1. Title of the first chapter

Definition 1 (Simple random walk in $\mathbb{Z}$). *Let $\{X_n\}_{n=0}^{+\infty}$ be a sequence of independent and identically distributedrandom variables with values in $\{-1,+1\}$. That $\forall n \in \mathbb{N}$ satisfy $\mathsf{P}(X_n = 1) = p \in (0,1)$ and $\mathsf{P}(X_n = -1) = 1 - p = q$. Let $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$. We call the pair $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$* Simple random walk in $\mathbb{Z}$. *In case that $p = q = \frac{1}{2}$ we call the pair $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$* Symmetric simple random walk in $\mathbb{Z}$.

Remark. Very often we refer to n as time, X_i as i-th step and S_n as position in time n. In simple random walk in $\mathbb{Z}$ we refer to $X_i = +1$ as i-th step was rightwards.

Definition 2 (Set of possible positions). *Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk. We call the set $A_n = \{z \in \mathbb{Z}; |z| \leq n, \frac{z+n}{2} \in \mathbb{Z}\}$ set of all possible positions of random walk $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ in time n.*

Theorem 1 (Probability of position x in time n). *Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk and A_n its set of possible positions.*

$$\mathsf{P}(S_n = x) = \begin{cases} \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}} & \text{for } x \in A_n \ , \\ 0, & \text{for } x \notin A_n. \end{cases}$$

Proof. Let us define new variables $r_i = \mathbf{1}_{[X_i=1]}, l_i = \mathbf{1}_{[X_i=-1]}, R_n = \sum_{i=1}^{n} r_i, L_n = \sum_{i=1}^{n} l_i$. r_i can be interpreted as indicator wether i-th step was rightwards. Then R_n is number of rightwards steps and L_n is number of leftwards steps. We can easily see that $R_n + L_n = n$ and $R_n - L_n = S_n$. Therefore we get by adding these two equations $R_n = \frac{S_n+n}{2}$.

r_i has alternative distribution with parameter p (*Alt(p)*). Therefore R_n as a sum of independent and identically distributedrandom variables with *Alt(p)* has binomial distribution with parameters n and p (*Bi(n,p)*). Therefore we get $\mathsf{P}(R_n = x) = \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}$. Where we define $\binom{a}{x} := 0$ for $a \in \mathbb{N}, x \in \mathbb{R} \setminus \mathbb{N}$, $x < 0$, $x > n$. Therefore we get $\mathsf{P}(S_n = x) = \mathsf{P}\left(R_n = \frac{x+n}{2}\right) = \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}$. □

Lemma 2 (Spatial homogeneity). *$\mathsf{P}(S_n = j | S_0 = a) = \mathsf{P}(S_n = j + b | S_0 = a + b) \forall b \in \mathbb{Z}$*

Proof. $\mathsf{P}(S_n = j | S_0 = a) = \mathsf{P}\left(\sum_{i=1}^{n} X_i = j - a\right) = \mathsf{P}\left(\sum_{i=1}^{n} X_i = (j+b) - (a+b)\right) = \mathsf{P}(S_n = j + b | S_0 = a + b)$. □

Lemma 3 (Temporal homogeneity). *$\mathsf{P}(S_n = j | S_0 = a) = \mathsf{P}(S_{n+m} = j | S_m = a) \forall m \in \mathbb{N}$*

Proof. $\mathsf{P}(S_n = j | S_0 = a) = \mathsf{P}\left(\sum_{i=1}^{n} X_i = j - a\right) = \mathsf{P}\left(\sum_{i=m+1}^{m+n} X_i = j - a\right) = \mathsf{P}(S_{n+m} = j | S_m = a)$. □

Lemma 4 (Markov property)**.** *Let $n \geq m$ and $a_i \in \mathbb{Z}$. Then $\mathsf{P}(S_n = j | S_0 = a_0, S_1 = a_1, \ldots, S_m = a_m) = \mathsf{P}(S_n = j | S_m = a_m)$*

Proof. Once S_m is known, then distribution of S_n depends only on steps $X_{m+1}, X_{m+2}, \ldots X_n$ and therefore cannot be dependent on any information concerning values $X_1, X_2, \ldots, X_m - 1$ and accordingly $S_1, S_2, \ldots, S_m - 1$. □

Remark. In symmetric random walk, everything can be counted by number of possible paths from point to point.

Definition 3 (Number of possible paths)**.** *Let $N_n(a, b)$ be number of posssible paths of random walk $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$from point $(0, a)$ to point (n, b) and $N_n^x(a, b)$ be number of possible paths from point $(0, a)$ to point (n, b) that visit point (z, x) for some $z \in \{0, \ldots n\}$.*

Theorem 5. *Let $a, b \in \mathbb{Z}, n \in \mathbb{N}$ then $N_n(a, b) = \binom{n}{\frac{1}{2}(n+b-a)}$.*

Proof. Let us choose a path from $(0, a)$ to (n, b) and let α be number of rightwards steps and β be number of leftwards steps. Then $\alpha + \beta = n$ and $\alpha - \beta = b - a$. By adding these two equations we get that $\alpha = \frac{1}{2}(n + b - a)$. The number of possible paths is the number of ways of picking α rightwards steps from n steps. Therefore we get $N_n(a, b) = \binom{n}{\alpha} = \binom{n}{\frac{1}{2}(n+b-a)}$. □

Theorem 6 (Reflection principle)**.** *Let $a, b > 0$, then $N_n^0(a, b) = N_n(-a, b)$.*

Proof. Each path from $(0, -a)$ to (n, b) has to intersect x-axis at least once at some point. Let k be the time of earlies intersection with x-axis. By reflexing the segment from $(0, -a)$ to $(k, 0)$ in the x-axis, we get a path from point $(0, a)$ to (n, b) which visits 0 at point k. Because reflection is bijective operation, we get the correspondence between the collections of such paths. □

Definition 4 (Return to origin)**.** *Let $S_0 = 0$ $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$be a random walk. Then if $\exists k \in \mathbb{N}$ such that $S_k = 0$ then we say that in k-th step occured return to origin.*

Theorem 7 (Ballot theorem)**.** *Let $n, b \in N$ Number of paths from point $(0, 0)$ to point (n, b) which do not return to origin is equal to $\frac{b}{n} N_n(0, b)$*

Proof. Let us call N the number of paths we are refering to. Because the path ends at point (n, b), the first step has to be rightwards. Therefore we now have $N = N_{n-1}(1, b) - N_{n-1}^0(1, b) = N_{n-1}(1, b) - N_{n-1}(-1, b)$. The last equation was aquired using Ballot theorem (7). We now have:

$$N_{n-1}(1,b) - N_{n-1}(-1,b) = \binom{n-1}{\frac{n}{2} + \frac{b}{2} - 1} - \binom{n-1}{\frac{n}{2} + \frac{b}{2}} = \frac{(n-1)!}{(\frac{n}{2} + \frac{b}{2} - 1)!(\frac{n}{2} - \frac{b}{2})!} - \frac{(n-1)!}{(\frac{n}{2} + \frac{b}{2})!(\frac{n}{2} - \frac{b}{2} - 1}$$

□

1.1 Title of the first subchapter of the first chapter

1.2 Title of the second subchapter of the first chapter

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A. Attachments

A.1 First Attachment