1. Title of the second chapter

1.1 Problem chapter 9 Feller

Definition 1 (δ, ε)**.** *Let* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a symmetric random walk .* $\delta_n(k)$ *shall denote* $\mathsf{P}\left(\sum\limits_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_n = 0\right)$, $\varepsilon_n^r(k)$ *shall denote* $\mathsf{P}\left(\sum\limits_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_1, S_2, \ldots$
$\varepsilon_n^{r,+}(k)$ *shall denote* $\mathsf{P}\left(\sum\limits_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_1, S_2, \ldots, S_{r-1} > 0, S_r = 0, S_n = 0\right)$,
$\varepsilon_n^{r,-}(k)$ *shall denote* $\mathsf{P}\left(\sum\limits_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_1, S_2, \ldots, S_{r-1} < 0, S_r = 0, S_n = 0\right)$.

Lemma 1 (Factorization of $\delta_{2n}(2k)$)**.** $\delta_{2n}(2k) = \frac{1}{2}\sum\limits_{r=1}^{n}\left(f_{2r}\delta_{2n-2r}(2k-2r) + f_{2r}\delta_{2n-2r}(2r)\right)$.

Proof. Because $S_{2n} = 0$ a return to origin must have happened. Let $2r$ the time of first return to origin, where $r \in \{1, 2, \ldots, n\}$. By the law of total probability: $\delta_{2n}(2k) = \mathsf{P}\left(\sum\limits_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_{2n} = 0\right) = \sum\limits_{r=1}^{n} \mathsf{P}\left(\sum\limits_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} \neq\right.$
which can be again by the law of total probability factorized as: $\sum\limits_{r=1}^{n} \mathsf{P}\left(\sum\limits_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1,\right.$
$\sum\limits_{r=1}^{n} \mathsf{P}\left(\sum\limits_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0 S_{2n} = 0\right) + \sum\limits_{r=1}^{n} \mathsf{P}\left(\sum\limits_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} =\right.$
$\sum\limits_{r=1}^{n} \varepsilon_{2n}^{2r,-}(2k)$. Now let us calculate $\varepsilon_{2n}^{2r,+}(2k)$. $\varepsilon_{2n}^{2r,+}(2k) = \mathsf{P}\left(\sum\limits_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, \right.$
$*a\, \mathsf{P}\left(\sum\limits_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n} = 0 \mid S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right) \mathsf{P}\left(S_1, S_2, \ldots, S_{2r-1} > 0, S\right.$
$\mathsf{P}\left(\sum\limits_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k - 2r, S_{2n} = 0 \,\middle|\, S_{2r} = 0\right) \frac{1}{2} f_{2r} = *b\, \mathsf{P}\left(\sum\limits_{i=1}^{2n-2r} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k - 2r\right.$
$*c\delta_{2n-2r}(2k-2r)\frac{1}{2}f_{2r}$. Similarly $\varepsilon_{2n}^{2r,-}(2k) = \mathsf{P}\left(\sum\limits_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} < 0, S\right.$
$*a\, \mathsf{P}\left(\sum\limits_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n} = 0 \mid S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right) \mathsf{P}\left(S_1, S_2, \ldots, S_{2r-1} < 0, S\right.$
$\mathsf{P}\left(\sum\limits_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n} = 0 \,\middle|\, S_{2r} = 0\right) \frac{1}{2} f_{2r} = *b\, \mathsf{P}\left(\sum\limits_{i=1}^{2n-2r} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n-2r} =\right.$
$*c\delta_{2n-2r}(2k)\frac{1}{2}f_{2r}$. Therefore $\delta_{2n}(2k) = \frac{1}{2}\sum\limits_{r=1}^{n} f_{2r}\delta_{2n-2r}(2k-2r) + \frac{1}{2}\sum\limits_{r=1}^{n} f_{2r}\delta_{2n-2r}(2k) =$
$\frac{1}{2}\sum\limits_{r=1}^{n}\left(f_{2r}\delta_{2n-2r}(2k-2r) + f_{2r}\delta_{2n-2r}(2r)\right)$ □

Theorem 2 (Equidistributional theorem)**.** *Let* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a symmetric random walk and* $n \in \mathbb{N}$*, then* $\forall k, l \in \{0, 1, \ldots, n\} : \delta_{2n}(2k) = \delta_{2n}(2l) = \frac{u_{2n}}{n+1}$.

Proof. Let us prove this statement by induction in n. In case that $n = 1$ we have two options for k. Either $k = 0$ or $k = 1$. $\delta_2(0) = \mathsf{P}(S_1 < 0, S_2 = 0) = \frac{1}{2}f_2 = *a\frac{1}{2}u_2\frac{1}{2-1} = \frac{u_2}{2}\delta_2(2) = \mathsf{P}(S_1 > 0, S_2 = 0) = \frac{1}{2}f_2 = \frac{u_2}{2}$.

Let the statment be true for all $l \leq n-1$. In that case $\delta_{2(n-l)}(2k) = \frac{u_{2(n-l)}}{n-l+1}$. We want to show that $\delta_{2n} = \frac{u_{2n}}{n+1}$. Let us calculate δ_{2n}. $\delta_{2n} = *b\frac{1}{2}\sum\limits_{r=1}^{n}\left(f_{2r}\delta_{2n-2r}(2k-2r) + f_{2r}\delta_{2n-2r}\right.$
$\frac{1}{2}\sum\limits_{r=1}^{n}\left(f_{2r}u_{2n-2r}\frac{1}{n-r+1} + f_{2r}u_{2n-2r}\frac{1}{n-r+1}\right) = \sum\limits_{r=1}^{n}\left(\frac{f_{2r}u_{2n-2r}}{n-r+1}\right)$ □