ONE LIMIT DISTRIBUTION FOR A RANDOM WALK ON THE LINE

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The present paper is devoted to the study of the asymptotic behavior of the "tails" of the distribution of the random series

$$\psi=\eta_1+\eta_2\xi_1+\eta_3\xi_1\xi_2+\ldots+\eta_n\xi_1\cdot\ldots\cdot\xi_{n-1}+\ldots,$$

arising in a limit theorem for the product of independent random elements of the group of linear transformations of the line (see U. Grenander [1] and A. Grincevičius [6]), in schemes of random walks on the simplest geometric objects, i.e., straight lines (see V. Maksimov [5]), and also in the theory of semi-Markov processes of birth and death (see G. Lev [2, 3]).

In what follows we shall assume that the random matrices

$$\begin{pmatrix}\xi_j & \eta_j\\ 0 & 1\end{pmatrix},\quad j=1,\ 2,\ 3,\ \ldots$$

are independent and identically distributed, and that $\xi_j > 0$ with probability 1. In this paper we shall prove

THEOREM 1. Let $\mathbf{M}\xi_1^\alpha<1$, $\mathbf{M}\xi_1^\beta<\infty$ and $\mathbf{P}\{\eta_1>x\}\sim x^{-\alpha}L(x)$ as $x\to\infty$ where $L(x)$ is a slowly varying function and $0<\alpha<\beta$. Then

$$\mathbf{P}\{\psi>x\}\sim x^{-\alpha}L(x)\sum_{n=0}^{\infty}(\mathbf{M}\xi_1^\alpha)^n.$$

THEOREM 2. Let $\mathbf{M}\xi_1^\beta=1$, $\mathbf{P}\{|\eta_1|>x\}=o(x^{-\alpha})$ as $x\to\infty$, where $0<\beta<\alpha$, $\mathbf{M}\xi_1^\beta\ln\xi_1<\infty$ and random variable ξ_1 is not degenerate. Then,

a) if random variable $\ln\xi_1$ has a nonarithmetic distribution, then there exists limit

$$\lim_{x\to\infty}\frac{\mathbf{P}\{\psi>x\}}{x^{-\beta}}=C_1,$$

where constant C_1 is known beforehand to be positive if, for some real number c, $\mathbf{P}\{\eta_1\geqslant c(1-\xi_1)\}=1$ and $\mathbf{P}\{\eta_1>c(1-\xi_1)\}>0$;

b) if random variable $\ln\xi_1$ has an arithmetic distribution with step λ then, with the possible exception of a no more than countable number of real values of h, there exists a limit $\lim\limits_{x\to\infty}\frac{\mathbf{P}\{\psi>e^{h+\lambda n}\}}{e^{-(h+\lambda n)\beta}}=C_2(h)$, where $C_2(h)$ is a periodic function with period λ; if, moreover, $\mathbf{P}\{\eta_1\geqslant 0\}=1$ and $\mathbf{P}\{\eta_1>0\}>0$, then the limit exists for all real h, and function $C_2(h)$ is strictly positive.

In the proofs of Theorems 1 and 2 we shall use the ideas in the proofs of the analogous results of G. Lev [3] for the probability of degeneracy of semi-Markov processes of birth, but, in the paper of G. Lev, on the one hand, there is assumed, and essentially utilized, the independence of the sequences (in our notation) η_j, j = 1, 2, 3, . . ., and ξ_k, k = 1, 2, 3, while, on the other hand, in the paper of G. Lev [3, p. 785], there is an erroneous application of the renewal theorem to the entire line and, in connection with this,

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there is no separate consideration of the cases of arithmetic and nonarithmetic distributions of the random variable $\ln \xi_1$.

In the proof of Theorem 1 we shall need two lemmas.

LEMMA 1. Let $\mathbf{M}\xi_1^\beta < \infty$, $\mathbf{P}\{\eta_1 > x\} \sim x^{-\alpha}L(x)$, where $L(x)$ is a slowly varying function, as $x \to \infty$, $0 < \alpha < \beta$, and $\lim\limits_{x\to\infty} \frac{\mathbf{P}\{\psi_1 > x\}}{x^{-\alpha}L(x)} = C$, where random variable ψ_1 does not depend on random matrix $\begin{pmatrix} \xi_1 & \eta_1 \\ 0 & 1 \end{pmatrix}$. Then,

$$\mathbf{P}\{\eta_1 + \xi_1\psi_1 > x\} \sim (1 + C\cdot\mathbf{M}\xi_1^\alpha)\, x^{-\alpha}L(x) \quad \text{as} \quad x \to \infty.$$

Proof. We shall show that

$$\lim_{x\to\infty} \frac{\mathbf{P}\{\xi_1\psi_1 > x\}}{x^{-\alpha}L(x)} = C\cdot\mathbf{M}\xi_1^\alpha. \tag{1}$$

Indeed, by virtue of the independence of random variables ξ_1 and ψ_1 we have

$$\lim_{x\to\infty} \frac{\mathbf{P}\{\xi_1\psi_1 > x\}}{x^{-\alpha}L(x)} = \lim_{x\to\infty} \int \frac{\mathbf{P}\{\psi_1 y > x\}}{x^{-\alpha}L(x)} F_{\xi_1}\{dy\} = \lim_{x\to\infty} \int_0^{\frac{x}{N}} \frac{\mathbf{P}\{\psi_1 y > x\}}{x^{-\alpha}L(x)} F_{\xi_1}\{dy\} \tag{2}$$

for any positive number N, since

$$\lim_{x\to\infty} \int_{\frac{x}{N}}^{\infty} \frac{\mathbf{P}\{\psi_1 y > x\}}{x^{-\alpha}L(x)} F_{\xi_1}\{dy\} \leqslant \lim_{x\to\infty} \frac{\mathbf{P}\left\{\xi_1 \geqslant \frac{x}{N}\right\}}{x^{-\alpha}L(x)} = 0$$

by virtue of the conditions of the lemma and the inequality

$$x^\beta \mathbf{P}\{\xi_1 > x\} \leqslant \mathbf{M}\xi_1^\beta < \infty.$$

The integrand on the right of Eq. (2) tends as $x \to \infty$, to Cy^α, so that it is only necessary to justify the transition to the limit under the integral sign.

We shall construct a majorant.

If $0 < y < 1$, then $\frac{\mathbf{P}\{\psi_1 y > x\}}{x^{-\alpha}L(x)} \leqslant \frac{\mathbf{P}\{\psi_1 > x\}}{x^{-\alpha}L(x)} < C_4$ for all x greater than some fixed value.

If, however, $1 \leqslant y < \frac{x}{N}$, we then use the representation of a slowly varying function (see W. Feller [4], Chap. 8, §9),

$$L(x) = a(x)\exp\left\{\int_1^x \frac{\varepsilon(t)}{t}\,dt\right\},$$

where $a(x) \to C_3 > 0$, $\varepsilon(x) \to 0$ as $x \to \infty$. We can choose positive number N such that $x \geqslant N$, $\frac{C_3}{2} < a(x) < 2C_3$, while $|\varepsilon(x)| < \frac{\beta-\alpha}{2}$. Now we have

$$\frac{\mathbf{P}\{\psi_1 y > x\}}{x^{-\alpha}L(x)} = \frac{\mathbf{P}\left\{\psi_1 > \frac{x}{y}\right\}}{\left(\frac{x}{y}\right)^{-\alpha} L\left(\frac{x}{y}\right)} \cdot y^\alpha \cdot \frac{L\left(\frac{x}{y}\right)}{L(x)} \leqslant C_4 y^\alpha \frac{L\left(\frac{x}{y}\right)}{L(x)} \leqslant$$

$$\leqslant 4C_4 y^\alpha \exp\left\{-\int_{\frac{x}{y}}^{x} \frac{\varepsilon(t)}{t}\,dt\right\} \leqslant 4C_4 y^\alpha \exp\left\{\int_{\frac{x}{y}}^{x} \frac{dt}{t}\cdot\frac{\beta-\alpha}{2}\right\} = 4C_4 y^{\alpha - \frac{\beta-\alpha}{2}} \leqslant 4C_4 y^\beta.$$

Thus,

$$\frac{\mathbf{P}\{\psi_1 y > x\}}{x^{-\alpha}L(x)} < g(y),$$

where

$$g(y)=\begin{cases} C_4, & \text{if } \quad 0<y<1, \\ 4C_4 y^{\beta}, & \text{if } \quad y\geqslant 1 \end{cases}$$

for all x starting with some fixed one and y < x/N, so that, on the right of Eq. (2), we can go to the limit under the integral sign. Equation (1) is proven.

Furthermore, if x > 1, while 1 > ε > 0, then, on one hand,

$$\begin{aligned}&\mathbf{P}\{\eta_1+\xi_1\psi_1>x\}\geqslant\mathbf{P}\{\eta_1>x(1+\varepsilon),\ |\xi_1\psi_1|<x\varepsilon\}+\\&+\mathbf{P}\{|\eta_1|<x\varepsilon,\ \xi_1\psi_1>x(1+\varepsilon)\}=\mathbf{P}\{\eta_1>x(1+\varepsilon)+\\&+\mathbf{P}\{\xi_1\psi_1>x(1+\varepsilon)\}-\mathbf{P}\{\eta_1>x(1+\varepsilon),\ |\xi_1\psi_1|\geqslant x\varepsilon\}-\\&-\mathbf{P}\{|\eta_1|>x\varepsilon,\ \xi_1\psi_1>x(1+\varepsilon)\}.\end{aligned}$$

Since

$$\frac{\mathbf{P}\{\eta_1>x(1+\varepsilon),\ |\xi_1\psi_1|\geqslant x\varepsilon\}}{x^{-\alpha}L(x)}\leqslant\frac{\mathbf{P}\{\xi_1>x^{\frac{\alpha+\beta}{2\beta}}\}}{x^{-\alpha}L(x)}+\frac{\mathbf{P}\{\eta_1>x(1+\varepsilon)\}\cdot\mathbf{P}\{|\psi_1|\geqslant x^{\frac{\beta-\alpha}{2\beta}}\varepsilon\}}{x^{-\alpha}L(x)}\to 0$$

as x → ∞, and analogously

$$\frac{\mathbf{P}\{|\eta_1|>x\varepsilon,\ \xi_1\psi_1>x(1+\varepsilon)\}}{x^{-\alpha}L(x)}\to 0,$$

then

$$\limsup_{x\to\infty}\frac{\mathbf{P}\{\eta_1+\xi_1\psi_1>x\}}{x^{-\alpha}L(x)}\geqslant\frac{1+C\mathbf{M}\xi_1^{\alpha}}{(1+\varepsilon)^{\alpha}}. \tag{3}$$

On the other hand,

$$\mathbf{P}\{\eta_1+\xi_1\psi_1>x\}\leqslant\mathbf{P}\{\eta_1>x(1-\varepsilon)\}+\mathbf{P}\{\xi_1\psi_1>x(1-\varepsilon)\}+\mathbf{P}\{\eta_1>x\varepsilon,\ \xi_1\psi_1>x\varepsilon\},$$

whence, we obtain an inequality contrary to (3) (see the proof analogous to the one here: W. Feller [4, Chap. 8, §9]). Proof of Lemma 1 is completed.

LEMMA 2. Let $\mathbf{M}\xi_1^{\alpha}<1$, $\alpha>0$. Then, $\mathbf{M}\ (1+\xi_1+\ldots+\xi_1\cdot\ldots\cdot\xi_n+\ldots)^{\alpha}<\infty$.

Proof. If $\alpha\geqslant 1$, then, by virtue of the triangle inequality in space L_{α},

$$\mathbf{M}(1+\xi_1+\ldots+\xi_1\cdot\ldots\cdot\xi_n)^{\alpha}\leqslant\left[1+(\mathbf{M}\xi_1^{\alpha})^{\frac{1}{\alpha}}+\ldots+(\mathbf{M}\xi_1^{\alpha})^{\frac{n}{\alpha}}\right]^{\alpha};$$

consequently the series $1+\xi_1+\ldots+\xi_1\cdot\ldots\cdot\xi_n+\ldots$ converges in metric L_{α} and

$$\mathbf{M}(1+\xi_1+\ldots+\xi_1\cdot\ldots\cdot\xi_n+\ldots)^{\alpha}\leqslant\left[1+(\mathbf{M}\xi_1^{\alpha})^{\frac{1}{\alpha}}+\ldots+(\mathbf{M}\xi_1^{\alpha})^{\frac{n}{\alpha}}+\ldots\right]^{\alpha}<\infty.$$

If α < 1 then

$$\mathbf{M}(1+\xi_1+\ldots+\xi_1\cdot\ldots\cdot\xi_n+\ldots)^{\alpha}\leqslant\mathbf{M}(1+\xi_1^{\alpha}+\ldots+\xi_1^{\alpha}\cdot\ldots\cdot\xi_n^{\alpha}+\ldots)<\infty.$$

Proof of Theorem 1. By induction as x → ∞

$$\mathbf{P}\{\eta_1+\eta_2\xi_1+\ldots+\eta_n\xi_1\cdot\ldots\cdot\xi_{n-1}>x\}\sim x^{-\alpha}L(x)\,[1+\mathbf{M}\xi_1^{\alpha}+\ldots+(\mathbf{M}\xi_1^{\alpha})^{n-1}]. \tag{4}$$

Since $\mathbf{M}\xi_1^{\alpha}<1$ and $\mathbf{M}\xi_1^{\beta}<\infty$, we can find real numbers $\delta\in(0,\alpha)$ and b > 1 such that $\mathbf{M}(b\xi_1)^{\alpha+\delta}<1$.

Furthermore,

$$\begin{aligned}&\mathbf{P}\{\eta_1+\eta_2\xi_1+\ldots+\eta_n\xi_1\cdot\ldots\cdot\xi_{n-1}+\ldots>x\}\leqslant\\&\leqslant\sum_{k=1}^{\infty}\mathbf{P}\left\{\eta_k\xi_1\cdot\ldots\cdot\xi_{k-1}>\frac{b-1}{b^k}x,\ \xi_1\cdot\ldots\xi_{k-1}b^{k-1}\leqslant\frac{(b-1)x}{bN}\right\}+\\&+\mathbf{P}\left\{\sup(1,\ b\xi_1,\ \ldots,\ b^{k-1}\xi_1\cdot\ldots\cdot\xi_{k-1},\ \ldots)>\frac{x(b-1)}{Nb}\right\}.\end{aligned}$$

By Lemma 2,

$$\frac{\mathbf{P}\left\{\sup(1,\ b\xi_1,\ \ldots,\ b^k\xi_1\cdot\ldots\cdot\xi_k,\ \ldots)>\frac{x(b-1)}{Nb}\right\}}{x^{-\alpha}L(x)}\leqslant\frac{\mathbf{P}\left\{1+b\xi_1+\ldots+b^k\xi_1\cdot\ldots\cdot\xi_k+\ldots>\frac{x(b-1)}{Nb}\right\}}{x^{-\alpha}L(x)}=\frac{O(x^{-\alpha-\delta})}{x^{-\alpha}L(x)}\to 0$$

as $x \to \infty$.

Just the same as in the proof of Lemma 1, we can choose number $N > 0$ such that, when $x > N$ and $\frac{b-1}{b^k} \leqslant y \leqslant \frac{x(b-1)}{Nb^k}$, the following inequality is met

$$\frac{\mathbf{P}\left\{\eta_k y > \frac{b-1}{b^k} x\right\}}{x^{-\alpha} L(x)} \leqslant 4 C_4 \left(\frac{yb^k}{b-1}\right)^{\alpha+\delta},$$

while, if $0 < y < b-1/b^k$,

$$\frac{\mathbf{P}\left\{\eta_k y > \frac{b-1}{b^k} x\right\}}{x^{-\alpha} L(x)} \leqslant 4 C_4 \left(\frac{yb^k}{b-1}\right)^{\alpha-\delta}.$$

Therefore,

$$\limsup_{x\to\infty} \frac{\mathbf{P}\{\eta_1+\eta_2\xi_1+\ldots+\eta_n\xi_1\cdot\ldots\cdot\xi_{n-1}+\ldots>x\}}{x^{-\alpha}L(x)} \leqslant$$

$$\leqslant \limsup_{x\to\infty} \sum_{k=1}^{\infty} \int_0^{\frac{x(b-1)}{Nb^k}} \frac{\mathbf{P}\left\{\eta_k y > \frac{b-1}{b^k} x\right\}}{x^{-\alpha} L(x)} F_{\xi_1\cdot\ldots\cdot\xi_{k-1}}\{dy\} \leqslant$$

$$\leqslant \sum_{k=1}^{\infty} 4C_4\left[\mathbf{M}\left(\frac{\xi_1\cdot\ldots\cdot\xi_{k-1}b^k}{b-1}\right)^{\alpha+\delta} + \mathbf{M}\left(\frac{\xi_1\cdot\ldots\cdot\xi_{k-1}b^k}{b-1}\right)^{\alpha-\delta}\right] = C_5 < \infty. \tag{5}$$

Using (4) and (5) we can easily show (just as in the proof of Lemma 1) that, for any natural n

$$\limsup_{x\to\infty} \frac{\mathbf{P}\{\eta_1+\eta_2\xi_1+\ldots+\eta_n\xi_1\cdot\ldots\cdot\xi_{n-1}+\ldots>x\}}{x^{-\alpha}L(x)} \leqslant$$

$$\leqslant [1+\mathbf{M}\xi_1^\alpha+\ldots+(\mathbf{M}\xi_1^\alpha)^{n-1}+C_5(\mathbf{M}\xi_1^\alpha)^n],$$

while

$$\liminf_{x\to\infty} \frac{\mathbf{P}\{\psi>x\}}{x^{-\alpha}L(x)} \geqslant [1+\mathbf{M}\xi_1^\alpha+\ldots+(\mathbf{M}\xi_1^\alpha)^{n-1}].$$

Whence also follows the assertion of Theorem 1.

Theorem 2 is of great interest since its conditions automatically hold if the bearer of random element $\begin{pmatrix}\xi_1 & \eta_1\\ 0 & 1\end{pmatrix}$ is lumped in some compact group if linear transformations of the line (in representation as the group of all real matrices of the form $\begin{pmatrix}X & Y\\ 0 & 1\end{pmatrix}$ $X>0$), $\mathbf{P}\{\xi_1>1\}>0$ and (as is necessary for the convergence almost certainly of the series ψ) $\mathbf{M} \ln \xi_1 < 0$.

We mention also that when there hold, both the conditions of Theorem 1 and the conditions of Theorem 2, series ψ converges almost certainly (see A. Grincevičius [6]).

Proof of Theorem 2. We have $\psi = \eta_1 + \xi_1\psi_1$, where random variable ψ_1 is distributed the same as ψ and does not depend on the random matrix $\begin{pmatrix}\xi_1 & \eta_1\\ 0 & 1\end{pmatrix}$. Therefore, for $x > 0$,

$$\mathbf{P}\{\psi>x\}=\mathbf{P}\{\xi_1\psi_1>x\}+\mathbf{P}\{x-\eta_1<\xi_1\psi_1\leqslant x\}-\mathbf{P}\{x<\xi_1\psi_1\leqslant x-\eta_1\}. \tag{6}$$

From Eq. (6) by an obvious transformation we obtain that, when $x > 0$,

$$\frac{1}{x}\int_0^x t^\beta \mathbf{P}\{\psi>t\}\,dt=\frac{1}{x}\int_0^x t^\beta \mathbf{P}\{\xi_1\psi_1>t\}\,dt+$$

$$+\frac{1}{x}\int_0^x t^\beta(\mathbf{P}\{t-\eta_1<\xi_1\psi_1\leqslant t\}-\mathbf{P}\{t<\xi_1\psi_1\leqslant t-\eta_1\})\,dt. \tag{7}$$

We use the notation

$$Z(x)=\frac{1}{x}\int_0^x t^\beta \mathbf{P}\{\psi>t\}\,dt$$

and

$$z(x)=\frac{1}{x}\int_0^x t^\beta[\mathbf{P}\{t-\eta_1<\xi_1\psi_1\leqslant x\}-\mathbf{P}\{t<\xi_1\psi_1\leqslant t-\eta_1\}]\,dt;$$

and then identity (7) gives

$$Z(x)=\int_0^\infty Z\left(\frac{x}{y}\right)y^\beta F_\xi\{dy\}+z(x). \tag{8}$$

It is easy to see that identity (8) is the renewal equation (see W. Feller [4]) for the multiplicative group of positive real numbers. By the change of variables $x = e^s$, we take this renewal equation to the usual form

$$Z(e^s)=\int_{-\infty}^{\infty} Z(e^{s-t})\,F\{dt\}+z(e^s), \tag{9}$$

where the probability distribution function is

$$F(t)=\int_{-\infty}^{t} e^{t\beta}F_{\ln\xi_1}\{dt\}.$$

(We note that $F(+\infty)=\int_{-\infty}^{+\infty} e^{t\beta}F_{\ln\xi_1}\{dt\}=\mathbf{M}\xi_1^\beta=1.$)

We now show that function $z(e^s)$ is directly Riemann-integrable (see W. Feller [4]). Let δ be a positive number less than 1 such that $\delta\alpha > \beta$; then

$$0\leqslant\int_0^x t^\beta\mathbf{P}\{t-\eta_1<\xi_1\psi_1\leqslant t\}\,dt\leqslant\int_0^x t^\beta\mathbf{P}\{\eta_1>t^\delta\}\,dt+\int_0^x t^\beta\mathbf{P}\{t-t^\delta<\xi_1\psi_1\leqslant t\}\,dt. \tag{10}$$

We now estimate each of the integrals on the right of inequality (10):

$$\int_0^x t^\beta\mathbf{P}\{\eta_1>t^\delta\}\,dt=O\left(\int_1^x t^\beta t^{-\delta\alpha}\,dt+1\right)=o(x^{1-\varepsilon_1})\quad\text{as}\quad x\to\infty$$

for some positive number $\varepsilon_1 < 1$.

Since

$$\mathbf{M}|\eta_1|^\beta=\beta\int_0^\infty x^{\beta-1}\mathbf{P}\{|\eta_1|>x\}\,dx=O\left(1+\int_1^\infty x^{\beta-1}x^{-\alpha}\,dx\right)=O(1)$$

and, for any positive number $\varepsilon_2<\beta$ $\mathbf{M}\xi_1^{\beta-\varepsilon_2}<1$, then, just as in Lemma 2, we readily show that $\mathbf{M}|\xi_1\psi_1|^{\beta-\varepsilon_2}<\infty$, so that, consequently,

$$\mathbf{P}\{\xi_1\psi_1>x\}=o(x^{-\beta+\varepsilon_2})\quad\text{as}\quad x\to\infty.$$

Now, using the previous inequality, the inequality

$$\begin{aligned}t^\beta-(t-t^\delta)^\beta(1-\delta t^{\delta-1})&=\delta(t-t^\delta)^\beta t^{\delta-1}+t^\beta-(t-t^\delta)^\beta=\\&=\delta(t-t^\delta)^\beta t^{\delta-1}+(y^\beta)'_{y=t-\Theta t^\delta}\cdot t^\delta=O(t^{\beta+\delta-1}),\end{aligned}$$

when $t\geqslant x_0$, for positive number x_0 such that, when $x\geqslant x_0$, $x^\delta\leqslant\frac{x}{2}$, and, making the change of variable $t = y - y^\delta$ in one of the following integrals, we obtain

$$\int_0^x t^\beta\mathbf{P}\{t-t^\delta<\xi_1\psi_1\leqslant t\}\,dt=\int_0^x t^\beta\mathbf{P}\{\xi_1\psi_1>t-t^\delta\}\,dt-$$

$$-\int_0^{x-x^\delta} t^\beta \mathbf{P}\{\xi_1\psi_1>t\}\,dt-\int_{x-x^\delta}^{x} t^\beta \mathbf{P}\{\xi_1\psi_1>t\}\,dt\leqslant$$

$$\leqslant\int_0^x t^\beta \mathbf{P}\{\xi_1\psi_1>t-t^\delta\}\,dt-\int_1^x (y-y^\delta)^\beta \mathbf{P}\{\xi_1\psi_1>y-y^\delta\}(1-\delta y^{\delta-1})dy=$$

$$=O(1)+\int_1^x [t^\beta-(t-t^\delta)^\beta(1-\delta t^{\delta-1})]\,\mathbf{P}\{\xi_1\psi_1>t-t^\delta\}\,dt=$$

$$=O\left(1+\int_{x_0}^x t^{\beta+\delta-1}\mathbf{P}\{\xi_1\psi_1>t-t^\delta\}\,dt\right)=O\left(1+\int_{x_0}^x t^{\beta+\delta-1}\cdot t^{-\beta+\varepsilon_1}dt\right)=$$

$$=O(1+x^{\delta+\varepsilon_1}).$$

Thus, we have shown that, for some positive number,

$$\varepsilon_3\int_0^x t^\beta \mathbf{P}\{t-\eta_1<\xi_1\psi_1\leqslant t\}\,dt=o(x^{1-\varepsilon_3})\quad\text{as}\quad x\to\infty.$$

Similarly, we estimate the integral

$$\int_0^x t^\beta \mathbf{P}\{t<\xi_1\psi_1\leqslant t-\eta_1\}\,dt,$$

therefore, by virtue of the definition of function $z(x)$ we find that, as $x\to\infty$, for some positive number ε_4

$$z(x)=o(x^{-\varepsilon_4}).\tag{11}$$

On the other hand, as $x\to 0$,

$$|z(x)|\leqslant\frac{2\int_0^x t^\beta\,dt}{x}=\frac{2x^\beta}{\beta+1}.\tag{12}$$

From inequalities (11) and (12) readily follows the direct Riemann-integrability of the function $z(e^s)$, so that, from the renewal theorem on the entire line it now follows (see W. Feller [4, Chap. 11, §9, Theorem 1], and the proof of Theorem 2 in Chap. 11, §1) that if the distribution of random variable $\ln\xi_1$ is nonarithmetic then, as $s\to\infty$,

$$Z(e^s)\to\frac{1}{\mu}\int_{-\infty}^{\infty} z(e^s)\,ds=C_1,$$

where

$$\mu=\int_{-\infty}^{\infty} tF\{dt\}=\mathbf{M}\xi_1^\beta\ln\xi_1<\infty.$$

Whence,

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x t^\beta \mathbf{P}\{\psi>t\}\,dt=C_1$$

and, consequently,

$$\lim_{x\to\infty}\frac{\mathbf{P}\{\psi>x\}}{x^{-\beta}}=C_1$$

(see W. Feller [4, Chap. 13, §5, proof of lemma] or G. Lev [7, p. 51]).

We now assume that random variable η_1 is (almost certainly) nonnegative and, with positive probability, is greater than zero. In this case, Eq. (6) takes the form

$$\mathbf{P}\{\psi>x\}=\mathbf{P}\{\xi_1\psi_1>x\}+\mathbf{P}\{x-\eta_1<\xi_1\psi_1\leqslant x\}.\tag{13}$$

It is not difficult to obtain

$$\mu C_1 = \int_{-\infty}^{\infty} \frac{1}{e^s} \left[\int_0^{e^s} t^\beta \mathbf{P}\{t-\eta_1 < \xi_1 \psi_1 \leqslant t\} dt\right] ds = \lim_{N\to\infty} \int_0^N \frac{1}{x^2} \left[\int_0^x t^\beta \mathbf{P}\{t-\eta_1 < \xi_1 \psi_1 \leqslant t\} dt\right] dx =$$

$$= \lim_{N\to\infty} \int_0^N \int_t^N \frac{1}{x^2} dx\, t^\beta \mathbf{P}\{t-\eta_1 < \xi_1 \psi_1 \leqslant t\} dt = \lim_{N\to\infty} \int_0^N t^{\beta-1} \mathbf{P}\{t-\eta_1 < \xi_1 \psi_1 \leqslant t\} dt -$$

$$- \lim_{N\to\infty} \frac{1}{N} \int_0^N t^\beta [\mathbf{P}\{\psi > t\} - \mathbf{P}\{\xi_1 \psi_1 > t\}] dt = \int_0^\infty t^{\beta-1} \mathbf{P}\{t-\eta_1 < \xi_1 \psi_1 \leqslant t\} dt,$$

therefore equation $C_1 = 0$ means identically equal to zero for all $x > 0$

$$\mathbf{P}\{x-\eta_1 < \xi_1 \psi_1 \leqslant x\} \equiv 0$$

and, consequently, $\mathbf{P}\{\psi > x\} \equiv P\{\xi_1 \psi_1 > x\}$ for all $x > 0$.

The last identity can hold only if $\mathbf{P}\{\psi > 0\} = 0$, or $\mathbf{P}\{\xi_1 = 1\} = 1$, and both these equations contradict the conditions of the theorem.

If $\mathbf{P}\{\eta_1 \geqslant c(1-\xi_1)\} = 1$ and $\mathbf{P}\{\eta_1 > c(1-\xi_1)\} > 0$, then, obviously, $\psi = c + [\eta_1 - c(1-\xi_1)] + [\eta_2 - c(1-\xi_2)]\xi_1 + [\eta_3 - c(1-\xi_3)]\xi_1\xi_2 + \ldots$ and, consequently,

$$\lim_{x\to\infty} \frac{\mathbf{P}\{\psi > x\}}{x^{-\beta}} = \lim_{x\to\infty} \frac{\mathbf{P}\{\psi - c > x\}}{x^{-\beta}} = C_1 > 0;$$

with this selection, the case of a nonarithmetic distribution of the random variable $\ln \xi_1$ is completely finished.

We now turn to the case of an arithmetic distribution of random variable $\ln \xi_1$ with step λ. The renewal equation of (9) gives, for any real number h,

$$\lim_{n\to\infty} Z(e^{h+\lambda n}) = \frac{1}{\mu} \sum_{k=-\infty}^{\infty} z(e^{h+\lambda k}) = C(h). \tag{14}$$

For two real numbers $h_1 \geqslant h_2$ we have

$$e^{h_1} Z(e^{h_1+\lambda n}) - e^{h_2} Z(e^{h_2+\lambda n}) \to e^{h_1} C(h_1) - e^{h_2} C(h_2)$$

as $n \to \infty$, or

$$\lim_{n\to\infty} e^{-hn} \int_{e^{h_2+\lambda n}}^{e^{h_1+\lambda n}} t^\beta \mathbf{P}\{\psi > t\} dt = e^{h_1} C(h_1) - e^{h_2} C(h_2).$$

By the change of variable $t = e^{\lambda n} \cdot s$ we obtain

$$\lim_{n\to\infty} \int_{e^{h_2}}^{e^{h_1}} e^{\lambda n \beta} s^\beta \mathbf{P}\{\psi > e^{\lambda n} \cdot s\} ds = e^{h_1} C(h_1) - e^{h_2} C(h_2). \tag{15}$$

By virtue of the monotonicity of function $\mathbf{P}\{\psi > x\}$ it follows from this that, for fixed s, the integrand in (15) remains bounded. In particular, there exists a positive number C_3 such that

$$x^\beta \mathbf{P}\{\psi > x\} < C_3 \quad \text{for all} \quad x > 0. \tag{16}$$

By the selection theorem (see W. Feller [4]), there exists a sequence $n_1, n_2, \ldots \to \infty$ such that, when n runs through it, then $e^{\lambda n \beta} s^\beta \mathbf{P}\{\psi > e^{\lambda n} s\} \to q(s)$ at all points of continuity of the latter. It follows from (15) that

$$\int_{e^{h_2}}^{e^{h_1}} q(s) ds = e^{h_1} C(h_1) - e^{h_2} C(h_2). \tag{17}$$

As the limit of nonincreasing functions, function $q(s)/s^\beta$ is also nonincreasing, so that, consequently, function q(s) has no more than a countable number of points of discontinuity. Equation (17) uniquely defines function q(s) at all points of continuity, which entails its independence on the selection of sequence $n_1, n_2, \ldots$. From this we conclude that, with the possible exception of a countable number of values of h, there exists the limit

$$e^{(h+\lambda n)\beta} \mathbf{P}\{\psi > e^{h+\lambda n}\} \to q(e^h) = C_2(h) \text{ as } n \to \infty.$$

If, now, η_1 is a nonnegative random variable, then the strict positivity of function $C_2(h)$ is proven the same as in the case of the nonarithmetic distribution of random variable $\ln \xi_1$. We now show that the limit exists for all real h.

From identity (13) we readily obtain the renewal equation

$$Z_1(e^s) = \int_{-\infty}^{\infty} Z_1(e^{s-t}) F\{dt\} + z_1(e^s),$$

where

$$Z_1(x) = x^\beta \mathbf{P}\{\psi > x\}, \text{ while } z_1(x) = x^\beta \mathbf{P}\{x - \eta_1 < \xi_1 \psi_1 \leqslant x\} =$$
$$= x^\beta [\mathbf{P}\{\psi > x\} - \mathbf{P}\{\xi_1 \psi_1 > x\}].$$

We should emphasize the difference between the renewal equations of (18) and (9), amounting to the following, that function $z(e^s)$ of Eq. (9) is directly Riemann-integrable which has not been proven for function $z_1(e^s)$ of Eq. (18).

However, in the case of an arithmetic distribution of random variable $\ln \xi_1$, for justifying the convergence of $\lim_{n\to\infty} Z_1(e^{h+\lambda n}) = C_1(h)$ it suffices to prove only the (absolute) convergence of the series

$$\sum_{k=-\infty}^{\infty} z_1(e^{h+\lambda k}),$$

and, by virtue of the nonnegativity of the terms of the series, it suffices to show the boundedness (uniformly in N) of the partial sums

$$\sum_{k=-\infty}^{N} z_1(e^{h+\lambda k}), \quad N = 1, 2, \ldots .$$

We now prove this. We have

$$\sum_{k=-\infty}^{N} z_1(e^{h+\lambda k}) = \sum_{k=-\infty}^{N} e^{\beta(h+\lambda k)} [\mathbf{P}\{\psi > e^{h+\lambda k}\} - \mathbf{P}\{\psi_1 \xi_1 > e^{h+\lambda k}\}] =$$
$$= \sum_{k=-\infty}^{N} e^{\beta(h+\lambda k)} \mathbf{P}\{\psi > e^{h+\lambda k}\} - \sum_{k=-\infty}^{N} e^{\beta(h+\lambda k)} \mathbf{P}\left\{\psi_1 > e^{h+\lambda\left(k - \frac{\ln \xi_1}{\lambda}\right)}\right\}.$$

Changing the index of summation in the second sum we obtain

$$\sum_{k=-\infty}^{N} z_1(e^{h+\lambda k}) = \sum_{k=-\infty}^{N} e^{\beta(h+\lambda k)} \mathbf{P}\{\psi > e^{h+\lambda k}\} -$$
$$- \mathbf{M} \sum_{k=-\infty}^{N - \frac{\ln \xi_1}{\lambda}} e^{\beta(h+\lambda k+\ln \xi_1)} \mathbf{P}\{\psi > e^{h+\lambda k}\} =$$
$$= \sum_{k=-\infty}^{N} e^{\beta(h+\lambda k)} \mathbf{P}\{\psi > e^{h+\lambda k}\} - \mathbf{M}\xi_1^\beta \sum_{k=-\infty}^{N} e^{\beta(h+\lambda k)} \mathbf{P}\{\psi > e^{h+\lambda k}\} +$$
$$+ \mathbf{M} \sum_{k=-\infty}^{N} e^{\beta(h+\lambda k+\ln \xi_1)} \mathbf{P}\{\psi > e^{h+\lambda k}\} -$$

$$-\mathbf{M}\sum_{k=-\infty}^{N-\frac{\ln\xi_1}{\lambda}} e^{\beta(h+\lambda k+\ln\xi_1)}\mathbf{P}\{\psi>e^{h+\lambda k}\}=$$

$$=\mathbf{M}\Big[\sum_{k=-\infty}^{N} e^{\beta(h+\lambda k+\ln\xi_1)}\mathbf{P}\{\psi>e^{h+\lambda k}\}-$$

$$-\sum_{k=-\infty}^{N-\frac{\ln\xi_1}{\lambda}} e^{\beta(h+\lambda k+\ln\xi_1)}\mathbf{P}\{\psi>e^{h+\lambda k}\}\Big]=$$

$$=\mathbf{M}\sum_{N-\frac{\ln\xi_1}{\lambda}<k\leqslant N} e^{\beta(h+\lambda k+\ln\xi_1)}\mathbf{P}\{\psi>e^{h+\lambda k}\}-$$

$$-\mathbf{M}\sum_{N<k\leqslant N-\frac{\ln\xi_1}{\lambda}} e^{\beta(h+\lambda k+\ln\xi_1)}\mathbf{P}\{\psi>e^{h+\lambda k}\}.$$

Using inequality (16), we find that the right side of the last equation is no greater than

$$C_3\mathbf{M}\sum_{N-\frac{\ln\xi_1}{\lambda}<k\leqslant N}\xi_1^\beta=C_3\mathbf{M}\frac{\ln^+\xi_1}{\lambda}\xi_1^\beta<\infty$$

by virtue of the conditions of the theorem.

The periodicity of function $C_2(h)$ is obvious. With this, Theorem 2 is completely proven.

With weaker conditions than in Theorem 2, we have the true

Assertion 1. Let $\mathbf{M}\xi_1^\beta=1$, $\mathbf{P}\{\eta_1>x\}=0(x^{-\alpha})$ as $x\to\infty$, where $0<\beta<\alpha$ and random variable ξ_1 is nondegenerate. Then, $\mathbf{P}\{\psi>x\}=0(x^{-\beta})$ for all $x>0$.

Assertion 1 is an immediate corollary to Theorem 2 and its proof.

We note that constant C_1 and function $C_2(h)$ in Theorem 2 can be identically equal to zero; more than that, the case is possible when $\psi<N$ almost certainly for some real number $N<\infty$.

Assertion 2. Let random series ψ converge almost everywhere. In order that $\mathbf{P}\{\psi>N\}=0$ for some real number $N<\infty$, it is necessary and sufficient that there exist real numbers $N_1\leqslant N$ such that $\mathbf{P}\{\eta_1+N_1\xi_1\leqslant N_1\}=1$.

Proof. Let $\mathbf{P}\{\psi>N\}=0$. We assume that $\mathbf{P}\{\eta_1+\xi_1N_1>N_1\}>0$ where N_1 is an exact upper bound to the carrier of random variable ψ. Then, there exist positive numbers Q and ε such that $\mathbf{P}\{\eta_1+\xi_1N_1>N_1+\varepsilon,\ \xi_1<Q\}>0$, while, since random elements $\begin{pmatrix}\xi_1 & \eta_1\\ 0 & 1\end{pmatrix}$ and ψ_1 are independent and

$$\mathbf{P}\Big\{\psi_1>N_1-\frac{\varepsilon}{2Q}\Big\}=\mathbf{P}\Big\{\psi>N_1-\frac{\varepsilon}{2Q}\Big\}>0,$$

then also

$$\mathbf{P}\Big\{\psi>N_1+\frac{\varepsilon}{2}\Big\}=\mathbf{P}\Big\{\eta_1+\xi_1\psi_1>N_1+\frac{\varepsilon}{2}\Big\}\geqslant$$
$$\geqslant\mathbf{P}\Big\{\eta_1+\xi_1\Big(\psi_1+\frac{\varepsilon}{2Q}\Big)>N_1+\varepsilon,\ \xi_1<Q\Big\}\geqslant$$
$$\geqslant\mathbf{P}\{\eta_1+\xi_1N_1>N_1+\varepsilon,\ \xi_1<Q\}\cdot\mathbf{P}\Big\{\psi_1+\frac{\varepsilon}{2Q}>N_1\Big\}>0.$$

We have arrived at a contradiction, so that $\mathbf{P}\{\eta_1+\xi_1N_1\leqslant N_1\}=1$. The converse assertion is obvious. Assertion 2 has been proven.

We study analogously the asymptotic behavior of $\mathbf{P}\{\psi < x\}$ as $x \to -\infty$.

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