, φ_M where we simply set the function to 0 for all points outside the interval $(0,1)$. As $u_h \in V_h$, we can also decompose the solution of 7 into these basis functions and obtain a linear system $A\vec{U} = F$, where $A_{ij} = a(\varphi_j, \varphi_i)$, $F_j = \int_0^1 f\varphi_j dx = \sum_{K\in\mathcal{T}_h} \int_K f\varphi_j$.

To derive the linear system we must find an expression for A_{ij} and F_j. From the definition of the basis functions, one can easily obtain the derivatives

$$\varphi_i'(x) = \begin{cases} \frac{1}{h_i}, & x \in K_i \\ \frac{1}{h_{i+1}}, & x \in K_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

We then get:

$$A_{ij} = a(\varphi_j, \varphi_i) = \int_0^1 \alpha\varphi_j'\varphi_i' - b\varphi_j\varphi_i' + c\varphi_j\varphi_i dx = \begin{cases} -\frac{\alpha}{h_i} - \frac{b}{2} + \frac{ch_i}{6}, & j = i-1 \\ \alpha(\frac{1}{h_i} + \frac{1}{h_{i+1}}) + \frac{c}{3}(h_i + h_{i+1}), & j = i \\ -\frac{\alpha}{h_{i+1}} + \frac{b}{2} + \frac{ch_{i+1}}{6}, & j = i+1 \end{cases}.$$

Where we simply perform the integrals over each interval where the basis functions are defined to get the cases. Note that we get a tridiagonal matrix, which is the way we have implemented the assembly. Not immediately obvious, we get that the first and last rows of the matrix A are defined as follows:

$$A_{0j} = \begin{cases} \frac{\alpha}{h_1} + \frac{b}{2} + \frac{ch_1}{3}, & j = 0 \\ -\frac{\alpha}{h_1} + \frac{b}{2} + \frac{ch_1}{6}, & j = 1 \end{cases}, \quad A_{Mj} = \begin{cases} -\frac{\alpha}{h_M} - \frac{b}{2} + \frac{ch_M}{6}, & j = M-1 \\ \frac{\alpha}{h_M} - \frac{b}{2} + \frac{ch_M}{3}, & j = M \end{cases}.$$

F_j has to be found by a quadrature approximation in most cases. In our implementation we used the Trapezoidal rule, which is of second order, to get a possibility of second order convergence of our method: $F_j = \sum_{K\in\mathcal{T}_h} \int_K f\varphi_j \approx \frac{0+f_j\cdot 1}{2}h_j + \frac{f_j\cdot 1+0}{2}h_{j+1} = \frac{h_j+h_{j+1}}{2}f_j$. This happens as $\varphi_i(x_j) = 0 \ \forall \ x_j : j \neq i$. We define $F_0 = \frac{h_1}{2}f_0$.

The method described above has been implemented in the jupyter notebook attached to the project, and we attach figures of the solution in two cases, as well as the stiffness matrix in the first case.

3.2 Testing

We test our method using the test functions $u_1(x) = x(1-x)$ and $u_2(x) = \sin(3\pi x)$. By imposing exact solutions we have to calculate the corresponding $f = -\alpha u_{xx} + bu_x + cu$ from 4. We find that

$$\begin{cases} f_1(x) = 2\alpha + b(1-2x) + cx(1-x) \\ f_2(x) = \sin(3\pi x)(\alpha(3\pi)^2 + c) + b3\pi\cos(3\pi x) \end{cases}$$

In both examples we measure the errors in both H^1 and L^2 as we reduce the intervals between the equidistant grid points. Testing with these functions, we find the numerical approximation 1 and obtain the log-log plots 2.

We thus have second order convergence in L^2 and first order convergence in H_0^1 for both test functions, which is exactly the theoretical convergence.

3.3 Error bound in H^1

To find an error bound of our Galerkin approximation for solving problem 7, Galerkin orthogonality and Cea's lemma are used. If u and u_h are the solutions of the infinite and finite dimensional problems respectively, then $a(u - u_h, v_h) = 0 \forall v_h \in V_h$ (Galerkin orthogonality) [1, s.18] This result comes from the bilinear property of a, $a(u, v_h) = F(v_h)$ and $a(u_h, v_h) = F(v_h)$:

$$a(u - u_h, v_h) = a(u, v_h) - a(u_h, v_h) = F(v_h) - F(v_h) = 0 \tag{8}$$

Cea's lemma puts a restriction on $\|u-u_h\|_{H^1}$ through continuity and coercivity of a:

$$\begin{aligned}
&\|u-u_h\|^2_{H^1} \overset{(6)}{\leq} \frac{1}{C}|a(u-u_h,u-u_h)| = \frac{1}{C}|a(u-u_h,u-v_h+v_h-u_h)| \\
&\frac{1}{C}|a(u-u_h,u-v_h)+a(u-u_h,v_h-u_h)| \overset{(8)}{=} \frac{1}{C}|a(u-u_h,u-v_h)+0| \\
&\overset{(5)}{\leq} \frac{K}{C}\|u-u_h\|_{H^1}\|u-v_h\|_{H^1}\forall v_h \in V_h \\
&\implies \|u-u_h\|_{H^1} \leq \frac{K}{C}\underset{v_h\in V_h}{inf}\|u-v_h\|_{H^1}, C = min\{\frac{\alpha}{2}, c-\frac{|b|^2}{2\alpha}\}
\end{aligned} \tag{9}$$

Since the equation holds for all $v_h \in V_h$, it also holds for the infimum. We have the following upper bound for linear interpolation error of functions in H_0^1:

$$\|e\|_{H^1} = \|v-I_hv\|_{H^1} \leq h\sqrt{2}\|v''\|_{L^2}, v\in H_0^1 \tag{10}$$

where I_hv denotes the linear interpolation of v in the space $X_h^1 \cap H_0^1$. h is the maximum distance between two adjoining nodes in $\mathcal{T}_h$.

With all of the results above, we can find an error bound the Galerkin approximation.

$$\begin{aligned}
&\|u-u_h\|_{H^1} \overset{(9)}{\leq} \frac{K}{C}\underset{v_h\in V_h}{inf}\|u-v_h\|_{H^1}, I_hu \in V_h \\
&\leq \frac{K}{C}\|u-I_hU\|_{H^1} \\
&\overset{(10)}{\leq} \frac{K}{C}\sqrt{2}h\|u''\|_{L^2(0,1)},\ \ C = min\{\frac{\alpha}{2}, c-\frac{|b|^2}{2\alpha}\}
\end{aligned} \tag{11}$$

One can see from the plots in figure 2 that the error bound holds. However, in our case it is two to three orders of magnitude greater than the actual error.

3.4 Non-smooth test functions

We will here test the method on non-smooth test functions. Consider the functions

$$w_1(x) = \begin{cases} 2x & x\in(0,\frac{1}{2}) \\ 2(1-x) & x\in(\frac{1}{2},1) \end{cases}$$

$$w_2(x) = x-|x|^{\frac{2}{3}}$$

Note that both of these have singular derivatives at respectively $x=\frac{1}{2}$ and $x=0$. We must then use their weak derivatives. The weak x-derivative $g=\nabla f$ of f in the domain $(0,1)$ must satisfy

$$\int_0^1 gvdx = -\int_0^1 fv_xdx \quad \forall v\in C_0^1(0,1) \tag{12}$$

where $C_0^1(0,1)$ is the set of continuously differentiable functions on $(0,1)$ with 0 boundary conditions. The weak derivative is thus a generalization of the derivative defined by the integration by parts identity.

Let's study $w_1(x)$ first. We suggest the following weak derivative:

$$\nabla w_1(x) = \begin{cases} 2 & x\in[0,\frac{1}{2}] \\ -2 & x\in(\frac{1}{2},1] \end{cases}$$

Then

$$\int_0^1 \nabla w_1 v = 2\int_0^{\frac{1}{2}} v - 2\int_{\frac{1}{2}}^1 x v_x v = 2[xv_x]_0^{\frac{1}{2}} - 2\int_0^{\frac{1}{2}} xv_x - 2[xv_x]_{\frac{1}{2}}^1 + 2\int_{\frac{1}{2}}^1 xv_x =$$
$$2v(\frac{1}{2}) - 2\int_0^{\frac{1}{2}} xv_x + 2\int_{\frac{1}{2}}^1 xv_x = -2\int_0^{\frac{1}{2}} xv_x - 2\int_{\frac{1}{2}}^1 (1-x)v_x = -\int_0^1 w_1 v_x$$

$\nabla w_1(x)$ is thus a weak derivative of $w_1(x)$. Consider further the possibility of a weak derivative $\nabla^2 w_1$ of ∇w_1. then

$$\int_0^1 \nabla^2 w_1 v = -2\int_0^{\frac{1}{2}} v + 2\int_{\frac{1}{2}}^1 v = -4v(\frac{1}{2})$$

The only function which satisfies this identity for all $v \in C_0^1(0,1)$ is the Dirac delta-function $-4\delta(x-\frac{1}{2})$, which is not in L^2. w_1 is thus in $H_0^1(0,1)$ (as w_1 and ∇w_1 obviously is in L^2), but not in $H_0^2(0,1)$. As it is then impossible to define f, we find $F(v)$ via $a(w_1, v)$:

$$F(v) = a(w_1, v) = \int_0^1 (\alpha \nabla w_1 v_x - b w_1 v_x + c w_1 v)dx =$$
$$2\alpha(\int_0^{\frac{1}{2}} v_x - \int_{\frac{1}{2}}^1 v_x) - 2b(\int_0^{\frac{1}{2}} xv_x + \int_{\frac{1}{2}}^1 (1-x)v_x) + 2c(\int_0^{\frac{1}{2}} xv + \int_{\frac{1}{2}}^1 (1-x)v)$$
$$= 4\alpha v(\frac{1}{2}) + 2b(\int_0^{\frac{1}{2}} v - \int_{\frac{1}{2}}^1 v) + 2c(\int_0^{\frac{1}{2}} xv + \int_{\frac{1}{2}}^1 (1-x)v)$$

Similiarly for w_2, we suggest $\nabla w_2 = 1 - \frac{2}{3}x^{-\frac{1}{3}}$ being a weak derivative on $(0,1)$. We see that

$$\int_0^1 \nabla w_2 dx = [x - x^{\frac{2}{3}}]_0^1 = 0, \text{ and}$$
$$\int_0^1 \nabla w_2 v dx = \int_0^1 (v - \frac{2}{3}x^{-\frac{1}{3}} v)dx = -\int_0^1 (x - x^{\frac{2}{3}})v_x dx = -\int_0^1 w_2 v_x dx$$

satisfying 12 and $\nabla w_2 \in L^2(0,1)$. Further, consider the weak derivative $\nabla^2 w_s$ of ∇w_2. From 12:

$$\int_0^1 \nabla^2 w_2 v dx = -\int_0^1 \nabla w_2 v_x dx = -\int_0^1 v_x dx + \int_0^1 \frac{2}{3}x^{-\frac{1}{3}} v_x dx = \int_0^1 \frac{2}{9}x^{-\frac{4}{3}} v dx,$$

which is only satisfied for all $v \in C^1$ if $\nabla^2 w_2 = \frac{2}{9}x^{-\frac{4}{3}}$. This is not in L^2. Then also $w_2 \in H_0^1(0,1)$, $w_2 \notin H_0^2(0,1)$. Again we find $F(v) = a(w_2, v) = \int_0^1 (\alpha \nabla w_2 v_x - b w_2 v_x + c w_2 v)dx$.

Applying our method on these test functions, we find the log-log plots 3. For w_1, we get respectively super-linear and sub-linear convergence in L^2 and H_0^1 (1.49 and 0.55, respectively). For w_2, we get sub-linear convergence in both L^2 and H_0^1 (0.64 and 0.25, respectively). For both functions we converge slower in H_0^1 than in L^2, in accordance with theory. For w_1, we use the weak derivative to find a weak solution to the problem, so we should not expect second order convergence as the result obtaining this as an upper bound depends on the second derivative of the solution.

For w_2, we get integrals containing $\varphi_0(x)$ which are not well defined as the weak derivative of w_2 is undefined at $x = 0$. To compute the integrals $F(\varphi_i)$, $i = 0, \ldots, M$ at these points, we change the lower bound from 0 to $\frac{x_1}{10}$ so the lower bound of the integral linearly converges to 0. In other words we include a linear error term in our calculations and in this case cannot expect second order convergence in L^2. In both cases we compute integrals by interpolation, adding another source of error.

3.5 Non-equidistant nodes

Let $f(x) = x^{-\frac{1}{4}},\ x \in (0,1)$. Since $\|f(x)\|_{L^2} = \int_0^1 |f(x)|^2 dx = \int_0^1 \frac{1}{\sqrt{x}} dx = [2x^{\frac{1}{2}}]_0^2 = 2\sqrt{2} < \infty$, $f(x) \in L^2(0,1)$, and $F(v) = \int_0^1 fv dx$ is well-defined.

For a fixed number of nodes, we want to check whether we get a smaller error for grids with more nodes near $x = 0$, $x = 1$ than for equispaced grids when we have the same total number of nodes. Numerically, we do this by transforming the gridpoints $x_j \to g(x_j)$, $g(x) = \frac{1}{2}\left(1 + \sin\left(x\pi - \frac{\pi}{2}\right)\right)$, shifting all nodes from the center to the endpoints. We find that we have both a lower error and greater rate of convergence when we shift the nodes (5). Incidentally, we get close to second order convergence in L^2 and first order convergence in H_0^1 when we shift the nodes. As an aside, we in this part plot the error as a function of the average step length as we get elements of differing lengths when we shift the nodes as described. This result is as expected as when we solve for u, we find that the function changes rapidly near $x = 0$ & $x = 1$, which means we should expect a greater error in these areas.

4 Conclusion

The weak formulation of 3 could be found by multiplying with the test function v, and integrating over $\Omega = (0, 1)$. Then $F(v)$ and $a(u, v)$ from 4 are both linear and continuous (bounded). $a(u, v)$ is also coercive if $\alpha \geq 0, b \neq 0, c \geq 0$ are constant and $c > \frac{|b|^2}{2\alpha}$. The assumptions for the Lax-Milgram theorem are then satisfied.

After implementation of the FEM of 3, we've found empirically (with equidistant nodes) that it converges in $H_0^1(0, 1)$ with rate 1 in 3.2. This is in accordance with the theory in 3.3. However, the convergence rate in L^2 is 2. As we are more interested in our approximation being close to the solution u of 3, and not their derivatives, the L^2 is more important.

Further, we've shown that our method also works with non-smooth test functions. In our examples, the original problem 4 is then not defined, as the double derivative term has a singularity. We then solve the problem with only the weak formulation. In both our examples, while still converging, we find a lower rate of convergence.

Lastly, we've tested our method on a left hand side f with a singularity on the border, and try a different method of spacing our nodes around this singularity. We find that we get faster convergence with nodes more closely spaced around the singularity, i.e. when f changes rapidly.

5 Who did what?

1. Heidi: Gårding inequality, proof of existence, error bound.
2. Elias: The weak formulation and it's properties, proofs for non-smooth potential exact solutions, summary and conculsion
3. Max: Numerical implementations and plots.

References

[1] C. Curry. Tma4212 part 2: Introduction to finite element methods. 2018.

Appendix

Figure 1: Numerical solutions of test functions

(a) $f_1(x)$ (b) $f_2(x)$

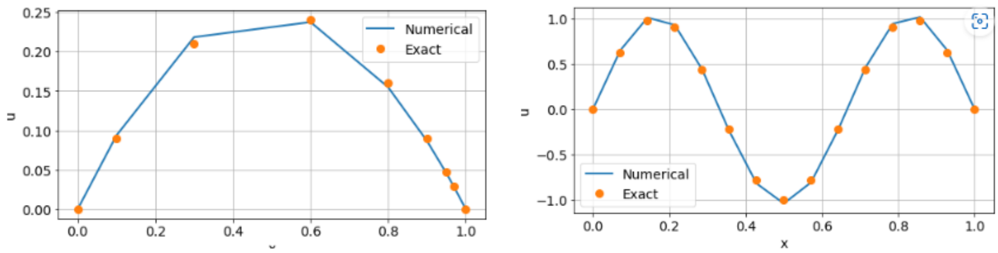

Figure 2: Convergence in L^2 and H^1

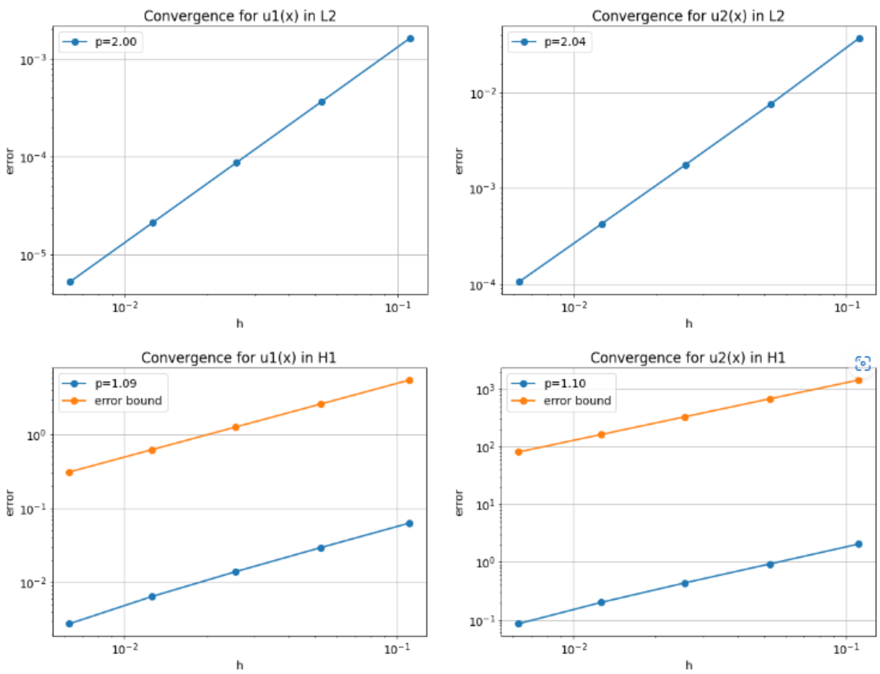

Figure 3: Convergence for non-smooth exact solutions

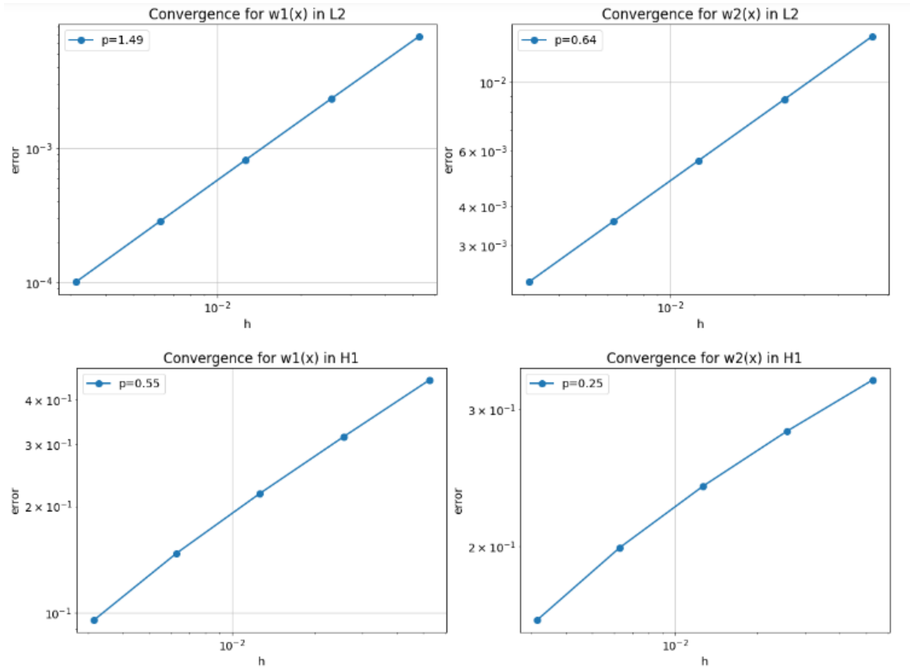

Figure 4: Shifted nodes

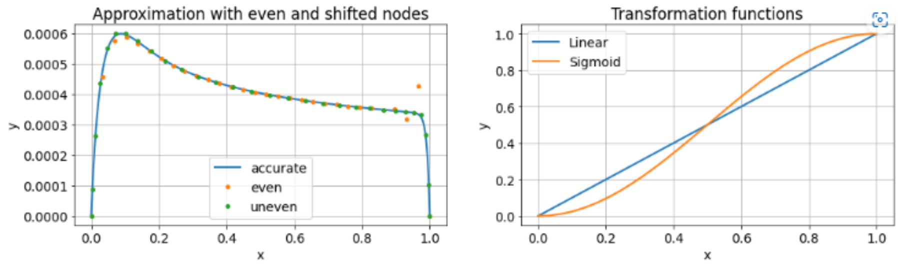

Figure 5: Convergence with shifted nodes