

1. Title of the first chapter

Definition 1 (Simple random walk in \mathbb{Z}). Let $\{X_n\}_{n=0}^{+\infty}$ be a sequence of independent and identically distributed random variables with values in $\{-1, +1\}$. That $\forall n \in \mathbb{N}$ satisfy $P(X_n = 1) = p \in (0, 1)$ and $P(X_n = -1) = 1 - p = q$. Let $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$. We call the pair $(\{S_n\}_{n=0}^{+\infty}, p)$ Simple random walk in \mathbb{Z} . In case that $p = q = \frac{1}{2}$ we call the pair $(\{S_n\}_{n=0}^{+\infty}, p)$ Symmetric simple random walk in \mathbb{Z} .

Remark. Very often we refer to n as time, X_i as i -th step and S_n as position in time n . In simple random walk in \mathbb{Z} we refer to $X_i = +1$ as i -th step was rightwards. If not stated otherwise, we

Definition 2 (Set of possible positions). Let $(\{S_n\}_{n=0}^{+\infty}, p)$ be a random walk. We call the set $A_n = \{z \in \mathbb{Z}; |z| \leq n, \frac{z+n}{2} \in \mathbb{Z}\}$ set of all possible positions of random walk $(\{S_n\}_{n=0}^{+\infty}, p)$ in time n .

Theorem 1 (Probability of position x in time n). Let $(\{S_n\}_{n=0}^{+\infty}, p)$ be a random walk and A_n its set of possible positions.

$$P(S_n = x) = \begin{cases} \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}} & \text{for } x \in A_n, \\ 0, & \text{for } x \notin A_n. \end{cases}$$

Proof. Let us define new variables $r_i = \mathbf{1}_{[X_i=1]}, l_i = \mathbf{1}_{[X_i=-1]}, R_n = \sum_{i=1}^n r_i, L_n = \sum_{i=1}^n l_i$. r_i can be interpreted as indicator whether i -th step was rightwards. Then R_n is number of rightwards steps and L_n is number of leftwards steps. We can easily see that $R_n + L_n = n$ and $R_n - L_n = S_n$. Therefore we get by adding these two equations $R_n = \frac{S_n + n}{2}$.

r_i has alternative distribution with parameter p ($Alt(p)$). Therefore R_n as a sum of independent and identically distributed random variables with $Alt(p)$ has binomial distribution with parameters n and p ($Bi(n, p)$). Therefore we get $P(R_n = x) = \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}$. Where we define $\binom{a}{x} := 0$ for $a \in \mathbb{N}, x \in \mathbb{R} \setminus \mathbb{N}, x < 0, x > n$. Therefore we get $P(S_n = x) = P(R_n = \frac{x+n}{2}) = \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}$. \square

Lemma 2 (Spatial homogeneity). $P(S_n = j | S_0 = a) = P(S_n = j + b | S_0 = a + b) \forall b \in \mathbb{Z}$

Proof. $P(S_n = j | S_0 = a) = P\left(\sum_{i=1}^n X_i = j - a\right) = P\left(\sum_{i=1}^n X_i = (j + b) - (a + b)\right) = P(S_n = j + b | S_0 = a + b)$. \square

Lemma 3 (Temporal homogeneity). $P(S_n = j | S_0 = a) = P(S_{n+m} = j | S_m = a) \forall m \in \mathbb{N}$

Proof. $P(S_n = j | S_0 = a) = P\left(\sum_{i=1}^n X_i = j - a\right) = P\left(\sum_{i=m+1}^{m+n} X_i = j - a\right) = P(S_{n+m} = j | S_m = a)$. \square

Lemma 4 (Markov property). *Let $n \geq m$ and $a_i \in \mathbb{Z}$. Then $P(S_n = j | S_0 = a_0, S_1 = a_1, \dots, S_m = a_m) = P(S_n = j | S_m = a_m)$*

Proof. Once S_m is known, then distribution of S_n depends only on steps $X_{m+1}, X_{m+2}, \dots, X_n$ and therefore cannot be dependent on any information concerning values X_1, X_2, \dots, X_m and accordingly S_1, S_2, \dots, S_m . \square

Remark. In symmetric random walk, everything can be counted by number of possible paths from point to point.

Definition 3 (Number of possible paths). *Let $N_n(a, b)$ be number of possible paths of random walk $(\{S_n\}_{n=0}^{+\infty}, p)$ from point $(0, a)$ to point (n, b) and $N_n^x(a, b)$ be number of possible paths from point $(0, a)$ to point (n, b) that visit point (z, x) for some $z \in \{0, \dots, n\}$.*

Theorem 5. *Let $a, b \in \mathbb{Z}, n \in \mathbb{N}$ then $N_n(a, b) = \binom{n}{\frac{1}{2}(n+b-a)}$.*

Proof. Let us choose a path from $(0, a)$ to (n, b) and let α be number of rightwards steps and β be number of leftwards steps. Then $\alpha + \beta = n$ and $\alpha - \beta = b - a$. By adding these two equations we get that $\alpha = \frac{1}{2}(n + b - a)$. The number of possible paths is the number of ways of picking α rightwards steps from n steps. Therefore we get $N_n(a, b) = \binom{n}{\alpha} = \binom{n}{\frac{1}{2}(n+b-a)}$. \square

Theorem 6 (Reflection principle). *Let $a, b > 0$, then $N_n^0(a, b) = N_n(-a, b)$.*

Proof. Each path from $(0, -a)$ to (n, b) has to intersect x -axis at least once at some point. Let k be the time of earliest intersection with x -axis. By reflexing the segment from $(0, -a)$ to $(k, 0)$ in the x -axis, we get a path from point $(0, a)$ to (n, b) which visits 0 at point k . Because reflection is bijective operation on sets of paths, we get the correspondence between the collections of such paths. \square

Definition 4 (Return to origin). *Let $S_0 = 0$ $(\{S_n\}_{n=0}^{+\infty}, p)$ be a random walk. Then if $\exists k \in \mathbb{N}$ such that $S_k = 0$ then we say that in k -th step occurred return to origin.*

Theorem 7 (Ballot theorem). *Let $n, b \in \mathbb{N}$ Number of paths from point $(0, 0)$ to point (n, b) which do not return to origin is equal to $\frac{b}{n} N_n(0, b)$*

Proof. Let us call N the number of paths we are referring to. Because the path ends at point (n, b) , the first step has to be rightwards. Therefore we now have $N = N_{n-1}(1, b) - N_{n-1}^0(1, b) = N_{n-1}(1, b) - N_{n-1}(-1, b)$. The last equation was acquired using Ballot theorem (6). We now have:

$$N_{n-1}(1, b) - N_{n-1}(-1, b) = \binom{n-1}{\frac{n}{2} + \frac{b}{2} - 1} - \binom{n-1}{\frac{n}{2} + \frac{b}{2}} = \frac{(n-1)!}{(\frac{n}{2} + \frac{b}{2} - 1)!(\frac{n}{2} - \frac{b}{2})!} - \frac{(n-1)!}{(\frac{n}{2} + \frac{b}{2})!(\frac{n}{2} - \frac{b}{2} - 1)!}$$

\square

Definition 5. $M_n = \max\{S_i, i \in \{0, 1, \dots, n\}\}$

Theorem 8 (Probability of maximum in time n). *Let $(\{S_n\}_{n=0}^{+\infty}, p)$ be a random walk.*

$$P(M_n \geq r, S_n = b) = \begin{cases} P(S_n = b) & \text{for } b \geq r, \\ P(S_n = 2r - b) \left(\frac{q}{p}\right)^{r-b}, & \text{for otherwise.} \end{cases}$$

Proof. Let us firstly consider the easier case in which $b \geq r$. Because we defined M_n as $\max\{S_i, i \in \{0, 1, \dots, n\}\}$ we get that $M_n \geq b \geq r$ therefore $[M_n \geq r] \subset [S_n = b]$ therefore we get $P(M_n \geq r, S_n = b) = P(S_n = b)$. Let $r \geq 1, b < r$. $N_n^r(0, b)$ stands for number of paths from point $(0, 0)$ to point (n, b) which reach up to r . Let $k \in \{0, 1, \dots, n\}$ denote the first time we reach r . By reflection principle (6), we can reflex the segment from (k, r) to (n, b) in the axis: $y = r$. Therefore we now have path from $(0, 0)$ to $(n, 2r - b)$ and we get that $N_n^r(0, b) = N_n(0, 2r - b)$. $P(S_n = b, M_n \geq r) = N_n^r(0, b)p^{\frac{n+b}{2}}q^{\frac{n-b}{2}} = N_n(0, 2r - b)p^{\frac{n+(2r-b)}{2}}q^{\frac{n-(2r-b)}{2}}p^{\frac{n+b}{2}}q^{\frac{n-b}{2}}p^{b-r}q^{r-b} = \left(\frac{q}{p}\right)^{r-b}P(S_n = 2r - b)$. \square

Definition 6 (Walk rechang new maximum at particular time). Let $b > 0$. $f_b(n)$ denotes the probability that we reach new maximum b at time n . $f_b(n) = P(M_{n-1} = S_{n-1} = b - 1, S_n = b)$

Theorem 9 (Probability of reaching new maximum b in time n). Let $b > 0$ then $f_b(n) = \frac{b}{n} P(S_n = b)$.

Proof. $f_b = P(M_{n-1} = S_{n-1} = b - 1, S_n = b) = pP(M_{n-1} = S_{n-1} = b - 1) = p(P(M_{n-1} \geq b - 1, S_{n-1} = b - 1) - P(M_{n-1} \geq b, S_{n-1} = b - 1)) = p(P(S_{n-1} = b - 1) - \frac{q}{p}P(S_{n-1} = b)) = pP(S_{n-1} = b - 1) - qP(S_{n-1} = b) = \binom{n-1}{\frac{n}{2}+\frac{b}{2}-1}p^{\frac{n}{2}+\frac{b}{2}}q^{\frac{n}{2}-\frac{b}{2}} - \binom{n-1}{\frac{n}{2}+\frac{b}{2}}p^{\frac{n}{2}+\frac{b}{2}}q^{\frac{n}{2}-\frac{b}{2}} = p^{\frac{n}{2}+\frac{b}{2}}q^{\frac{n}{2}-\frac{b}{2}}\left(\frac{(n-1)!}{(\frac{n}{2}+\frac{b}{2}-1)!(\frac{n}{2}-\frac{b}{2})!} - \frac{(n-1)!}{(\frac{n}{2}+\frac{b}{2})!(\frac{n}{2}-\frac{b}{2}-1)!}\right) = p^{\frac{n}{2}+\frac{b}{2}}q^{\frac{n}{2}-\frac{b}{2}}\left(\frac{(n-1)!}{(\frac{n}{2}+\frac{b}{2})!(\frac{n}{2}-\frac{b}{2})!}\right)\left(\frac{1}{\frac{n}{2}-\frac{b}{2}} - \frac{1}{\frac{n}{2}+\frac{b}{2}}\right) = p^{\frac{n}{2}+\frac{b}{2}}q^{\frac{n}{2}-\frac{b}{2}}\frac{b}{n}\left(\frac{n!}{(\frac{n}{2}+\frac{b}{2})!(\frac{n}{2}-\frac{b}{2})!}\right) = \frac{b}{n}p^{\frac{n}{2}+\frac{b}{2}}q^{\frac{n}{2}-\frac{b}{2}}\binom{n}{\frac{n}{2}+\frac{b}{2}} = \frac{b}{n}P(S_n = b)$. \square

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A. Attachments

A.1 First Attachment