A Verified Certificate Checker for Finite-Precision Error Bounds in Coq and HOL4

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ARM

Finite-Precision Computations have errors

$0.2 + 0.1$ $\quad\quad$ $\widetilde{0.2} \mathbin{\tilde{+}} \widetilde{0.1}$

$0.3 \neq 0.30000000000000004$

roundoff error

Does it matter?

Finite-Precision Computations have errors

$f(x) = 0.2 + x$

$\tilde{f}(\tilde{x}) = \widetilde{0.2} \mathbin{\tilde{+}} \tilde{x}$

roundoff error

$$\boxed{\max_{x \in [a,b]} | f(x) - \tilde{f}(\tilde{x}) | \leq \varepsilon}$$

Did Daisy compute
a correct roundoff error ε?

$$\max_{x \in [a,b]} |\, f(x) - \tilde{f}(\tilde{x}) \,| \leq \varepsilon$$

f: real valued function
$P(x)$: input constraints

$\tilde{f}$: finite-precision function
ε: roundoff error

>8k Lines of Scala Code

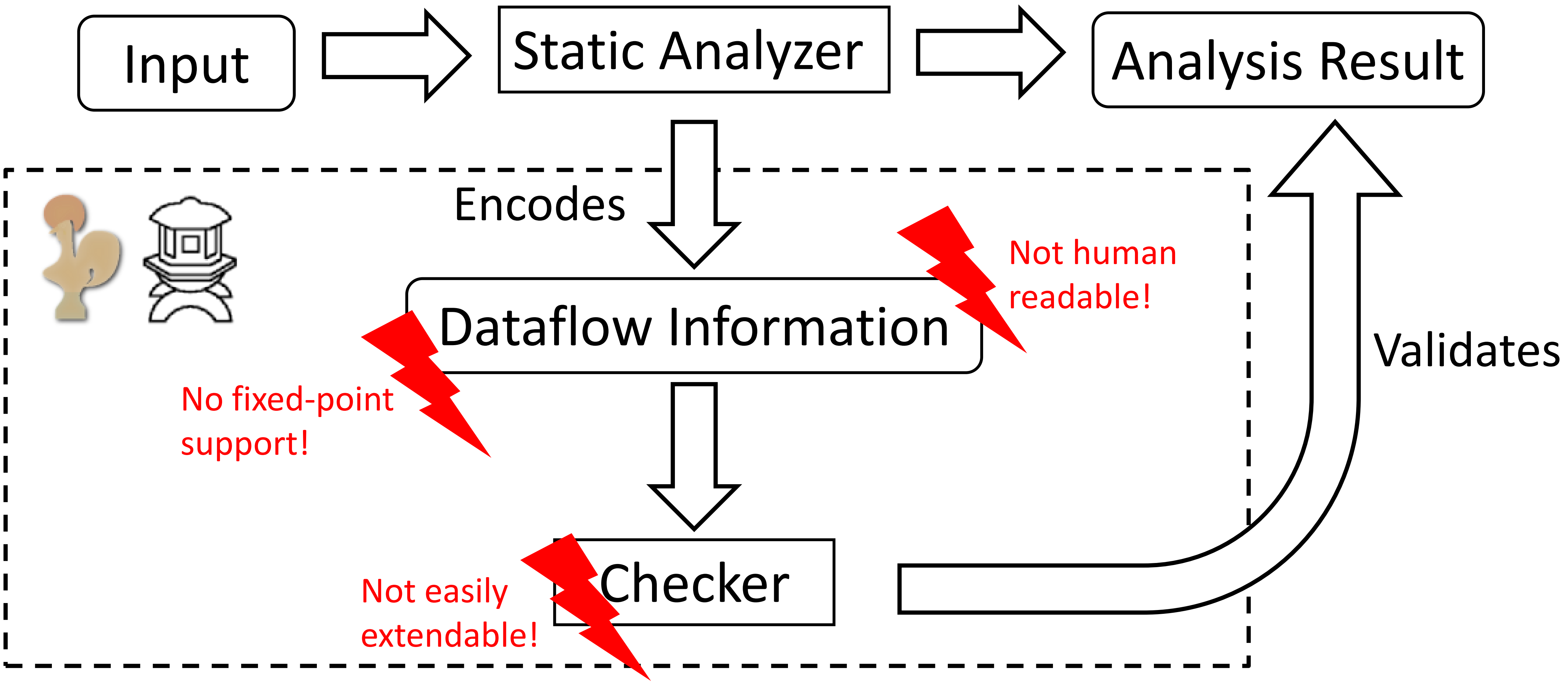
Did Daisy compute
a correct roundoff error ε?

$$\max_{x \in [a,b]} |\, f(x) - \tilde{f}(\tilde{x}) \,| \leq \varepsilon$$

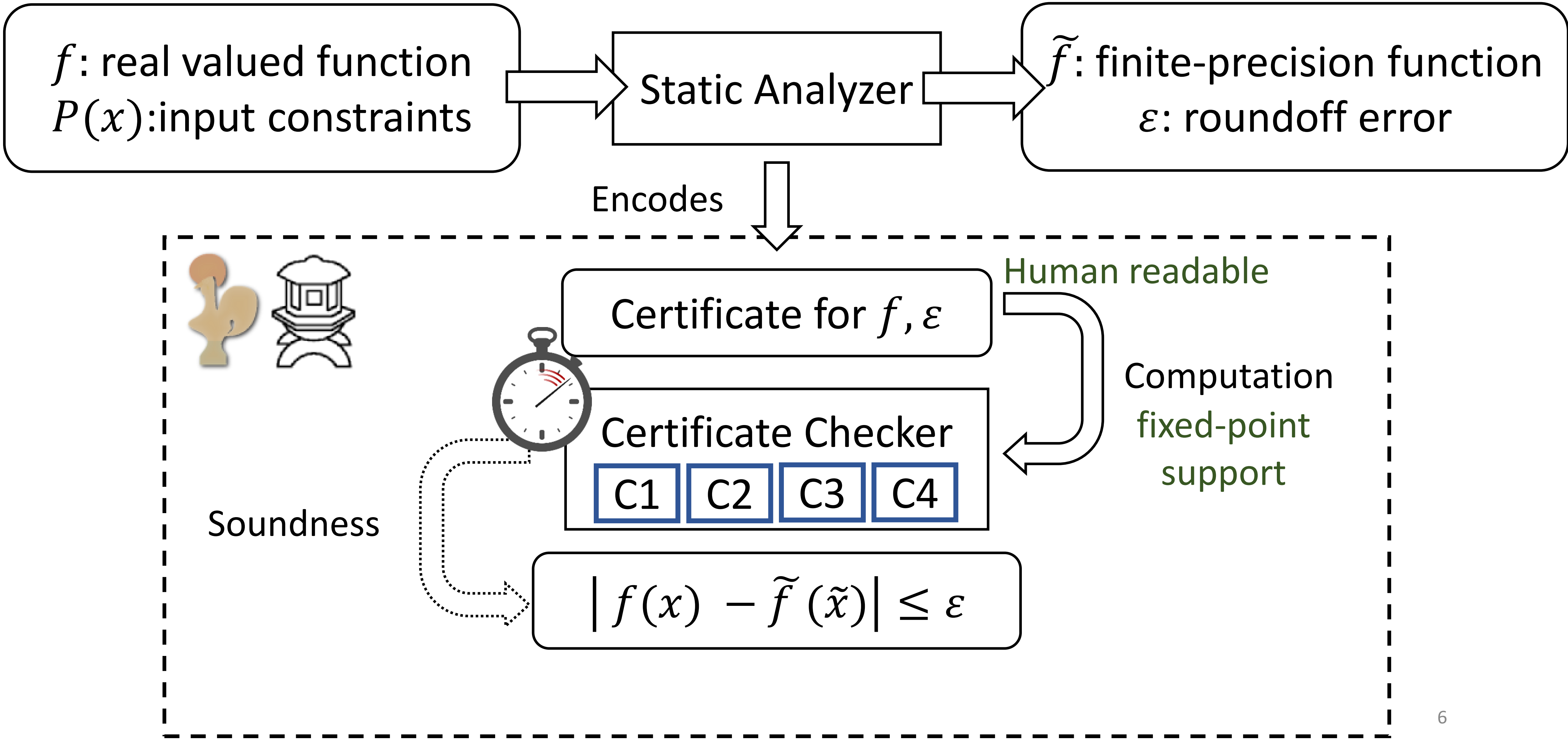
Verification

Validation

Certificate Checking



FloVer: A Checker For Finite-Precision Error Bounds



Contributions

FloVer, a certificate checker for finite-precision error bounds

- checks floating-point and fixed-point error bounds

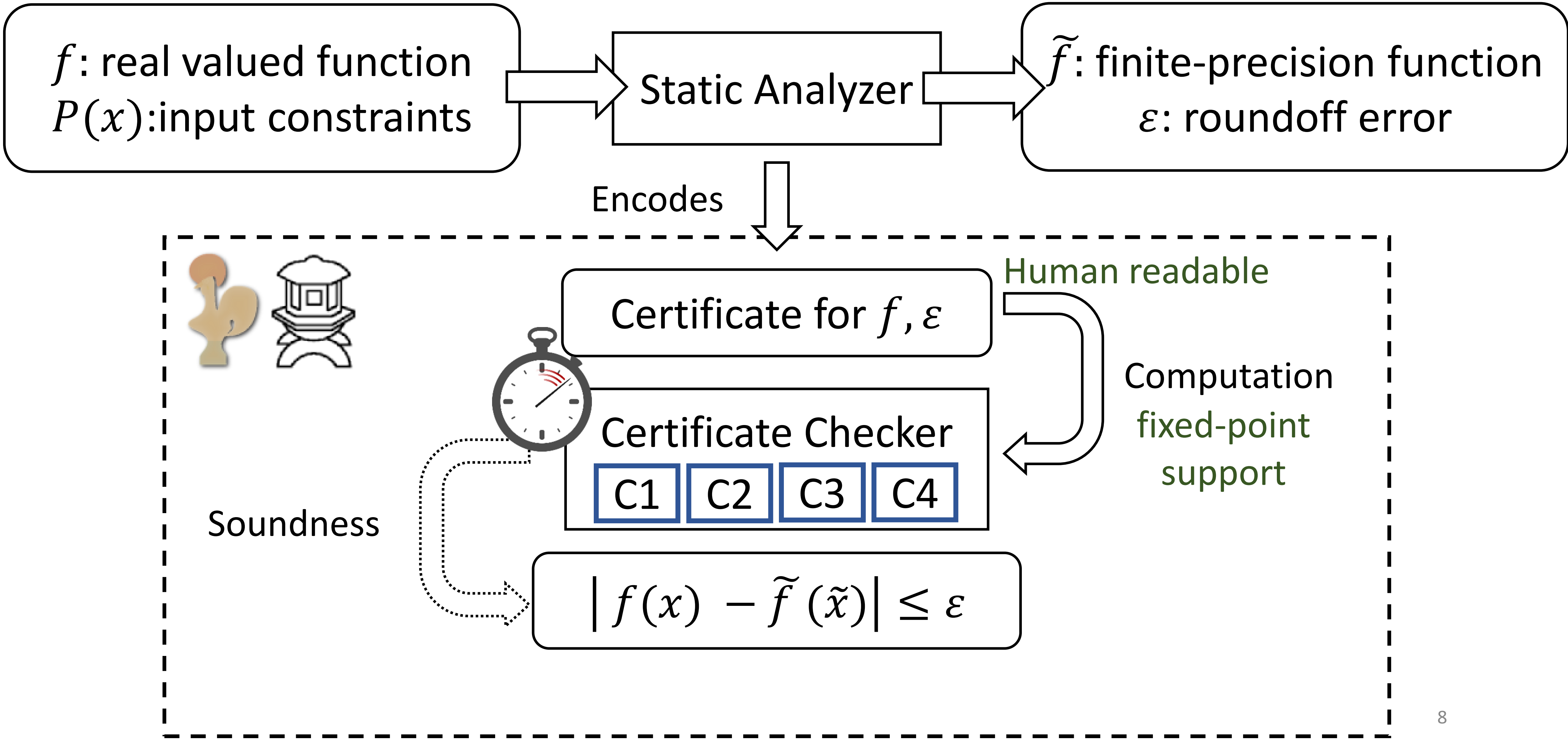
  →soundness proof with respect to IEEE754 semantics

- build modular with separate validators

  →uses interval and affine arithmetic

- supports arithmetic (+,-,*,/), let-bindings and fused-multiply-add
- supports mixed-precision (different types for operations)
- runs fast with verified extracted binary

FloVer: A Checker For Finite-Precision Error Bounds



f: real valued function
$P(x)$: input constraints

Certificate
f: real valued function
$P(x)$: input constraints
R: range analysis result
E: error analysis result

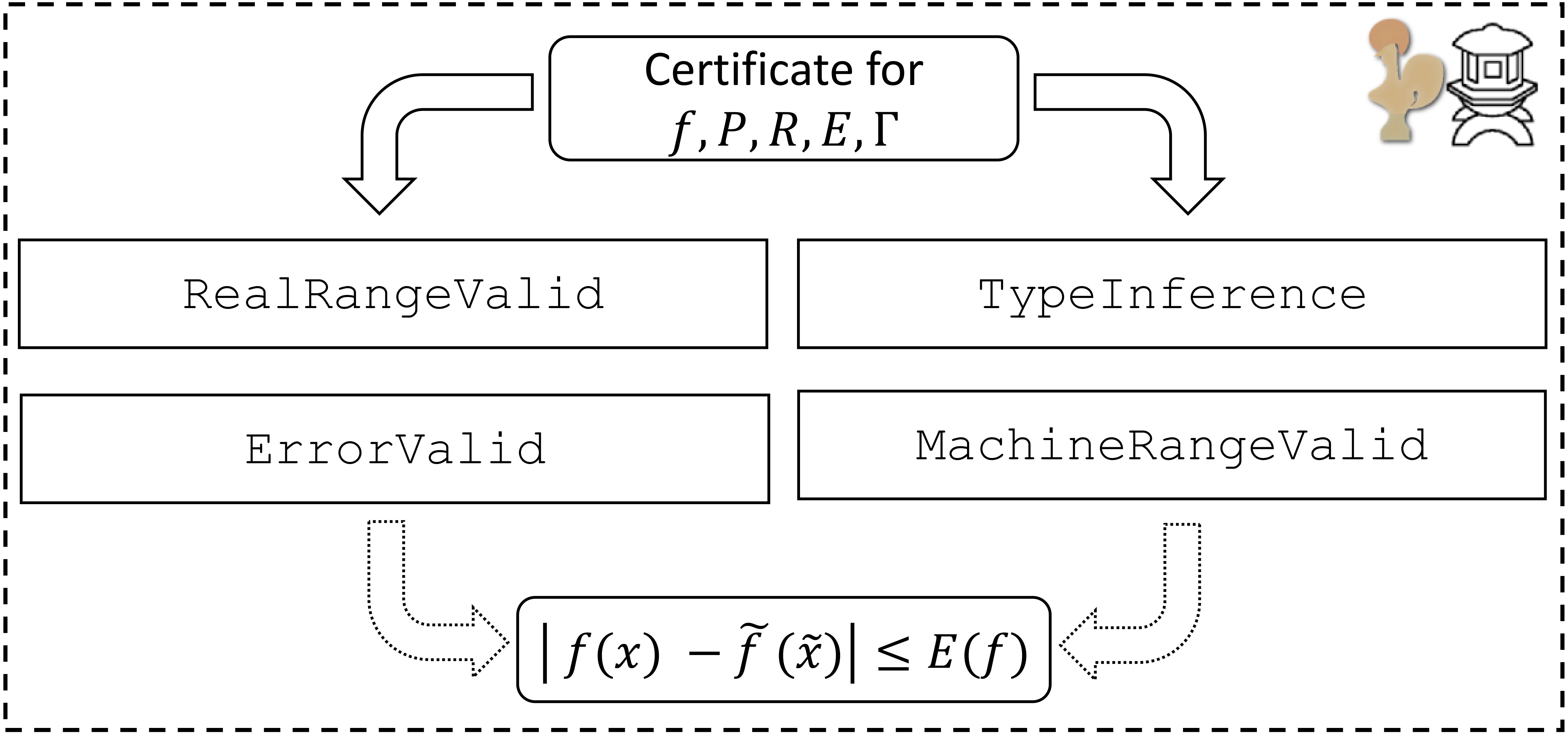
Daisy

real-valued
range analysis R

finite-precision
error analysis E

$\tilde{f}$: finite-precision function
ε: roundoff error

FloVer: A Checker For Finite-Precision Error Bounds



Checking Range Analysis

$$\texttt{RealRangeValid}(0.2, P, R) = 0.2 \subseteq R(0.2)$$
$$\texttt{RealRangeValid}(x, P, R) = P(x) \subseteq R(x)$$
$$\texttt{RealRangeValid}(0.2 + x, P, R) = R(0.2) +^{RA} R\ (x) \subseteq R(0.2 + x)$$

Certificate

$f\ (x) = 0.2 + x$

$P(x)$: input constraints

R: range analysis

E: error analysis

Γ: variable types

Soundness theorem:

$$\texttt{RealRangeValid}(f, P, R) \wedge$$
$$\forall x, (E\ x) \in P\ (x) \Longrightarrow$$
$$(f, E) \downarrow v \wedge v \in R(f)$$

FloVer: A Checker For Finite-Precision Error Bounds

Certificate for
f, P, R, E, Γ

`RealRangeValid` ✓

`TypeInference`

`ErrorValid`

`MachineRangeValid`

$$|f(x) - \tilde{f}(\tilde{x})| \leq E(f)$$

An Abstraction for Floating-Point Computations

$$\tilde{f}(\tilde{x}) = \widetilde{0.2} \,\tilde{+}\, \tilde{x}$$

$$= \left(\widetilde{0.2} + \tilde{x}\right) * (1 + \delta)$$

$$= \left(\widetilde{0.2} + \tilde{x}\right) + \underbrace{\left(\widetilde{0.2} + \tilde{x}\right) * \delta}_{\text{error for + operation}}$$

more errors contributed by operands

IEEE754 abstraction

$$\exists\, \delta .\, a \,\tilde{+}\, b = (a + b) * (1 + \delta)$$

where $|\delta| \leq \varepsilon$

$$\tilde{x} = x * (1 + \delta)$$

Computing a Roundoff Error

We want to bound

$$\left|(0.2 + x) - \left(\widetilde{0.2} \mathbin{\tilde{+}} \tilde{x}\right)\right|$$

IEEE754 abstraction

$$\boxed{\exists \delta .\, a \mathbin{\tilde{+}} b = (a + b) * (1 + \delta)}$$

where $|\delta| \leq \varepsilon$

$$\leq \left|\left(0.2 - \widetilde{0.2}\right) + (x - \tilde{x}) + \left(\widetilde{0.2} + \tilde{x}\right) * \delta\right|$$

$$\leq \underbrace{err_1 + err_2}_{\text{propagation error}} + \underbrace{\left|\left(\widetilde{0.2} + \tilde{x}\right) * \delta\right|}_{\text{roundoff error } (e_{new})}$$

operand errors:

$$\left|0.2 - \widetilde{0.2}\right| \leq err_1$$

$$|x - \tilde{x}| \leq err_2$$

Checking Roundoff Error Bounds

$$\texttt{ErrorValid}(0.2, R, \Gamma, E) = 0.2 * \varepsilon \leq E(0.2)$$

$$\texttt{ErrorValid}(x, R, \Gamma, E) = maxAbs\ (R(x)) * \varepsilon \leq E(x)$$

$$\texttt{ErrorValid}(0.2 + x, R, \Gamma, E) = E(0.2) + E(x) + e_{new} \leq E(0.2 + x)$$

Certificate

$f\ (x) = 0.2 + x$

$P(x)$: input constraints

R: range analysis

E: error analysis

Γ: variable types

Soundness Theorem:

$$\textbf{ErrorValid}(f, E, R) \land$$

$$\textbf{RangeValid}(f, P, R) \land$$

$$(\forall x, E\ x \in P(x)) \Longrightarrow$$

$$(f, E) \downarrow v \land \left(\tilde{f}, E\right) \downarrow \tilde{v} \land\ |v - \tilde{v}| \leq E\ (f)$$

Supported abstract domains

Interval Arithmetic

e $\in [a, b]$ where $a \leq e \leq b$

non-relational, easy to implement

Affine Arithmetic

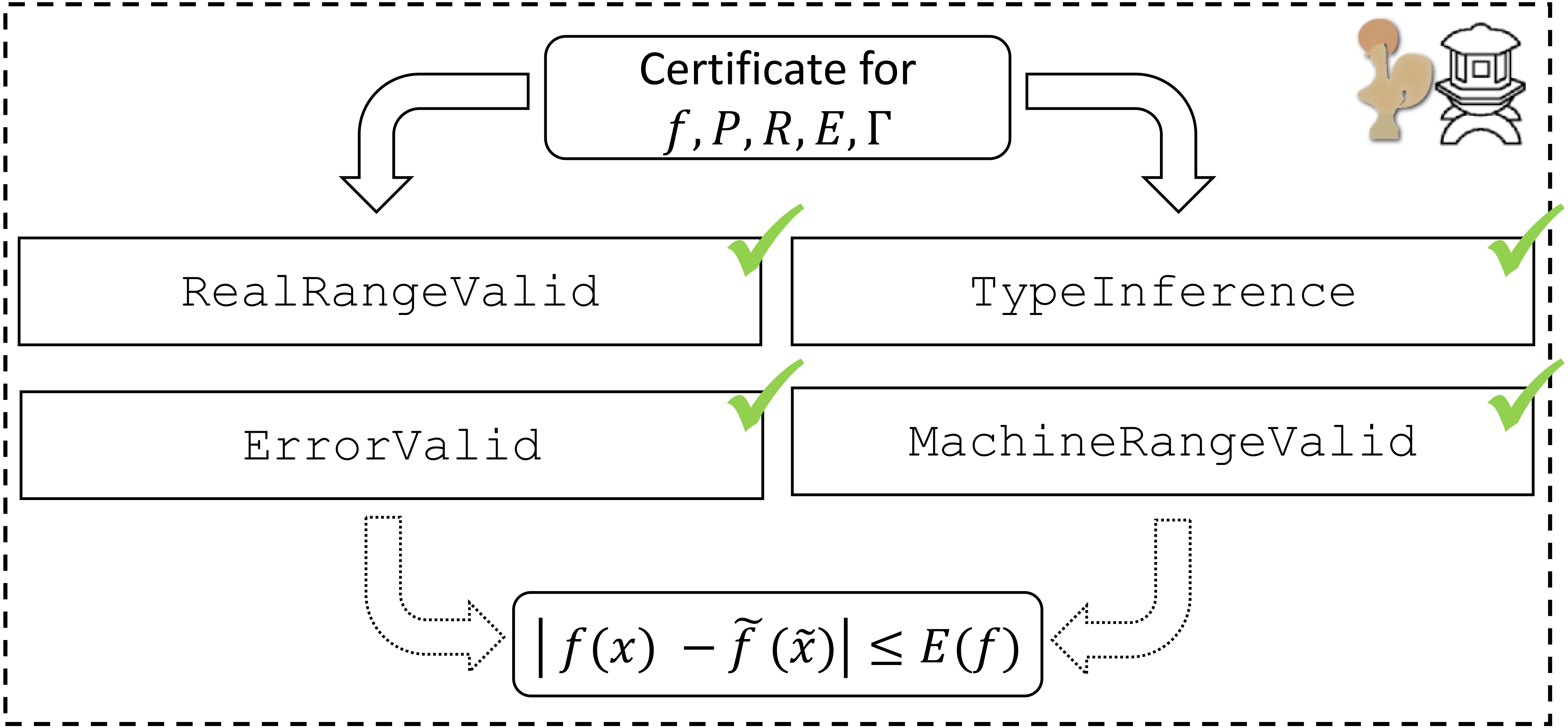
e $\in x_0 + x_1 * \varepsilon_1 + \cdots + x_n * \varepsilon_n$
where $\mathrm{x}_0 + \sum_{i=1} x_i * -1 \leq e \leq x_0 + \sum_{i=1}^{n} x_i$

relational, complex to implement
Track linear, approx. non-linear

Monotonicity of operations:

$$e_1 \in rep_1 \wedge e_2 \in rep_2 \Longrightarrow e_1 \circ e_2 \in (rep_1 \circ^{RA} rep_2)$$

FloVer: A Checker For Finite-Precision Error Bounds



Type inference in FloVer

uniform precision

mixed precision

Annotate variables and constants

Unused

Certificate

f: real valued function

$P(x)$: input constraints

R: range analysis

E: error analysis

Γ: variable types

But we want $\tilde{f}$ to be

```
f(x:single) =
  let y:double = x + 3.0
  in y * 1.3
```

result type ?

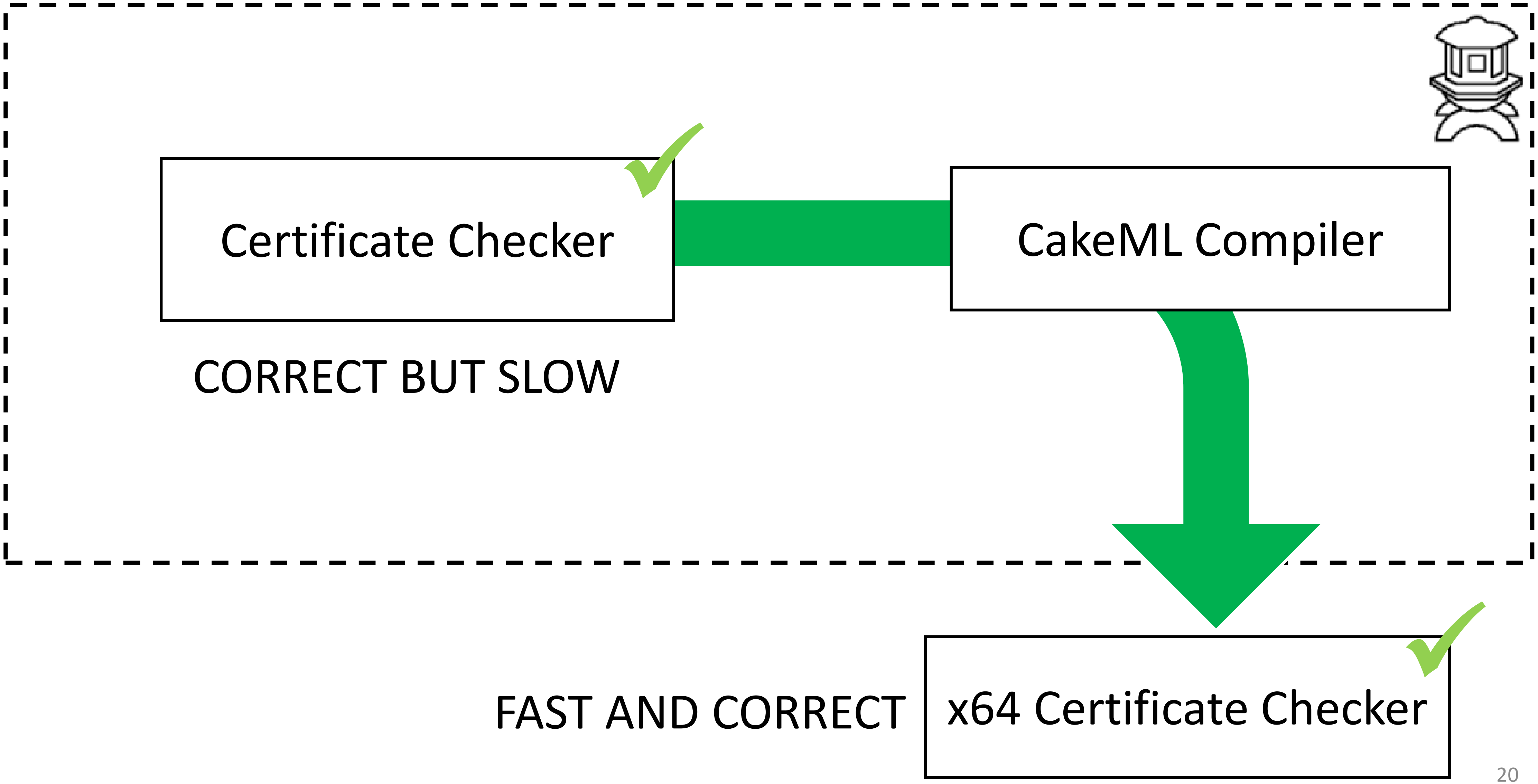
`TypeInference`(f, Γ) infers a global type map M_T for f

Soundness of FloVer

Let f be a real-valued function, E a real-valued environment, $\tilde{E}$ its finite-precision counterpart, P a precondition constraining the free variables of f, Γ a map from all free variables of f to a precision, R a range analysis result, Γ a type-map and E an error analysis result. Then

$$
\begin{gathered}
E \sim_{(E,\mathcal{V},\mathcal{D},\Gamma)} \tilde{E} \land \\
\texttt{TypeInference}(\Gamma, f) = \Gamma \land \texttt{RealRangeValid}(f, P, R) \land \\
\texttt{MachineRangeValid}(f, \Gamma, R, E) \land \\
\texttt{ErrorValid}(f, \Gamma, R, E) \Rightarrow \\
\exists\, v\, \tilde{v}_1\, m_1 .\ (f, E, \Gamma) \Downarrow (v, \infty) \land (\tilde{f}, \tilde{E}, \Gamma) \Downarrow (\tilde{v}_1, m_1) \land \\
(\forall \tilde{v}_2\, m_2 .\ (\tilde{f}, \tilde{E}, \Gamma) \Downarrow (v_2, m_2) \Rightarrow |v - \tilde{v}_2| \leq E(f))
\end{gathered}
$$

Extraction using the CakeML compiler toolchain



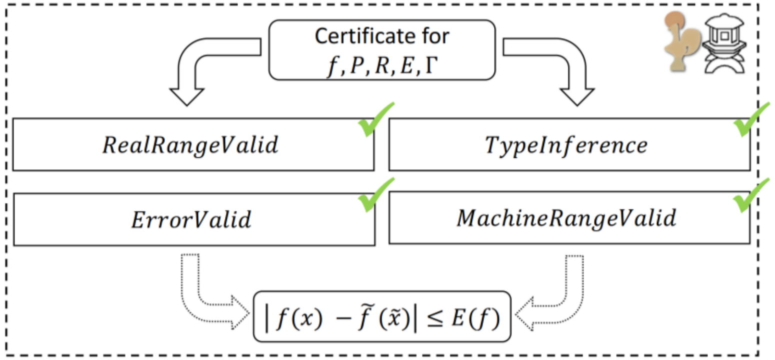
Experiments

Measure end-to-end running times for **18 benchmarks** from literature

Benchmark	# ops.	Daisy	Coq		HOL4	Binary Interval	
			Interval	Affine	Interval	CakeML	OCaml
ballBeam	7	4.62	3.50	3.26	89.04	<0.01	0.02
doppler	17	4.86	5.28	12.21	610.67	0.05	0.02
bspline	28	4.21	4.61	4.07	298.44	0.03	0.08
…	…	…	…	…	…	…	…
traincar1	36	4.85	10.87	9.84	932.93	0.07	0.11
floudas	99	7.76	13.99	12.76	565.68	0.14	0.27
Traincar4	269	10.60	116.94	115.38	17429.30	1.10	0.77

Complexity

FloVer



- Supporting **floating-point** and **fixed-point** arithmetic
- **Runs fully automatically** in Coq and HOL4
- Mixed-precision
- Extract a **verified binary** with CakeML
- **Proven sound** for IEEE754 floating-point semantics
- Code on https://gitlab.mpi-sws.org/AVA/Flover

Questions?