1. Title of the first chapter

Definition 1 (Simple random walk in $\mathbb{Z}$)
Let $\{X_n\}_{n=0}^{+\infty}$ be a sequence of independent and identically distributedrandom variables with values in $\{-1,+1\}$. That $\forall n \in \mathbb{N}$ satisfy $\mathsf{P}(X_n = 1) = p \in (0,1)$ and $\mathsf{P}(X_n = -1) = 1 - p = q$. Let $S_0 = 0$ and $S_n = \sum\limits_{i=1}^{n} X_i$. We call the pair $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ Simple random walk in $\mathbb{Z}$. *In case that $p = q = \frac{1}{2}$ we call the pair $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$* Symmetric simple random walk in $\mathbb{Z}$.

Remark. Very often we refer to n as time , X_i as i-th step and S_n as position in time n. In simple random walk in $\mathbb{Z}$ we refer to $X_i = +1$ as i-th step was rightwards. If not stated otherwise, we assume that $S_0 = 0$.

Definition 2 (Set of possible positions)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$be a random walk . We call the set $A_n = \{z \in \mathbb{Z}; |z| \leq n, \frac{z+n}{2} \in \mathbb{Z}\}$ set of all possible positions of random walk $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$in time n.

Theorem 1 (Probability of position x in time n)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$be a random walk and A_n its set of possible positions.

$$\mathsf{P}(S_n = x) = \begin{cases} \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}} & \textit{for } x \in A_n \ , \\ 0, & \textit{for } x \notin A_n. \end{cases}$$

Proof. Let us define new variables $r_i = \mathbf{1}_{[X_i=1]}, l_i = \mathbf{1}_{[X_i=-1]}, R_n = \sum\limits_{i=1}^{n} r_i, L_n = \sum\limits_{i=1}^{n} l_i$. r_i can be interpreted as indicator wether i-th step was rightwards. Then R_n is number of rightwards steps and L_n is number of leftwards steps. We can easily see that $R_n + L_n = n$ and $R_n - L_n = S_n$. Therefore we get by adding these two equations $R_n = \frac{S_n+n}{2}$.

r_i has alternative distribution with parameter p *Alt(p)*. Therefore R_n as a sum of independent and identically distributedrandom variables with *Alt(p)* has binomial distribution with parameters n and p (*Bi(n,p)*). Therefore we get $\mathsf{P}(R_n = x) = \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}$. Where we define $\binom{a}{x} := 0$ for $a \in \mathbb{N}, x \in \mathbb{R} \setminus \mathbb{N}$, $x < 0$, $x > n$. Therefore we get $\mathsf{P}(S_n = x) = \mathsf{P}\left(R_n = \frac{x+n}{2}\right) = \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}$. ☐

Lemma 2 (Spatial homogeneity)
$\mathsf{P}(S_n = j \mid S_0 = a) = \mathsf{P}(S_n = j + b \mid S_0 = a + b)\, \forall b \in \mathbb{Z}$

Proof. $\mathsf{P}(S_n = j \mid S_0 = a) = \mathsf{P}\left(\sum\limits_{i=1}^{n} X_i = j - a\right)$
$= \mathsf{P}\left(\sum\limits_{i=1}^{n} X_i = (j+b) - (a+b)\right) = \mathsf{P}(S_n = j + b \mid S_0 = a + b)$. ☐

Lemma 3 (Temporal homogeneity)
$\mathsf{P}(S_n = j \mid S_0 = a) = \mathsf{P}(S_{n+m} = j \mid S_m = a)\, \forall m \in \mathbb{N}$

Proof. $\mathsf{P}(S_n = j \mid S_0 = a) = \mathsf{P}\left(\sum\limits_{i=1}^{n} X_i = j - a\right)$
$= \mathsf{P}\left(\sum\limits_{i=m+1}^{m+n} X_i = j - a\right) = \mathsf{P}(S_{n+m} = j \mid S_m = a)$. ☐

Lemma 4 (Markov property)
Let $n \geq m$ and $a_i \in \mathbb{Z}, i \in \mathbb{N}$. Then $P(S_n = j \mid S_0 = a_0, S_1 = a_1, \ldots, S_m = a_m) = P(S_n = j \mid S_m = a_m)$

Proof. Once S_m is known, then distribution of S_n depends only on steps $X_{m+1}, X_{m+2}, \ldots X_n$ and therefore cannot be dependent on any information concerning values $X_1, X_2, \ldots, X_m - 1$ and accordingly $S_1, S_2, \ldots, S_m - 1$. $\square$

Remark. In symmetric random walk, everything can be counted by number of possible paths from point to point.

Definition 3 (Number of possible paths)
Let $N_n(a,b)$ be number of posssible paths of random walk $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$from point $(0,a)$ to point (n,b) and $N_n^x(a,b)$ be number of possible paths from point $(0,a)$ to point (n,b) that visit point (z,x) for some $z \in \{0, \ldots n\}$.

Theorem 5
Let $a,b \in \mathbb{Z}, n \in \mathbb{N}$ then $N_n(a,b) = \binom{n}{\frac{1}{2}(n+b-a)}$.

Proof. Let us choose a path from $(0,a)$ to (n,b) and let α be number of rightwards steps and β be number of leftwards steps. Then $\alpha + \beta = n$ and $\alpha - \beta = b - a$. By adding these two equations we get that $\alpha = \frac{1}{2}(n+b-a)$. The number of possible paths is the number of ways of picking α rightwards steps from n steps. Therefore we get $N_n(a,b) = \binom{n}{\alpha} = \binom{n}{\frac{1}{2}(n+b-a)}$. $\square$

Theorem 6 (Reflection principle)
Let $a,b > 0$, then $N_n^0(a,b) = N_n(-a,b)$.

Proof. Each path from $(0,-a)$ to (n,b) has to intersect x-axis at least once at some point. Let k be the time of earliest intersection with x-axis. By reflexing the segment from $(0,-a)$ to $(k,0)$ in the x-axis, we get a path from point $(0,a)$ to (n,b) which visits 0 at point k. Because reflection is bijective operation on sets of paths, we get the correspondence between the collections of such paths. $\square$

Definition 4 (Return to origin)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$be a random walk . Then if $\exists k \in \mathbb{N}$ such that $S_k = 0$ then we say that in k-th step occured return to origin. Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$be a random walk . Then if $S_1, S_2, \ldots, S_{2n-1} \neq 0$ and $S_{2n} = 0$

Theorem 7 (Ballot theorem)
Let $n,b \in N$ Number of paths from point $(0,0)$ to point (n,b) which do not return to origin is equal to $\frac{b}{n}N_n(0,b)$

Proof. Let us call N the number of paths we are refering to. Because the path ends at point (n,b), the first step has to be rightwards. Therefore we now have $N = N_{n-1}(1,b) - N_{n-1}^0(1,b) = N_{n-1}(1,b) - N_{n-1}(-1,b)$. The last equation was aquired using Reflection principle (6). We now have:$N_{n-1}(1,b) - N_{n-1}(-1,b) = \binom{n-1}{\frac{n}{2}+\frac{b}{2}-1} - \binom{n-1}{\frac{n}{2}+\frac{b}{2}} = \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!} - \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!} = \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}\right)\left(\frac{n}{2}-\frac{b}{2}-1\right)!} - \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}\right)\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!} = \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!}\left(\frac{1}{\frac{n}{2}-\frac{b}{2}} - \frac{1}{\frac{n}{2}+\frac{b}{2}}\right) = \frac{1}{n}\frac{n!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!}\left(\frac{\left(\frac{n}{2}+\frac{b}{2}-\frac{n}{2}+\frac{b}{2}\right)}{\left(\frac{n}{2}-\frac{b}{2}\right)\left(\frac{n}{2}+\frac{b}{2}\right)}\right) \frac{b}{n}\frac{n!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!} = \frac{b}{n}\binom{n}{\left(\frac{n}{2}+\frac{b}{2}\right)} = \frac{b}{n}N_n(0,b)$ $\square$

Definition 5
$M_n = \max\{S_i, i \in \{0,1,\ldots,n\}\}$

Theorem 8 (Probability of maximum up to time n)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a random walk .*

$$\mathsf{P}(M_n \geq r, S_n = b) = \begin{cases} \mathsf{P}(S_n = b) & \text{for } b \geq r \ , \\ \mathsf{P}(S_n = 2r - b)\left(\frac{q}{p}\right)^{r-b}, & \text{for otherwise.} \end{cases}$$

Proof. Let us firstly consider the easier case in which $b \geq r$. Because we defined M_n as $\max\{S_i, i \in \{0,1,\ldots,n\}\}$ we get that $M_n \geq b \geq r$ therefore $[M_n \geq r] \subset [S_n = b]$ therefore we get $\mathsf{P}(M_n \geq r, S_n = b) = \mathsf{P}(S_n = b)$. Let $r \geq 1, b < r$. $N_n^r(0,b)$ stands for number of paths from point $(0,0)$ to point (n,b) which reach up to r. Let $k \in \{0,1,\ldots,n\}$ denote the first time we reach r. By reflection principle (6) , we can reflex the segment from (k,r) to (n,b) in the axis:$y = r$. Therefore we now have path from $(0,0)$ to $(n, 2r-b)$ and we get that $N_n^r(0,b) = N_n(0, 2r-b)$. $\mathsf{P}(S_n = b, M_n \geq r) = N_n^r(0,b)\, p^{\frac{n+b}{2}} q^{\frac{n-b}{2}} = N_n(0, 2r-b)\, p^{\frac{n+(2r-b)}{2}} q^{\frac{n-(2r-b)}{2}} p^{\frac{n+b}{2}} q^{\frac{n-b}{2}} p^{b-r} q^{r-b} = \left(\frac{q}{p}\right)^{r-b} \mathsf{P}(S_n = 2r-b)$. □

Definition 6 (Walk reaching new maximum at particular time)
Let $b > 0$. $f_b(n)$ *denotes the probability that we reach new maximum* b *in time* n. $f_b(n) = \mathsf{P}(M_{n-1} = S_{n-1} = b-1, S_n = b)$

Theorem 9 (Probability of reaching new maximum b in time n)
Let $b > 0$ *then* $f_b(n) = \frac{b}{n}\mathsf{P}(S_n = b)$.

Proof. $f_b = \mathsf{P}(M_{n-1} = S_{n-1} = b-1, S_n = b) = p\,\mathsf{P}(M_{n-1} = S_{n-1} = b-1)$
$= p\left(\mathsf{P}(M_{n-1} \geq b-1, S_{n-1} = b-1) - \mathsf{P}(M_{n-1} \geq b, S_{n-1} = b-1)\right)$
$= p\left(\mathsf{P}(S_{n-1} = b-1) - \frac{q}{p}\mathsf{P}(S_{n-1} = b+1)\right)$
$= p\,\mathsf{P}(S_{n-1} = b-1) - q\,\mathsf{P}(S_{n-1} = b+1)$
$= \binom{n-1}{\frac{n}{2}+\frac{b}{2}-1} p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} - \binom{n-1}{\frac{n}{2}+\frac{b}{2}} p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} = p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \left(\frac{(n-1)!}{(\frac{n}{2}+\frac{b}{2}-1)!(\frac{n}{2}-\frac{b}{2})!} - \frac{(n-1)!}{(\frac{n}{2}+\frac{b}{2})!(\frac{n}{2}-\frac{b}{2}-1)!}\right) =$
$p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \left(\frac{(n-1)!}{(\frac{n}{2}+\frac{b}{2})!(\frac{n}{2}-\frac{b}{2})!}\right)\left(\frac{1}{\frac{n}{2}-\frac{b}{2}} - \frac{1}{\frac{n}{2}+\frac{b}{2}}\right) = p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \frac{b}{n}\left(\frac{n!}{(\frac{n}{2}+\frac{b}{2})!(\frac{n}{2}-\frac{b}{2})!}\right)$
$= \frac{b}{n} p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \binom{n}{\frac{n}{2}+\frac{b}{2}} = \frac{b}{n}\mathsf{P}(S_n = b)$. □

Theorem 10 (XXXMean number of visits to b before returning to origin in symmetric random walk)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a symmetric random walk . Mean number* μ_b *of visits of the walk to point* b *before returning to origin is equal to 1.*

Proof. aa □

Definition 7 (Return to origin)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a random walk . Let* $k \in \mathbb{N}$. *We say a* return to origin *occurred in time* $2k$ *if* $S_{2k} = 0$. *The probability that in time* $2k$ *occured a return to origin shall be denoted by* u_{2k}. *We say that in time* $2k$ *occurred* first return to origin *if* $S_1, S_2, \ldots S_{2k-1} \neq 0$ *and* $S_{2k} = 0$. *The probability that in time* $2k$ *occured first return to origin shall be denoted by* f_{2k}. *By definition* $f_0 = 0$. *Let* $\alpha 2n(2k)$ *denote* $u_{2k}u_{2(n-k)}$

Lemma 11 (Binomial identity)
Let $n, k \in \mathbb{N}, n > k : \binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{n}\binom{n}{k}$

Proof. $\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-k-1)!} - \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{1}{k} - \frac{1}{n-k}\right)$
$= \frac{1}{n}\frac{n!}{(k-1)!(n-k-1)!}\frac{n-2k}{k(n-k)} = \frac{n-2k}{n}\frac{n!}{k!(n-k)!} = \frac{n-2k}{n}\binom{n}{k}$ □

Lemma 12 (Main lemma)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a symmetrical random walk. Then* $P(S_1 \cdot S_2, \cdot, \ldots, S_{2n} \neq 0) = P(S_{2n} = 0)$.

Proof. $\mathsf{P}(S_1 \cdot S_2, \cdot, \ldots, S_{2n} \neq 0) = \sum_{i=-\infty}^{+\infty} \mathsf{P}(S_1 \cdot S_2, \cdot, \ldots, S_{2n-1} \neq 0, S_{2n} = 2i) = \sum_{i=-n}^{n} \mathsf{P}(S_1 \cdot S_2, \cdot, \ldots, S_{2n-1} \neq 0, S_{2n} = 2i) = 2 \cdot \sum_{i=1}^{n} \mathsf{P}(S_1 \cdot S_2, \cdot, \ldots, S_{2n-1} \neq 0, S_{2n} = 2i) = 2\sum_{i=1}^{n} \frac{2i}{2n}\mathsf{P}(S_{2n} = 2k) = 2\sum_{i=1}^{n}\frac{2i}{2n}\binom{2n}{n+i}2^{-2n} = 2 \cdot 2^{-2n}\sum_{i=1}^{n}\left(\binom{2n-1}{m+k-1} - \binom{2n-1}{m+k}\right) = 2 \cdot 2^{-2n}\binom{2n-1}{n} = 2^{-2n}\frac{2n}{n}\binom{2n-1}{n} = 2^{-2n}\frac{2n(2n-1)!}{m(m-1)!m!} = 2^{-2n}\frac{(2n)!}{m!m!} = 2^{-2n}\binom{2n}{n} = \mathsf{P}(S_2 n = 0)$ □

Theorem 13
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a symmetric random walk . The probability that the last return to origin up to time* $2n$ *occurred in time* $2k$ *is* $P(S_{2k} = 0)\, P\left(S_{2(n-k)} = 0\right)$.

Proof. $\alpha 2n\,(2k) = u_{2k}u_{2(n-k)} = \mathsf{P}(S_{2k} = 0)\,\mathsf{P}(S_{2k+1} \cdot S_{2k+2} \cdot, \ldots, S_{2n} \neq 0 \mid S_{2k} = 0) = \mathsf{P}(S_{2k} = 0)\,\mathsf{P}\left(S_1 \cdot S_2 \cdot, \ldots, S_{2(n-k)} \neq 0\right) = \mathsf{P}(S_{2k} = 0)\,\mathsf{P}\left(S_{2(n-k)} = 0\right)$ □

Theorem 14
Let $b \in Z$. $P(S_1 \cdot S_2 \cdot, \ldots, S_n \neq 0, S_n = b) = \frac{|b|}{n} P(S_n = b)$.

Proof. Let us Without loss of generalityassume that $b > 0$. In that case, first step has to be rightwards $(X_1 = +1)$. Now we have path from point $(1, 1)$ to point (n, b) that does not return to origin. By Ballot theorem 7 there are $\frac{b}{n}N_n(0, b)$ such paths. Each path consists of $\frac{n+b}{2}$ rightwards steps and $\frac{n-b}{2}$ leftwards steps. Therefore $\mathsf{P}(S_1 \cdot S_2 \cdot, \ldots, S_n \neq 0, S_n = b) = \frac{b}{n}N_n(0, b)\, p^{\frac{n+b}{2}} q^{\frac{n-b}{2}} = \frac{b}{n}\mathsf{P}(S_n = b)$. Case $b < 0$ is identical. □

Lemma 15
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a symmetric random walk .* $P(S_1, S_2, \ldots, S_{2n} > 0) = \frac{1}{2}P(S_{2n} = 0) = \frac{1}{2}u_{2n}$.

Proof. Because $S_i > 0 \forall i \in \mathbb{N}$ the first step has to be rightwards $(X_1 = S_1 = 1)$. Therefore we get $\mathsf{P}(S_1, S_2, \ldots, S_{2n} > 0) = \sum_{r=1}^{n-1}\mathsf{P}(S_1, S_2, \ldots, S_{2n} > 0, S_{2n} = 2r)$.
The r-th term follows equation: $\mathsf{P}(S_1, S_2, \ldots, S_{2n} > 0, S_{2n} = 2r)$
$= \mathsf{P}(X_1 = 1, S_2, \ldots, S_{2n} > 0, S_{2n} = 2r)$
$= \frac{1}{2}\mathsf{P}(S_2, S_3, \ldots, S_{2n} > 0, S_{2n} = 2r)$
$= \frac{1}{2}(\mathsf{P}(S_{2n} = 2r) - \mathsf{P}(S_2 \cdot S_3 \cdot \ldots \cdot S_{2n} = 0, S_{2n} = 2r))$
$= \frac{1}{2}\left(\frac{1}{2}^{2n-1}N_{2n-1}(1, 2r) - \frac{1}{2}^{2n-1}N^0_{2n-1}(1, 2r)\right)$
$= \frac{1}{2}\frac{1}{2}^{2n-1}\left(N_{2n-1}(1, 2r) - N^0_{2n-1}(1, 2r)\right)$
$= \frac{1}{2}\frac{1}{2}^{2n-1}(N_{2n-1}(1, 2r) - N_{2n-1}(-1, 2r))$

$= \frac{1}{2}\frac{1}{2}^{2n-1}\left(\binom{2n-1}{n+r-1} - \binom{2n-1}{n+r}\right)$. Because of the fact that the negative parts of r-th terms cancel against the positive parts of $(r+1)$-st terms and the sume reduces to just $\frac{1}{2}\frac{1}{2}^{2n-1}\binom{2n-1}{n} = \frac{1}{2}\cdot 2 \cdot \frac{1}{2}^{2n}\binom{2n-1}{n} = \frac{1}{2}\frac{1}{2}^{2n}\frac{2(2n-1)!}{n!(n-1)!} = \frac{1}{2}\frac{1}{2}^{2n}\frac{(2n)!}{n!n!} = \frac{1}{2}\frac{1}{2}^{2n}\binom{2n}{n} = \frac{1}{2}\,\mathsf{P}\left(S_{2n}=0\right) = \frac{1}{2}u_{2n}$. □

Theorem 16 (No return=return)
$P\left(S_1, S_2, \ldots, S_{2n} \neq 0\right) = P\left(S_{2n} = 0\right) = u_{2n}$

Proof. The event $[S_1, S_2, \ldots, S_{2n} \neq 0]$ can be split into two disjoint events: $= [S_1, S_2, \ldots, S_{2n} < 0] \cup [S_1, S_2, \ldots, S_{2n} > 0]$. By previous theorem (15) we get that probability of both of them is $\frac{1}{2}u_{2n}$. Because the the events are disjoint we can sum their probabilities and we get the result. □

Corollary. $\mathsf{P}\left(S_1, S_2, \ldots, S_{2n} \geq 0\right) = \mathsf{P}\left(S_{2n} = 0\right) = u_{2n}$

Proof. $\frac{1}{2}u_{2n} = \mathsf{P}\left(S_1, S_2, \ldots, S_{2n} > 0\right) = \mathsf{P}\left(X_1 = 1, S_2, S_3 \ldots, S_{2n} \geq 1\right)$
$= \frac{1}{2}\,\mathsf{P}\left(S_2, S_3 \ldots, S_{2n} \geq 1 \mid S_1 = 1\right)$
$= \frac{1}{2}\,\mathsf{P}\left(S_1, S_2 \ldots, S_{2n-1} \geq 1 \mid S_0 = 1\right)$
$= \frac{1}{2}\,\mathsf{P}\left(S_1, S_2 \ldots, S_{2n-1} \geq 0\right)$
$= \frac{1}{2}\,\mathsf{P}\left(S_1, S_2 \ldots, S_{2n} \geq 0\right)$. Therefore $\mathsf{P}\left(S_1, S_2 \ldots, S_{2n} \geq 0\right) = u_{2n}$. □

Theorem 17
$f_{2n} = u_{2n-2} - u_{2n}$

Proof. The event $[S_1, S_2, \ldots S_{2n-1} \neq 0]$ can be split into two disjoint events: $[S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} = 0]$ and $[S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} \neq 0]$. Therefore $\mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0\right) = \mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} = 0\right) + \mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} \neq 0\right)$. Therefore we get $f_{2n} = \mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} = 0\right) = \mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0\right) - \mathsf{P}\left(S_1, S_2, \ldots, S_{2n} \neq 0\right)$. Because $2n-1$ is odd. $\mathsf{P}\left(S_{2n-1} = 0\right) = 0$. Therefore the first term is equal to $\mathsf{P}\left(S_1, S_2, \ldots S_{2n-2} \neq 0\right)$ which is by 16 equal to u_{2n-2}. Second term is by 16 equal to u_{2n}. Therefore we get the result. □

Corollary. $f_{2n} = \frac{1}{2n-1}u_{2n}$

Proof. $u_{2n-2} = \frac{1}{2}^{2n-2}\binom{2n-2}{n-1} = 4 \cdot \frac{1}{2}^{2n}\frac{(2n-2)!}{(n-1)!(n-1)!} = \frac{4n^2}{(2n)(2n-1)}\frac{1}{2}^{2n}\binom{2n}{n} = \frac{2n}{2n-1}u_{2n}$. Therefore $u_{2n-2} - u_{2n} = u_{2n}\left(\frac{2n}{2n-1} - 1\right) = u_{2n}\frac{1}{2n-1}$. □

Theorem 18 (Arcsine law for last visits)
Let $k, n \in \mathbb{N}, k \leq n$. The probability that up to time $2n$ a return to origin occured in time $2k$ is given by $\alpha_{2n}\left(2k\right) = u_{2n}u_{2(n-k)}$.

Proof. The probability involved can be rewritten as:
$\mathsf{P}\left(S_{2k+1}, S_{2k+2}, \ldots, S_{2n} \neq 0, S_{2k} = 0\right)$
$= *** \,\mathsf{P}\left(S_{2k+1}, S_{2k+2}, \ldots, S_{2n} \neq 0 \mid S_{2k} = 0\right)\mathsf{P}\left(S_{2k} = 0\right)$
$= * \,\mathsf{P}\left(S_1, S_2, \ldots, S_{2(n-k)} \neq 0\right)\mathsf{P}\left(S_{2k} = 0\right)$
$= ** \,u_{2(n-k)}u_{2k}$ □

Definition 8 (Time spend on the positive and negative sides)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk . We say that the walk spent τ time units of n on the positive side if $\sum_{i=1}^{n}\mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = \tau$. Let $\beta_n\left(\tau\right)$ denote the probability of such an event. We say that the walk spent ζ time units of n on the negative side if $\sum_{i=1}^{n}\mathbf{1}_{[S_i<0 \vee S_{i-1}<0]} = \zeta$.

Theorem 19 (Arcsine law for sojourn times-OWN PROOF)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a symmetric random walk . Then* $\beta_{2n}(2k) = \alpha_{2n}(2k)$.

Proof. Firstly let us start with degenerate cases. *f$\beta_{2n}(2n)$
$= \mathsf{P}\left(\sum\limits_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2n\right) = \mathsf{P}(S_1, S_2, \ldots, S_{2n} \geq 0) = *u_{2n}$.

By symmetry $\beta_{2n}(0) = \beta_{2n}(2n) = u_{2n}$. Let $1 \leq k \leq v-1$, where $0 \leq v \leq n$.
For such k stands equation: $\beta_{2n}(2k) = \mathsf{P}\left(\sum\limits_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k\right)$
$= *a \sum\limits_{r=1}^{n} \mathsf{P}\left(\sum\limits_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} \neq 0, S_{2r} = 0\right)$
$= *b \sum\limits_{r=1}^{n} \mathsf{P}\left(\sum\limits_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right)$
$+ \sum\limits_{r=1}^{n} \mathsf{P}\left(\sum\limits_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right)$
$= *c \sum\limits_{r=1}^{n} \mathsf{P}\left(\sum\limits_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k \;\middle|\; S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right)$
$\mathsf{P}(S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0)$
$+ \sum\limits_{r=1}^{n} \mathsf{P}\left(\sum\limits_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k \;\middle|\; S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right)$
$\mathsf{P}(S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0)$
$= *d \sum\limits_{r=1}^{n} \frac{1}{2} f_{2r} \mathsf{P}\left(\sum\limits_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k \;\middle|\; S_{2r} = 0\right)$
$+ \sum\limits_{r=1}^{n} \frac{1}{2} f_{2r} \mathsf{P}\left(\sum\limits_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k - 2r \;\middle|\; S_{2r} = 0\right)$
$= *e \sum\limits_{r=1}^{n} \frac{1}{2} f_{2r} \mathsf{P}\left(\sum\limits_{i=1}^{2n-2r} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k\right) + \sum\limits_{r=1}^{n} \frac{1}{2} f_{2r} \mathsf{P}\left(\sum\limits_{i=1}^{2n-2r} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k - 2r\right) =$
$\sum\limits_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k) + \sum\limits_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k-2r)$ Now let us proceed by induction.
Case for $v = 1$ is trivial because it implies degenerous case from *f. Let the statment be true for $v \leq n-1$, then $\sum\limits_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k) + \sum\limits_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k-2r)$
$= *g \sum\limits_{r=1}^{n} \frac{1}{2} f_{2r} \alpha_{2n-2r}(2k) + \sum\limits_{r=1}^{n} \frac{1}{2} f_{2r} \alpha_{2n-2r}(2k-2r)$
$= *h \sum\limits_{r=1}^{n} \frac{1}{2} f_{2r} u_{2k} u_{2n-2r-2k} + \sum\limits_{r=1}^{n} \frac{1}{2} f_{2r} u_{2k-2r} u_{2n-2k}$
$= *i \frac{1}{2} u_{2k} \sum\limits_{r=1}^{n} f_{2r} u_{2n-2r-2k} + \frac{1}{2} u_{2n-2k} \sum\limits_{r=1}^{n} f_{2r} u_{2k-2r} u_{2n-2k}$
$= *j \frac{1}{2} u_{2n-2k} u_{2k} + \frac{1}{2} u_{2n-2k} u_{2k} = u_{2n-2k} u_{2k}$
$= *h \alpha_{2n}(2k)$

□

Definition 9 (Change of a sign)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a random walk . We say that in time n occurred a change of sign if if* $S_{n-1} \cdot S_{n+1} = -1$ *in other words if* $(S_{n-1} = +1 \wedge S_{n+1} = -1) \vee (S_{n-1} = -1 \wedge S_{n+1} = +1)$. *We shall denote the probability that up to time n occured r changes of sign by* $\xi_{r,n}$.

Theorem 20 (Change of a sign)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a symmetric random walk . The probability* $\xi_{r,2n+1} = 2\,P(S_{2n+1} = 2r+1)$

Proof. Feller □

1.1 Problem chapter 9 Feller

Definition 10 (δ, ε)

Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a symmetric random walk .* $\delta_n(k)$ *shall denote* $\mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_n =\right.$

$\varepsilon_n^r(k)$ *shall denote* $\mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_1, S_2, \ldots, S_{r-1} \neq 0, S_r = 0, S_n = 0\right)$,

$\varepsilon_n^{r,+}(k)$ *shall denote* $\mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_1, S_2, \ldots, S_{r-1} > 0, S_r = 0, S_n = 0\right)$,

$\varepsilon_n^{r,-}(k)$ *shall denote* $\mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_1, S_2, \ldots, S_{r-1} < 0, S_r = 0, S_n = 0\right)$.

Lemma 21 (Factorization of $\delta_{2n}(2k)$)

$\delta_{2n}(2k) = \frac{1}{2}\sum_{r=1}^{n}\left(f_{2r}\delta_{2n-2r}(2k-2r) + f_{2r}\delta_{2n-2r}(2r)\right)$.

Proof. Because $S_{2n} = 0$ a return to origin must have happened. Let $2r$ the time of first return to origin, where $r \in \{1, 2, \ldots, n\}$. By the law of total probability:

$\delta_{2n}(2k)$

$= \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_{2n} = 0\right)$

$= \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} \neq 0, S_{2r} = 0 S_{2n} = 0\right)$ which can be again by the law of total probability factorized as:

$\sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} \neq 0, S_{2r} = 0 S_{2n} = 0\right)$

$= \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0 S_{2n} = 0\right)$

$+ \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0 S_{2n} = 0\right)$

$= \sum_{r=1}^{n} \varepsilon_{2n}^{2r,+}(2k) + \sum_{r=1}^{n} \varepsilon_{2n}^{2r,-}(2k)$.

Now let us calculate $\varepsilon_{2n}^{2r,+}(2k)$

$= \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0 S_{2n} = 0\right)$

$= *a\, \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n} = 0 \mid S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right)$

$\mathsf{P}(S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0)$

$= \mathsf{P}\left(\sum_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k - 2r, S_{2n} = 0 \,\middle|\, S_{2r} = 0\right) \frac{1}{2} f_{2r}$

$= *b\, \mathsf{P}\left(\sum_{i=1}^{2n-2r} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k - 2r, S_{2n-2r} = 0\right) \frac{1}{2} f_{2r}$

$= *c\delta_{2n-2r}(2k-2r)\frac{1}{2} f_{2r}$. Similarly $\varepsilon_{2n}^{2r,-}(2k)$

$= \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0 S_{2n} = 0\right)$

$= *a\, \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n} = 0 \mid S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right)$

$\mathsf{P}(S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0)$

$= \mathsf{P}\left(\sum_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n} = 0 \,\middle|\, S_{2r} = 0\right) \frac{1}{2} f_{2r}$

$= *b\, \mathsf{P}\left(\sum_{i=1}^{2n-2r} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n-2r} = 0\right) \frac{1}{2} f_{2r} = *c\delta_{2n-2r}(2k)\frac{1}{2} f_{2r}$.

Therefore $\delta_{2n}(2k) = \frac{1}{2}\sum\limits_{r=1}^{n} f_{2r}\delta_{2n-2r}(2k-2r) + \frac{1}{2}\sum\limits_{r=1}^{n} f_{2r}\delta_{2n-2r}(2k)$
$= \frac{1}{2}\sum\limits_{r=1}^{n}\left(f_{2r}\delta_{2n-2r}(2k-2r) + f_{2r}\delta_{2n-2r}(2r)\right)$ □

Theorem 22 (Equidistributional theorem-ALMOST COMPLETE OWN PROOF) *Let* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a symmetric random walk and* $n \in \mathbb{N}$, *then* $\forall k, l \in \{0, 1, \ldots, n\}$: $\delta_{2n}(2k) = \delta_{2n}(2l) = \frac{u_{2n}}{n+1}$.

Proof. Let us prove this statement by induction in n. In case that $n = 1$ we have two options for k. Either $k = 0$ or $k = 1$. $\delta_2(0) = \mathsf{P}(S_1 < 0, S_2 = 0) = \frac{1}{2}f_2 = *a\frac{1}{2}u_2\frac{1}{2-1} = \frac{u_2}{2}\delta_2(2) = \mathsf{P}(S_1 > 0, S_2 = 0) = \frac{1}{2}f_2 = \frac{u_2}{2}$.

Let the statment be true for all $l \leq n-1$. In that case $\delta_{2(n-l)}(2k) = \frac{u_{2(n-l)}}{n-l+1}$. We want to show that $\delta_{2n} = \frac{u_{2n}}{n+1}$.

Let us calculate δ_{2n}. $\delta_{2n} = *b\frac{1}{2}\sum\limits_{r=1}^{n}\left(f_{2r}\delta_{2n-2r}(2k-2r) + f_{2r}\delta_{2n-2r}(2r)\right) = \frac{1}{2}\sum\limits_{r=1}^{n}\left(f_{2r}u_{2n-2r}\frac{1}{n-r+1} + f_{2r}u_{2n-2r}\frac{1}{n-r+1}\right) = \sum\limits_{r=1}^{n}\left(\frac{f_{2r}u_{2n-2r}}{n-r+1}\right) =? \frac{u_{2n}}{n+1}$ Because???

□

Lemma 23
$\sum\limits_{r=1}^{n}\left(\frac{1}{n-r+1}\left(2^{-(2r-2)}\binom{2r-2}{r-1} - 2^{-2r}\binom{2r}{r}\right)2^{-(2n-2r)}\binom{2n-2r}{n-r}\right) = \frac{1}{n+1}2^{-2n}\binom{2n}{n}$

Proof. □