

1. Title of the first chapter

Definition 1 (Simple random walk in \mathbb{Z}). Let $\{X_n\}_{n=0}^{+\infty}$ be a sequence of independent and identically distributed random variables with values in $\{-1, +1\}$. That $\forall n \in \mathbb{N}$ satisfy $P(X_n = 1) = p \in (0, 1)$ and $P(X_n = -1) = 1 - p = q$. Let $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$. We call the pair $(\{S_n\}_{n=0}^{+\infty}, p)$ Simple random walk in \mathbb{Z} . In case that $p = q = \frac{1}{2}$ we call the pair $(\{S_n\}_{n=0}^{+\infty}, p)$ Symmetric simple random walk in \mathbb{Z} .

Remark. Very often we refer to n as time, X_i as i -th step and S_n as position in time n . In simple random walk in \mathbb{Z} we refer to $X_i = +1$ as i -th step was rightwards. If not stated otherwise, we

Definition 2 (Set of possible positions). Let $(\{S_n\}_{n=0}^{+\infty}, p)$ be a random walk. We call the set $A_n = \{z \in \mathbb{Z}; |z| \leq n, \frac{z+n}{2} \in \mathbb{Z}\}$ set of all possible positions of random walk $(\{S_n\}_{n=0}^{+\infty}, p)$ in time n .

Theorem 1 (Probability of position x in time n). Let $(\{S_n\}_{n=0}^{+\infty}, p)$ be a random walk and A_n its set of possible positions.

$$P(S_n = x) = \begin{cases} \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}} & \text{for } x \in A_n, \\ 0, & \text{for } x \notin A_n. \end{cases}$$

Proof. Let us define new variables $r_i = \mathbf{1}_{[X_i=1]}, l_i = \mathbf{1}_{[X_i=-1]}, R_n = \sum_{i=1}^n r_i, L_n = \sum_{i=1}^n l_i$. r_i can be interpreted as indicator whether i -th step was rightwards. Then R_n is number of rightwards steps and L_n is number of leftwards steps. We can easily see that $R_n + L_n = n$ and $R_n - L_n = S_n$. Therefore we get by adding these two equations $R_n = \frac{S_n + n}{2}$.

r_i has alternative distribution with parameter p $Alt(p)$. Therefore R_n as a sum of independent and identically distributed random variables with $Alt(p)$ has binomial distribution with parameters n and p ($Bi(n, p)$). Therefore we get $P(R_n = x) = \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}$. Where we define $\binom{a}{x} := 0$ for $a \in \mathbb{N}, x \in \mathbb{R} \setminus \mathbb{N}, x < 0, x > n$. Therefore we get $P(S_n = x) = P(R_n = \frac{x+n}{2}) = \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}$. \square

Lemma 2 (Spatial homogeneity). $P(S_n = j \mid S_0 = a) = P(S_n = j + b \mid S_0 = a + b) \forall b \in \mathbb{Z}$

Proof. $P(S_n = j \mid S_0 = a) = P\left(\sum_{i=1}^n X_i = j - a\right) = P\left(\sum_{i=1}^n X_i = (j + b) - (a + b)\right) = P(S_n = j + b \mid S_0 = a + b)$. \square

Lemma 3 (Temporal homogeneity). $P(S_n = j \mid S_0 = a) = P(S_{n+m} = j \mid S_m = a) \forall m \in \mathbb{N}$

Proof. $P(S_n = j \mid S_0 = a) = P\left(\sum_{i=1}^n X_i = j - a\right) = P\left(\sum_{i=m+1}^{m+n} X_i = j - a\right) = P(S_{n+m} = j \mid S_m = a)$. \square

Lemma 4 (Markov property). *Let $n \geq m$ and $a_i \in \mathbb{Z}$. Then $P(S_n = j \mid S_0 = a_0, S_1 = a_1, \dots, S_m = a_m) = P(S_n = j \mid S_m = a_m)$*

Proof. Once S_m is known, then distribution of S_n depends only on steps $X_{m+1}, X_{m+2}, \dots, X_n$ and therefore cannot be dependent on any information concerning values X_1, X_2, \dots, X_m and accordingly $S_1, S_2, \dots, S_m - 1$. \square

Remark. In symmetric random walk, everything can be counted by number of possible paths from point to point.

Definition 3 (Number of possible paths). *Let $N_n(a, b)$ be number of possible paths of random walk $(\{S_n\}_{n=0}^{+\infty}, p)$ from point $(0, a)$ to point (n, b) and $N_n^x(a, b)$ be number of possible paths from point $(0, a)$ to point (n, b) that visit point (z, x) for some $z \in \{0, \dots, n\}$.*

Theorem 5. *Let $a, b \in \mathbb{Z}, n \in \mathbb{N}$ then $N_n(a, b) = \binom{n}{\frac{1}{2}(n+b-a)}$.*

Proof. Let us choose a path from $(0, a)$ to (n, b) and let α be number of rightwards steps and β be number of leftwards steps. Then $\alpha + \beta = n$ and $\alpha - \beta = b - a$. By adding these two equations we get that $\alpha = \frac{1}{2}(n + b - a)$. The number of possible paths is the number of ways of picking α rightwards steps from n steps. Therefore we get $N_n(a, b) = \binom{n}{\alpha} = \binom{n}{\frac{1}{2}(n+b-a)}$. \square

Theorem 6 (Reflection principle). *Let $a, b > 0$, then $N_n^0(a, b) = N_n(-a, b)$.*

Proof. Each path from $(0, -a)$ to (n, b) has to intersect x -axis at least once at some point. Let k be the time of earliest intersection with x -axis. By reflexing the segment from $(0, -a)$ to $(k, 0)$ in the x -axis, we get a path from point $(0, a)$ to (n, b) which visits 0 at point k . Because reflection is bijective operation on sets of paths, we get the correspondence between the collections of such paths. \square

Definition 4 (Return to origin). *Let $(\{S_n\}_{n=0}^{+\infty}, p)$ be a random walk. Then if $\exists k \in \mathbb{N}$ such that $S_k = 0$ then we say that in k -th step occured return to origin. Let $(\{S_n\}_{n=0}^{+\infty}, p)$ be a random walk. Then if $S_1, S_2, \dots, S_{2n-1} \neq 0$ and $S_{2n} = 0$*

Theorem 7 (Ballot theorem). *Let $n, b \in \mathbb{N}$ Number of paths from point $(0, 0)$ to point (n, b) which do not return to origin is equal to $\frac{b}{n} N_n(0, b)$*

Proof. Let us call N the number of paths we are refering to. Because the path ends at point (n, b) , the first step has to be rightwards. Therefore we now have $N = N_{n-1}(1, b) - N_{n-1}^0(1, b) = N_{n-1}(1, b) - N_{n-1}(-1, b)$. The last equation was aquired using Reflection principle (6). We now have:

$$N_{n-1}(1, b) - N_{n-1}(-1, b) = \binom{n-1}{\frac{n}{2} + \frac{b}{2} - 1} - \binom{n-1}{\frac{n}{2} + \frac{b}{2}} = \frac{(n-1)!}{\left(\frac{n}{2} + \frac{b}{2} - 1\right)! \left(\frac{n}{2} - \frac{b}{2}\right)!} - \frac{(n-1)!}{\left(\frac{n}{2} + \frac{b}{2}\right)! \left(\frac{n}{2} - \frac{b}{2}\right)!}$$

\square

Definition 5. $M_n = \max\{S_i, i \in \{0, 1, \dots, n\}\}$

Theorem 8 (Probability of maximum up to time n). *Let $(\{S_n\}_{n=0}^{+\infty}, p)$ be a random walk.*

$$P(M_n \geq r, S_n = b) = \begin{cases} P(S_n = b) & \text{for } b \geq r, \\ P(S_n = 2r - b) \left(\frac{q}{p}\right)^{r-b}, & \text{for otherwise.} \end{cases}$$

Proof. Let us firstly consider the easier case in which $b \geq r$. Because we defined M_n as $\max\{S_i, i \in \{0, 1, \dots, n\}\}$ we get that $M_n \geq b \geq r$ therefore $[M_n \geq r] \subset [S_n = b]$ therefore we get $P(M_n \geq r, S_n = b) = P(S_n = b)$. Let $r \geq 1, b < r$. $N_n^r(0, b)$ stands for number of paths from point $(0, 0)$ to point (n, b) which reach up to r . Let $k \in \{0, 1, \dots, n\}$ denote the first time we reach r . By reflection principle (6), we can reflex the segment from (k, r) to (n, b) in the axis: $y = r$. Therefore we now have path from $(0, 0)$ to $(n, 2r - b)$ and we get that $N_n^r(0, b) = N_n(0, 2r - b)$. $P(S_n = b, M_n \geq r) = N_n^r(0, b) p^{\frac{n+b}{2}} q^{\frac{n-b}{2}} = N_n(0, 2r - b) p^{\frac{n+(2r-b)}{2}} q^{\frac{n-(2r-b)}{2}} p^{\frac{n+b}{2}} q^{\frac{n-b}{2}} p^{b-r} q^{r-b} = \left(\frac{q}{p}\right)^{r-b} P(S_n = 2r - b)$. \square

Definition 6 (Walk reaching new maximum at particular time). *Let $b > 0$. $f_b(n)$ denotes the probability that we reach new maximum b in time n . $f_b(n) = P(M_{n-1} = S_{n-1} = b - 1, S_n = b)$*

Theorem 9 (Probability of reaching new maximum b in time n). *Let $b > 0$ then $f_b(n) = \frac{b}{n} P(S_n = b)$.*

Proof. $f_b = P(M_{n-1} = S_{n-1} = b - 1, S_n = b) = p P(M_{n-1} = S_{n-1} = b - 1) = p (P(M_{n-1} \geq b - 1, S_{n-1} = b - 1) - P(M_{n-1} \geq b, S_{n-1} = b - 1)) = p (P(S_{n-1} = b - 1) - \frac{q}{p} P(S_{n-1} = b)) = p P(S_{n-1} = b - 1) - q P(S_{n-1} = b) = \left(\frac{n-1}{\frac{n}{2} + \frac{b}{2} - 1}\right) p^{\frac{n}{2} + \frac{b}{2}} q^{\frac{n}{2} - \frac{b}{2}} - \left(\frac{n-1}{\frac{n}{2} + \frac{b}{2}}\right) p^{\frac{n}{2} + \frac{b}{2}} q^{\frac{n}{2} - \frac{b}{2}} = p^{\frac{n}{2} + \frac{b}{2}} q^{\frac{n}{2} - \frac{b}{2}} \left(\frac{(n-1)!}{(\frac{n}{2} + \frac{b}{2} - 1)!(\frac{n}{2} - \frac{b}{2})!} - \frac{(n-1)!}{(\frac{n}{2} + \frac{b}{2})!(\frac{n}{2} - \frac{b}{2} - 1)!}\right) = p^{\frac{n}{2} + \frac{b}{2}} q^{\frac{n}{2} - \frac{b}{2}} \left(\frac{(n-1)!}{(\frac{n}{2} + \frac{b}{2})!(\frac{n}{2} - \frac{b}{2})!}\right) \left(\frac{1}{\frac{n}{2} - \frac{b}{2}} - \frac{1}{\frac{n}{2} + \frac{b}{2}}\right) = p^{\frac{n}{2} + \frac{b}{2}} q^{\frac{n}{2} - \frac{b}{2}} \frac{b}{n} \left(\frac{n!}{(\frac{n}{2} + \frac{b}{2})!(\frac{n}{2} - \frac{b}{2})!}\right) = \frac{b}{n} p^{\frac{n}{2} + \frac{b}{2}} q^{\frac{n}{2} - \frac{b}{2}} \binom{n}{\frac{n}{2} + \frac{b}{2}} = \frac{b}{n} P(S_n = b)$. \square

Theorem 10 (XXXMean number of visits to b before returning to origin in symmetric random walk). *Let $(\{S_n\}_{n=0}^{+\infty}, p)$ be a symmetric random walk. Mean number μ_b of visits of the walk to point b before returning to origin is equal to 1.*

Proof. aa \square

Definition 7 (Return to origin). *Let $(\{S_n\}_{n=0}^{+\infty}, p)$ be a random walk. Let $k \in \mathbb{N}$. We say a return to origin occurred in time $2k$ if $S_{2k} = 0$. The probability that in time $2k$ occurred a return to origin shall be denoted by u_{2k} . We say that in time $2k$ occurred first return to origin if $S_1, S_2, \dots, S_{2k-1} \neq 0$ and $S_{2k} = 0$. The probability that in time $2k$ occurred first return to origin shall be denoted by f_{2k} . By definition $f_0 = 0$. Let $\alpha_{2n}(2k)$ denote $u_{2k} u_{2(n-k)}$*

Lemma 11 (Binomial identity). *Let $n, k \in \mathbb{N}, n > k : \binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{n} \binom{n}{k}$*

Proof. $\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-k-1)!} - \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k-1)!} \left(\frac{1}{k} - \frac{1}{n-k}\right) = \frac{1}{n} \frac{n!}{(k-1)!(n-k-1)!} \frac{n-2k}{k(n-k)} = \frac{n-2k}{n} \frac{n!}{k!(n-k)!} = \frac{n-2k}{n} \binom{n}{k}$ \square

Lemma 12 (Main lemma). *Let $(\{S_n\}_{n=0}^{+\infty}, p)$ be a symmetrical random walk. Then $P(S_1 \cdot S_2 \cdot \dots \cdot S_{2n} \neq 0) = P(S_{2n} = 0)$*

$$\begin{aligned}
\text{Proof. } P(S_1 \cdot S_2, \cdot, \dots, S_{2n} \neq 0) &= \sum_{i=-\infty}^{+\infty} P(S_1 \cdot S_2, \cdot, \dots, S_{2n-1} \neq 0, S_{2n} = 2i) = \\
&= \sum_{i=-n}^n P(S_1 \cdot S_2, \cdot, \dots, S_{2n-1} \neq 0, S_{2n} = 2i) = 2 \cdot \sum_{i=1}^n P(S_1 \cdot S_2, \cdot, \dots, S_{2n-1} \neq 0, S_{2n} = 2i) = \\
&= 2 \sum_{i=1}^n \frac{2i}{2n} P(S_{2n} = 2k) = 2 \sum_{i=1}^n \frac{2i}{2n} \binom{2n}{n+i} 2^{-2n} = 2 \cdot 2^{-2n} \sum_{i=1}^n \left(\binom{2n-1}{m+k-1} - \binom{2n-1}{m+k} \right) = 2 \cdot \\
&= 2^{-2n} \binom{2n-1}{n} = 2^{-2n} \frac{2n}{n} \binom{2n-1}{n} = 2^{-2n} \frac{2n(2n-1)!}{m(m-1)!m!} = 2^{-2n} \frac{(2n)!}{m!m!} = 2^{-2n} \binom{2n}{n} = P(S_{2n} = 0) \quad \square
\end{aligned}$$

Theorem 13. Let $(\{S_n\}_{n=0}^{+\infty}, p)$ be a symmetric random walk. The probability that the last return to origin up to time $2n$ occurred in time $2k$ is $P(S_{2k} = 0) P(S_{2(n-k)} = 0)$.

$$\begin{aligned}
\text{Proof. } \alpha_{2n}(2k) &= u_{2k} u_{2(n-k)} = P(S_{2k} = 0) P(S_{2k+1} \cdot S_{2k+2}, \dots, S_{2n} \neq 0 \mid S_{2k} = 0) = \\
&= P(S_{2k} = 0) P(S_1 \cdot S_2, \dots, S_{2(n-k)} \neq 0) = P(S_{2k} = 0) P(S_{2(n-k)} = 0) \quad \square
\end{aligned}$$

Theorem 14. Let $b \in \mathbb{Z}$. $P(S_1 \cdot S_2, \dots, S_n \neq 0, S_n = b) = \frac{|b|}{n} P(S_n = b)$.

Proof. Let us Without loss of generality assume that $b > 0$. In that case, first step has to be rightwards ($X_1 = +1$). Now we have path from point $(1, 1)$ to point (n, b) that does not return to origin. By Ballot theorem 7 there are $\frac{b}{n} N_n(0, b)$ such paths. Each path consists of $\frac{n+b}{2}$ rightwards steps and $\frac{n-b}{2}$ leftwards steps. Therefore $P(S_1 \cdot S_2, \dots, S_n \neq 0, S_n = b) = \frac{b}{n} N_n(0, b) p^{\frac{n+b}{2}} q^{\frac{n-b}{2}} = \frac{b}{n} P(S_n = b)$. Case $b < 0$ is identical. \square

Lemma 15. Let $(\{S_n\}_{n=0}^{+\infty}, p)$ be a symmetric random walk. $P(S_1, S_2, \dots, S_{2n} > 0) = \frac{1}{2} P(S_{2n} = 0) = \frac{1}{2} u_{2n}$.

Proof. Because $S_i > 0 \forall i \in \mathbb{N}$ the first step has to be rightwards ($X_1 = S_1 = 1$). Therefore we get $P(S_1, S_2, \dots, S_{2n} > 0) = \sum_{r=1}^{n-1} P(S_1, S_2, \dots, S_{2n} > 0, S_{2n} = 2r)$.

The r -th term follows equation: $P(S_1, S_2, \dots, S_{2n} > 0, S_{2n} = 2r) = P(X_1 = 1, S_2, \dots, S_{2n} > 0, S_{2n} = 2r) = \frac{1}{2} P(S_2, S_3, \dots, S_{2n} > 0, S_{2n} = 2r) = \frac{1}{2} (P(S_{2n} = 2r) - P(S_2 \cdot S_3 \cdot \dots \cdot S_{2n} = 0, S_{2n} = 2r)) = \frac{1}{2} \left(\frac{1}{2} {}^{2n-1}N_{2n-1}(1, 2r) - \frac{1}{2} {}^{2n-1}N_{2n-1}^0(1, 2r) \right) = \frac{1}{2} \frac{1}{2} {}^{2n-1} \left(N_{2n-1}(1, 2r) - N_{2n-1}^0(1, 2r) \right) = \frac{1}{2} \frac{1}{2} {}^{2n-1} (N_{2n-1}(1, 2r) - N_{2n-1}(-1, 2r)) = \frac{1}{2} \frac{1}{2} {}^{2n-1} \left(\binom{2n-1}{n+r-1} - \binom{2n-1}{n+r} \right)$. Because of the fact that the negative parts of r -th terms cancel against the positive parts of $(r+1)$ -st terms and the sum reduces to just $\frac{1}{2} \frac{1}{2} {}^{2n-1} \binom{2n-1}{n} = \frac{1}{2} \cdot 2 \cdot \frac{1}{2} \binom{2n-1}{n} = \frac{1}{2} \frac{1}{2} \frac{2n}{n!} \frac{2(2n-1)!}{(n-1)!} = \frac{1}{2} \frac{1}{2} \frac{2n}{n!} \frac{(2n)!}{n!} = \frac{1}{2} \frac{1}{2} \binom{2n}{n} = \frac{1}{2} P(S_{2n} = 0) = \frac{1}{2} u_{2n}$. \square

Theorem 16 (No return=return). $P(S_1, S_2, \dots, S_{2n} \neq 0) = P(S_{2n} = 0) = u_{2n}$

Proof. The event $[S_1, S_2, \dots, S_{2n} \neq 0]$ can be split into two disjoint events: $= [S_1, S_2, \dots, S_{2n} < 0] \cup [S_1, S_2, \dots, S_{2n} > 0]$. By previous theorem (15) we get that probability of both of them is $\frac{1}{2} u_{2n}$. Because the events are disjoint we can sum their probabilities and we get the result. \square

Corollary. $P(S_1, S_2, \dots, S_{2n} \geq 0) = P(S_{2n} = 0) = u_{2n}$

Proof. $\frac{1}{2} u_{2n} = P(S_1, S_2, \dots, S_{2n} > 0) = P(X_1 = 1, S_2, S_3, \dots, S_{2n} \geq 1) = \frac{1}{2} P(S_2, S_3, \dots, S_{2n} \geq 1) = \frac{1}{2} P(S_1, S_2, \dots, S_{2n-1} \geq 1 \mid S_0 = 1) = \frac{1}{2} P(S_1, S_2, \dots, S_{2n-1} \geq 0) = \frac{1}{2} P(S_1, S_2, \dots, S_{2n} \geq 0)$. Therefore $P(S_1, S_2, \dots, S_{2n} \geq 0) = u_{2n}$. \square

Theorem 17 (XXX). $f_{2n} = u_{2n-2}u_{2n}$

Proof. The event $[S_1, S_2, \dots, S_{2n-1} \neq 0]$ can be split into two disjoint events: $[S_1, S_2, \dots, S_{2n-1} \neq 0, S_{2n} = 0]$ and $[S_1, S_2, \dots, S_{2n-1} \neq 0, S_{2n} \neq 0]$. Therefore $P(S_1, S_2, \dots, S_{2n-1} \neq 0) = P(S_1, S_2, \dots, S_{2n-1} \neq 0, S_{2n} = 0) + P(S_1, S_2, \dots, S_{2n-1} \neq 0, S_{2n} \neq 0)$. Therefore we get $f_{2n} = P(S_1, S_2, \dots, S_{2n-1} \neq 0, S_{2n} = 0) = P(S_1, S_2, \dots, S_{2n-1} \neq 0) - P(S_1, S_2, \dots, S_{2n} \neq 0)$. Because $2n - 1$ is odd. $P(S_{2n-1} = 0) = 0$. Therefore the first term is equal to $P(S_1, S_2, \dots, S_{2n-2} \neq 0)$ which is by 16 equal to u_{2n-2} . Second term is by 16 equal to u_{2n} . Therefore we get the result. \square

Corollary. $f_{2n} = \frac{1}{2n-1}u_{2n}$

Proof. $u_{2n-2} = \frac{1}{2} \frac{2^{2n-2} \binom{2n-2}{n-1}}{(n-1)!} = 4 \cdot \frac{1}{2} \frac{2^{2n} \frac{(2n-2)!}{(n-1)!(n-1)!}}{(2n)(2n-1)} = \frac{4n^2}{(2n)(2n-1)} \frac{1}{2} \frac{2^{2n} \binom{2n}{n}}{(2n-1)} = \frac{2n}{2n-1}u_{2n}$. Therefore $u_{2n-2} - u_{2n} = u_{2n} \left(\frac{2n}{2n-1} - 1 \right) = u_{2n} \frac{1}{2n-1}$. \square

Theorem 18 (Arcsine law for last visits). *Let $k, n \in \mathbb{N}, k \leq n$. The probability that up to time $2n$ a return to origin occured in time $2k$ is given by $\alpha_{2n}(2k) = u_{2n}u_{2(n-k)}$.*

Proof. The probability involved can be rewritten as: $P(S_{2k+1}, S_{2k+2}, \dots, S_{2n} \neq 0, S_{2k} = 0) = P(S_{2k+1}, S_{2k+2}, \dots, S_{2n} \neq 0 \mid S_{2k} = 0) P(S_{2k} = 0) = P(S_1, S_2, \dots, S_{2(n-k)} \neq 0) P(S_{2k} = 0) = u_{2(n-k)}u_{2k}$. \square

Definition 8 (Time spend on the positive and negative sides). *Let $(\{S_n\}_{n=0}^{+\infty}, p)$ be a random walk. We say that the walk spent τ time units of n on the positive side if $\sum_{i=1}^n \mathbf{1}_{[S_i > 0 \vee S_{i-1} > 0]} = \tau$. Let $\beta_n(\tau)$ denote the probability of such an event. We say that the walk spent ζ time units of n on the negative side if $\sum_{i=1}^n \mathbf{1}_{[S_i < 0 \vee S_{i-1} < 0]} = \zeta$.*

Theorem 19 (Arcsine law for sojourn times). *Let $(\{S_n\}_{n=0}^{+\infty}, p)$ be a symmetric random walk. Then $\beta_{2n}(2k) = \alpha_{2n}(2k)$.*

Proof. Firstly let us start with degenerate cases. $\beta_{2n}(2n) = P\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i > 0 \vee S_{i-1} > 0]} = 2n\right) = P(S_1, S_2, \dots, S_{2n} \geq 0) = u_{2n}$. By symmetry $\beta_{2n}(0) = \beta_{2n}(2n) = u_{2n}$.

$$\beta_{2n}(2k) = P\left(\sum_{i=1}^n \mathbf{1}_{[S_i > 0 \vee S_{i-1} > 0]} = 2k\right) = \sum_{r=1}^n P\left(\sum_{i=1}^n \mathbf{1}_{[S_i > 0 \vee S_{i-1} > 0]} = 2k, S_1, S_2, \dots, S_{2r-1} \neq 0, S_{2r} = 0\right)$$

\square

1.1 Title of the first subchapter of the first chapter

1.2 Title of the second subchapter of the first chapter

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