Introduction

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In the whole thesis binomial coefficients follow the rule: $\binom{n}{a} = 0 \text{if} a \notin \{0, 1, \ldots, n\}$

$\stackrel{LTP}{=}$ states that the equation comes frome the Law of total probability.
$\stackrel{\mathrm{T}a}{=}$ states that the equation comes from Theorem a
$\stackrel{\mathrm{L}a}{=}$ states that the equation comes from Lemma a
$\stackrel{\mathrm{D}a}{=}$ states that the equation comes from Lemma a
$\stackrel{\mathrm{def}}{=}$ states that the equation comes from the definition
$\stackrel{IA}{=}$ states that the equation comes from induction assumption
$\stackrel{GPR}{=}$ states that the equation comes from the general product rule

1. Basic definitions

Remark. First, let us properly introduce what a random walk is. After stating some of the basic definitions we will move to the core of this thesis which is to explore properties of occupation of a set times.

Definition 1 (Simple random walk in $\mathbb{Z}$)**.** Let $\{X_n\}_{n=0}^{+\infty}$ be a sequence of independent and identically distributed $\{-1,1\}$-valued random variables, that for some $p \in (0,1)$ and for $n \in \mathbb{N}$ satisfy $\mathsf{P}(X_n = 1) = p$ and $\mathsf{P}(X_n = -1) = 1 - p =: q$.

Let $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$. We call the pair $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *Simple random walk in* $\mathbb{Z}$.

If $p = q = \frac{1}{2}$, the pair $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ reduces to the element $\{S_n\}_{n=0}^{+\infty}$ which is called *Symmetric simple random walk in* $\mathbb{Z}$.

Remark. We often refer to n as time, X_i as the i-th step and S_n is the position of the walk in time n or after n steps. While talking about simple random walk in $\mathbb{Z}$ we say the i-th step was rightwards or more often upwards if $X_i = +1$ and that the i-th step was leftwards or downwards if $X_i = -1$. If it is not stated otherwise, we will always assume that $S_0 = 0$.

Remark. The most important characteristic of random walks is the probability of being in position x in time n. In order to calculate such probability we have to firstly find possible positions.

For example, it is impossible for the random walk to be in position x in time n if $x > n$ simply because there have not been enough steps to make it up to x. It is also impossible (given $S_0 = 0$) that after even number of steps the random walk is in odd-numbered position and vice versa. Therefore we define the set of possible positions.

Definition 2 (Set of possible positions)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk. We call the set $A_n = \{z \in \mathbb{Z}; |z| \leq n, \frac{z+n}{2} \in \mathbb{Z}\}$ *set of all possible positions* of random walk $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ in time n.

Theorem 1 (Probability of position x in time n)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a random walkthen for all* $n \in \mathbb{N}$

$$\mathsf{P}(S_n = x) = \begin{cases} \binom{n}{\frac{n+x}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}} & \text{for } x \in A_n\ , \\ 0, & \text{for } x \notin A_n. \end{cases}$$

Proof. Consider random variables $X_i^+ := \mathbf{1}_{[X_i=1]}$, and $X_i^- := \mathbf{1}_{[X_i=-1]}$, and define new random variables $R_n = \sum_{i=1}^{n} X_i^+, L_n = \sum_{i=1}^{n} X_i^-$. The random variable X_i^+ can be interpreted as indicator whether i-th step was rightwards. Then, R_n is number of rightwards steps and L_n is number of leftwards steps. We can easily see that $R_n + L_n = n$ and $R_n - L_n = S_n$. Therefore we get by adding these two equations $R_n = \frac{S_n+n}{2}$.

Clearly, X_i^+ has alternative distribution with parameter p (*Alt(p)*). Hence, R_n as a sum of independent and identically distributed random variables with distribution *Alt(p)* has binomial distribution with parameters n and p (*Bi(n,p)*).

Therefore we get $\mathsf{P}\left(R_n = x\right) = \binom{n}{x} p^x q^{n-x}$, where we define binomial coefficients as statet in preface. Finally, for $a \in A_n$ we get

$$\mathsf{P}\left(S_n = x\right) = \mathsf{P}\left(R_n = \frac{x+n}{2}\right) = \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}.$$

$\square$

Remark. Following are three simple lemmata that simplify many calculations in what follows. After proving them we can focus on the thesis goals.

Lemma 2 (Spatial homogeneity)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a random walk and* $n \in \mathbb{N}$ *then for all* $b, j \in \mathbb{Z}$

$$\mathsf{P}\left(S_n = j\right) = \mathsf{P}\left(S_n = j + b \mid S_0 = b\right)$$

Proof. For any $j, b \in \mathbb{Z}$ holds

$$\begin{aligned} \mathsf{P}\left(S_n = j\right) = \mathsf{P}\left(\sum_{i=1}^{n} X_i = j\right) &= \mathsf{P}\left(\sum_{i=1}^{n} X_i = (j+b) - b\right) \\ &= \mathsf{P}\left(S_n = j + b \mid S_0 = b\right). \end{aligned}$$

$\square$

Lemma 3 (Temporal homogeneity)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a random walk and* $n, m \in \mathbb{N}$. *Then for all* $j \in \mathbb{Z}$

$$\mathsf{P}\left(S_n = j\right) = \mathsf{P}\left(S_{n+m} = j \mid S_m = 0\right)$$

Proof. For any $j \in \mathbb{Z}$ and $m \in \mathbb{N}$

$$\begin{aligned} \mathsf{P}\left(S_n = j\right) = \mathsf{P}\left(\sum_{i=1}^{n} X_i = j\right) &= \mathsf{P}\left(\sum_{i=m+1}^{m+n} X_i = j\right) \\ &= \mathsf{P}\left(S_{n+m} = j \mid S_m = 0\right), \end{aligned}$$

where the second equality follows from identical distribution of $\{X_n\}_{n=1}^{+\infty}$. $\square$

Lemma 4 (Markov property)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a random walk,* $n, m \in \mathbb{N}, n \geq m, a_i \in \mathbb{Z}, i \in \mathbb{N}$ *such that* $\mathsf{P}\left(S_0 = a_0, S_1 = a_1, \ldots, S_m = a_m\right) > 0$. *Then*

$$\mathsf{P}\left(S_n = j \mid S_0 = a_0, S_1 = a_1, \ldots, S_m = a_m\right) = \mathsf{P}\left(S_n = j \mid S_m = a_m\right)$$

Proof. Because $\{X_n\}_{n=1}^{+\infty}$ is a sequence of independent variables, once S_m is known, then the conditional distribution of S_n depends only on steps $X_{m+1}, X_{m+2}, \ldots, X_n$ and therefore cannot depend on any information concerning values $X_1, X_2, \ldots, X_{m-1}$ and accordingly $S_1, S_2, \ldots, S_{m-1}$. $\square$

2. Number of paths+maxima

Remark. Once having stated basic definitions we may ask questions about occupation times of a set. Let $a > 0$. How many steps up to time n does our walk spend above a (in interval $[a, +\infty)$)? Similarly how many steps does the walk spend in interval $[-a, +\infty)$? We are going to answer these questions in following TODO: :section/chapter/subchapter.

Remark. The assumption $p = \frac{1}{2}$ simlifies the calculation of probabilities because we have 2^{-n} instead of $p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}$. Therefore each path of length n hes equal probability of 2^{-n}. Therefore the probabilities only depend on $\binom{n}{\frac{x+n}{2}}$ which can be more generalized like in the following definition.

Definition 3 (Number of possible paths)**.** Let $n \in \mathbb{N}, a, b \in \mathbb{Z}$. Let $N_n(a, b)$ be *number of posssible paths* of random walk $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ from point $(0, a)$ to point (n, b) and $N_n^x(a, b)$ be number of possible paths from point $(0, a)$ to point (n, b) that visit point (z, x) for some $z \in \{1, 2, \ldots, n\}$.

Theorem 5 (Number of possible paths)
Let $a, b \in \mathbb{Z}, n \in \mathbb{N}$ then

$$N_n(a, b) = \binom{n}{\frac{1}{2}(n + b - a)}.$$

Proof. Let us choose a path from the point $(0, a)$ to point (n, b) and let α be number of the rightwards steps and β be number of the leftwards steps. Then $\alpha + \beta = n$ and $\alpha - \beta = b - a$. By adding these two equations we get that $\alpha = \frac{1}{2}(n + b - a)$. The number of possible paths is the number of ways of picking α rightwards steps from n steps. Therefore we get $N_n(a, b) = \binom{n}{\alpha} = \binom{n}{\frac{1}{2}(n+b-a)}$. □

Theorem 6 (Reflection principle)
Let $a, b \in \mathbb{N}$, then

$$N_n^0(a, b) = N_n(-a, b).$$

Proof. Each path from $(0, -a)$ to (n, b) has to intersect $y = 0$-axis at least once at some point. Let k be the time of the first intersection with x-axis. By reflecting the segment from $(0, -a)$ to $(k, 0)$ in the x-axis and letting the segment from $(k, 0)$ to (n, b) stay the same as it was, we get a path from point $(0, a)$ to (n, b) which visits 0 in time k. Because reflection is a bijective operation on sets of paths, we get the correspondence between the collections of such paths. □

Remark. Following is the definition of return to origin which is a crucial term for this thesis. Let us come back to our question. While calculating the number of steps the walk spends in interval $[a, +\infty)$ we find the first passage through a and then set a as a new origin. However, our achievements concerning return to origin are in the latter subchapter.

Definition 4 (Return to origin)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk, $k \in \mathbb{N}$. We say a *return to origin* occurred in time $2k$ if $S_{2k} = 0$. The probability that in time $2k$ occurred a return to origin is denoted by u_{2k}. We say that *the first return to* *origin* occurred in time $2k$ if $S_1, S_2, \ldots S_{2k-1} \neq 0$ and $S_{2k} = 0$. The probability that in time $2k$ occurred the first return to origin is denoted by f_{2k}. We define $f_0 := 0$. Let $\alpha_{2n}(2k)$ denote $u_{2k}u_{2n-2k}$.

Theorem 7 (Ballot theorem)
Let $n, b \in \mathbb{N}$ Number of paths from point $(0,0)$ to point (n,b) which do not return to origin is equal to $\frac{b}{n}N_n(0,b)$

Proof. Let us call N the number of paths we are referring to. Because the path ends at point (n,b) and it does not return to origin, the first step has to be rightwards. Therefore we now have

$$N = N_{n-1}(1,b) - N_{n-1}^0(1,b) \stackrel{\text{T6}}{=} N_{n-1}(1,b) - N_{n-1}(-1,b).$$

Hence we get that:

$$\begin{aligned}
N &= N_{n-1}(1,b) - N_{n-1}(-1,b) \stackrel{\text{T5}}{=} \binom{n-1}{\frac{n}{2}+\frac{b}{2}-1} - \binom{n-1}{\frac{n}{2}+\frac{b}{2}} \\
&= \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!} - \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!} \\
&= \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}\right)\left(\frac{n}{2}-\frac{b}{2}-1\right)!} - \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}\right)\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!} \\
&= \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!}\left(\frac{1}{\frac{n}{2}-\frac{b}{2}} - \frac{1}{\frac{n}{2}+\frac{b}{2}}\right) \\
&= \frac{1}{n}\frac{n!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!}\left(\frac{\left(\frac{n}{2}+\frac{b}{2}-\frac{n}{2}+\frac{b}{2}\right)}{\left(\frac{n}{2}-\frac{b}{2}\right)\left(\frac{n}{2}+\frac{b}{2}\right)}\right) \\
&= \frac{b}{n}\frac{n!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!} = \frac{b}{n}\binom{n}{\left(\frac{n}{2}+\frac{b}{2}\right)} \stackrel{\text{T5}}{=} \frac{b}{n}N_n(0,b)
\end{aligned}$$

□

Remark. The name *Ballot theorem* comes from the question: In a ballot where candidate A receives p votes and candidate B receives q votes with $p > q$, what is the probability that A had been strictly ahead of B throughout the count?

Answer to this question can be derived from the previous theorem. In our case $b = p - q$ and $n = p + q$.

Definition 5 (Maximum and minimum)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk. $M_n^+ := \max\{S_i, i \in \{1,2,\ldots,n\}\}$ is called *maximum of random walk* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *up to time n* and $M_n^- := \min\{S_i, i \in \{1,2,\ldots,n\}\}$ is called *minimum of random walk* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *up to time n*. $M_n = \max\{M_n^+, -M_n^-\}$ is called *absolute maximum of random walk* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *up to time n.*

Theorem 8 (Probability of maximum up to time n)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk. Let $r \in \mathbb{N}, b \in \mathbb{Z}$then

$$\mathsf{P}\left(M_n^+ \geq r, S_n = b\right) = \begin{cases} \mathsf{P}(S_n = b) & \text{if } b \geq r\ , \\ \mathsf{P}(S_n = 2r - b)\left(\frac{q}{p}\right)^{r-b}, & \text{otherwise.} \end{cases}$$

Proof. Let us firstly consider the easier case in which $b \geq r$. Because we defined M_n^+ as $\max\{S_i, i \in \{1, 2, \dots, n\}\}$ we get that $M_n^+ \geq b \geq r$ therefore $[M_n^+ \geq r] \subset [S_n = b]$ hence we get $\mathsf{P}\left(M_n^+ \geq r, S_n = b\right) = \mathsf{P}\left(S_n = b\right)$.

Now let $r \geq 1, b < r$. $N_n^r(0, b)$ stands for number of paths from point $(0, 0)$ to point (n, b) which reach at least to r. Let $k \in \{1, 2, \dots, n\}$ denote the first time the walk reaches r. By reflection principle (6), we can reflect the segment from (k, r) to (n, b) in the axis $y = r$. Therefore we now have path from $(0, 0)$ to $(n, 2r - b)$ and we get that

$$N_n^r(0, b) = N_n(0, 2r - b) \text{ hence } \mathsf{P}\left(S_n = b, M_n^+ \geq r\right) = N_n^r(0, b)\, p^{\frac{n+b}{2}} q^{\frac{n-b}{2}} =$$
$$N_n(0, 2r - b)\, p^{\frac{n+(2r-b)}{2}} q^{\frac{n-(2r-b)}{2}} p^{b-r} q^{r-b} = \left(\frac{q}{p}\right)^{r-b} \mathsf{P}\left(S_n = 2r - b\right).$$

□

Definition 6 (Walk reaching new maximum at particular time)**.** TODO: -newline Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk $n, b \in \mathbb{N}$. We say that *the walk reached new maximum b in time n* if $M_{n-1}^+ = S_{n-1} = b - 1, S_n = b$. We denote such probability by $f_b(n)$.

Theorem 9 (Probability of reaching new maximum b in time n)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk $n, b \in \mathbb{N}$ then

$$f_b(n) = \frac{b}{n} \mathsf{P}\left(S_n = b\right).$$

Proof.

$$\begin{aligned}
f_b(n) &= \mathsf{P}\left(M_{n-1} = S_{n-1} = b - 1, S_n = b\right) = \mathsf{P}\left(M_{n-1} = S_{n-1} = b - 1, X_n = +1\right) \\
&= p\, \mathsf{P}\left(M_{n-1} = S_{n-1} = b - 1\right) \\
&= p\left(\mathsf{P}\left(M_{n-1} \geq b - 1, S_{n-1} = b - 1\right) - \mathsf{P}\left(M_{n-1} \geq b, S_{n-1} = b - 1\right)\right) \\
&\overset{\text{T8}}{=} p\left(\mathsf{P}\left(S_{n-1} = b - 1\right) - \frac{q}{p} \mathsf{P}\left(S_{n-1} = b + 1\right)\right) \\
&= p\, \mathsf{P}\left(S_{n-1} = b - 1\right) - q\, \mathsf{P}\left(S_{n-1} = b + 1\right) \\
&= \binom{n-1}{\frac{n}{2} + \frac{b}{2} - 1} p^{\frac{n}{2} + \frac{b}{2}} q^{\frac{n}{2} - \frac{b}{2}} - \binom{n-1}{\frac{n}{2} + \frac{b}{2}} p^{\frac{n}{2} + \frac{b}{2}} q^{\frac{n}{2} - \frac{b}{2}} \\
&= p^{\frac{n}{2} + \frac{b}{2}} q^{\frac{n}{2} - \frac{b}{2}} \left(\frac{(n-1)!}{\left(\frac{n}{2} + \frac{b}{2} - 1\right)!\left(\frac{n}{2} - \frac{b}{2}\right)!} - \frac{(n-1)!}{\left(\frac{n}{2} + \frac{b}{2}\right)!\left(\frac{n}{2} - \frac{b}{2} - 1\right)!}\right) \\
&= p^{\frac{n}{2} + \frac{b}{2}} q^{\frac{n}{2} - \frac{b}{2}} \left(\frac{(n-1)!}{\left(\frac{n}{2} + \frac{b}{2} - 1\right)!\left(\frac{n}{2} - \frac{b}{2} - 1\right)!}\right)\left(\frac{1}{\frac{n}{2} - \frac{b}{2}} - \frac{1}{\frac{n}{2} + \frac{b}{2}}\right) \\
&= p^{\frac{n}{2} + \frac{b}{2}} q^{\frac{n}{2} - \frac{b}{2}} \frac{b}{n} \left(\frac{n!}{\left(\frac{n}{2} + \frac{b}{2}\right)!\left(\frac{n}{2} - \frac{b}{2}\right)!}\right) = \frac{b}{n} p^{\frac{n}{2} + \frac{b}{2}} q^{\frac{n}{2} - \frac{b}{2}} \binom{n}{\frac{n}{2} + \frac{b}{2}} = \frac{b}{n} \mathsf{P}\left(S_n = b\right).
\end{aligned}$$

The fourth equation follows from the fact that the event $[M_{n-1} \geq b - 1]$ can be split into two disjoint events: $[M_{n-1} \geq b - 1] = [M_{n-1} \geq b] \cup [M_{n-1} = b - 1]$.

Hence: $\mathsf{P}\left(M_{n-1} = b - 1\right) = \mathsf{P}\left(M_{n-1} \geq b - 1\right) - \mathsf{P}\left(M_{n-1} \geq b\right)$. The same applies for the probability $\mathsf{P}\left(M_{n-1} = b - 1, S_{n-1} = b - 1\right)$ □

3. Returns to the origin

Remark. Following is quite a tricky identity concerning binomial numbers which may not be obvious to the reader at first sight. Therefore we have decided to state it as a lemma.

Lemma 10 (Binomial identity)
Let $n, k \in \mathbb{N}, n > k$ then

$$\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{n}\binom{n}{k}$$

Proof.

$$\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{(n-1)!}{k!\,(n-k-1)!} - \frac{(n-1)!}{(k-1)!\,(n-k)!}$$
$$= \frac{(n-1)!}{(k-1)!\,(n-k-1)!}\left(\frac{1}{k} - \frac{1}{n-k}\right)$$
$$= \frac{1}{n}\frac{n!}{(k-1)!\,(n-k-1)!}\frac{n-2k}{k\,(n-k)} = \frac{n-2k}{n}\frac{n!}{k!\,(n-k)!} = \frac{n-2k}{n}\binom{n}{k}$$

□

Remark. Thanks to the previous lemma we are able to prove following theorem. After proving the theorem and two corrolaries stated as lemmata we will be finally able to answer our first question.

Theorem 11 (Probability of no return up to n is equal to return in time n)
Let $\{S_n\}_{n=0}^{+\infty}$ be a symmetric random walk , then

$$\mathsf{P}\left(S_1, S_2, \ldots, S_{2n} \neq 0\right) = \mathsf{P}\left(S_{2n} = 0\right).$$

Proof.

$$\mathsf{P}\left(S_1, S_2, \ldots, S_{2n} \neq 0\right) \overset{LTP}{=} \sum_{i=-n}^{n} \mathsf{P}\left(S_1, S_2, \ldots, S_{2n} \neq 0, S_{2n} = 2i\right)$$
$$= 2\sum_{i=1}^{n} \mathsf{P}\left(S_1, S_2, \ldots, S_{2n} \neq 0, S_{2n} = 2i\right)$$
$$\overset{\text{T7}}{=} 2\sum_{i=1}^{n}\frac{2i}{2n}\mathsf{P}\left(S_{2n} = 2i\right) = 2\sum_{i=1}^{n}\frac{2i}{2n}\binom{2n}{n-i}2^{-2n}$$
$$\overset{\text{L10}}{=} 2\cdot 2^{-2n}\sum_{i=1}^{n}\left(\binom{2n-1}{n-i} - \binom{2n-1}{n-i-1}\right) = 2\cdot 2^{-2n}\binom{2n-1}{n}$$
$$= 2^{-2n}\frac{2n}{n}\binom{2n-1}{n-1} = 2^{-2n}\binom{2n}{n} = \mathsf{P}\left(S_{2n} = 0\right).$$

The second equation follows from the fact that the random walk is symmetric and the sixth equation follows from the fact that it is a telescopic sum. □

Lemma 12 (Probability of positive path)
Let $\{S_n\}_{n=0}^{+\infty}$ *be a symmetric random walk.*

$$\mathsf{P}\left(S_1, S_2, \ldots, S_{2n} > 0\right) = \frac{1}{2}\,\mathsf{P}\left(S_{2n} = 0\right).$$

Proof.

$$\mathsf{P}\left(S_1, S_2, \ldots, S_{2n} > 0\right) \overset{LTP}{=} \sum_{r=1}^{n} \mathsf{P}\left(S_1, S_2, \ldots, S_{2n} > 0, S_{2n} = 2r\right).$$

The r-th term follows equation:

$$\begin{aligned}
&\mathsf{P}\left(S_1, S_2, \ldots, S_{2n} > 0, S_{2n} = 2r\right) = \mathsf{P}\left(X_1 = 1, S_2, \ldots, S_{2n} > 0, S_{2n} = 2r\right)\\
&= \frac{1}{2}\,\mathsf{P}\left(S_2, S_3, \ldots, S_{2n} > 0, S_{2n} = 2r \mid S_1 = 1\right)\\
&= \frac{1}{2}\left(\mathsf{P}\left(S_{2n} = 2r \mid S_1 = 1\right) - \mathsf{P}\left(S_2 \cdot S_3 \cdot \ldots \cdot S_{2n-1} = 0, S_{2n} = 2r \mid S_1 = 1\right)\right)\\
&= \frac{1}{2}\left(2^{-(2n-1)} N_{2n-1}(1, 2r) - 2^{-(2n-1)} N^0_{2n-1}(1, 2r)\right)\\
&= \frac{1}{2} 2^{-(2n-1)}\left(N_{2n-1}(1, 2r) - N^0_{2n-1}(1, 2r)\right)\\
&\overset{\text{T6}}{=} \frac{1}{2} 2^{-(2n-1)}\left(N_{2n-1}(1, 2r) - N_{2n-1}(-1, 2r)\right)\\
&= \frac{1}{2} 2^{-(2n-1)}\left(\binom{2n-1}{n+r-1} - \binom{2n-1}{n+r}\right)
\end{aligned}$$

The first equation comes from the disjoint decomposition:

$[S_{2n} = 2r] = [S_1 \cdot S_2 \cdot \ldots \cdot S_{2n-1} \neq 0, S_{2n} = 2r] \cup [S_1 \cdot S_2 \cdot \ldots \cdot S_{2n-1} = 0, S_{2n} = 2r]$.

Because of the fact that the negative parts of r-th terms cancel against the positive parts of $(r+1)$-st terms and the sum reduces to

$$\begin{aligned}
&\frac{1}{2} 2^{-(2n-1)} \binom{2n-1}{n} = \frac{1}{2} 2^{-2n} \cdot 2 \binom{2n-1}{n} = \frac{1}{2} 2^{-2n} \frac{2n}{n} \binom{2n-1}{n}\\
&= \frac{1}{2} 2^{-2n} \binom{2n}{n} = \frac{1}{2}\,\mathsf{P}\left(S_{2n} = 0\right).
\end{aligned}$$

□

Lemma 13 (Probability of non-negative path)
Let $\{S_n\}_{n=0}^{+\infty}$ *be a symmetric random walk, then*

$$\mathsf{P}\left(S_1, S_2, \ldots, S_{2n} \geq 0\right) = \mathsf{P}\left(S_{2n} = 0\right)$$

Proof.

$$\begin{aligned}
&\frac{1}{2}\,\mathsf{P}\left(S_{2n} = 0\right) \overset{\text{L12}}{=} \mathsf{P}\left(S_1, S_2, \ldots, S_{2n} > 0\right) = \mathsf{P}\left(X_1 = 1, S_2, S_3 \ldots, S_{2n} \geq 1\right)\\
&\overset{LTP}{=} \mathsf{P}\left(S_2, S_3 \ldots, S_{2n} \geq 1 \mid S_1 = 1\right) \mathsf{P}\left(X_1 = 1\right)\\
&= \frac{1}{2}\,\mathsf{P}\left(S_2, S_3 \ldots, S_{2n} \geq 1 \mid S_1 = 1\right)\\
&\overset{\text{L3}}{=} \frac{1}{2}\,\mathsf{P}\left(S_1, S_2 \ldots, S_{2n-1} \geq 1 \mid S_0 = 1\right) \overset{\text{L2}}{=} \frac{1}{2}\,\mathsf{P}\left(S_1, S_2 \ldots, S_{2n-1} \geq 0\right)\\
&= \frac{1}{2}\,\mathsf{P}\left(S_1, S_2 \ldots, S_{2n} \geq 0\right).
\end{aligned}$$

The last equation comes from the fact that

$$[S_1, S_2 \ldots, S_{2n-1} \geq 0] = [S_1, S_2 \ldots, S_{2n-1} \geq 1] = [S_1, S_2 \ldots, S_{2n} \geq 0].$$

□

Theorem 14 (Probability of position x in time n without returning to origin) *Let* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a random walk and* $x \in \mathbb{Z}$ *then*

$$\mathsf{P}\left(S_1, S_2, \ldots, S_n \neq 0, S_n = x\right) = \frac{|x|}{n} \mathsf{P}\left(S_n = x\right).$$

Proof. Let us without loss of generality assume that $x > 0$. In that case, first step has to be rightwards $(X_1 = +1)$. Now we have path from point $(1, 1)$ to point (n, x) that does not return to origin. By Ballot theorem (7) there are $\frac{x}{n} N_n(0, x)$ such paths. Each path consists of $\frac{n+x}{2}$ rightwards steps and $\frac{n-x}{2}$ leftwards steps. Therefore $\mathsf{P}\left(S_1 \cdot S_2 \cdot, \ldots, S_n \neq 0, S_n = x\right) = \frac{x}{n} N_n(0, x)\, p^{\frac{n+x}{2}} q^{\frac{n-x}{2}} = \frac{x}{n} \mathsf{P}\left(S_n = x\right)$. Case $x < 0$ is identical. □

Theorem 15 (First return as difference of returns) *Let* $\{S_n\}_{n=0}^{+\infty}$ *be a symmetric random walk, then*

$$f_{2n} = u_{2n-2} - u_{2n}$$

Proof. The event $[S_1, S_2, \ldots S_{2n-1} \neq 0]$ is the union of two disjoint events:

$$[S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} = 0] \cup [S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} \neq 0].$$

Hence we get

$$\begin{aligned} f_{2n} &= \mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} = 0\right) \\ &= \mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0\right) - \mathsf{P}\left(S_1, S_2, \ldots, S_{2n} \neq 0\right). \end{aligned}$$

Because $2n - 1$ is odd. $\mathsf{P}\left(S_{2n-1} = 0\right) = 0$. Therefore the first term is equal to $\mathsf{P}\left(S_1, S_2, \ldots S_{2n-2} \neq 0\right)$ which is by 11 equal to u_{2n-2}. Second term is by 11 equal to u_{2n}. Therefore we get the result. □

4. Own proofs

Lemma 16
Let $\{S_n\}_{n=0}^{+\infty}$ *be a symmetric random walk, then*

$$f_{2n} = \frac{1}{2n-1} u_{2n}$$

Proof.

$$\begin{aligned} u_{2n-2} &= 2^{-(2n-2)} \binom{2n-2}{n-1} = 4 \cdot 2^{-2n} \frac{(2n-2)!}{(n-1)!\,(n-1)!} \\ &= \frac{4n^2}{(2n)\,(2n-1)} 2^{-2n} \binom{2n}{n} = \frac{2n}{2n-1} u_{2n}. \end{aligned}$$

Therefore,

$$f_{2n} \overset{\text{L15}}{=} u_{2n-2} - u_{2n} = u_{2n} \left(\frac{2n}{2n-1} - 1 \right) = u_{2n} \frac{1}{2n-1}.$$

□

Lemma 17 (Decomposition of f_n)
Let $\{S_n\}_{n=0}^{+\infty}$ *be a symmetric random walk.*

$$\mathsf{P}\left(S_{2n} = 0\right) = \sum_{r=1}^{n} f_{2r} u_{2n-2r}$$

Proof.

$$\begin{aligned} &\mathsf{P}\left(S_{2n} = 0\right) \overset{LTP}{=} \sum_{r=1}^{n} \mathsf{P}\left(S_{2n} = 0, S_1, S_2, \ldots, S_{2r-1} \neq 0, S_{2r} = 0\right) \\ &\overset{GPR}{=} \sum_{r=1}^{n} \mathsf{P}\left(S_{2n} = 0 \mid S_1, S_2, \ldots, S_{2r-1} \neq 0 S_{2r} = 0\right) \cdot \\ &\mathsf{P}\left(S_1, S_2, \ldots, S_{2r-1} \neq 0, S_{2r} = 0\right) \\ &\overset{\text{L4,D4}}{=} \sum_{r=1}^{n} \mathsf{P}\left(S_{2n} = 0 \mid S_{2r} = 0\right) f_{2r} \overset{\text{L3}}{=} \sum_{r=1}^{n} \mathsf{P}\left(S_{2n-2r} = 0\right) f_{2r} \overset{\text{D4}}{=} \sum_{r=1}^{n} u_{2n-2r} f_{2r}. \end{aligned}$$

□

Theorem 18 (Arcsine law for last visits)
Let $\{S_n\}_{n=0}^{+\infty}$ *be a symmetric random walk* $k, n \in \mathbb{N}, k \leq n$. *The probability that up to time* $2n$ *the last return to origin occurred in time* $2k$ *is given by*

$$\mathsf{P}\left(S_{2k} = 0\right) \mathsf{P}\left(S_{2n-2k} = 0\right).$$

Proof. The probability involved can be rewritten as:

$$\begin{aligned} &\mathsf{P}\left(S_{2k} = 0, S_{2k+1}, S_{2k+2}, \ldots, S_{2n} \neq 0\right) \\ &\overset{GPR}{=} \mathsf{P}\left(S_{2k+1}, S_{2k+2}, \ldots, S_{2n} \neq 0 \mid S_{2k} = 0\right) \mathsf{P}\left(S_{2k} = 0\right) \\ &\overset{\text{L3}}{=} \mathsf{P}\left(S_1, S_2, \ldots, S_{2n-2k} \neq 0\right) \mathsf{P}\left(S_{2k} = 0\right) \overset{\text{T11}}{=} u_{2n-2k} u_{2k} \end{aligned}$$

□

Definition 7 (Time spend on the positive and negative sides)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk. We say that *the walk spent* τ *time units of* n *on the positive side* if $\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = \tau$. Let T_n denote $\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]}$ and $\beta_n(\tau)$ denote $\mathsf{P}(T_n = \tau)$. Similarly, we say that *the walk spent* ζ *time units of* n *on the negative side* if $\sum_{i=1}^{n} \mathbf{1}_{[S_i<0 \vee S_{i-1}<0]} = \zeta$.

Theorem 19 (Arcsine law for sojourn times-OWN PROOF)
Let $\{S_n\}_{n=0}^{+\infty}$ *be a symmetric random walk. Then*

$$\beta_{2n}(2k) = \alpha_{2n}(2k) = \mathsf{P}(S_{2n} = 0)\,\mathsf{P}(S_{2n-2k} = 0).$$

Proof. First let us start with degenerate cases.

$$\beta_{2n}(2n) = \mathsf{P}(T_{2n} = 2n) \overset{\text{L13}}{=} \mathsf{P}(S_1, S_2, \ldots, S_{2n} \geq 0) = u_{2n}.$$

By symmetry $\beta_{2n}(0) = \beta_{2n}(2n)$.

Let us fix $v \leq n$ and let $1 \leq k \leq v-1$, then the following equation holds:

$$\beta_{2n}(2k) \overset{\text{D7}}{=} \mathsf{P}(T_{2n} = 2k) \overset{LTP}{=} \sum_{r=1}^{n} \mathsf{P}(T_{2n} = 2k, S_1, S_2, \ldots, S_{2r-1} \neq 0, S_{2r} = 0)$$
$$= \sum_{r=1}^{n} \mathsf{P}(T_{2n} = 2k, S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0)$$
$$+ \sum_{r=1}^{n} \mathsf{P}(T_{2n} = 2k, S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0)$$
$$\overset{GPR}{=} \sum_{r=1}^{n} \mathsf{P}(T_{2n} = 2k \mid S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0) \cdot$$
$$\mathsf{P}(S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0)$$
$$+ \sum_{r=1}^{n} \mathsf{P}(T_{2n} = 2k \mid S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0) \cdot$$
$$\mathsf{P}(S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0)$$
$$\overset{\text{L4}}{=} \sum_{r=1}^{n} \mathsf{P}(T_{2n} = 2k \mid S_{2r} = 0)\,\mathsf{P}(S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0)$$
$$+ \sum_{r=1}^{n} \mathsf{P}(T_{2n} = 2k \mid S_{2r} = 0)\,\mathsf{P}(S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0)$$
$$\overset{*}{=} \sum_{r=1}^{n} \frac{1}{2} f_{2r}\,\mathsf{P}(T_{2n} - T_{2r} = 2k \mid S_{2r} = 0)$$
$$+ \sum_{r=1}^{n} \frac{1}{2} f_{2r}\,\mathsf{P}(T_{2n} - T_{2r} = 2k - 2r \mid S_{2r} = 0)$$
$$\overset{\text{L3}}{=} \sum_{r=1}^{n} \frac{1}{2} f_{2r}\,\mathsf{P}(T_{2n-2r} = 2k) + \sum_{r=1}^{n} \frac{1}{2} f_{2r}\,\mathsf{P}(T_{2n-2r} = 2k - 2r)$$
$$= \sum_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k) + \sum_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k - 2r).$$

The third equation comes from the disjoint decomposition of

$$[S_1, S_2, \ldots, S_{2r-1} \neq 0] = [S_1, S_2, \ldots, S_{2r-1} > 0] \cup [S_1, S_2, \ldots, S_{2r-1} < 0]$$

and the equation marked $*$ comes from using the condition that up to time $2r$ the steps were on the positive/negative sides.

Now let us proceed by induction. Case for $v = 1$ is trivial because it implies the degenerate case. Let the statement be true for all $v \leq n-1$, then

$$\begin{aligned}
\beta_{2n}(2k) &= \sum_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k) + \sum_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k-2r) \\
&\overset{IA}{=} \sum_{r=1}^{n} \frac{1}{2} f_{2r} \alpha_{2n-2r}(2k) + \sum_{r=1}^{n} \frac{1}{2} f_{2r} \alpha_{2n-2r}(2k-2r) \\
&\overset{\text{D4}}{=} \sum_{r=1}^{n} \frac{1}{2} f_{2r} u_{2k} u_{2n-2r-2k} + \sum_{r=1}^{n} \frac{1}{2} f_{2r} u_{2k-2r} u_{2n-2k} \\
&= \frac{1}{2} u_{2k} \sum_{r=1}^{n} f_{2r} u_{2n-2r-2k} + \frac{1}{2} u_{2n-2k} \sum_{r=1}^{n} f_{2r} u_{2k-2r} \\
&\overset{\text{L17}}{=} \frac{1}{2} u_{2n-2k} u_{2k} + \frac{1}{2} u_{2n-2k} u_{2k} = u_{2n-2k} u_{2k} \overset{\text{D4}}{=} \alpha_{2n}(2k).
\end{aligned}$$

□

Definition 8 (Time spent above a). Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk. We say that *the walk spent z time units up to time n in the interval* $[a, +\infty)$ if $\sum_{i=1}^{n} \mathbf{1}_{[S_i > a \vee S_{i-1} > a]} = z$. Let $T_n(a)$ denote $\sum_{i=1}^{n} \mathbf{1}_{[S_i > a \vee S_{i-1} > a]}$.

Theorem 20 (Probability questioned TODO:)
Let $\{S_n\}_{n=0}^{+\infty}$ be a symmetric random walk, let $a > 0$. Then

$$\mathsf{P}(T_{2n}(2a) = 2z) =$$

Proof.

$$\begin{aligned}
&\mathsf{P}(T_{2n}(2a) = 2z) \\
&\overset{GPR}{=} \sum_{i=1}^{n} (T_{2n}(2a) = 2z \mid S_1, S_2, \ldots, S_{2i-1} < 2a, S_{2i} = 2a)\, \mathsf{P}(S_1, S_2, \ldots, S_{2i-1} < 2a, S_{2i} = 2a) \\
&=
\end{aligned}$$

□

4.1 Problem chapter 9 Feller-není dokončeno, zkontrolováno ani upraveno do čitelnější podoby

Definition 9 $(\delta_n, \varepsilon_n^{r,\pm})$. Let $\{S_n\}_{n=0}^{+\infty}$ be a symmetric random walk.

$$\delta_n(k) \text{ shall denote } \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i > 0 \vee S_{i-1} > 0]} = k, S_n = 0\right),$$

$$\varepsilon_n^{r}(k) \text{ shall denote } \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i > 0 \vee S_{i-1} > 0]} = k, S_1, S_2, \ldots, S_{r-1} \neq 0, S_r = 0, S_n = 0\right),$$

$$\varepsilon_n^{r,+}(k) \text{ shall denote } \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i > 0 \vee S_{i-1} > 0]} = k, S_1, S_2, \ldots, S_{r-1} > 0, S_r = 0, S_n = 0\right),$$

$$\varepsilon_n^{r,-}(k) \text{ shall denote } \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i > 0 \vee S_{i-1} > 0]} = k, S_1, S_2, \ldots, S_{r-1} < 0, S_r = 0, S_n = 0\right).$$

Lemma 21 (Factorization of $\delta_{2n}(2k)$)
$\delta_{2n}(2k) = \frac{1}{2}\sum_{r=1}^{n}\left(f_{2r}\delta_{2n-2r}(2k-2r) + f_{2r}\delta_{2n-2r}(2r)\right)$.

Proof. Because $S_{2n}=0$ a return to origin must have happened. Let $2r$ the time of first return to origin, where $r \in \{1,2,\ldots,n\}$. By the law of total probability:
$\delta_{2n}(2k) \overset{D9}{=} \mathsf{P}\left(\sum_{i=1}^{2n}\mathbf{1}_{[S_i>0\vee S_{i-1}>0]} = k, S_{2n}=0\right)$
$\overset{LTP}{=} \sum_{r=1}^{n}\mathsf{P}\left(\sum_{i=1}^{2n}\mathbf{1}_{[S_i>0\vee S_{i-1}>0]} = 2k, S_1,S_2,\ldots,S_{2r-1}\neq 0, S_{2r}=0, S_{2n}=0\right)$
$\overset{D9}{=} \sum_{r=1}^{n}\varepsilon_{2n}^{2k} \overset{*}{=} \sum_{r=1}^{n}\mathsf{P}\left(\sum_{i=1}^{2n}\mathbf{1}_{[S_i>0\vee S_{i-1}>0]} = 2k, S_1,S_2,\ldots,S_{2r-1}>0, S_{2r}=0, S_{2n}=0\right)$
$+\sum_{r=1}^{n}\mathsf{P}\left(\sum_{i=1}^{2n}\mathbf{1}_{[S_i>0\vee S_{i-1}>0]} = 2k, S_1,S_2,\ldots,S_{2r-1}<0, S_{2r}=0, S_{2n}=0\right)$
$=\sum_{r=1}^{n}\varepsilon_{2n}^{2r,+}(2k) + \sum_{r=1}^{n}\varepsilon_{2n}^{2r,-}(2k)$.

Where $*$ comes from the disjoint decomposition $[S_1,S_2,\ldots,S_{2r-1}\neq 0] = [S_1,S_2,\ldots,S_{2r-1}>0]\cup[S_1,S_2,\ldots,S_{2r-1}<0]$.

Now let us calculate $\varepsilon_{2n}^{2r,+}(2k)$
$=\mathsf{P}\left(\sum_{i=1}^{2n}\mathbf{1}_{[S_i>0\vee S_{i-1}>0]} = 2k, S_1,S_2,\ldots,S_{2r-1}>0, S_{2r}=0, S_{2n}=0\right)$
$\overset{GPR}{=} \mathsf{P}\left(\sum_{i=1}^{2n}\mathbf{1}_{[S_i>0\vee S_{i-1}>0]} = 2k, S_{2n}=0 \mid S_1,S_2,\ldots,S_{2r-1}>0, S_{2r}=0\right)$
$\mathsf{P}(S_1,S_2,\ldots,S_{2r-1}>0, S_{2r}=0)$
$\overset{*}{=} \mathsf{P}(T_{2n}-T_{2r}=2k-2r, S_{2n}=0 \mid S_{2r}=0)\,\mathsf{P}(S_1,S_2,\ldots,S_{2r-1}>0,S_{2r}=0)$
$\overset{**}{=} \mathsf{P}(T_{2n}-T_{2r}=2k-2r, S_{2n}=0 \mid S_{2r}=0)\,\frac{1}{2}f_2r$
$\overset{L3}{=} \mathsf{P}(T_{2n-2r}=2k-2r, S_{2n-2r}=0)\,\frac{1}{2}f_{2r}$
$\overset{D9}{=} \delta_{2n-2r}(2k-2r)\,\frac{1}{2}f_{2r}$.

Where $*$ comes from Lemma (4) and using the condition.

Where $**$ comes from the fact that $f_{2r} \overset{D4}{=} \mathsf{P}(S_1,S_2,\ldots,S_{2r-1}\neq 0, S_{2r}=0) = \mathsf{P}(S_1=1,S_2,\ldots,S_{2r-1}>0,S_{2r}=0)+\mathsf{P}(S_1=-1,S_2,\ldots,S_{2r-1}<0,S_{2r}=0)$ and $\mathsf{P}(S_1=-1,S_2,\ldots,S_{2r-1}<0,S_{2r}=0) = \mathsf{P}(S_1=1,S_2,\ldots,S_{2r-1}>0,S_{2r}=0)$ because of symmetry. Hence $\mathsf{P}(S_1=1,S_2,\ldots,S_{2r-1}>0,S_{2r}=0)=\frac{1}{2}f_{2r}$.

Similarly $\varepsilon_{2n}^{2r,+}(2k)$
$=\mathsf{P}\left(\sum_{i=1}^{2n}\mathbf{1}_{[S_i>0\vee S_{i-1}>0]} = 2k, S_1,S_2,\ldots,S_{2r-1}<0, S_{2r}=0, S_{2n}=0\right)$
$=\mathsf{P}\left(\sum_{i=1}^{2n}\mathbf{1}_{[S_i>0\vee S_{i-1}>0]} = 2k, S_{2n}=0 \mid S_1,S_2,\ldots,S_{2r-1}<0, S_{2r}=0\right)$
$\mathsf{P}(S_1,S_2,\ldots,S_{2r-1}<0, S_{2r}=0)$
$=\mathsf{P}(T_{2n}-T_{2r}=2k, S_{2n}=0 \mid S_{2r}=0)\,\mathsf{P}(S_1,S_2,\ldots,S_{2r-1}<0,S_{2r}=0)$
$=\mathsf{P}(T_{2n}-T_{2r}=2k, S_{2n}=0 \mid S_{2r}=0)\,\frac{1}{2}f_2r$
$=\mathsf{P}(T_{2n-2r}=2k, S_{2n-2r}=0)\,\frac{1}{2}f_{2r}$
$=\delta_{2n-2r}(2k)\,\frac{1}{2}f_{2r}$.

Therefore $\delta_{2n}(2k) = \frac{1}{2}\sum_{r=1}^{n}f_{2r}\delta_{2n-2r}(2k-2r) + \frac{1}{2}\sum_{r=1}^{n}f_{2r}\delta_{2n-2r}(2k)$
$=\frac{1}{2}\sum_{r=1}^{n}\left(f_{2r}\delta_{2n-2r}(2k-2r)+f_{2r}\delta_{2n-2r}(2r)\right)$ □

Theorem 22 (Equidistributional theorem-ALMOST COMPLETE OWN PROOF)
Let $\{S_n\}_{n=0}^{+\infty}$ *be a symmetric random walk and* $n\in\mathbb{N}$, *then* $\forall k,l \in \{0,1,2,\ldots,n\}$:
$\delta_{2n}(2k) = \delta_{2n}(2l) = \frac{u_{2n}}{n+1}$.

Proof. Let us prove this statement by induction in n. In case that $n = 1$ we have two options for k. Either $k = 0$ or $k = 1$. $\delta_2(0) = \mathsf{P}(S_1 = -1, S_2 = 0) = \frac{1}{2}u_2 = \mathsf{P}(S_1 = +1, S_2 = 0) = \delta_2(2)$.

Let the statement be true for all $l \leq n - 1$. In that case $\delta_{2(n-l)}(2k) = \frac{u_{2(n-l)}}{n-l+1} \forall k \in \{1, 2, \ldots, n-l\}$. We want to show that $\delta_{2n} = \frac{u_{2n}}{n+1}$.

Let us calculate $\delta_{2n} \overset{\text{L21}}{=} \frac{1}{2}\sum_{r=1}^{n}\left(f_{2r}\delta_{2n-2r}(2k-2r) + f_{2r}\delta_{2n-2r}(2r)\right)$

$\overset{IA}{=} \frac{1}{2}\sum_{r=1}^{n}\left(f_{2r}u_{2n-2r}\frac{1}{n-r+1} + f_{2r}u_{2n-2r}\frac{1}{n-r+1}\right) = \sum_{r=1}^{n}\frac{f_{2r}u_{2n-2r}}{n-r+1} \overset{\text{SNAD TO DOKAZU L23}}{=} \frac{u_{2n}}{n+1}$

□

Lemma 23 (Sum of binomials-POTŘEBUJU DOKÁZAT)
$\sum_{r=1}^{n}\frac{f_{2r}u_{2n-2r}}{n-r+1} = \frac{u_{2n}}{n+1}$

Proof. $f_{2r}u_{2n-2r} \overset{\text{L16}}{=} \frac{1}{2r-1}u_{2r}u_{2n-2r} \overset{\text{D4}}{=} \frac{1}{2r-1}2^{-2r}\binom{2r}{r}2^{-(2n-2r)}\binom{2n-2r}{n-r}$.

Therefore $\sum_{r=1}^{n}\frac{f_{2r}u_{2n-2r}}{n-r+1} = \sum_{r=1}^{n}\frac{1}{2r-1}\frac{1}{n-r+1}2^{-2n}\binom{2r}{r}\binom{2n-2r}{n-r} \overset{???}{=} \frac{1}{n+1}2^{-2n}\binom{2n}{n}$ □

5. Simple random walk in more dimensions

Definition 10 (Type II random walk in $\mathbb{Z}^m$)**.** Let $m \in \mathbb{N}$. $\forall n \in \mathbb{N}$, let $X_n = \left(x_n^1 \quad x_n^2 \quad \ldots, x_n^m\right)^T$, where $\{x_n^i\}_{i=1}^m$ are $\forall n \in \mathbb{N}$ independent.

Let $\forall i \in \{1, 2, \ldots, m\} x_n^i$ have values in $\{-1, +1\}$ with probabilities $\mathsf{P}\left(x_n^i = +1\right) = p_i \in (0, 1)$ and $\mathsf{P}\left(x_n^i = -1\right) = 1 - p_i =: q_i \in (0, 1)$.

Let $\{X_n\}_{n=0}^{+\infty}$ be a sequence of independent and identically distributed random variables. Let $S_0 = \mathbf{0}$ and $\forall n \in \mathbb{N} : \mathbf{S_n} = \sum_{i=1}^{n} X_i$ and $\mathbf{p} = \left(p_1, p_2, \ldots, p_m\right)^{\mathbf{T}}$. Then the pair $\left(\{\mathbf{S_n}\}_{n=0}^{+\infty}, \mathbf{p}\right)$ is called *Type II random walk in* $\mathbb{Z}^m$.

If $\forall i \in \{1, 2, \ldots, m\} : p_i = q_i = \frac{1}{2}$ we call the element $\{\mathbf{S_n}\}_{n=0}^{+\infty}$ *Symmetric type II random walk* $\mathbb{Z}^m$.

Remark. Type II random walk can be interpreted as m simple random walks in $\mathbb{Z}$ happening at a time, each of them parallel to an axis of $\mathbb{Z}^m$.

Theorem 24
Let $m \in \mathbb{N}$ and $\left(\{\mathbf{S_n}\}_{n=0}^{+\infty}, \mathbf{p}\right)$ be a Type II random walk in $\mathbb{Z}^m$. Let $\mathbf{y} = \left(y_1, y_2, \ldots, y_m\right)^T \in \mathbb{Z}^m$. Then following equation stands:

$$\mathsf{P}\left(S_n = x\right) = \begin{cases} \prod_{i=1}^m \binom{n}{\frac{y_i+n}{2}} p_i^{\frac{n+y_i}{2}} q_i^{\frac{n-y_i}{2}}, & \text{if } \forall i \in \{1, 2, \ldots, m\} : y_i \in A_n, \\ 0, & \text{if } \exists i \in \{1, 2, \ldots, m\} : y_i \notin A_n. \end{cases}$$

Where A_n is from definition (2)

Proof. $\mathsf{P}\left(\mathbf{S_n} = \mathbf{y}\right) = \mathsf{P}\left(S_n^1 = y_1, S_n^2 = y_2, \ldots, S_n^m = y_m\right) \overset{\perp\!\!\!\perp}{=} \prod_{i=1}^{m} \mathsf{P}\left(S_n^i = y_i\right)$
$= \prod_{i=1}^{m} \binom{n}{\frac{y_i+n}{2}} p_i^{\frac{n+y_i}{2}} q_i^{\frac{n-y_i}{2}}$. The second equation comes from the independence of $\{\mathbf{S_i}\}_{i=1}^m$ which comes easily from independence of X_n^i. The second equation comes from Theorem (1). $\square$

Remark. Due to the the aim of this thesis which is researching occupation time of a set of random walks we are going to concern only on symmetric random walks.

Definition 11 (Orthant)**.** Let $m \in \mathbb{Z}$. Then $O \subset \mathbb{Z}^m$ is called an *open orthant in* $\mathbb{Z}^m$ if $\forall o := \left(o_1, o_2, \ldots, o_m\right)^T \in O, \forall i \in \{1, 2, \ldots, m\} : o_i \varepsilon_i > 0$, where $\varepsilon_i \in \{-1, +1\}$.

$C \subset \mathbb{Z}^m$ is called a *closed orthant in* $\mathbb{Z}^m$ if $\forall c := \left(c_1, c_2, \ldots, c_m\right)^T \in C, \forall i \in \{1, 2, \ldots, m\} : c_i \varepsilon_i \geq 0$, where $\varepsilon_i \in \{-1, +1\}$.

Remark. The statement $\mathbf{x} > \mathbf{y}$ will mean $\forall i \in \{1, 2, \ldots, m\} : x_i > y_i$. Same applies to $<, \leq, \geq$.

Theorem 25 (Probability of being in an open orthant)
Let $\{\mathbf{S_n}\}_{n=0}^{+\infty}$ be a Symmetric type II random walk in $\mathbb{Z}^m$. Let O be an open orthant in Z^m. $\mathsf{P}\left(\mathbf{S_n} \in O\right) = \left(\frac{1}{2} u_{2n}\right)^m$.

Proof. Without loss of generality we can assume that in the definition of O we choose $\forall i \in \{1, 2, \dots, m\} \varepsilon_i := +1$ then: $\mathsf{P}(\mathbf{S_n} \in O) = \mathsf{P}(S_n^1 > 0, S_n^2 > 0, \dots, S_n^m > 0) = \prod_{i=1}^m \mathsf{P}(S_n^i > 0) = (\mathsf{P}(S_n^i > 0))^m = \left(\frac{1}{2} u_{2n}\right)^m$. Where the last two equations come from the identical distribution of S_n^i and Theorem (12). □

Theorem 26 (Probability of being in a closed orthant)
Let $\{\mathbf{S_n}\}_{n=0}^{+\infty}$ be a Symmetric type II random walk in $\mathbb{Z}^m$. Let C be a closed orthant in Z^m. $\mathsf{P}(\mathbf{S_n} \in C) = (u_{2n})^m$.

Proof. The proof is very similar to previous proof. Without loss of generality we can again assume that in the definition of C we choose $\forall i \in \{1, 2, \dots, m\} \varepsilon_i := +1$ then: $\mathsf{P}(\mathbf{S_n} \in C) = \mathsf{P}(S_n^1 \geq 0, S_n^2 \geq 0, \dots, S_n^m \geq 0) = \prod_{i=1}^m \mathsf{P}(S_n^i \geq 0) = (\mathsf{P}(S_n^i \geq 0))^m = (u_{2n})^m$. Where the last two equations come from the identical distribution of S_n^i and Lemma (13). □

Theorem 27 (Zákon iterovaného logaritmu)
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