1. Basic definitions

Remark. First, let us properly introduce what a random walk is. After stating some of the basic definitions we will move to the core of this thesis which is to explore properties of occupation of a set times.

Definition 1 (Simple random walk in $\mathbb{Z}$)**.** Let $\{X_n\}_{n=0}^{+\infty}$ be a sequence of independent and identically distributed $\{-1, 1\}$-valued random variables, that for some $p \in (0, 1)$ and for $n \in \mathbb{N}$ satisfy $\mathsf{P}(X_n = 1) = p$ and $\mathsf{P}(X_n = -1) = 1 - p =: q$.

Let $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$. We call the pair $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *Simple random walk in* $\mathbb{Z}$.

If $p = q = \frac{1}{2}$, the pair $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ reduces to the element $\{S_n\}_{n=0}^{+\infty}$ which is called *Symmetric simple random walk in* $\mathbb{Z}$.

Remark. Very often we refer to n as time , X_i as the i-th step and S_n as position of the walk in time n or after n steps. While referring to simple random walk in $\mathbb{Z}$ we refer to $X_i = +1$ as i-th step was rightwards or more often upwards and to $X_i = -1$ as i-th step was leftwards or downwards. If it is not stated otherwise, we will always assume that $S_0 = 0$.

Remark. The most important element in random walks is the probability of being in position x in time n. In order to calculate such probability we have to firstly define what are even possible positions.

For example it is impossible for the random walk to be in position x in time n if $x > n$ simply because there have not been enough steps to make it up to x. It is also impossible (given the preposition that $S_0 = 0$) that after even number of steps the random walk is in odd-numbered position and vice versa. Therefore we define the set of possible positions.

Definition 2 (Set of possible positions)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk. We call the set $A_n = \{z \in \mathbb{Z}; |z| \leq n, \frac{z+n}{2} \in \mathbb{Z}\}$ *set of all possible positions* of random walk $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ in time n.

Theorem 1 (Probability of position x in time n)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a random walk and* A_n *its set of possible positions.*

$$\mathsf{P}(S_n = x) = \begin{cases} \binom{n}{\frac{n+x}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}} & \textit{for } x \in A_n \ , \\ 0, & \textit{for } x \notin A_n. \end{cases}$$

Remark. While having the definition of set of possible positions it is easy to prove the theorem by finding random variable with alternative distribution in each step. By summing them we get a variable with binomial distribution and then we simply modify the result to get desired probability.

Proof. Consider random variables $\mathbf{1}_{[X_i=1]}$, and $\mathbf{1}_{[X_i=-1]}$, and define new random variables $R_n = \sum_{i=1}^{n} \mathbf{1}_{[X_i=1]}, L_n = \sum_{i=1}^{n} \mathbf{1}_{[X_i=-1]}$. The random variable $\mathbf{1}_{[X_i=1]}$ can be interpreted as indicator whether i-th step was rightwards. Then, R_n is number of rightwards steps and L_n is number of leftwards steps. We can easily see that

$R_n + L_n = n$ and $R_n - L_n = S_n$. Therefore we get by adding these two equations $R_n = \frac{S_n+n}{2}$.

Clearly, $\mathbf{1}_{[X_i=1]}$ has alternative distribution with parameter p (*Alt(p)*). Hence, R_n as a sum of independent and identically distributed random variables with distribution *Alt(p)* has binomial distribution with parameters n and p (*Bi(n,p)*). Therefore we get $\mathsf{P}(R_n = x) = \binom{n}{x} p^x q^{n-x}$, where we define binomial coefficients as statet in preface. Finally, for $a \in A_n$ we get

$$\mathsf{P}(S_n = x) = \mathsf{P}\left(R_n = \frac{x+n}{2}\right) = \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}.$$

□

Remark. Following are three simple lemmata that simplify many calculations in the rest of thesis. After proving them we can ask ourselves questions regarding the thesis aims.

Lemma 2 (Spatial homogeneity)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk and $n \in \mathbb{N}, a, b, j \in \mathbb{Z}$. Then for all $b \in \mathbb{Z}$

$$\mathsf{P}(S_n = j \mid S_0 = a) = \mathsf{P}(S_n = j + b \mid S_0 = a + b)$$

Proof. For any $j, a, b \in \mathbb{Z}$ holds

$$\begin{aligned}\mathsf{P}(S_n = j \mid S_0 = a) &= \mathsf{P}\left(\sum_{i=1}^{n} X_i = j - a\right) = \mathsf{P}\left(\sum_{i=1}^{n} X_i = (j+b) - (a+b)\right) \\ &= \mathsf{P}(S_n = j + b \mid S_0 = a + b).\end{aligned}$$

□

Lemma 3 (Temporal homogeneity)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk and $n, m \in \mathbb{N}, a, j \in \mathbb{Z}$. Then for all $m \in \mathbb{N}$

$$\mathsf{P}(S_n = j \mid S_0 = a) = \mathsf{P}(S_{n+m} = j \mid S_m = a)$$

Proof. For any $j, a \in \mathbb{Z}$ and $m \in \mathbb{N}$

$$\begin{aligned}\mathsf{P}(S_n = j \mid S_0 = a) &= \mathsf{P}\left(\sum_{i=1}^{n} X_i = j - a\right) = \mathsf{P}\left(\sum_{i=m+1}^{m+n} X_i = j - a\right) \\ &= \mathsf{P}(S_{n+m} = j \mid S_m = a),\end{aligned}$$

where the second equality follows from identical distribution of $\{X_n\}_{n=1}^{+\infty}$. □

Lemma 4 (Markov property)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk, $n, m \in \mathbb{N}, n \geq m, a_i \in \mathbb{Z}, i \in \mathbb{N}$ such that $\mathsf{P}(S_0 = a_0, S_1 = a_1, \ldots, S_m = a_m) > 0$. Then

$$\mathsf{P}(S_n = j \mid S_0 = a_0, S_1 = a_1, \ldots, S_m = a_m) = \mathsf{P}(S_n = j \mid S_m = a_m)$$

Proof. Because $\{X_n\}_{n=1}^{+\infty}$ is a sequence of independent variables, once S_m is known, then distribution of S_n depends only on steps
$X_{m+1}, X_{m+2}, \ldots X_n$ and therefore cannot depend on any information concerning values $X_1, X_2, \ldots, X_{m-1}$ and accordingly $S_1, S_2, \ldots, S_{m-1}$. □

2. Number of way+maxima

Remark. Once having stated basic definitions we may succeed to ask ourselves questions about occupation of a set times. Let $a > 0$. How many steps up to time n does our walk spend above a (in interval $[a, +\infty)$)? Similarly how many steps does the walk spend in interval $[-a, +\infty)$? We are going to answer these questions in following pages.

Remark. While calculating probabilities in symmetric random walks the fact that $p = q = \frac{1}{2}$ simplifies calculating because in stead of $p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}$ we now have 2^{-n}. Therefore the probabilities only depend on $\binom{n}{\frac{x+n}{2}}$ which can be more generalized as it is in the following definition.

Definition 3 (Number of possible paths)**.** Let $N_n(a, b)$ be *number of posssible paths* of random walk $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ from point $(0, a)$ to point (n, b) and $N_n^x(a, b)$ be number of possible paths from point $(0, a)$ to point (n, b) that visit point (z, x) for some $z \in \{1, 2, \ldots, n\}$.

Theorem 5 (Number of possible paths)
Let $a, b \in \mathbb{Z}, n \in \mathbb{N}$ *then*

$$N_n(a, b) = \binom{n}{\frac{1}{2}(n + b - a)}.$$

Proof. Let us choose a path from the point $(0, a)$ to point (n, b) and let α be number of rightwards steps and β be number of leftwards steps. Then $\alpha + \beta = n$ and $\alpha - \beta = b - a$. By adding these two equations we get that $\alpha = \frac{1}{2}(n + b - a)$. The number of possible paths is the number of ways of picking α rightwards steps from n steps. Therefore we get $N_n(a, b) = \binom{n}{\alpha} = \binom{n}{\frac{1}{2}(n+b-a)}$. $\square$

Theorem 6 (Reflection principle)
Let $a, b \in \mathbb{N}$, *then*

$$N_n^0(a, b) = N_n(-a, b).$$

Proof. Each path from $(0, -a)$ to (n, b) has to intersect $y = 0$-axis at least once at some point. Let k be the time of earliest intersection with x-axis. By reflecting the segment from $(0, -a)$ to $(k, 0)$ in the x-axis and letting the segment from $(k, 0)$ to (n, b) stay the same as it was, we get a path from point $(0, a)$ to (n, b) which visits 0 in time k. Because reflection is a bijective operation on sets of paths, we get the correspondence between the collections of such paths. $\square$

Remark. Following is the definition of return to origin which is a crucial term for this thesis. Let us come back to our question. While calculating the number of steps the walk spends in interval $[a, +\infty)$ we calculate our first passage through a and then set a as a new origin. However, our achievements concerning return to origin are in the latter subchapter.

Definition 4 (Return to origin)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk, $k \in \mathbb{N}$. We say a *return to origin* occurred in time $2k$ if $S_{2k} = 0$. The probability that in time $2k$ occurred a return to origin shall be denoted by u_{2k}. We say that in time $2k$ occurred *first return to origin* if $S_1, S_2, \dots S_{2k-1} \neq 0$ and $S_{2k} = 0$. The probability that in time $2k$ occurred first return to origin shall be denoted by f_{2k}. We define $f_0 := 0$. Let $\alpha_{2n}(2k)$ denote $u_{2k}u_{2n-2k}$.

Theorem 7 (Ballot theorem)
Let $n, b \in N$ Number of paths from point $(0,0)$ to point (n,b) which do not return to origin is equal to $\frac{b}{n}N_n(0,b)$

Proof. Let us call N the number of paths we are referring to. Because the path ends at point (n,b) and it does not return to origin, the first step has to be rightwards. Therefore we now have

$$N = N_{n-1}(1,b) - N_{n-1}^0(1,b) \overset{\text{T6}}{=} N_{n-1}(1,b) - N_{n-1}(-1,b).$$

Hence we get that:

$$\begin{aligned} N &= N_{n-1}(1,b) - N_{n-1}(-1,b) \overset{\text{T5}}{=} \binom{n-1}{\frac{n}{2}+\frac{b}{2}-1} - \binom{n-1}{\frac{n}{2}+\frac{b}{2}} \\ &= \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!} - \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!} \\ &= \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}\right)\left(\frac{n}{2}-\frac{b}{2}-1\right)!} - \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}\right)\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!} \\ &= \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!}\left(\frac{1}{\frac{n}{2}-\frac{b}{2}} - \frac{1}{\frac{n}{2}+\frac{b}{2}}\right) \\ &= \frac{1}{n}\frac{n!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!}\left(\frac{\left(\frac{n}{2}+\frac{b}{2}-\frac{n}{2}+\frac{b}{2}\right)}{\left(\frac{n}{2}-\frac{b}{2}\right)\left(\frac{n}{2}+\frac{b}{2}\right)}\right) \\ &= \frac{b}{n}\frac{n!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!} = \frac{b}{n}\binom{n}{\left(\frac{n}{2}+\frac{b}{2}\right)} \overset{\text{T5}}{=} \frac{b}{n}N_n(0,b) \end{aligned}$$

□

Remark. The name *Ballot theorem* comes from the question: In a ballot where candidate A receives p votes and candidate B receives q votes with $p > q$, what is the probability that A had been strictly ahead of B throughout the whole count?

Answer to this question can be derived from the previous theorem. In our case $b = p - q$ and $n = p + q$.

Definition 5 (Maximum and minimum)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk. $M_n^+ := \max\{S_i, i \in \{1,2,\dots,n\}\}$ is called *maximum of random walk* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *up to time n* and $M_n^- := \min\{S_i, i \in \{1,2,\dots,n\}\}$ is called *minimum of random walk* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *up to time n*. $M_n = \max\{M_n^+, -M_n^-\}$ is called *absolute maximum of random walk* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *up to time n.*

Theorem 8 (Probability of maximum up to time n)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk.

$$\mathsf{P}\left(M_n^+ \geq r, S_n = b\right) = \begin{cases} \mathsf{P}(S_n = b) & \text{for } b \geq r\ , \\ \mathsf{P}(S_n = 2r - b)\left(\frac{q}{p}\right)^{r-b}, & \text{for otherwise.} \end{cases}$$

Proof. Let us firstly consider the easier case in which $b \geq r$. Because we defined M_n^+ as $\max\{S_i, i \in \{1, 2, \ldots, n\}\}$ we get that $M_n^+ \geq b \geq r$ therefore $[M_n^+ \geq r] \subset [S_n = b]$ hence we get $\mathsf{P}\left(M_n^+ \geq r, S_n = b\right) = \mathsf{P}\left(S_n = b\right)$.

Now let $r \geq 1, b < r$. $N_n^r(0, b)$ stands for number of paths from point $(0, 0)$ to point (n, b) which reach up to r. Let $k \in \{1, 2, \ldots, n\}$ denote the first time tha walk reaches r. By reflection principle (6), we can reflect the segment from (k, r) to (n, b) in the axis:$y = r$. Therefore we now have path from $(0, 0)$ to $(n, 2r - b)$ and we get that

$$N_n^r(0, b) = N_n(0, 2r - b) \text{ hence } \mathsf{P}\left(S_n = b, M_n^+ \geq r\right) = N_n^r(0, b)\, p^{\frac{n+b}{2}} q^{\frac{n-b}{2}} =$$
$$N_n(0, 2r - b)\, p^{\frac{n+(2r-b)}{2}} q^{\frac{n-(2r-b)}{2}} p^{b-r} q^{r-b} = \left(\frac{q}{p}\right)^{r-b} \mathsf{P}\left(S_n = 2r - b\right).$$

□

Definition 6 (Walk reaching new maximum at particular time). Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk $n, b \in \mathbb{N}$. We say that *the walk reached new maximum b in time n* if $M_{n-1}^+ = S_{n-1} = b - 1, S_n = b$. We denote such probability by $f_b(n)$.

Theorem 9 (Probability of reaching new maximum b in time n)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk $n, b \in \mathbb{N}$ then

$$f_b(n) = \frac{b}{n} \mathsf{P}\left(S_n = b\right).$$

Proof.

$$\begin{aligned}
f_b &= \mathsf{P}\left(M_{n-1} = S_{n-1} = b - 1, S_n = b\right) = \mathsf{P}\left(M_{n-1} = S_{n-1} = b - 1, X_n = +1\right) \\
&= p\,\mathsf{P}\left(M_{n-1} = S_{n-1} = b - 1\right) \\
&\overset{*}{=} p\left(\mathsf{P}\left(M_{n-1} \geq b - 1, S_{n-1} = b - 1\right) - \mathsf{P}\left(M_{n-1} \geq b, S_{n-1} = b - 1\right)\right) \\
&\overset{\text{T8}}{=} p\left(\mathsf{P}\left(S_{n-1} = b - 1\right) - \frac{q}{p}\mathsf{P}\left(S_{n-1} = b + 1\right)\right) \\
&= p\,\mathsf{P}\left(S_{n-1} = b - 1\right) - q\,\mathsf{P}\left(S_{n-1} = b + 1\right) \\
&= \binom{n-1}{\frac{n}{2} + \frac{b}{2} - 1} p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} - \binom{n-1}{\frac{n}{2} + \frac{b}{2}} p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \\
&= p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \left(\frac{(n-1)!}{\left(\frac{n}{2} + \frac{b}{2} - 1\right)!\left(\frac{n}{2} - \frac{b}{2}\right)!} - \frac{(n-1)!}{\left(\frac{n}{2} + \frac{b}{2}\right)!\left(\frac{n}{2} - \frac{b}{2} - 1\right)!}\right) \\
&= p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \left(\frac{(n-1)!}{\left(\frac{n}{2} + \frac{b}{2} - 1\right)!\left(\frac{n}{2} - \frac{b}{2} - 1\right)!}\right)\left(\frac{1}{\frac{n}{2} - \frac{b}{2}} - \frac{1}{\frac{n}{2} + \frac{b}{2}}\right) \\
&= p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \frac{b}{n}\left(\frac{n!}{\left(\frac{n}{2} + \frac{b}{2}\right)!\left(\frac{n}{2} - \frac{b}{2}\right)!}\right) = \frac{b}{n} p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \binom{n}{\frac{n}{2} + \frac{b}{2}} = \frac{b}{n}\mathsf{P}\left(S_n = b\right).
\end{aligned}$$

Where $*$ comes from the fact that the event $[M_{n-1} \geq b - 1]$ can be split into two disjoint events: $[M_{n-1} \geq b - 1] = [M_{n-1} \geq b] \cup [M_{n-1} = b - 1]$.

Hence: $\mathsf{P}\left(M_{n-1} = b - 1\right) = \mathsf{P}\left(M_{n-1} \geq b - 1\right) - \mathsf{P}\left(M_{n-1} \geq b\right)$. The same applies for the probability $\mathsf{P}\left(M_{n-1} = b - 1, S_{n-1} = b - 1\right)$ □

3. Returns to origin

Remark. Following is a simple identity concerning binomial numbers stated as a lemma. The lemma is stated in the thesis because we did not find it obvious.

Lemma 10 (Binomial identity)
Let $n, k \in \mathbb{N}, n > k$ then following equation holds

$$\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{n}\binom{n}{k}$$

Proof.

$$\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{(n-1)!}{k!\,(n-k-1)!} - \frac{(n-1)!}{(k-1)!\,(n-k)!}$$
$$= \frac{(n-1)!}{(k-1)!\,(n-k-1)!}\left(\frac{1}{k} - \frac{1}{n-k}\right)$$
$$= \frac{1}{n}\frac{n!}{(k-1)!\,(n-k-1)!}\frac{n-2k}{k\,(n-k)} = \frac{n-2k}{n}\frac{n!}{k!\,(n-k)!} = \frac{n-2k}{n}\binom{n}{k}$$

□

Remark. Thanks to the previous lemma we are able to prove following theorem. After proving the theorem and two corrolaries stated as lemmata we will be finally able to answer our first question.

Theorem 11 (Probability of no return up to n is equal to return in time n)
Let $\{S_n\}_{n=0}^{+\infty}$ be a symmetric random walk , then

$$\mathsf{P}\left(S_1, S_2, \ldots, S_{2n} \neq 0\right) = \mathsf{P}\left(S_{2n} = 0\right).$$

Proof.

$$\mathsf{P}\left(S_1, S_2, \ldots, S_{2n} \neq 0\right) \overset{LTP}{=} \sum_{i=-n}^{n} \mathsf{P}\left(S_1, S_2, \ldots, S_{2n} \neq 0, S_{2n} = 2i\right)$$
$$\overset{*}{=} 2\sum_{i=1}^{n} \mathsf{P}\left(S_1, S_2, \ldots, S_{2n} \neq 0, S_{2n} = 2i\right) \overset{\text{T7}}{=} 2\sum_{i=1}^{n}\frac{2i}{2n}\mathsf{P}\left(S_{2n} = 2i\right) = 2\sum_{i=1}^{n}\frac{2i}{2n}\binom{2n}{n-i}2^{-2n}$$
$$\overset{\text{L10}}{=} 2\cdot 2^{-2n}\sum_{i=1}^{n}\left(\binom{2n-1}{n-i} - \binom{2n-1}{n-i-1}\right) \overset{**}{=} 2\cdot 2^{-2n}\binom{2n-1}{n}$$
$$= 2^{-2n}\frac{2n}{n}\binom{2n-1}{n-1} = 2^{-2n}\binom{2n}{n} = \mathsf{P}\left(S_{2n} = 0\right).$$

Where $*$ comes from the fact that the random walk is symmetric and $**$ comes from the fact that the positive part of i-th term cancels against the negative part of $i+1$-st term. □

Lemma 12 (Probability of being strictly above origin)
Let $\{S_n\}_{n=0}^{+\infty}$ be a symmetric random walk.

$$\mathsf{P}\left(S_1, S_2, \ldots, S_{2n} > 0\right) = \frac{1}{2}\mathsf{P}\left(S_{2n} = 0\right).$$

Proof.

$$\mathsf{P}\left(S_1, S_2, \ldots, S_{2n} > 0\right) \overset{LTP}{=} \sum_{r=1}^{n} \mathsf{P}\left(S_1, S_2, \ldots, S_{2n} > 0, S_{2n} = 2r\right).$$

The r-th term follows equation:

$$\begin{aligned}
&\mathsf{P}\left(S_1, S_2, \ldots, S_{2n} > 0, S_{2n} = 2r\right) = \mathsf{P}\left(X_1 = 1, S_2, \ldots, S_{2n} > 0, S_{2n} = 2r\right)\\
&= \frac{1}{2}\,\mathsf{P}\left(S_2, S_3, \ldots, S_{2n} > 0, S_{2n} = 2r \mid S_1 = 1\right)\\
&\overset{*}{=} \frac{1}{2}\left(\mathsf{P}\left(S_{2n} = 2r \mid S_1 = 1\right) - \mathsf{P}\left(S_2 \cdot S_3 \cdot \ldots \cdot S_{2n-1} = 0, S_{2n} = 2r \mid S_1 = 1\right)\right)\\
&= \frac{1}{2}\left(2^{-(2n-1)} N_{2n-1}\left(1, 2r\right) - 2^{-(2n-1)} N^0_{2n-1}\left(1, 2r\right)\right)\\
&= \frac{1}{2} 2^{-(2n-1)}\left(N_{2n-1}\left(1, 2r\right) - N^0_{2n-1}\left(1, 2r\right)\right)\\
&\overset{\text{T6}}{=} \frac{1}{2} 2^{-(2n-1} \left(N_{2n-1}\left(1, 2r\right) - N_{2n-1}\left(-1, 2r\right)\right)\\
&= \frac{1}{2} 2^{-(2n-1)}\left(\binom{2n-1}{n+r-1} - \binom{2n-1}{n+r}\right).
\end{aligned}$$

Where $*$ comes from the disjoint decomposition:

$$[S_{2n} = 2r] = [S_1 \cdot S_2 \cdot \ldots \cdot S_{2n-1} \neq 0, S_{2n} = 2r] \cup [S_1 \cdot S_2 \cdot \ldots \cdot S_{2n-1} = 0, S_{2n} = 2r].$$

Because of the fact that the negative parts of r-th terms cancel against the positive parts of $(r+1)$-st terms and the sum reduces to just

$$\begin{aligned}
&\frac{1}{2} 2^{-(2n-1)}\binom{2n-1}{n} = \frac{1}{2} 2 2^{-2n}\binom{2n-1}{n} = \frac{1}{2} 2^{-2n}\frac{2n}{n}\binom{2n-1}{n}\\
&= \frac{1}{2} 2^{-2n}\binom{2n}{n} = \frac{1}{2}\,\mathsf{P}\left(S_{2n} = 0\right).
\end{aligned}$$

□

Lemma 13 (Probability of being above or in origin)
Let $\{S_n\}_{n=0}^{+\infty}$ be a symmetric random walk. The following equation holds:

$$\mathsf{P}\left(S_1, S_2, \ldots, S_{2n} \geq 0\right) = \mathsf{P}\left(S_{2n} = 0\right)$$

Proof.

$$\begin{aligned}
&\frac{1}{2}\,\mathsf{P}\left(S_{2n} = 0\right) \overset{\text{L12}}{=} \mathsf{P}\left(S_1, S_2, \ldots, S_{2n} > 0\right) = \mathsf{P}\left(X_1 = 1, S_2, S_3 \ldots, S_{2n} \geq 1\right)\\
&\overset{LTP}{=} \mathsf{P}\left(S_2, S_3 \ldots, S_{2n} \geq 1 \mid S_1 = 1\right)\mathsf{P}\left(X_1 = 1\right) = \frac{1}{2}\,\mathsf{P}\left(S_2, S_3 \ldots, S_{2n} \geq 1 \mid S_1 = 1\right)\\
&\overset{\text{L3}}{=} \frac{1}{2}\,\mathsf{P}\left(S_1, S_2 \ldots, S_{2n-1} \geq 1 \mid S_0 = 1\right) \overset{\text{L2}}{=} \frac{1}{2}\,\mathsf{P}\left(S_1, S_2 \ldots, S_{2n-1} \geq 0\right)\\
&= \frac{1}{2}\,\mathsf{P}\left(S_1, S_2 \ldots, S_{2n} \geq 0\right).
\end{aligned}$$

Where the last equation comes from the fact that $[S_1, S_2 \ldots, S_{2n-1} \geq 0] = [S_1, S_2 \ldots, S_{2n-1} \geq 1] = [S_1, S_2 \ldots, S_{2n} \geq 0]$. □

Theorem 14 (Probability of position x in time n without returning to origin) *Let* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a random walk and* $x \in \mathbb{Z}$ *then the following equation holds.*

$$\mathsf{P}\left(S_1, S_2, \ldots, S_n \neq 0, S_n = x\right) = \frac{|x|}{n}\,\mathsf{P}\left(S_n = x\right).$$

Proof. Let us without loss of generality assume that $x > 0$. In that case, first step has to be rightwards $(X_1 = +1)$. Now we have path from point $(1,1)$ to point (n, x) that does not return to origin. By Ballot theorem 7 there are $\frac{x}{n} N_n(0, x)$ such paths. Each path consists of $\frac{n+x}{2}$ rightwards steps and $\frac{n-x}{2}$ leftwards steps. Therefore $\mathsf{P}\left(S_1 \cdot S_2 \cdot, \ldots, S_n \neq 0, S_n = x\right) = \frac{x}{n} N_n(0, x)\, p^{\frac{n+x}{2}} q^{\frac{n-x}{2}} = \frac{x}{n}\,\mathsf{P}\left(S_n = x\right)$. Case $x < 0$ is identical. □

Theorem 15
Let $\{S_n\}_{n=0}^{+\infty}$ *be a symmetric random walk. The probability that the last return to origin up to time* $2n$ *occurred in time* $2k$ *is* $\mathsf{P}\left(S_{2k} = 0\right)\mathsf{P}\left(S_{2n-2k} = 0\right)$.

Proof.

$$\mathsf{P}\left(S_{2k+1}, S_{2k+2}, \ldots, S_{2n} \neq 0, S_{2k} = 0\right) \overset{GPR}{=} \mathsf{P}\left(S_{2k+1}, S_{2k+2}, \ldots, S_{2n} \neq 0 \mid S_{2k} = 0\right)\mathsf{P}\left(S_{2k} = 0\right)$$
$$\overset{\text{L3}}{=} \mathsf{P}\left(S_{2k} = 0\right)\mathsf{P}\left(S_1, S_2, \ldots, S_{2n-2k} \neq 0\right) \overset{\text{T11}}{=} \mathsf{P}\left(S_{2k} = 0\right)\mathsf{P}\left(S_{2n-2k} = 0\right).$$

□

Theorem 16 (First return as difference of returns)
Let $\{S_n\}_{n=0}^{+\infty}$ *be a symmetric random walk. The following equation holds:*

$$f_{2n} = u_{2n-2} - u_{2n}$$

Proof. The event $[S_1, S_2, \ldots S_{2n-1} \neq 0]$ can be split into two disjoint events: $[S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} = 0] \cup [S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} \neq 0]$. Hence we get

$$\begin{aligned} f_{2n} &= \mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} = 0\right) \\ &= \mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0\right) - \mathsf{P}\left(S_1, S_2, \ldots, S_{2n} \neq 0\right). \end{aligned}$$

Because $2n-1$ is odd. $\mathsf{P}\left(S_{2n-1} = 0\right) = 0$. Therefore the first term is equal to $\mathsf{P}\left(S_1, S_2, \ldots S_{2n-2} \neq 0\right)$ which is by 11 equal to u_{2n-2}. Second term is by 11 equal to u_{2n}. Therefore we get the result. □

4. Own proofs

Lemma 17
Let $\{S_n\}_{n=0}^{+\infty}$ *be a symmetric random walk. The following equation holds:*

$$f_{2n} = \frac{1}{2n-1} u_{2n}$$

Proof.

$$\begin{aligned} u_{2n-2} &= 2^{-(2n-2)} \binom{2n-2}{n-1} = 4 \cdot 2^{-2n} \frac{(2n-2)!}{(n-1)!\,(n-1)!} \\ &= \frac{4n^2}{(2n)\,(2n-1)} 2^{-2n} \binom{2n}{n} = \frac{2n}{2n-1} u_{2n}. \end{aligned}$$

Therefore

$$f_{2n} \overset{\text{L16}}{=} u_{2n-2} - u_{2n} = u_{2n} \left(\frac{2n}{2n-1} - 1 \right) = u_{2n} \frac{1}{2n-1}.$$

□

Lemma 18 (Decomposition of f_n)
Let $\{S_n\}_{n=0}^{+\infty}$ *be a symmetric random walk.*

$$\mathsf{P}\left(S_{2n} = 0\right) = \sum_{r=1}^{n} f_{2r} u_{2n-2r}$$

Proof.

$$\begin{aligned} &\mathsf{P}\left(S_{2n} = 0\right) \overset{LTP}{=} \sum_{r=1}^{n} \mathsf{P}\left(S_{2n} = 0, S_1, S, 2, \ldots, S_{2r-1} \neq 0, S_{2r} = 0\right) \\ &\overset{GPR}{=} \sum_{r=1}^{n} \mathsf{P}\left(S_{2n} = 0 \mid S_1, S, 2, \ldots, S_{2r-1} \neq 0 S_{2r} = 0\right) \mathsf{P}\left(S_1, S, 2, \ldots, S_{2r-1} \neq 0, S_{2r} = 0\right) \\ &\overset{\text{L4,D4}}{=} \sum_{r=1}^{n} \mathsf{P}\left(S_{2n} = 0 \mid S_{2r} = 0\right) f_{2r} \overset{\text{L3}}{=} \sum_{r=1}^{n} \mathsf{P}\left(S_{2n-2r} = 0\right) f_{2r} \overset{\text{D4}}{=} \sum_{r=1}^{n} u_{2n-2r} f_{2r}. \end{aligned}$$

□

Theorem 19 (Arcsine law for last visits)
Let $\{S_n\}_{n=0}^{+\infty}$ *be a symmetric random walk* $k, n \in \mathbb{N}, k \leq n$. *The probability that up to time* $2n$ *the last return to origin occurred in time* $2k$ *is given by*

$$\mathsf{P}\left(S_{2n} = 0\right) \mathsf{P}\left(S_{2n-2k} = 0\right).$$

Proof. The probability involved can be rewritten as:

$$\begin{aligned} &\mathsf{P}\left(S_{2k} = 0, S_{2k+1}, S_{2k+2}, \ldots, S_{2n} \neq 0\right) \\ &\overset{GPR}{=} \mathsf{P}\left(S_{2k+1}, S_{2k+2}, \ldots, S_{2n} \neq 0 \mid S_{2k} = 0\right) \mathsf{P}\left(S_{2k} = 0\right) \\ &\overset{\text{L3}}{=} \mathsf{P}\left(S_1, S_2, \ldots, S_{2n-2k} \neq 0\right) \mathsf{P}\left(S_{2k} = 0\right) \overset{\text{T11}}{=} u_{2n-2k} u_{2k} \end{aligned}$$

□

Definition 7 (Time spend on the positive and negative sides)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk. We say that *the walk spent* τ *time units of* n *on the positive side* if $\sum\limits_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = \tau$. Let $\beta_n(\tau)$ denote the probability of such an event. Similarly, we say that *the walk spent* ζ *time units of* n *on the negative side* if $\sum\limits_{i=1}^{n} \mathbf{1}_{[S_i<0 \vee S_{i-1}<0]} = \zeta$.

Theorem 20 (Arcsine law for sojourn times-OWN PROOF)
Let $\{S_n\}_{n=0}^{+\infty}$ *be a symmetric random walk. Then*

$$\beta_{2n}(2k) = \alpha_{2n}(2k) = \mathsf{P}(S_{2n} = 0)\,\mathsf{P}(S_{2n-2k} = 0).$$

Proof. Firstly let us start with degenerate cases.

$$\beta_{2n}(2n) = \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2n\right) \overset{\text{L13}}{=} \mathsf{P}(S_1, S_2, \ldots, S_{2n} \geq 0) = u_{2n}.$$

By symmetry $\beta_{2n}(0) = \beta_{2n}(2n)$.

Let $1 \leq k \leq v-1$, where $0 \leq v \leq n$. For such k following equation holds:

$$\begin{aligned}
&\beta_{2n}(2k) \overset{\text{D7}}{=} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k\right) \\
&\overset{LTP}{=} \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} \neq 0, S_{2r} = 0\right) \\
&\overset{*}{=} \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right) \\
&+ \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right) \\
&\overset{GPR}{=} \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k \,\middle|\, S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right) \\
&\mathsf{P}(S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0) \\
&+ \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k \,\middle|\, S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right) \\
&\mathsf{P}(S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0) \\
&\overset{**}{=} \sum_{r=1}^{n} \frac{1}{2} f_{2r}\, \mathsf{P}\left(\sum_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k \,\middle|\, S_{2r} = 0\right) \\
&+ \sum_{r=1}^{n} \frac{1}{2} f_{2r}\, \mathsf{P}\left(\sum_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k - 2r \,\middle|\, S_{2r} = 0\right) \\
&\overset{\text{L3}}{=} \sum_{r=1}^{n} \frac{1}{2} f_{2r}\, \mathsf{P}\left(\sum_{i=1}^{2n-2r} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k\right) \\
&+ \sum_{r=1}^{n} \frac{1}{2} f_{2r}\, \mathsf{P}\left(\sum_{i=1}^{2n-2r} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k - 2r\right) \\
&= \sum_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k) + \sum_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k - 2r).
\end{aligned}$$

Where $*$ comes from the disjoint decomposition of $[S_1, S_2, \ldots, S_{2r-1} \neq 0] = [S_1, S_2, \ldots, S_{2r-1} > 0] \cup [S_1, S_2, \ldots, S_{2r-1} < 0]$ and $**$ comes from using the condition that up to time $2r$ the steps were on the positive/negative sides.

Now let us proceed by induction. Case for $v = 1$ is trivial because it implies the degenerate case. Let the statement be true for $v \leq n-1$, then

$$\begin{aligned}
&\sum_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k) + \sum_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k-2r) \\
&\overset{IA}{=} \sum_{r=1}^{n} \frac{1}{2} f_{2r} \alpha_{2n-2r}(2k) + \sum_{r=1}^{n} \frac{1}{2} f_{2r} \alpha_{2n-2r}(2k-2r) \\
&\overset{\text{D4}}{=} \sum_{r=1}^{n} \frac{1}{2} f_{2r} u_{2k} u_{2n-2r-2k} + \sum_{r=1}^{n} \frac{1}{2} f_{2r} u_{2k-2r} u_{2n-2k} \\
&= \frac{1}{2} u_{2k} \sum_{r=1}^{n} f_{2r} u_{2n-2r-2k} + \frac{1}{2} u_{2n-2k} \sum_{r=1}^{n} f_{2r} u_{2k-2r} \\
&\overset{\text{L18}}{=} \frac{1}{2} u_{2n-2k} u_{2k} + \frac{1}{2} u_{2n-2k} u_{2k} = u_{2n-2k} u_{2k} \overset{\text{D4}}{=} \alpha_{2n}(2k).
\end{aligned}$$

□

4.1 Problem chapter 9 Feller-není dokončeno, zkontrolováno ani upraveno do čitelnější podoby

Definition 8 ($\delta_n, \varepsilon_n^{r,\pm}$)**.** Let $\{S_n\}_{n=0}^{+\infty}$ be a symmetric random walk.

$\delta_n(k)$ shall denote $\mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_n = 0\right)$,

$\varepsilon_n^{r}(k)$ shall denote $\mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_1, S_2, \ldots, S_{r-1} \neq 0, S_r = 0, S_n = 0\right)$,

$\varepsilon_n^{r,+}(k)$ shall denote $\mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_1, S_2, \ldots, S_{r-1} > 0, S_r = 0, S_n = 0\right)$,

$\varepsilon_n^{r,-}(k)$ shall denote $\mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_1, S_2, \ldots, S_{r-1} < 0, S_r = 0, S_n = 0\right)$.

Lemma 21 (Factorization of $\delta_{2n}(2k)$)
$\delta_{2n}(2k) = \frac{1}{2} \sum_{r=1}^{n} \left(f_{2r} \delta_{2n-2r}(2k-2r) + f_{2r} \delta_{2n-2r}(2r)\right)$.

Proof. Because $S_{2n} = 0$ a return to origin must have happened. Let $2r$ the time of first return to origin, where $r \in \{1, 2, \ldots, n\}$. By the law of total probability:

$$\begin{aligned}
&\delta_{2n}(2k) \overset{\text{D8}}{=} \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_{2n} = 0\right) \\
&\overset{LTP}{=} \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} \neq 0, S_{2r} = 0, S_{2n} = 0\right) \\
&\overset{\text{D8}}{=} \sum_{r=1}^{n} \varepsilon_{2n}^{2k} \overset{*}{=} \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0, S_{2n} = 0\right) \\
&+ \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0, S_{2n} = 0\right) \\
&= \sum_{r=1}^{n} \varepsilon_{2n}^{2r,+}(2k) + \sum_{r=1}^{n} \varepsilon_{2n}^{2r,-}(2k).
\end{aligned}$$

Where $*$ comes from the disjoint decomposition $[S_1, S_2, \ldots, S_{2r-1} \neq 0] = [S_1, S_2, \ldots, S_{2r-1} > 0] \cup [S_1, S_2, \ldots, S_{2r-1} < 0]$.

Now let us calculate $\varepsilon_{2n}^{2r,+}(2k)$

$= \mathsf{P}\left(\sum\limits_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0, S_{2n} = 0\right)$

$\overset{GPR}{=} \mathsf{P}\left(\sum\limits_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n} = 0 \mid S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right)$

$\mathsf{P}\left(S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right)$

$\overset{*}{=} \mathsf{P}\left(\sum\limits_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k - 2r, S_{2n} = 0 \,\middle|\, S_{2r} = 0\right) \mathsf{P}\left(S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right)$

$\overset{**}{=} \mathsf{P}\left(\sum\limits_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k - 2r, S_{2n} = 0 \,\middle|\, S_{2r} = 0\right) \frac{1}{2} f_2 r$

$\overset{\text{L3}}{=} \mathsf{P}\left(\sum\limits_{i=1}^{2n-2r} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k - 2r, S_{2n-2r} = 0\right) \frac{1}{2} f_{2r}$

$\overset{\text{D8}}{=} \delta_{2n-2r}(2k - 2r) \frac{1}{2} f_{2r}$.

Where $*$ comes from Lemma (4) and using the condition.

Where $**$ comes from the fact that $f_{2r} \overset{\text{D4}}{=} \mathsf{P}\left(S_1, S_2, \ldots, S_{2r-1} \neq 0, S_{2r} = 0\right) = \mathsf{P}\left(S_1 = 1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right) + \mathsf{P}\left(S_1 = -1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right)$ and $\mathsf{P}\left(S_1 = -1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right) = \mathsf{P}\left(S_1 = 1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right)$ because of symmetry. Hence $\mathsf{P}\left(S_1 = 1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right) = \frac{1}{2} f_{2r}$.

Similarly $\varepsilon_{2n}^{2r,+}(2k)$

$= \mathsf{P}\left(\sum\limits_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0, S_{2n} = 0\right)$

$= \mathsf{P}\left(\sum\limits_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n} = 0 \mid S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right)$

$\mathsf{P}\left(S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right)$

$= \mathsf{P}\left(\sum\limits_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n} = 0 \,\middle|\, S_{2r} = 0\right) \mathsf{P}\left(S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right)$

$= \mathsf{P}\left(\sum\limits_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n} = 0 \,\middle|\, S_{2r} = 0\right) \frac{1}{2} f_2 r$

$= \mathsf{P}\left(\sum\limits_{i=1}^{2n-2r} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n-2r} = 0\right) \frac{1}{2} f_{2r}$

$= \delta_{2n-2r}(2k) \frac{1}{2} f_{2r}$.

Therefore $\delta_{2n}(2k) = \frac{1}{2} \sum\limits_{r=1}^{n} f_{2r} \delta_{2n-2r}(2k - 2r) + \frac{1}{2} \sum\limits_{r=1}^{n} f_{2r} \delta_{2n-2r}(2k)$

$= \frac{1}{2} \sum\limits_{r=1}^{n} \left(f_{2r} \delta_{2n-2r}(2k - 2r) + f_{2r} \delta_{2n-2r}(2r)\right)$ □

Theorem 22 (Equidistributional theorem-ALMOST COMPLETE OWN PROOF) *Let $\{S_n\}_{n=0}^{+\infty}$ be a symmetric random walk and $n \in \mathbb{N}$, then $\forall k, l \in \{0, 1, 2, \ldots, n\}$: $\delta_{2n}(2k) = \delta_{2n}(2l) = \frac{u_{2n}}{n+1}$.*

Proof. Let us prove this statement by induction in n. In case that $n = 1$ we have two options for k. Either $k = 0$ or $k = 1$. $\delta_2(0) = \mathsf{P}(S_1 = -1, S_2 = 0) = \frac{1}{2} u_2 = \mathsf{P}(S_1 = +1, S_2 = 0) = \delta_2(2)$.

Let the statement be true for all $l \leq n - 1$. In that case $\delta_{2(n-l)}(2k) = \frac{u_{2(n-l)}}{n-l+1} \forall k \in \{1, 2, \ldots, n - l\}$. We want to show that $\delta_{2n} = \frac{u_{2n}}{n+1}$.

Let us calculate $\delta_{2n} \overset{\text{L21}}{=} \frac{1}{2} \sum\limits_{r=1}^{n} \left(f_{2r} \delta_{2n-2r}(2k - 2r) + f_{2r} \delta_{2n-2r}(2r)\right)$

$\overset{IA}{=} \frac{1}{2} \sum\limits_{r=1}^{n} \left(f_{2r} u_{2n-2r} \frac{1}{n-r+1} + f_{2r} u_{2n-2r} \frac{1}{n-r+1}\right) = \sum\limits_{r=1}^{n} \frac{f_{2r} u_{2n-2r}}{n-r+1} \overset{\text{SNAD TO DOKAZU L23}}{=} \frac{u_{2n}}{n+1}$

$\square$

Lemma 23 (Sum of binomials-POTŘEBUJU DOKÁZAT)
$\sum\limits_{r=1}^{n} \frac{f_{2r}u_{2n-2r}}{n-r+1} = \frac{u_{2n}}{n+1}$

Proof. $f_{2r}u_{2n-2r} \overset{\text{L17}}{=} \frac{1}{2r-1}u_{2r}u_{2n-2r} \overset{\text{D4}}{=} \frac{1}{2r-1}2^{-2r}\binom{2r}{r}2^{-(2n-2r)}\binom{2n-2r}{n-r}$.

Therefore $\sum\limits_{r=1}^{n} \frac{f_{2r}u_{2n-2r}}{n-r+1} = \sum\limits_{r=1}^{n} \frac{1}{2r-1}\frac{1}{n-r+1}2^{-2n}\binom{2r}{r}\binom{2n-2r}{n-r} \overset{???}{=} \frac{1}{n+1}2^{-2n}\binom{2n}{n}$ $\square$

5. Simple random walk in more dimensions

Definition 9 (Type II random walk in $\mathbb{Z}^m$)**.** Let $m \in \mathbb{N}$. $\forall n \in \mathbb{N}$, let $X_n = \left(x_n^1 \quad x_n^2 \quad \dots, x_n^m\right)^T$, where $\{x_n^i\}_{i=1}^m$ are $\forall n \in \mathbb{N}$ independent.

Let $\forall i \in \{1, 2, \dots, m\} x_n^i$ have values in $\{-1, +1\}$ with probabilities $\mathsf{P}\left(x_n^i = +1\right) = p_i \in (0, 1)$ and $\mathsf{P}\left(x_n^i = -1\right) = 1 - p_i =: q_i \in (0, 1)$.

Let $\{X_n\}_{n=0}^{+\infty}$ be a sequence of independent and identically distributed random variables. Let $S_0 = \mathbf{0}$ and $\forall n \in \mathbb{N} : \mathbf{S_n} = \sum\limits_{i=1}^{n} X_i$ and $\mathbf{p} = \left(p_1, p_2, \dots, p_m\right)^{\mathbf{T}}$. Then the pair $\left(\{\mathbf{S_n}\}_{n=0}^{+\infty}, \mathbf{p}\right)$ is called *Type II random walk in* $\mathbb{Z}^m$.

If $\forall i \in \{1, 2, \dots, m\} : p_i = q_i = \frac{1}{2}$ we call the element $\{\mathbf{S_n}\}_{n=0}^{+\infty}$ *Symmetric type II random walk* $\mathbb{Z}^m$.

Remark. Type II random walk can be interpreted as m simple random walks in $\mathbb{Z}$ happening at a time, each of them parallel to an axis of $\mathbb{Z}^m$.

Theorem 24
Let $m \in \mathbb{N}$ *and* $\left(\{\mathbf{S_n}\}_{n=0}^{+\infty}, \mathbf{p}\right)$ *be a Type II random walk in* $\mathbb{Z}^m$. *Let* $\mathbf{y} = \left(y_1, y_2, \dots, y_m\right)^T \in \mathbb{Z}^m$. *Then following equation stands:*

$$\mathsf{P}\left(S_n = x\right) = \begin{cases} \prod_{i=1}^m \binom{n}{\frac{y_i+n}{2}} p_i^{\frac{n+y_i}{2}} q_i^{\frac{n-y_i}{2}}, & \text{if } \forall i \in \{1, 2, \dots, m\} : y_i \in A_n, \\ 0, & \text{if } \exists i \in \{1, 2, \dots, m\} : y_i \notin A_n. \end{cases}$$

Where A_n *is from definition (2)*

Proof. $\mathsf{P}\left(\mathbf{S_n} = \mathbf{y}\right) = \mathsf{P}\left(S_n^1 = y_1, S_n^2 = y_2, \dots, S_n^m = y_m\right) \stackrel{\perp\!\!\!\perp}{=} \prod\limits_{i=1}^{m} \mathsf{P}\left(S_n^i = y_i\right)$
$= \prod\limits_{i=1}^{m} \binom{n}{\frac{y_i+n}{2}} p_i^{\frac{n+y_i}{2}} q_i^{\frac{n-y_i}{2}}$. The second equation comes from the independence of $\{\mathbf{S_i}\}_{i=1}^m$ which comes easily from independence of X_n^i. The second equation comes from Theorem (1). ☐

Remark. Due to the the aim of this thesis which is researching occupation time of a set of random walks we are going to concern only on symmetric random walks.

Definition 10 (Orthant)**.** Let $m \in \mathbb{Z}$. Then $O \subset \mathbb{Z}^m$ is called an *open orthant in* $\mathbb{Z}^m$ if $\forall o := \left(o_1, o_2, \dots, o_m\right)^T \in O, \forall i \in \{1, 2, \dots, m\} : o_i \varepsilon_i > 0$, where $\varepsilon_i \in \{-1, +1\}$.

$C \subset \mathbb{Z}^m$ is called a *closed orthant in* $\mathbb{Z}^m$ if $\forall c := \left(c_1, c_2, \dots, c_m\right)^T \in C, \forall i \in \{1, 2, \dots, m\} : c_i \varepsilon_i \geq 0$, where $\varepsilon_i \in \{-1, +1\}$.

Remark. The statement $\mathbf{x} > \mathbf{y}$ will mean $\forall i \in \{1, 2, \dots, m\} : x_i > y_i$. Same applies to $<, \leq, \geq$.

Theorem 25 (Probability of being in an open orthant)
Let $\{\mathbf{S_n}\}_{n=0}^{+\infty}$ *be a Symmetric type II random walk in* $\mathbb{Z}^m$. *Let* O *be an open orthant in* Z^m. $\mathsf{P}\left(\mathbf{S_n} \in O\right) = \left(\frac{1}{2} u_{2n}\right)^m$.

Proof. Without loss of generality we can assume that in the definition of O we choose $\forall i \in \{1,2,\ldots,m\}\varepsilon_i := +1$ then: $\mathsf{P}\left(\mathbf{S_n} \in O\right) = \mathsf{P}\left(S_n^1 > 0, S_n^2 > 0, \ldots, S_n^m > 0\right) = \prod_{i=1}^m \mathsf{P}\left(S_n^i > 0\right) = \left(\mathsf{P}\left(S_n^i > 0\right)\right)^m = \left(\frac{1}{2}u_{2n}\right)^m$. Where the last two equations come from the identical distribution of S_n^i and Theorem (12). □

Theorem 26 (Probability of being in a closed orthant)
Let $\{\mathbf{S_n}\}_{n=0}^{+\infty}$ be a Symmetric type II random walk in $\mathbb{Z}^m$. Let C be a closed orthant in Z^m. $\mathsf{P}\left(\mathbf{S_n} \in C\right) = \left(u_{2n}\right)^m$.

Proof. The proof is very similar to previous proof. Without loss of generality we can again assume that in the definition of C we choose $\forall i \in \{1,2,\ldots,m\}\varepsilon_i := +1$ then: $\mathsf{P}\left(\mathbf{S_n} \in C\right) = \mathsf{P}\left(S_n^1 \geq 0, S_n^2 \geq 0, \ldots, S_n^m \geq 0\right) = \prod_{i=1}^m \mathsf{P}\left(S_n^i \geq 0\right) = \left(\mathsf{P}\left(S_n^i \geq 0\right)\right)^m = \left(u_{2n}\right)^m$. Where the last two equations come from the identical distribution of S_n^i and Lemma (13). □

Theorem 27 (Zákon iterovaného logaritmu)
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