1. Simple random walk in one dimension

Definition 1 (Simple random walk in $\mathbb{Z}$)**.** Let $\{X_n\}_{n=0}^{+\infty}$ be a sequence of independent and identically distributed $\{-1, 1\}$-valued random variables, that for some $p \in (0,1)$ and for $n \in \mathbb{N}$ satisfy $\mathsf{P}\left(X_n = 1\right) = p$ and $\mathsf{P}\left(X_n = -1\right) = 1 - p =: q$.

Let $S_0 = 0$ and $S_n = \sum\limits_{i=1}^{n} X_i$. We call the pair $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *Simple random walk in $\mathbb{Z}$.*

If $p = q = \frac{1}{2}$, $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ is called *Symmetric simple random walk in $\mathbb{Z}$.*

Remark. Very often we refer to n as time, X_i as the i-th step and S_n as position of the walk in time n. In simple random walk in $\mathbb{Z}$ we refer to $X_i = +1$ as i-th step was rightwards and to $X_i = -1$ as i-th step was leftwards. If not stated otherwise, we assume that $S_0 = 0$.

Definition 2 (Set of possible positions)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$be a random walk . We call the set $A_n = \{z \in \mathbb{Z}; |z| \leq n, \frac{z+n}{2} \in \mathbb{Z}\}$ *set of all possible positions* of random walk $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$in time n.

Theorem 1 (Probability of position x in time n)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$be a random walk and A_n its set of possible positions.

$$\mathsf{P}\left(S_n = x\right) = \begin{cases} \binom{n}{\frac{n+x}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}} & \text{for } x \in A_n \ , \\ 0, & \text{for } x \notin A_n. \end{cases}$$

Proof. Consider random variables $\mathbf{1}_{[X_i=1]}$, and $\mathbf{1}_{[X_i=-1]}$, and define new random variables $R_n = \sum\limits_{i=1}^{n} \mathbf{1}_{[X_i=1]}, L_n = \sum\limits_{i=1}^{n} \mathbf{1}_{[X_i=-1]}$. The random variable $\mathbf{1}_{[X_i=1]}$ can be interpreted as indicator whether i-th step was rightwards. Then, R_n is number of rightwards steps and L_n is number of leftwards steps. We can easily see that $R_n + L_n = n$ and $R_n - L_n = S_n$. Therefore we get by adding these two equations $R_n = \frac{S_n + n}{2}$.

Clearly, $\mathbf{1}_{[X_i=1]}$ has alternative distribution with parameter p (*Alt(p)*). Hence, R_n as a sum of independent and identically distributed random variables with distribution *Alt(p)* has binomial distribution with parameters n and p (*Bi(n,p)*). Therefore we get $\mathsf{P}\left(R_n = x\right) = \binom{n}{x} p^x q^{n-x}$, where we define $\binom{a}{x} := 0$ for $a \in \mathbb{N}, x \notin \{0, 1, \ldots, n\}$. Finally, for $a \in A_n$ we get

$$\mathsf{P}\left(S_n = x\right) = \mathsf{P}\left(R_n = \frac{x+n}{2}\right) = \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}.$$

□

Lemma 2 (Spatial homogeneity)
Let $n \in \mathbb{N}, a, b, j \in \mathbb{Z}$. Then for all $b \in \mathbb{Z}$

$$\mathsf{P}\left(S_n = j \mid S_0 = a\right) = \mathsf{P}\left(S_n = j + b \mid S_0 = a + b\right)$$

Proof. For any $j, a, b \in \mathbb{Z}$ holds

$$\mathsf{P}(S_n = j \mid S_0 = a) = \mathsf{P}\left(\sum_{i=1}^{n} X_i = j - a\right) = \mathsf{P}\left(\sum_{i=1}^{n} X_i = (j+b) - (a+b)\right)$$
$$= \mathsf{P}(S_n = j + b \mid S_0 = a + b).$$

$\square$

Lemma 3 (Temporal homogeneity)
Let $n, m \in \mathbb{N}, a, j \in \mathbb{Z}$. Then for all $m \in \mathbb{N}$

$$\mathsf{P}(S_n = j \mid S_0 = a) = \mathsf{P}(S_{n+m} = j \mid S_m = a)$$

Proof. For any $j, a \in \mathbb{Z}$ and $m \in \mathbb{N}$

$$\mathsf{P}(S_n = j \mid S_0 = a) = \mathsf{P}\left(\sum_{i=1}^{n} X_i = j - a\right) = \mathsf{P}\left(\sum_{i=m+1}^{m+n} X_i = j - a\right)$$
$$= \mathsf{P}(S_{n+m} = j \mid S_m = a),$$

where the second equality follows from identical distribution of $\{X_n\}_{n=1}^{+\infty}$. $\square$

Lemma 4 (Markov property)
Let $m, n \in \mathbb{N}, n \geq m$ and $a_i \in \mathbb{Z}, i \in \mathbb{N}$. Then $\mathsf{P}(S_n = j \mid S_0 = a_0, S_1 = a_1, \ldots, S_m = a_m) =$ $\mathsf{P}(S_n = j \mid S_m = a_m)$

Proof. Once S_m is known, then distribution of S_n depends only on steps $X_{m+1}, X_{m+2}, \ldots X_n$ and therefore cannot be dependent on any information concerning values $X_1, X_2, \ldots, X_{m-1}$ and accordingly $S_1, S_2, \ldots, S_{m-1}$. $\square$

Remark. **check english** -přepsat nějak líp In symmetric random walk, everything can be counted by number of possible paths from point to point.

Definition 3 (Number of possible paths)**.** Let $N_n(a, b)$ be *number of posssible paths* of random walk $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$from point $(0, a)$ to point (n, b) and $N_n^x(a, b)$ be number of possible paths from point $(0, a)$ to point (n, b) that visit point (z, x) for some $z \in \{1, 2, \ldots, n\}$.

Theorem 5
Let $a, b \in \mathbb{Z}, n \in \mathbb{N}$ then $N_n(a, b) = \binom{n}{\frac{1}{2}(n+b-a)}$.

Proof. Let us choose a path from $(0, a)$ to (n, b) and let α be number of rightwards steps and β be number of leftwards steps. Then $\alpha + \beta = n$ and $\alpha - \beta = b - a$. By adding these two equations we get that $\alpha = \frac{1}{2}(n + b - a)$. The number of possible paths is the number of ways of picking α rightwards steps from n steps. Therefore we get $N_n(a, b) = \binom{n}{\alpha} = \binom{n}{\frac{1}{2}(n+b-a)}$. $\square$

Theorem 6 (Reflection principle)
Let $a, b > 0$, then $N_n^0(a, b) = N_n(-a, b)$.

Proof. Each path from $(0,-a)$ to (n,b) has to intersect $y=0$-axis at least once at some point. Let k be the time of earliest intersection with x-axis. By reflexing the segment from $(0,-a)$ to $(k,0)$ in the x-axis and letting the segment from $(k,0)$ to (n,b) be the same, we get a path from point $(0,a)$ to (n,b) which visits 0 at point k. Because reflection is a bijective operation on sets of paths, we get the correspondence between the collections of such paths. □

Definition 4 (Return to origin)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$be a random walk . Let $k \in \mathbb{N}$. We say a *return to origin* occurred in time $2k$ if $S_{2k}=0$. The probability that in time $2k$ occured a return to origin shall be denoted by u_{2k}. We say that in time $2k$ occurred *first return to origin* if $S_1, S_2, \dots S_{2k-1} \neq 0$ and $S_{2k}=0$. The probability that in time $2k$ occured first return to origin shall be denoted by f_{2k}. By definition $f_0=0$. Let $\alpha 2n\,(2k)$ denote $u_{2k}u_{2(n-k)}$

Theorem 7 (Ballot theorem)
Let $n, b \in N$ *Number of paths from point* $(0,0)$ *to point* (n,b) *which do not return to origin is equal to* $\frac{b}{n}N_n\,(0,b)$

Proof. Let us call N the number of paths we are refering to. Because the path ends at point (n,b), the first step has to be rightwards. Therefore we now have $N = N_{n-1}\,(1,b) - N^0_{n-1}\,(1,b) \stackrel{\text{T6}}{=} N_{n-1}\,(1,b) - N_{n-1}\,(-1,b)\,.$

Therefore we now have:$N_{n-1}\,(1,b) - N_{n-1}\,(-1,b) = \binom{n-1}{\frac{n}{2}+\frac{b}{2}-1} - \binom{n-1}{\frac{n}{2}+\frac{b}{2}} = \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!} - \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!} = \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}\right)\left(\frac{n}{2}-\frac{b}{2}-1\right)!} - \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}\right)\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!} = \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!}\left(\frac{1}{\frac{n}{2}-\frac{b}{2}} - \frac{1}{\frac{n}{2}+\frac{b}{2}}\right) = \frac{1}{n}\frac{n!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!}\left(\frac{\left(\frac{n}{2}+\frac{b}{2}-\frac{n}{2}+\frac{b}{2}\right)}{\left(\frac{n}{2}-\frac{b}{2}\right)\left(\frac{n}{2}+\frac{b}{2}\right)}\right) = \frac{b}{n}\frac{n!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!} = \frac{b}{n}\binom{n}{\left(\frac{n}{2}+\frac{b}{2}\right)} = \frac{b}{n}N_n\,(0,b)$ □

Remark. The name *Ballot theorem* comes from the question: In a ballot where candidate A receives p votes and candidate B receives q votes with $p>q$, what is the probability that A will be strictly ahead of B throughout the count?

Definition 5. $M_n^+ = \max\{S_i, i \in \{1,2,\dots,n\}\}$, $M_n^- = \max\{-S_i, i \in \{1,2,\dots,n\}\}$, $M_n^A = \max M_n^+, M_n^-$

Theorem 8 (Probability of maximum up to time n)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$*be a random walk .*

$$\mathsf{P}\left(M_n^+ \geq r, S_n = b\right) = \begin{cases} \mathsf{P}\,(S_n = b) & \text{for } b \geq r\ , \\ \mathsf{P}\,(S_n = 2r-b)\left(\frac{q}{p}\right)^{r-b}, & \text{for otherwise.} \end{cases}$$

Proof. Let us firstly consider the easier case in which $b \geq r$. Because we defined M_n^+ as $\max\{S_i, i \in \{1,2,\dots,n\}\}$ we get that $M_n^+ \geq b \geq r$ therefore $[M_n^+ \geq r] \subset [S_n = b]$ therefore we get $\mathsf{P}\,(M_n^+ \geq r, S_n = b) = \mathsf{P}\,(S_n = b)$.

Now let $r \geq 1, b < r$. $N_n^r\,(0,b)$ stands for number of paths from point $(0,0)$ to point (n,b) which reach up to r. Let $k \in \{1,2,\dots,n\}$ denote the first time we reach r. By reflection principle (6) , we can reflex the segment from (k,r) to (n,b) in the axis:$y=r$. Therefore we now have path from $(0,0)$ to $(n,2r-b)$ and we get that $N_n^r\,(0,b) = N_n\,(0,2r-b)$. $\mathsf{P}\,(S_n = b, M_n^+ \geq r) = N_n^r\,(0,b)\,p^{\frac{n+b}{2}}q^{\frac{n-b}{2}} = N_n\,(0,2r-b)\,p^{\frac{n+(2r-b)}{2}}q^{\frac{n-(2r-b)}{2}}p^{\frac{n+b}{2}}q^{\frac{n-b}{2}}p^{b-r}q^{r-b} = \left(\frac{q}{p}\right)^{r-b}\mathsf{P}\,(S_n = 2r-b)\,.$ □

Definition 6 (Walk reaching new maximum at particular time)**.** Let $b > 0$. $f_b(n)$ denotes the probability that we reach new maximum b in time n. $f_b(n) = \mathsf{P}(M_{n-1} = S_{n-1} = b-1, S_n = b)$

Theorem 9 (Probability of reaching new maximum b in time n)
Let $b > 0$ then $f_b(n) = \frac{b}{n}\mathsf{P}(S_n = b)$.

Proof. $f_b = \mathsf{P}(M_{n-1} = S_{n-1} = b-1, S_n = b) = \mathsf{P}(M_{n-1} = S_{n-1} = b-1, X_n = +1) = p\,\mathsf{P}(M_{n-1} = S_{n-1} = b-1)$
$\overset{*}{=} p\left(\mathsf{P}(M_{n-1} \geq b-1, S_{n-1} = b-1) - \mathsf{P}(M_{n-1} \geq b, S_{n-1} = b-1)\right)$
$\overset{\text{T8}}{=} p\left(\mathsf{P}(S_{n-1} = b-1) - \frac{q}{p}\mathsf{P}(S_{n-1} = b+1)\right)$
$= p\,\mathsf{P}(S_{n-1} = b-1) - q\,\mathsf{P}(S_{n-1} = b+1)$
$= \binom{n-1}{\frac{n}{2}+\frac{b}{2}-1} p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} - \binom{n-1}{\frac{n}{2}+\frac{b}{2}} p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} = p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \left(\frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!} - \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!}\right) = p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \left(\frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!}\right)\left(\frac{1}{\frac{n}{2}-\frac{b}{2}} - \frac{1}{\frac{n}{2}+\frac{b}{2}}\right) = p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \frac{b}{n}\left(\frac{n!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!}\right)$
$= \frac{b}{n} p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \binom{n}{\frac{n}{2}+\frac{b}{2}} = \frac{b}{n}\mathsf{P}(S_n = b)$. Where $*$ comes from the fact that the event $[M_{n-1} \geq b-1]$ can be split into two disjoint events: $[M_{n-1} \geq b-1] = [M_{n-1} \geq b] \cup [M_{n-1} = b-1]$. Therefore $\mathsf{P}(M_{n-1} \geq b-1) = \mathsf{P}(M_{n-1} \geq b) + \mathsf{P}(M_{n-1} = b-1)$. Hence: $\mathsf{P}(M_{n-1} = b-1) = \mathsf{P}(M_{n-1} \geq b-1) - \mathsf{P}(M_{n-1} \geq b)$. The same applies for the probability $\mathsf{P}(M_{n-1} = b-1, S_{n-1} = b-1)$ □

Theorem 10 (XXXMean number of visits to b before returning to origin in symmetric random walk)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a symmetric random walk . Mean number μ_b of visits of the walk to point b before returning to origin is equal to 1.

Proof. aa □

Lemma 11 (Binomial identity)
Let $n, k \in \mathbb{N}, n > k$: $\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{n}\binom{n}{k}$

Proof. $\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-k-1)!} - \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{1}{k} - \frac{1}{n-k}\right)$
$= \frac{1}{n}\frac{n!}{(k-1)!(n-k-1)!}\frac{n-2k}{k(n-k)} = \frac{n-2k}{n}\frac{n!}{k!(n-k)!} = \frac{n-2k}{n}\binom{n}{k}$ □

Lemma 12 (Main lemma)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a symmetrical random walk. Then $\mathsf{P}(S_1, S_2, \ldots, S_{2n} \neq 0) = \mathsf{P}(S_{2n} = 0)$.

Proof. $\mathsf{P}(S_1, S_2, \ldots, S_{2n} \neq 0) \overset{LTP}{=} \sum_{i=-\infty}^{+\infty} \mathsf{P}(S_1, S_2, \ldots, S_{2n} \neq 0, S_{2n} = 2i) = \sum_{i=-n}^{n} \mathsf{P}(S_1, S_2, \ldots, S_2$
$2 \cdot \sum_{i=1}^{n} \mathsf{P}(S_1, S_2, \ldots, S_{2n} \neq 0, S_{2n} = 2i) \overset{\text{T7}}{=} 2\sum_{i=1}^{n} \frac{2i}{2n}\mathsf{P}(S_{2n} = 2i) = 2\sum_{i=1}^{n} \frac{2i}{2n}\binom{2n}{n+i}2^{-2n} \overset{\text{L11}}{=} 2 \cdot 2^{-2n}\sum_{i=1}^{n}\left(\binom{2n-1}{n-i} - \binom{2n-1}{n-i-1}\right) \overset{**}{=} 2 \cdot 2^{-2n}\binom{2n-1}{n} = 2^{-2n}\frac{2n}{n}\binom{2n-1}{n} = 2^{-2n}\frac{2n(2n-1)!}{m(m-1)!m!} = 2^{-2n}\frac{(2n)!}{m!m!} = 2^{-2n}\binom{2n}{n} = \mathsf{P}(S_2 n = 0)$. Where $*$ comes from the fact that the random walk is symmetric and $**$ comes from the fact that the positive part of i-th term cancels against the negative part of $i+1$-st term. □

Theorem 13
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a symmetric random walk . The probability that the last return to origin up to time $2n$ occurred in time $2k$ is $\mathsf{P}(S_{2k} = 0)\,\mathsf{P}\left(S_{2(n-k)} = 0\right)$.

Proof. $\alpha 2n\left(2k\right)=u_{2k}u_{2(n-k)}=\mathsf{P}\left(S_{2k}=0\right)\mathsf{P}\left(S_{2k+1},S_{2k+2},\ldots,S_{2n}\neq 0\mid S_{2k}=0\right)=$ $\mathsf{P}\left(S_{2k}=0\right)\mathsf{P}\left(S_1,S_2,\ldots,S_{2(n-k)}\neq 0\right)=\mathsf{P}\left(S_{2k}=0\right)\mathsf{P}\left(S_{2(n-k)}=0\right)$ □

Theorem 14
Let $b\in Z$. $\mathsf{P}\left(S_1,S_2,\ldots,S_n\neq 0,S_n=b\right)=\frac{|b|}{n}\mathsf{P}\left(S_n=b\right)$.

Proof. Let us without loss of generality assume that $b>0$. In that case, first step has to be rightwards $\left(X_1=+1\right)$. Now we have path from point $(1,1)$ to point (n,b) that does not return to origin. By Ballot theorem 7 there are $\frac{b}{n}N_n\left(0,b\right)$ such paths. Each path consists of $\frac{n+b}{2}$ rightwards steps and $\frac{n-b}{2}$ leftwards steps. Therefore $\mathsf{P}\left(S_1\cdot S_2\cdot,\ldots,S_n\neq 0,S_n=b\right)=\frac{b}{n}N_n\left(0,b\right)p^{\frac{n+b}{2}}q^{\frac{n-b}{2}}=\frac{b}{n}\mathsf{P}\left(S_n=b\right)$. Case $b<0$ is identical. □

Lemma 15
Let $\left(\{S_n\}_{n=0}^{+\infty},p\right)$ be a symmetric random walk . $\mathsf{P}\left(S_1,S_2,\ldots,S_{2n}>0\right)=\frac{1}{2}\mathsf{P}\left(S_{2n}=0\right)=\frac{1}{2}u_{2n}$.

Proof. Because $S_i>0\forall i\in\mathbb{N}$ the first step has to be rightwards $\left(X_1=S_1=1\right)$. Therefore we get $\mathsf{P}\left(S_1,S_2,\ldots,S_{2n}>0\right)=\sum_{r=1}^{n}\mathsf{P}\left(S_1,S_2,\ldots,S_{2n}>0,S_{2n}=2r\right)$. The r-th term follows equation: $\mathsf{P}\left(S_1,S_2,\ldots,S_{2n}>0,S_{2n}=2r\right)$
$=\mathsf{P}\left(X_1=1,S_2,\ldots,S_{2n}>0,S_{2n}=2r\right)$
$=\frac{1}{2}\mathsf{P}\left(S_2,S_3,\ldots,S_{2n}>0,S_{2n}=2r\mid S_1=1\right)$
$\stackrel{*}{=}\frac{1}{2}\left(\mathsf{P}\left(S_{2n}=2r\mid S_1=1\right)-\mathsf{P}\left(S_2,S_3,\ldots,S_{2n}=0,S_{2n}=2r\mid S_1=1\right)\right)$
$=\frac{1}{2}\left(\frac{1}{2}^{2n-1}N_{2n-1}\left(1,2r\right)-\frac{1}{2}^{2n-1}N_{2n-1}^{0}\left(1,2r\right)\right)$
$=\frac{1}{2}\frac{1}{2}^{2n-1}\left(N_{2n-1}\left(1,2r\right)-N_{2n-1}^{0}\left(1,2r\right)\right)$
$\stackrel{\mathrm{T6}}{=}\frac{1}{2}\frac{1}{2}^{2n-1}\left(N_{2n-1}\left(1,2r\right)-N_{2n-1}\left(-1,2r\right)\right)$
$=\frac{1}{2}\frac{1}{2}^{2n-1}\left(\binom{2n-1}{n+r-1}-\binom{2n-1}{n+r}\right)$. Where $*$ comes from decomposition: $[S_{2n}=2r]=[S_{2n}=2r,S_1\cdot S_2\cdot\ldots\cdot S_2n\neq 0]\cup[S_{2n}=2r,S_1\cdot S_2\cdot\ldots\cdot S_2n=0]$.

Because of the fact that the negative parts of r-th terms cancel against the positive parts of $(r+1)$-st terms and the sume reduces to just $\frac{1}{2}\frac{1}{2}^{2n-1}\binom{2n-1}{n}=\frac{1}{2}\cdot 2\cdot\frac{1}{2}^{2n}\binom{2n-1}{n}=\frac{1}{2}\frac{1}{2}^{2n}\frac{2(2n-1)!}{n!(n-1)!}=\frac{1}{2}\frac{1}{2}^{2n}\frac{(2n)!}{n!n!}=\frac{1}{2}\frac{1}{2}^{2n}\binom{2n}{n}=\frac{1}{2}\mathsf{P}\left(S_{2n}=0\right)=\frac{1}{2}u_{2n}$. □

Theorem 16 (No return=return)
$\mathsf{P}\left(S_1,S_2,\ldots,S_{2n}\neq 0\right)=\mathsf{P}\left(S_{2n}=0\right)=u_{2n}$

Proof. The event $[S_1,S_2,\ldots,S_{2n}\neq 0]$ can be split into two disjoint events: $=[S_1,S_2,\ldots,S_{2n}<0]\cup[S_1,S_2,\ldots,S_{2n}>0]$. By previous theorem (15) we get that probability of both of them is $\frac{1}{2}u_{2n}$. Because the the events are disjoint we can sum their probabilities and we get the result. □

Lemma 17
$\mathsf{P}\left(S_1,S_2,\ldots,S_{2n}\geq 0\right)=\mathsf{P}\left(S_{2n}=0\right)=u_{2n}$

Proof. $\frac{1}{2}u_{2n}=\mathsf{P}\left(S_1,S_2,\ldots,S_{2n}>0\right)=\mathsf{P}\left(X_1=1,S_2,S_3\ldots,S_{2n}\geq 1\right)$
$\stackrel{nasobeni}{=}\mathsf{P}\left(S_1=1\right)\mathsf{P}\left(S_2,S_3\ldots,S_{2n}\geq 1\mid S_1=1\right)$
$=\frac{1}{2}\mathsf{P}\left(S_2,S_3\ldots,S_{2n}\geq 1\mid S_1=1\right)$
$\stackrel{\mathrm{L3}}{}\frac{1}{2}\mathsf{P}\left(S_1,S_2\ldots,S_{2n-1}\geq 1\mid S_0=1\right)$

$\overset{\text{L2}}{}\frac{1}{2}\,\mathsf{P}\left(S_1, S_2 \ldots, S_{2n-1} \geq 0\right)$
$= \frac{1}{2}\,\mathsf{P}\left(S_1, S_2 \ldots, S_{2n} \geq 0\right)$. Because $[S_{2n-1} \geq 0] \Rightarrow [S_{2n-1} \geq 1] \Rightarrow [S_{2n} \geq 0]$
Therefore $\mathsf{P}\left(S_1, S_2 \ldots, S_{2n} \geq 0\right) = u_{2n}$. □

Theorem 18
$f_{2n} = u_{2n-2} - u_{2n}$

Proof. The event $[S_1, S_2, \ldots S_{2n-1} \neq 0]$ can be split into two disjoint events: $[S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} = 0]$ and $[S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} \neq 0]$. Therefore $\mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0\right) = \mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} = 0\right) + \mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} \neq 0\right)$. Therefore we get $f_{2n} = \mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} = 0\right) = \mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0\right) - \mathsf{P}\left(S_1, S_2, \ldots, S_{2n} \neq 0\right)$. Because $2n-1$ is odd. $\mathsf{P}\left(S_{2n-1} = 0\right) = 0$. Therefore the first term is equal to $\mathsf{P}\left(S_1, S_2, \ldots S_{2n-2} \neq 0\right)$ which is by 16 equal to u_{2n-2}. Second term is by 16 equal to u_{2n}. Therefore we get the result. □

Lemma 19
$f_{2n} = \frac{1}{2n-1} u_{2n}$

Proof. $u_{2n-2} = \frac{1}{2}^{2n-2}\binom{2n-2}{n-1} = 4 \cdot \frac{1}{2}^{2n} \frac{(2n-2)!}{(n-1)!(n-1)!} = \frac{4n^2}{(2n)(2n-1)} \frac{1}{2}^{2n}\binom{2n}{n} = \frac{2n}{2n-1} u_{2n}$.
Therefore $u_{2n-2} - u_{2n} = u_{2n}\left(\frac{2n}{2n-1} - 1\right) = u_{2n}\frac{1}{2n-1}$. □

Lemma 20 (Decomposition of f_n)
$u_{2n} = \sum\limits_{r=1}^{n} f_{2r} u_{2n-2r}$

Proof. $u_{2n} \overset{\text{D4}}{=} \mathsf{P}\left(S_{2n} = 0\right) \overset{LTP}{=} \sum\limits_{r=1}^{n} \mathsf{P}\left(S_{2n} = 0, S_1, S, 2, \ldots, S_{2r-1} \neq 0 S_{2r} = 0\right) \overset{nasobeni}{=}$
$\sum\limits_{r=1}^{n} \mathsf{P}\left(S_{2n} = 0 \mid S_1, S, 2, \ldots, S_{2r-1} \neq 0 S_{2r} = 0\right) \mathsf{P}\left(S_1, S, 2, \ldots, S_{2r-1} \neq 0 S_{2r} = 0\right) =$
$\sum\limits_{r=1}^{n} \mathsf{P}\left(S_{2n} = 0 \mid S_{2r} = 0\right) f_{2r} \overset{\text{L3}}{=} \sum\limits_{r=1}^{n} u_{2n-2r} f_{2r}$. □

Theorem 21 (Arcsine law for last visits)
Let $k, n \in \mathbb{N}, k \leq n$. The probability that up to time $2n$ the last return to origin occured in time $2k$ is given by $\alpha_{2n}(2k) = u_{2n} u_{2(n-k)}$.

Proof. The probability involved can be rewritten as:
$\quad \mathsf{P}\left(S_{2k+1}, S_{2k+2}, \ldots, S_{2n} \neq 0, S_{2k} = 0\right)$
$\overset{nasobeni}{=} \mathsf{P}\left(S_{2k+1}, S_{2k+2}, \ldots, S_{2n} \neq 0 \mid S_{2k} = 0\right) \mathsf{P}\left(S_{2k} = 0\right)$
$\overset{\text{L3}}{=} \mathsf{P}\left(S_1, S_2, \ldots, S_{2(n-k)} \neq 0\right) \mathsf{P}\left(S_{2k} = 0\right)$
$\overset{\text{T16}}{=} u_{2(n-k)} u_{2k}$

□

Definition 7 (Time spend on the positive and negative sides). Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk . We say that the walk spent τ time units of n on the positive side if $\sum\limits_{i=1}^{n} \mathbf{1}_{[S_i > 0 \vee S_{i-1} > 0]} = \tau$. Let $\beta_n(\tau)$ denote the probability of such an event. We say that the walk spent ζ time units of n on the negative side if $\sum\limits_{i=1}^{n} \mathbf{1}_{[S_i < 0 \vee S_{i-1} < 0]} = \zeta$.

Theorem 22 (Arcsine law for sojourn times-OWN PROOF)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a symmetric random walk . Then $\beta_{2n}(2k) = \alpha_{2n}(2k)$.

Proof. Firstly let us start with degenerate cases. $\beta_{2n}(2n)$
$= \mathsf{P}\left(\sum\limits_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2n\right) \stackrel{\text{L17}}{=} \mathsf{P}(S_1, S_2, \ldots, S_{2n} \geq 0) = *u_{2n}$. By symmetry $\beta_{2n}(0) = \beta_{2n}(2n) = u_{2n}$.

Let $1 \leq k \leq v-1$, where $0 \leq v \leq n$. For such k stands equation:

$$\beta_{2n}(2k) \stackrel{\text{D7}}{=} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k\right)$$
$$\stackrel{LTP}{=} \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} \neq 0, S_{2r} = 0\right)$$
$$\stackrel{*}{=} \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right)$$
$$+ \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right)$$
$$\stackrel{nasobeni}{=} \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k \,\middle|\, S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right)$$
$$\mathsf{P}(S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0)$$
$$+ \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k \,\middle|\, S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right)$$
$$\mathsf{P}(S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0)$$
$$\stackrel{**}{=} \sum_{r=1}^{n} \tfrac{1}{2} f_{2r}\, \mathsf{P}\left(\sum_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k \,\middle|\, S_{2r} = 0\right)$$
$$+ \sum_{r=1}^{n} \tfrac{1}{2} f_{2r}\, \mathsf{P}\left(\sum_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k-2r \,\middle|\, S_{2r} = 0\right)$$
$$\stackrel{\text{L3}}{=} \sum_{r=1}^{n} \tfrac{1}{2} f_{2r}\, \mathsf{P}\left(\sum_{i=1}^{2n-2r} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k\right) + \sum_{r=1}^{n} \tfrac{1}{2} f_{2r}\, \mathsf{P}\left(\sum_{i=1}^{2n-2r} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k-2r\right) =$$
$\sum\limits_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k) + \sum\limits_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k-2r)$. Where $*$ comes from the disjoint decomposition of $[S_1, S_2, \ldots, S_{2r-1} \neq 0] = [S_1, S_2, \ldots, S_{2r-1} > 0] \cup [S_1, S_2, \ldots, S_{2r-1} < 0]$ and $**$ comes from using the condition that up to time $2r$ the steps were on the positive/negative sides.

Now let us proceed by induction. Case for $v = 1$ is trivial because it implies degenerous case from *f. Let the statment be true for $v \leq n-1$, then

$$\sum_{r=1}^{n} \tfrac{1}{2} f_{2r} \beta_{2n-2r}(2k) + \sum_{r=1}^{n} \tfrac{1}{2} f_{2r} \beta_{2n-2r}(2k-2r)$$
$$\stackrel{IA}{=} \sum_{r=1}^{n} \tfrac{1}{2} f_{2r} \alpha_{2n-2r}(2k) + \sum_{r=1}^{n} \tfrac{1}{2} f_{2r} \alpha_{2n-2r}(2k-2r)$$
$$\stackrel{\text{D4}}{=} \sum_{r=1}^{n} \tfrac{1}{2} f_{2r} u_{2k} u_{2n-2r-2k} + \sum_{r=1}^{n} \tfrac{1}{2} f_{2r} u_{2k-2r} u_{2n-2k}$$
$$= \tfrac{1}{2} u_{2k} \sum_{r=1}^{n} f_{2r} u_{2n-2r-2k} + \tfrac{1}{2} u_{2n-2k} \sum_{r=1}^{n} f_{2r} u_{2k-2r} u_{2n-2k}$$
$$\text{L20} \quad \tfrac{1}{2}\, u_{2n-2k} u_{2k} + \tfrac{1}{2} u_{2n-2k} u_{2k} = u_{2n-2k} u_{2k}$$
$$\stackrel{\text{D4}}{=} \alpha_{2n}(2k).$$

□

Definition 8 (Change of a sign)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$be a random walk . We say that in time n occurred a change of sign if if $S_{n-1} \cdot S_{n+1} = -1$ in other words if $(S_{n-1} = +1 \wedge S_{n+1} = -1) \vee (S_{n-1} = -1 \wedge S_{n+1} = +1)$. We shall denote the probability that up to time n occured r changes of sign by $\xi_{r,n}$.

Theorem 23 (Change of a sign)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$*be a symmetric random walk . The probability* $\xi_{r,2n+1} = 2\,\mathsf{P}(S_{2n+1} = 2r+1)$

Proof. Feller □

1.1 Problem chapter 9 Feller-není dokončeno ani zkotrolováno

Definition 9 ($\delta_n, \varepsilon_n^{r,\pm}$)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$be a symmetric random walk . $\delta_n(k)$ shall denote $\mathsf{P}\left(\sum\limits_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_n = 0\right)$, $\varepsilon_n^r(k)$ shall denote $\mathsf{P}\left(\sum\limits_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_1, S\right.$
$\varepsilon_n^{r,+}(k)$ shall denote $\mathsf{P}\left(\sum\limits_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_1, S_2, \dots, S_{r-1} > 0, S_r = 0, S_n = 0\right)$,
$\varepsilon_n^{r,-}(k)$ shall denote $\mathsf{P}\left(\sum\limits_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_1, S_2, \dots, S_{r-1} < 0, S_r = 0, S_n = 0\right)$.

Lemma 24 (Factorization of $\delta_{2n}(2k)$)
$\delta_{2n}(2k) = \frac{1}{2}\sum\limits_{r=1}^{n}\left(f_{2r}\delta_{2n-2r}(2k-2r) + f_{2r}\delta_{2n-2r}(2r)\right)$.

Proof. Because $S_{2n} = 0$ a return to origin must have happened. Let $2r$ the time of first return to origin, where $r \in \{1, 2, \dots, n\}$. By the law of total probability:

$$\delta_{2n}(2k) \overset{\text{D9}}{=} \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_{2n} = 0\right)$$
$$\overset{LTP}{=} \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \dots, S_{2r-1} \neq 0, S_{2r} = 0, S_{2n} = 0\right)$$
$$\overset{\text{D9}}{=} \sum_{r=1}^{n} \varepsilon_{2n}^{2k} \overset{*}{=} \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \dots, S_{2r-1} > 0, S_{2r} = 0, S_{2n} = 0\right)$$
$$+ \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \dots, S_{2r-1} < 0, S_{2r} = 0, S_{2n} = 0\right)$$
$$= \sum_{r=1}^{n} \varepsilon_{2n}^{2r,+}(2k) + \sum_{r=1}^{n} \varepsilon_{2n}^{2r,-}(2k).$$

Where $*$ comes from the disjoint decomposition $[S_1, S_2, \dots, S_{2r-1} \neq 0] = [S_1, S_2, \dots, S_{2r-1} > 0] \cup [S_1, S_2, \dots, S_{2r-1} < 0]$.

Now let us calculate $\varepsilon_{2n}^{2r,+}(2k)$

$$= \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \dots, S_{2r-1} > 0, S_{2r} = 0, S_{2n} = 0\right)$$
$$\overset{nasobeni}{=} \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n} = 0 \mid S_1, S_2, \dots, S_{2r-1} > 0, S_{2r} = 0\right)$$
$$\mathsf{P}\left(S_1, S_2, \dots, S_{2r-1} > 0, S_{2r} = 0\right)$$
$$\overset{*}{=} \mathsf{P}\left(\sum_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k-2r, S_{2n} = 0 \;\middle|\; S_{2r} = 0\right) \mathsf{P}\left(S_1, S_2, \dots, S_{2r-1} > 0, S_{2r} = 0\right)$$
$$\overset{**}{=} \mathsf{P}\left(\sum_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k-2r, S_{2n} = 0 \;\middle|\; S_{2r} = 0\right) \tfrac{1}{2} f_2 r$$
$$\overset{\text{L3}}{=} \mathsf{P}\left(\sum_{i=1}^{2n-2r} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k-2r, S_{2n-2r} = 0\right) \tfrac{1}{2} f_{2r}$$
$$\overset{\text{D9}}{=} \delta_{2n-2r}(2k-2r)\, \tfrac{1}{2} f_{2r}.$$

Where $*$ comes from Lemma (4) and using the condition.

Where $**$ comes from the fact that $f_{2r} \overset{\text{D4}}{=} \mathsf{P}(S_1, S_2, \dots, S_{2r-1} \neq 0, S_{2r} = 0) = \mathsf{P}(S_1 = 1, S_2, \dots, S_{2r-1} > 0, S_{2r} = 0) + \mathsf{P}(S_1 = -1, S_2, \dots, S_{2r-1} < 0, S_{2r} = 0)$ and $\mathsf{P}(S_1 = -1, S_2, \dots, S_{2r-1} < 0, S_{2r} = 0) = \mathsf{P}(S_1 = 1, S_2, \dots, S_{2r-1} > 0, S_{2r} = 0)$ because of symmetry. Hence $\mathsf{P}(S_1 = 1, S_2, \dots, S_{2r-1} > 0, S_{2r} = 0) = \frac{1}{2} f_{2r}$.

Similarly $\varepsilon_{2n}^{2r,+}(2k)$

$$= \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \dots, S_{2r-1} < 0, S_{2r} = 0, S_{2n} = 0\right)$$

$= \mathsf{P}\left(\sum\limits_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n} = 0 \mid S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right)$
$\mathsf{P}\left(S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right)$
$= \mathsf{P}\left(\sum\limits_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n} = 0 \;\middle|\; S_{2r} = 0\right) \mathsf{P}\left(S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right)$
$= \mathsf{P}\left(\sum\limits_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n} = 0 \;\middle|\; S_{2r} = 0\right) \frac{1}{2} f_2 r$
$= \mathsf{P}\left(\sum\limits_{i=1}^{2n-2r} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n-2r} = 0\right) \frac{1}{2} f_{2r}$
$= \delta_{2n-2r}\left(2k\right) \frac{1}{2} f_{2r}$.

Therefore $\delta_{2n}\left(2k\right) = \frac{1}{2} \sum\limits_{r=1}^{n} f_{2r}\delta_{2n-2r}\left(2k-2r\right) + \frac{1}{2} \sum\limits_{r=1}^{n} f_{2r}\delta_{2n-2r}\left(2k\right)$
$= \frac{1}{2} \sum\limits_{r=1}^{n} \left(f_{2r}\delta_{2n-2r}\left(2k-2r\right) + f_{2r}\delta_{2n-2r}\left(2r\right)\right)$ □

Theorem 25 (Equidistributional theorem-ALMOST COMPLETE OWN PROOF)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a symmetric random walk and* $n \in \mathbb{N}$*, then* $\forall k, l \in \{0, 1, 2, \ldots, n\}$:
$\delta_{2n}\left(2k\right) = \delta_{2n}\left(2l\right) = \frac{u_{2n}}{n+1}$.

Proof. Let us prove this statement by induction in n. In case that $n = 1$ we have two options for k. Either $k = 0$ or $k = 1$. $\delta_2\left(0\right) = \mathsf{P}\left(S_1 = -1, S_2 = 0\right) = \frac{1}{2}u_2 = \mathsf{P}\left(S_1 = +1, S_2 = 0\right) = \delta_2\left(2\right)$.

Let the statment be true for all $l \leq n-1$. In that case $\delta_{2(n-l)}\left(2k\right) = \frac{u_{2(n-l)}}{n-l+1} \forall k \in \{1, 2, \ldots, n-l\}$. We want to show that $\delta_{2n} = \frac{u_{2n}}{n+1}$.

Let us calculate $\delta_{2n} \overset{\text{L24}}{=} \frac{1}{2} \sum\limits_{r=1}^{n} \left(f_{2r}\delta_{2n-2r}\left(2k-2r\right) + f_{2r}\delta_{2n-2r}\left(2r\right)\right)$
$\overset{\underline{IA}}{=} \frac{1}{2} \sum\limits_{r=1}^{n} \left(f_{2r}u_{2n-2r}\frac{1}{n-r+1} + f_{2r}u_{2n-2r}\frac{1}{n-r+1}\right) = \sum\limits_{r=1}^{n} \frac{f_{2r}u_{2n-2r}}{n-r+1} \overset{\text{SNAD TO DOKAZU L26}}{=} \frac{u_{2n}}{n+1}$
□

Lemma 26 (Sum of binomials-POTŘEBUJU DOKÁZAT)
$\sum\limits_{r=1}^{n} \frac{f_{2r}u_{2n-2r}}{n-r+1} = \frac{u_{2n}}{n+1}$

Proof. $f_{2r}u_{2n-2r} \overset{\text{L19}}{=} \frac{1}{2r-1}u_{2r}u_{2n-2r} \overset{\text{D4}}{=} \frac{1}{2r-1}2^{-2r}\binom{2r}{r}2^{-(2n-2r)}\binom{2n-2r}{n-r}$.

Therefore $\sum\limits_{r=1}^{n} \frac{f_{2r}u_{2n-2r}}{n-r+1} = \sum\limits_{r=1}^{n} \frac{1}{2r-1}\frac{1}{n-r+1}2^{-2n}\binom{2r}{r}\binom{2n-2r}{n-r} \overset{???}{=} \frac{1}{n+1}2^{-2n}\binom{2n}{n}$ □

2. Simple random walk in more dimensions

Definition 10 (Type II random walk in $\mathbb{Z}^m$)**.** Let $m \in \mathbb{N}$. $\forall n \in \mathbb{N}$, let $X_n = \begin{pmatrix} x_n^1 & x_n^2 & \dots, x_n^m \end{pmatrix}^T$, where $\{x_n^i\}_{i=1}^m$ are $\forall n \in \mathbb{N}$ independent.

Let $\forall i \in \{1, 2, \dots, m\} x_n^i$ have values in $\{-1, +1\}$ with probabilities $\mathsf{P}(x_n^i = +1) = p_i \in (0, 1)$ and $\mathsf{P}(x_n^i = -1) = 1 - p_i =: q_i \in (0, 1)$.

Let $\{X_n\}_{n=0}^{+\infty}$ be a sequence of independent and identically distributed random variables. Let $S_0 = \mathbf{0}$ and $\forall n \in \mathbb{N} : \mathbf{S_n} = \sum_{i=1}^{n} X_i$ and $\mathbf{p} = \begin{pmatrix} p_1, p_2, \dots, p_m \end{pmatrix}^{\mathbf{T}}$. Then the pair $\left(\{\mathbf{S_n}\}_{n=0}^{+\infty}, \mathbf{p}\right)$ is called *Type II random walk in* $\mathbb{Z}^m$.

If $\forall i \in \{1, 2, \dots, m\} : p_i = q_i = \frac{1}{2}$ we call the element $\{\mathbf{S_n}\}_{n=0}^{+\infty}$ *Symmetric type II random walk* $\mathbb{Z}^m$.

Remark. Type II random walk can be interpreted as m simple random walks in $\mathbb{Z}$ happening at a time, each of them parallel to an axis of $\mathbb{Z}^m$.

Theorem 27
Let $m \in \mathbb{N}$ and $\left(\{\mathbf{S_n}\}_{n=0}^{+\infty}, \mathbf{p}\right)$ be a Type II random walk in $\mathbb{Z}^m$. Let $\mathbf{y} = \begin{pmatrix} y_1, y_2, \dots, y_m \end{pmatrix}^T \in \mathbb{Z}^m$. Then following equation stands:

$$\mathsf{P}(S_n = x) = \begin{cases} \prod_{i=1}^m \binom{n}{\frac{y_i+n}{2}} p_i^{\frac{n+y_i}{2}} q_i^{\frac{n-y_i}{2}}, & \textit{if } \forall i \in \{1, 2, \dots, m\} : y_i \in A_n, \\ 0, & \textit{if } \exists i \in \{1, 2, \dots, m\} : y_i \notin A_n. \end{cases}$$

Where A_n is from definition (2)

Proof. $\mathsf{P}(\mathbf{S_n} = \mathbf{y}) = \mathsf{P}(S_n^1 = y_1, S_n^2 = y_2, \dots, S_n^m = y_m) \overset{\perp\!\!\!\perp}{=} \prod_{i=1}^{m} \mathsf{P}(S_n^i = y_i)$ $= \prod_{i=1}^{m} \binom{n}{\frac{y_i+n}{2}} p_i^{\frac{n+y_i}{2}} q_i^{\frac{n-y_i}{2}}$. The second equation comes from the independency of $\{\mathbf{S_i}\}_{i=1}^m$ which comes easily from independency of X_n^i. The second equation comes from Theorem (1). □

Remark. Due to the the aim of this thesis which is reasearching occupation time of a set of random walks we are going to concern only on symmetric random walks.

Definition 11 (Orthant)**.** Let $m \in \mathbb{Z}$. Then $O \subset \mathbb{Z}^m$ is called an *open orthant in* $\mathbb{Z}^m$ if $\forall o := \begin{pmatrix} o_1, o_2, \dots, o_m \end{pmatrix}^T \in O, \forall i \in \{1, 2, \dots, m\} : o_i \varepsilon_i > 0$, where $\varepsilon_i \in \{-1, +1\}$.

$C \subset \mathbb{Z}^m$ is called a *closed orthant in* $\mathbb{Z}^m$ if $\forall c := \begin{pmatrix} c_1, c_2, \dots, c_m \end{pmatrix}^T \in C, \forall i \in \{1, 2, \dots, m\} : c_i \varepsilon_i \geq 0$, where $\varepsilon_i \in \{-1, +1\}$.

Remark. The statement $\mathbf{x} > \mathbf{y}$ will mean $\forall i \in \{1, 2, \dots, m\} : x_i > y_i$. Same applies to $<, \leq, \geq$.

Theorem 28 (Probability of being in an open orthant)
Let $\{\mathbf{S_n}\}_{n=0}^{+\infty}$ be a Symmetric type II random walk in $\mathbb{Z}^m$. Let O be an open orthant in Z^m. $\mathsf{P}(\mathbf{S_n} \in O) = \left(\frac{1}{2} u_{2n}\right)^m$.

Proof. Without loss of generality we can assume that in the definition of O we choose $\forall i \in \{1, 2, \dots, m\} \varepsilon_i := +1$ then: $\mathsf{P}(\mathbf{S_n} \in O) = \mathsf{P}(S_n^1 > 0, S_n^2 > 0, \dots, S_n^m > 0) = \prod_{i=1}^m \mathsf{P}(S_n^i > 0) = (\mathsf{P}(S_n^i > 0))^m = \left(\frac{1}{2} u_{2n}\right)^m$. Where the last two equations come from the identical distribution of S_n^i and Theorem (15). □

Theorem 29 (Probability of being in a closed orthant)
Let $\{\mathbf{S_n}\}_{n=0}^{+\infty}$ be a Symmetric type II random walk in $\mathbb{Z}^m$. Let C be a closed orthant in Z^m. $\mathsf{P}(\mathbf{S_n} \in C) = (u_{2n})^m$.

Proof. The proof is very similar to previous proof. Without loss of generality we can again assume that in the definition of C we choose $\forall i \in \{1, 2, \dots, m\} \varepsilon_i := +1$ then: $\mathsf{P}(\mathbf{S_n} \in C) = \mathsf{P}(S_n^1 \geq 0, S_n^2 \geq 0, \dots, S_n^m \geq 0) = \prod_{i=1}^m \mathsf{P}(S_n^i \geq 0) = (\mathsf{P}(S_n^i \geq 0))^m = (u_{2n})^m$. Where the last two equations come from the identical distribution of S_n^i and Lemma (17). □

Theorem 30 (Zákon iterovaného logaritmu)
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