1. Multi dimensional random walk

Definition 1 (Type II random walk in $\mathbb{Z}^m$)
Let $m \in \mathbb{N}$. $\forall n \in \mathbb{N}$, let $X_n = \left(x_1^x \quad x_2^x \quad \dots, x_m^x\right)^T$, where $\{x_i^x\}_{i=1}^m$ are $\forall n \in \mathbb{N}$ independent. ***check english*** *NEVÍM JAK TO NAPSAT LÍP Let $\forall i \in \{1, 2, \dots, m\} x_i^x$ have values in $\{-1, +1\}$ with probabilities $\mathsf{P}(x_i^x = +1) = p_i \in (0, 1)$ and $\mathsf{P}(x_i^x = -1) = q_i = 1 - p_i \in (0, 1)$.*

Let $\{X_n\}_{n=0}^{+\infty}$ be a sequence of independent and identically distributed random variables. Let $S_0 = \mathbf{0}$ and $\forall n \in \mathbb{N} : \mathbf{S_n} = \sum\limits_{i=1}^{n} X_i$ and $\mathbf{p} = \left(p_1, p_2, \dots, p_m\right)^{\mathbf{T}}$. Then the pair $\left(\{\mathbf{S_n}\}_{n=0}^{+\infty}, \mathbf{p}\right)$ is called Type II random walk in $\mathbb{Z}^m$.

If $\forall i \in \{1, 2, \dots, m\} : p_i = q_i = \frac{1}{2}$ we call the ***check english*** *element $\{\mathbf{S_n}\}_{n=0}^{+\infty}$* Symmetric type II random walk $\mathbb{Z}^m$.

Remark. Type II random walk can be interpreted as m simple random walks in $\mathbb{Z}$ happening at a time, each of them parallel to an axis of $\mathbb{Z}^m$.

Theorem 1
Let $m \in \mathbb{N}$ and $\left(\{\mathbf{S_n}\}_{n=0}^{+\infty}, \mathbf{p}\right)$ be a Type II random walk in $\mathbb{Z}^m$. Let $\mathbf{y} = \left(y_1, y_2, \dots, y_m\right)^T \in \mathbb{Z}^m$. Then following equation stands:

$$\mathsf{P}(S_n = x) = \begin{cases} \prod_{i=1}^{m} \binom{n}{\frac{y_i+n}{2}} p_i^{\frac{n+y_i}{2}} q_i^{\frac{n-y_i}{2}}, & \text{if } \forall i \in \{1, 2, \dots, m\} : y_i \in A_n, \\ 0, & \text{if } \exists i \in \{1, 2, \dots, m\} : y_i \notin A_n. \end{cases}$$

Proof. $\mathsf{P}(\mathbf{S_n} = \mathbf{y}) = \mathsf{P}(S_1^x = y_1, S_2^x = y_2, \dots, S_m^x = y_m) \overset{\perp\!\!\!\perp}{=} \prod\limits_{i=1}^{m} \mathsf{P}(S_n^x = y_i)$
$= \prod\limits_{i=1}^{m} \binom{n}{\frac{y_i+n}{2}} p_i^{\frac{n+y_i}{2}} q_i^{\frac{n-y_i}{2}}$. The second equation comes from the independency of $\{\mathbf{S_i}\}_{i=1}^m$ which comes easily from independency of X_n^i. The second equation comes from the theorem of probability of position and time from the first chapter (**??**). $\square$

Remark. Due to the the aim of this thesis which is reasearching occupation time of a set of random walks we are going to concern only on symmetric random walks.

Definition 2 (Orthant-je to potřeba?)
Let $m \in \mathbb{Z}$. Then $O \subset \mathbb{Z}^m$ is called an open orthant in $\mathbb{Z}^m$ *if $\forall o \in O, \forall i \in \{1, 2, \dots, m\} : o = \left(o_1, o_2, \dots, o_m\right)^T : o_i\varepsilon_i > 0$, where $\varepsilon_i \in \{-1, +1\}$. $C \subset \mathbb{Z}^m$ is called a* closed orthant in $\mathbb{Z}^m$ *if $\forall c \in C, \forall i \in \{1, 2, \dots, m\} : o = \left(o_1, o_2, \dots, o_m\right)^T : o_i\varepsilon_i \geq 0$, where $\varepsilon_i \in \{-1, +1\}$.*

Remark. The statement $\mathbf{x} > \mathbf{y}$ will mean $\forall i \in \{1, 2, \dots, m\} : x_i > y_i$. Same applies to $<, \leq, \geq$.

Theorem 2 (Probability of being in an open orthant)
Let $\{\mathbf{S_n}\}_{n=0}^{+\infty}$ *be a Symmetric type II random walk in* $\mathbb{Z}^m$. *Let* O *be an open orthant in* Z^m. $\mathsf{P}(\mathbf{S_n} \in C) = \left(\frac{1}{2}u_{2n}\right)^m$.

Proof. Without loss of generality we can assume that in the definition of C we choose $\forall i \in \{1, 2, \ldots, m\}\varepsilon_i := +1$ then: $\mathsf{P}(\mathbf{S_n} \in O) = \mathsf{P}(S_1^x > 0, S_2^x > 0, \ldots, S_m^x > 0) = \prod_{i=1}^m \mathsf{P}(S_i^x > 0) = (\mathsf{P}(S_n^x > 0))^m = \left(\frac{1}{2}u_{2n}\right)^m$. Where the last equation comes from Lemma **??**. □

Theorem 3 (Probability of being in a closed orthant)
Let $\{\mathbf{S_n}\}_{n=0}^{+\infty}$ *be a Symmetric type II random walk in* $\mathbb{Z}^m$. *Let* C *be a closed orthant in* Z^m. $\mathsf{P}(\mathbf{S_n} \in C) = (u_{2n})^m$.

Proof. Without loss of generality we can again assume that in the definition of C we choose $\forall i \in \{1, 2, \ldots, m\}\varepsilon_i := +1$ then: $\mathsf{P}(\mathbf{S_n} \in C) = \mathsf{P}(S_1^x \geq 0, S_2^x \geq 0, \ldots, S_m^x \geq 0) = \prod_{i=1}^m \mathsf{P}(S_i^x \geq 0) = (\mathsf{P}(S_n^x \geq 0))^m = (u_{2n})^m$. Where the last equation comes from Lemma **??**. □