1. Title of the first chapter

Definition 1 (Simple random walk in $\mathbb{Z}$)**.** *Let $\{X_n\}_{n=0}^{+\infty}$ be a sequence of independent and identically distributedrandom variables with values in $\{-1,+1\}$. That $\forall n \in \mathbb{N}$ satisfy $P(X_n = 1) = p \in (0,1)$ and $P(X_n = -1) = 1 - p = q$. Let $S_0 = 0$ and $S_n = \sum\limits_{i=1}^{n} X_i$. We call the pair $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$* Simple random walk in $\mathbb{Z}$. *In case that $p = q = \frac{1}{2}$ we call the pair $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$* Symmetric simple random walk in $\mathbb{Z}$.

Remark. Very often we refer to n as time , X_i as i-th step and S_n as position in time n. In simple random walk in $\mathbb{Z}$ we refer to $X_i = +1$ as i-th step was rightwards. If not stated otherwise, we

Definition 2 (Set of possible positions)**.** *Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk. We call the set $A_n = \{z \in \mathbb{Z}; |z| \leq n, \frac{z+n}{2} \in \mathbb{Z}\}$ set of all possible positions of random walk $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ in time n.*

Theorem 1 (Probability of position x in time n)**.** *Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk and A_n its set of possible positions.*

$$P(S_n = x) = \begin{cases} \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}} & \text{for } x \in A_n \ , \\ 0, & \text{for } x \notin A_n. \end{cases}$$

Proof. Let us define new variables $r_i = \mathbf{1}_{[X_i=1]}, l_i = \mathbf{1}_{[X_i=-1]}, R_n = \sum\limits_{i=1}^{n} r_i, L_n = \sum\limits_{i=1}^{n} l_i$. r_i can be interpreted as indicator wether i-th step was rightwards. Then R_n is number of rightwards steps and L_n is number of leftwards steps. We can easily see that $R_n + L_n = n$ and $R_n - L_n = S_n$. Therefore we get by adding these two equations $R_n = \frac{S_n+n}{2}$.

r_i has alternative distribution with parameter p *Alt(p)* . Therefore R_n as a sum of independent and identically distributedrandom variables with *Alt(p)* has binomial distribution with parameters n and p (*Bi(n,p)*) . Therefore we get $\mathsf{P}(R_n = x) = \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}$. Where we define $\binom{a}{x} := 0$ for $a \in \mathbb{N}, x \in \mathbb{R} \setminus \mathbb{N}$, $x < 0$, $x > n$. Therefore we get $\mathsf{P}(S_n = x) = \mathsf{P}\left(R_n = \frac{x+n}{2}\right) = \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}$. $\square$

Lemma 2 (Spatial homogeneity)**.** $P(S_n = j \mid S_0 = a) = P(S_n = j + b \mid S_0 = a + b) \, \forall b \in \mathbb{Z}$

Proof. $\mathsf{P}(S_n = j \mid S_0 = a) = \mathsf{P}\left(\sum\limits_{i=1}^{n} X_i = j - a\right) = \mathsf{P}\left(\sum\limits_{i=1}^{n} X_i = (j+b) - (a+b)\right) = \mathsf{P}(S_n = j + b \mid S_0 = a + b)$. $\square$

Lemma 3 (Temporal homogeneity)**.** $P(S_n = j \mid S_0 = a) = P(S_{n+m} = j \mid S_m = a) \, \forall m \in \mathbb{N}$

Proof. $\mathsf{P}(S_n = j \mid S_0 = a) = \mathsf{P}\left(\sum\limits_{i=1}^{n} X_i = j - a\right) = \mathsf{P}\left(\sum\limits_{i=m+1}^{m+n} X_i = j - a\right) = \mathsf{P}(S_{n+m} = j \mid S_m =$ $\square$

Lemma 4 (Markov property)**.** *Let $n \geq m$ and $a_i \in \mathbb{Z}$. Then $\mathsf{P}(S_n = j \mid S_0 = a_0, S_1 = a_1, \ldots, S_m$ $\mathsf{P}(S_n = j \mid S_m = a_m)$*

Proof. Once S_m is known, then distribution of S_n depends only on steps $X_{m+1}, X_{m+2}, \ldots X_n$ and therefore cannot be dependent on any information concerning values $X_1, X_2, \ldots, X_m -$ 1 and accordingly $S_1, S_2, \ldots, S_m - 1$. □

Remark. In symmetric random walk, everything can be counted by number of possible paths from point to point.

Definition 3 (Number of possible paths)**.** *Let $N_n(a, b)$ be number of posssible paths of random walk $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$from point $(0, a)$ to point (n, b) and $N_n^x(a, b)$ be number of possible paths from point $(0, a)$ to point (n, b) that visit point (z, x) for some $z \in \{0, \ldots n\}$.*

Theorem 5. *Let $a, b \in \mathbb{Z}, n \in \mathbb{N}$ then $N_n(a, b) = \binom{n}{\frac{1}{2}(n+b-a)}$.*

Proof. Let us choose a path from $(0, a)$ to (n, b) and let α be number of rightwards steps and β be number of leftwards steps. Then $\alpha + \beta = n$ and $\alpha - \beta = b - a$. By adding these two equations we get that $\alpha = \frac{1}{2}(n + b - a)$. The number of possible paths is the number of ways of picking α rightwards steps from n steps. Therefore we get $N_n(a, b) = \binom{n}{\alpha} = \binom{n}{\frac{1}{2}(n+b-a)}$. □

Theorem 6 (Reflection principle)**.** *Let $a, b > 0$, then $N_n^0(a, b) = N_n(-a, b)$.*

Proof. Each path from $(0, -a)$ to (n, b) has to intersect x-axis at least once at some point. Let k be the time of earliest intersection with x-axis. By reflexing the segment from $(0, -a)$ to $(k, 0)$ in the x-axis, we get a path from point $(0, a)$ to (n, b) which visits 0 at point k. Because reflection is bijective operation on sets of paths, we get the correspondence between the collections of such paths. □

Definition 4 (Return to origin)**.** *Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$be a random walk. Then if $\exists k \in \mathbb{N}$ such that $S_k = 0$ then we say that in k-th step occured return to origin. Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$be a random walk. Then if $S_1, S_2, \ldots, S_{2n-1} \neq 0$ and $S_{2n} = 0$*

Theorem 7 (Ballot theorem)**.** *Let $n, b \in N$ Number of paths from point $(0, 0)$ to point (n, b) which do not return to origin is equal to $\frac{b}{n} N_n(0, b)$*

Proof. Let us call N the number of paths we are refering to. Because the path ends at point (n, b), the first step has to be rightwards. Therefore we now have $N = N_{n-1}(1, b) - N_{n-1}^0(1, b) = N_{n-1}(1, b) - N_{n-1}(-1, b)$. The last equation was aquired using Reflection principle (6) . We now have:

$$N_{n-1}(1, b) - N_{n-1}(-1, b) = \binom{n-1}{\frac{n}{2} + \frac{b}{2} - 1} - \binom{n-1}{\frac{n}{2} + \frac{b}{2}} = \frac{(n-1)!}{\left(\frac{n}{2} + \frac{b}{2} - 1\right)! \left(\frac{n}{2} - \frac{b}{2}\right)!} - \frac{(n-1)!}{\left(\frac{n}{2} + \frac{b}{2}\right)! \left(\frac{n}{2} - \frac{b}{2}\right.}$$

□

Definition 5. $M_n = \max\{S_i, i \in \{0, 1, \ldots, n\}\}$

Theorem 8 (Probability of maximum up to time n)**.** *Let* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a random walk.*

$$\mathsf{P}(M_n \geq r, S_n = b) = \begin{cases} \mathsf{P}(S_n = b) & \text{for } b \geq r \ , \\ \mathsf{P}(S_n = 2r - b) \left(\frac{q}{p}\right)^{r-b}, & \text{for otherwise.} \end{cases}$$

Proof. Let us firstly consider the easier case in which $b \geq r$. Because we defined M_n as $\max\{S_i, i \in \{0, 1, \ldots, n\}\}$ we get that $M_n \geq b \geq r$ therefore $[M_n \geq r] \subset [S_n = b]$ therefore we get $\mathsf{P}(M_n \geq r, S_n = b) = \mathsf{P}(S_n = b)$. Let $r \geq 1, b < r$. $N_n^r(0, b)$ stands for number of paths from point $(0, 0)$ to point (n, b) which reach up to r. Let $k \in \{0, 1, \ldots, n\}$ denote the first time we reach r. By reflection principle (6) , we can reflex the segment from (k, r) to (n, b) in the axis:$y = r$. Therefore we now have path from $(0, 0)$ to $(n, 2r - b)$ and we get that $N_n^r(0, b) = N_n(0, 2r - b)$. $\mathsf{P}(S_n = b, M_n \geq r) = N_n^r(0, b)\, p^{\frac{n+b}{2}} q^{\frac{n-b}{2}} = N_n(0, 2r - b)\, p^{\frac{n+(2r-b)}{2}} q^{\frac{n-(2r-b)}{2}} p^{\frac{n+b}{2}} q^{\frac{n-b}{2}} p^{b-r} q^{r-b} = \left(\frac{q}{p}\right)^{r-b} \mathsf{P}(S_n = 2r - b)\,.$ ☐

Definition 6 (Walk reaching new maximum at particular time)**.** *Let* $b > 0$. $f_b(n)$ *denotes the probability that we reach new maximum b in time n.* $f_b(n) = \mathsf{P}(M_{n-1} = S_{n-1} = b - 1, S_n = b)$

Theorem 9 (Probability of reaching new maximum b in time n)**.** *Let* $b > 0$ *then* $f_b(n) = \frac{b}{n} \mathsf{P}(S_n = b)\,.$

Proof. $f_b = \mathsf{P}(M_{n-1} = S_{n-1} = b - 1, S_n = b) = p\,\mathsf{P}(M_{n-1} = S_{n-1} = b - 1) = p\left(\mathsf{P}(M_{n-1} \geq b - 1, S_{n-1} = b - 1) - \mathsf{P}(M_{n-1} \geq b, S_{n-1} = b - 1)\right) = p\left(\mathsf{P}(S_{n-1} = b - 1) - \frac{q}{p}\mathsf{P}(S\right.$ $p\,\mathsf{P}(S_{n-1} = b - 1) - q\,\mathsf{P}(S_{n-1} = b + 1) = \binom{n-1}{\frac{n}{2}+\frac{b}{2}-1} p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} - \binom{n-1}{\frac{n}{2}+\frac{b}{2}} p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} = p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \left(\frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!} - \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!}\right) = p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \left(\frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!}\right)\left(\frac{1}{\frac{n}{2}-\frac{b}{2}} - \frac{1}{\frac{n}{2}+\frac{b}{2}}\right) = p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \frac{b}{n} \left(\frac{n!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!}\right) = \frac{b}{n} p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \binom{n}{\frac{n}{2}+\frac{b}{2}} = \frac{b}{n}\mathsf{P}(S_n = b)\,.$ ☐

Theorem 10 (XXXMean number of visits to b before returning to origin in symmetric random walk)**.** *Let* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a symmetric random walk. Mean number μ_b of visits of the walk to point b before returning to origin is equal to 1.*

Proof. aa ☐

Definition 7 (Return to origin)**.** *Let* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a random walk. Let* $k \in \mathbb{N}$. *We say a* return to origin *occurred in time* $2k$ *if* $S_{2k} = 0$. *The probability that in time* $2k$ *occured a return to origin shall be denoted by* u_{2k}. *We say that in time* $2k$ *occurred* first return to origin *if* $S_1, S_2, \ldots S_{2k-1} \neq 0$ *and* $S_{2k} = 0$. *The probability that in time* $2k$ *occured first return to origin shall be denoted by* f_{2k}. *By definition* $f_0 = 0$. *Let* $\alpha 2n(2k)$ *denote* $u_{2k} u_{2(n-k)}$

Lemma 11 (Binomial identity)**.** *Let* $n, k \in \mathbb{N}, n > k : \binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{n}\binom{n}{k}$

Proof. $\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-k-1)!} - \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{1}{k} - \frac{1}{n-k}\right) = \frac{1}{n}\frac{n!}{(k-1)!(n-k-1)!}\frac{n-2k}{k(n-k)} = \frac{n-2k}{n}\frac{n!}{k!(n-k)!} = \frac{n-2k}{n}\binom{n}{k}$ ☐

Lemma 12 (Main lemma)**.** *Let* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a symmetrical random walk. Then* $\mathsf{P}(S_1 \cdot S_2, \cdot, \ldots, S_{2n} \neq 0) = \mathsf{P}(S_{2n} = 0)$

Proof. $\mathsf{P}\left(S_1 \cdot S_2, \cdot, \ldots, S_{2n} \neq 0\right) = \sum_{i=-\infty}^{+\infty} \mathsf{P}\left(S_1 \cdot S_2, \cdot, \ldots, S_{2n-1} \neq 0, S_{2n} = 2i\right) = \sum_{i=-n}^{n} \mathsf{P}\left(S_1 \cdot S_2, \cdot, \ldots, S_{2n-1} \neq 0, S_{2n} = 2i\right) = 2 \cdot \sum_{i=1}^{n} \mathsf{P}\left(S_1 \cdot S_2, \cdot, \ldots, S_{2n-1} \neq 0, S_{2n} = 2i\right) = 2 \sum_{i=1}^{n} \frac{2i}{2n} \mathsf{P}\left(S_{2n} = 2k\right) = 2 \sum_{i=1}^{n} \frac{2i}{2n} \binom{2n}{n+i} 2^{-2n} = 2 \cdot 2^{-2n} \sum_{i=1}^{n} \left(\binom{2n-1}{m+k-1} - \binom{2n-1}{m+k}\right) = 2 \cdot 2^{-2n} \binom{2n-1}{n} = 2^{-2n} \frac{2n}{n} \binom{2n-1}{n} = 2^{-2n} \frac{2n(2n-1)!}{m(m-1)!m!} = 2^{-2n} \frac{(2n)!}{m!m!} = 2^{-2n} \binom{2n}{n} = \mathsf{P}\left(S_2 n = 0\right)$ □

Theorem 13. *Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a symmetric random walk. The probability that the last return to origin up to time $2n$ occurred in time $2k$ is $P\left(S_{2k} = 0\right) P\left(S_{2(n-k)} = 0\right)$.*

Proof. $\alpha 2n\left(2k\right) = u_{2k} u_{2(n-k)} = \mathsf{P}\left(S_{2k} = 0\right) \mathsf{P}\left(S_{2k+1} \cdot S_{2k+2} \cdot, \ldots, S_{2n} \neq 0 \mid S_{2k} = 0\right) = \mathsf{P}\left(S_{2k} = 0\right) \mathsf{P}\left(S_1 \cdot S_2 \cdot, \ldots, S_{2(n-k)} \neq 0\right) = \mathsf{P}\left(S_{2k} = 0\right) \mathsf{P}\left(S_{2(n-k)} = 0\right)$ □

Theorem 14. *Let $b \in Z$. $P\left(S_1 \cdot S_2 \cdot, \ldots, S_n \neq 0, S_n = b\right) = \frac{|b|}{n} P\left(S_n = b\right)$.*

Proof. Let us Without loss of generalityassume that $b > 0$. In that case, first step has to be rightwards $\left(X_1 = +1\right)$. Now we have path from point $(1, 1)$ to point (n, b) that does not return to origin. By Ballot theorem 7 there are $\frac{b}{n} N_n\left(0, b\right)$ such paths. Each path consists of $\frac{n+b}{2}$ rightwards steps and $\frac{n-b}{2}$ leftwards steps. Therefore $\mathsf{P}\left(S_1 \cdot S_2 \cdot, \ldots, S_n \neq 0, S_n = b\right) = \frac{b}{n} N_n\left(0, b\right) p^{\frac{n+b}{2}} q^{\frac{n-b}{2}} = \frac{b}{n} \mathsf{P}\left(S_n = b\right)$. Case $b < 0$ is identical. □

Lemma 15. *Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a symmetric random walk. $P\left(S_1, S_2, \ldots, S_{2n} > 0\right) = \frac{1}{2} P\left(S_{2n} = 0\right) = \frac{1}{2} u_{2n}$.*

Proof. Because $S_i > 0 \forall i \in \mathbb{N}$ the first step has to be rightwards $\left(X_1 = S_1 = 1\right)$. Therefore we get $\mathsf{P}\left(S_1, S_2, \ldots, S_{2n} > 0\right) = \sum_{r=1}^{n-1} \mathsf{P}\left(S_1, S_2, \ldots, S_{2n} > 0, S_{2n} = 2r\right)$. The r-th term follows equation: $\mathsf{P}\left(S_1, S_2, \ldots, S_{2n} > 0, S_{2n} = 2r\right) = \mathsf{P}\left(X_1 = 1, S_2, \ldots, S_{2n} > 0, S\right.$ $\frac{1}{2} \mathsf{P}\left(S_2, S_3, \ldots, S_{2n} > 0, S_{2n} = 2r\right) = \frac{1}{2}\left(\mathsf{P}\left(S_{2n} = 2r\right) - \mathsf{P}\left(S_2 \cdot S_3 \cdot \ldots \cdot S_{2n} = 0, S_{2n} = 2r\right)\right) = \frac{1}{2}\left(\frac{1}{2}^{2n-1} N_{2n-1}\left(1, 2r\right) - \frac{1}{2}^{2n-1} N^0_{2n-1}\left(1, 2r\right)\right) = \frac{1}{2}\frac{1}{2}^{2n-1}\left(N_{2n-1}\left(1, 2r\right) - N^0_{2n-1}\left(1, 2r\right)\right) = \frac{1}{2}\frac{1}{2}^{2n-1}\left(N_{2n-1}\left(1, 2r\right) - N_{2n-1}\left(-1, 2r\right)\right) = \frac{1}{2}\frac{1}{2}^{2n-1}\left(\binom{2n-1}{n+r-1} - \binom{2n-1}{n+r}\right)$. Because of the fact that the negative parts of r-th terms cancel against the positive parts of $(r+1)$-st terms and the sume reduces to just $\frac{1}{2}\frac{1}{2}^{2n-1}\binom{2n-1}{n} = \frac{1}{2} \cdot 2 \cdot \frac{1}{2}^{2n}\binom{2n-1}{n} = \frac{1}{2}\frac{1}{2}^{2n}\frac{2(2n-1)!}{n!(n-1)!} = \frac{1}{2}\frac{1}{2}^{2n}\frac{(2n)!}{n!n!} = \frac{1}{2}\frac{1}{2}^{2n}\binom{2n}{n} = \frac{1}{2}\mathsf{P}\left(S_{2n} = 0\right) = \frac{1}{2} u_{2n}$. □

Theorem 16 (No return=return)**.** *$P\left(S_1, S_2, \ldots, S_{2n} \neq 0\right) = P\left(S_{2n} = 0\right) = u_{2n}$*

Proof. The event $\left[S_1, S_2, \ldots, S_{2n} \neq 0\right]$ can be split into two disjoint events: $= \left[S_1, S_2, \ldots, S_{2n} < 0\right] \cup \left[S_1, S_2, \ldots, S_{2n} > 0\right]$. By previous theorem (15) we get that probability of both of them is $\frac{1}{2} u_{2n}$. Because the the events are disjoint we can sum their probabilities and we get the result. □

Corollary. $\mathsf{P}\left(S_1, S_2, \ldots, S_{2n} \geq 0\right) = \mathsf{P}\left(S_{2n} = 0\right) = u_{2n}$

Proof. $\frac{1}{2} u_{2n} = \mathsf{P}\left(S_1, S_2, \ldots, S_{2n} > 0\right) = \mathsf{P}\left(X_1 = 1, S_2, S_3 \ldots, S_{2n} \geq 1\right) = \frac{1}{2} \mathsf{P}\left(S_2, S_3 \ldots, S_{2n} \geq 1\right.$ $\frac{1}{2} \mathsf{P}\left(S_1, S_2 \ldots, S_{2n-1} \geq 1 \mid S_0 = 1\right) = \frac{1}{2} \mathsf{P}\left(S_1, S_2 \ldots, S_{2n-1} \geq 0\right) = \frac{1}{2} \mathsf{P}\left(S_1, S_2 \ldots, S_{2n} \geq 0\right)$. Therefore $\mathsf{P}\left(S_1, S_2 \ldots, S_{2n} \geq 0\right) = u_{2n}$. □

Theorem 17 (XXX)**.** $f_2 n = u_{2n-2} u_{2n}$

Proof. The event $[S_1, S_2, \dots S_{2n-1} \neq 0]$ can be split into two disjoint events: $[S_1, S_2, \dots S_{2n-1} \neq 0, S_{2n} = 0]$ and $[S_1, S_2, \dots S_{2n-1} \neq 0, S_{2n} \neq 0]$. Therefore $\mathsf{P}\left(S_1, S_2, \dots S_{2n-1} \neq 0\right) = \mathsf{P}\left(S_1, S_2, \dots S_{2n-1} \neq 0, S_{2n} = 0\right) + \mathsf{P}\left(S_1, S_2, \dots S_{2n-1} \neq 0, S_{2n} \neq 0\right)$. Therefore we get $f_{2n} = \mathsf{P}\left(S_1, S_2, \dots S_{2n-1} \neq 0, S_{2n} = 0\right) = \mathsf{P}\left(S_1, S_2, \dots S_{2n-1} \neq 0\right) - \mathsf{P}\left(S_1, S_2, \dots, S_{2n} \neq 0\right)$. Because $2n-1$ is odd. $\mathsf{P}\left(S_{2n-1} = 0\right) = 0$. Therefore the first term is equal to $\mathsf{P}\left(S_1, S_2, \dots S_{2n-2} \neq 0\right)$ which is by 16 equal to u_{2n-2}. Second term is by 16 equal to u_{2n}. Therefore we get the result. □

Corollary. $f_{2n} = \frac{1}{2n-1} u_{2n}$

Proof. $u_{2n-2} = \frac{1}{2}^{2n-2}\binom{2n-2}{n-1} = 4 \cdot \frac{1}{2}^{2n} \frac{(2n-2)!}{(n-1)!(n-1)!} = \frac{4n^2}{(2n)(2n-1)} \frac{1}{2}^{2n}\binom{2n}{n} = \frac{2n}{2n-1} u_{2n}$. Therefore $u_{2n-2} - u_{2n} = u_{2n}\left(\frac{2n}{2n-1} - 1\right) = u_{2n}\frac{1}{2n-1}$. □

Theorem 18 (Arcsine law for last visits)**.** *Let $k, n \in \mathbb{N}, k \leq n$. The probability that up to time $2n$ a return to origin occured in time $2k$ is given by $\alpha_{2n}(2k) = u_{2n} u_{2(n-k)}$.*

Proof. The probability involved can be rewritten as: $\mathsf{P}\left(S_{2k+1}, S_{2k+2}, \dots, S_{2n} \neq 0, S_{2k} = 0\right) =$ $*** \, \mathsf{P}\left(S_{2k+1}, S_{2k+2}, \dots, S_{2n} \neq 0 \mid S_{2k} = 0\right)\mathsf{P}\left(S_{2k} = 0\right) = * \, \mathsf{P}\left(S_1, S_2, \dots, S_{2(n-k)} \neq 0\right)\mathsf{P}\left(S_{2k} = 0\right.$ $** \, u_{2(n-k)} u_{2k}$ □

Definition 8 (Time spend on the positive and negative sides)**.** *Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk. We say that the walk spent τ time units of n on the positive side if $\sum_{i=1}^{n} \mathbf{1}_{[S_i > 0 \vee S_{i-1} > 0]} = \tau$. Let $\beta_n(\tau)$ denote the probability of such an event. We say that the walk spent ζ time units of n on the negative side if $\sum_{i=1}^{n} \mathbf{1}_{[S_i < 0 \vee S_{i-1} < 0]} = \zeta$.*

Theorem 19 (Arcsine law for sojourn times)**.** *Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a symmetric random walk. Then $\beta_{2n}(2k) = \alpha_{2n}(2k)$.*

Proof. Firstly let us start with degenerate cases. *f$\beta_{2n}(2n) = \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i > 0 \vee S_{i-1} > 0]} = 2n\right) = \mathsf{P}\left(S_1, S_2, \dots, S_{2n} \geq 0\right) = *u_{2n}$. By symmetry $\beta_{2n}(0) = \beta_{2n}(2n) = u_{2n}$. Let $1 \leq k \leq v-1$, where $0 \leq v \leq n$. For such k stands equation:

$$\beta_{2n}(2k) = \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i > 0 \vee S_{i-1} > 0]} = 2k\right) = *a \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i > 0 \vee S_{i-1} > 0]} = 2k, S_1, S_2, \dots, S_{2r-1} \neq 0,\right.$$

Now let us proceed by induction. Case for $v = 1$ is trivial because it implies degenerous case from *f. Let the statment be true for $v \leq n-1$, then $\sum_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k) + \sum_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k-2r) = *g \sum_{r=1}^{n} \frac{1}{2} f_{2r} \alpha_{2n-2r}(2k) + \sum_{r=1}^{n} \frac{1}{2} f_{2r} \alpha_{2n-2r}(2k-2r) =$ $*h \sum_{r=1}^{n} \frac{1}{2} f_{2r} u_{2k} u_{2n-2r-2k} + \sum_{r=1}^{n} \frac{1}{2} f_{2r} u_{2k-2r} u_{2n-2k} = *i \frac{1}{2} u_{2k} \sum_{r=1}^{n} f_{2r} u_{2n-2r-2k} + \frac{1}{2} u_{2n-2k} \sum_{r=1}^{n} f_{2r} u_{2k-2r} u_{2n}$ $*j \frac{1}{2} u_{2n-2k} u_{2k} + \frac{1}{2} u_{2n-2k} u_{2k} = u_{2n-2k} u_{2k} = *h \alpha_{2n}(2k)$

□

Definition 9 (Change of a sign)**.** *Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk. We say that in time n occurred a change of sign if if $S_{n-1} \cdot S_{n+1} = -1$ in other words if $(S_{n-1} = +1 \wedge S_{n+1} = -1) \vee (S_{n-1} = -1 \wedge S_{n+1} = +1)$*

Theorem 20 (Change of a sign)**.**

1.1 Title of the first subchapter of the first chapter

1.2 Title of the second subchapter of the first chapter