1. Title of the first chapter

Definition 1 (Simple random walk in $\mathbb{Z}$). *Let $\{X_n\}_{n=0}^{+\infty}$ be a sequence of independent and identically distributedrandom variables with values in $\{-1, +1\}$. That $\forall n \in \mathbb{N}$ satisfy $P(X_n = 1) = p \in (0, 1)$ and $P(X_n = -1) = 1 - p = q$. Let $S_0 = 0$ and $S_n = \sum\limits_{i=1}^{n} X_i$. We call the pair $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ Simple random walk in $\mathbb{Z}$. In case that $p = q = \frac{1}{2}$ we call the pair $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ Symmetric simple random walk in $\mathbb{Z}$.*

Remark. Very often we refer to n as time , X_i as i-th step and S_n as position in time n. In simple random walk in $\mathbb{Z}$ we refer to $X_i = +1$ as i-th step was rightwards. If not stated otherwise, we

Definition 2 (Set of possible positions). *Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk. We call the set $A_n = \{z \in \mathbb{Z}; |z| \leq n, \frac{z+n}{2} \in \mathbb{Z}\}$ set of all possible positions of random walk $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ in time n.*

Theorem 1 (Probability of position x in time n). *Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk and A_n its set of possible positions.*

$$P(S_n = x) = \begin{cases} \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}} & \text{for } x \in A_n \ , \\ 0, & \text{for } x \notin A_n. \end{cases}$$

Proof. Let us define new variables $r_i = \mathbf{1}_{[X_i=1]}, l_i = \mathbf{1}_{[X_i=-1]}, R_n = \sum\limits_{i=1}^{n} r_i, L_n = \sum\limits_{i=1}^{n} l_i$. r_i can be interpreted as indicator wether i-th step was rightwards. Then R_n is number of rightwards steps and L_n is number of leftwards steps. We can easily see that $R_n + L_n = n$ and $R_n - L_n = S_n$. Therefore we get by adding these two equations $R_n = \frac{S_n + n}{2}$.

r_i has alternative distribution with parameter p (*Alt(p)*). Therefore R_n as a sum of independent and identically distributedrandom variables with *Alt(p)* has binomial distribution with parameters n and p (*Bi(n,p)*). Therefore we get $\mathsf{P}(R_n = x) = \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}$. Where we define $\binom{a}{x} := 0$ for $a \in \mathbb{N}, x \in \mathbb{R} \setminus \mathbb{N}$, $x < 0, x > n$. Therefore we get $\mathsf{P}(S_n = x) = \mathsf{P}\left(R_n = \frac{x+n}{2}\right) = \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}$. $\square$

Lemma 2 (Spatial homogeneity). *$P(S_n = j|S_0 = a) = P(S_n = j + b|S_0 = a + b) \forall b \in \mathbb{Z}$*

Proof. $\mathsf{P}(S_n = j|S_0 = a) = \mathsf{P}\left(\sum\limits_{i=1}^{n} X_i = j - a\right) = \mathsf{P}\left(\sum\limits_{i=1}^{n} X_i = (j + b) - (a + b)\right) = \mathsf{P}(S_n = j + b|S_0 = a + b)$. $\square$

Lemma 3 (Temporal homogeneity). *$P(S_n = j|S_0 = a) = P(S_{n+m} = j|S_m = a) \forall m \in \mathbb{N}$*

Proof. $\mathsf{P}(S_n = j|S_0 = a) = \mathsf{P}\left(\sum\limits_{i=1}^{n} X_i = j - a\right) = \mathsf{P}\left(\sum\limits_{i=m+1}^{m+n} X_i = j - a\right) = \mathsf{P}(S_{n+m} = j|S_m = a)$. $\square$

Lemma 4 (Markov property). *Let $n \geq m$ and $a_i \in \mathbb{Z}$. Then $P(S_n = j | S_0 = a_0, S_1 = a_1, \ldots, S_m = a_m) = P(S_n = j | S_m = a_m)$*

Proof. Once S_m is known, then distribution of S_n depends only on steps $X_{m+1}, X_{m+2}, \ldots X_n$ and therefore cannot be dependent on any information concerning values $X_1, X_2, \ldots, X_m - 1$ and accordingly $S_1, S_2, \ldots, S_m - 1$. $\square$

Remark. In symmetric random walk, everything can be counted by number of possible paths from point to point.

Definition 3 (Number of possible paths). *Let $N_n(a, b)$ be number of posssible paths of random walk $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ from point $(0, a)$ to point (n, b) and $N_n^x(a, b)$ be number of possible paths from point $(0, a)$ to point (n, b) that visit point (z, x) for some $z \in \{0, \ldots n\}$.*

Theorem 5. *Let $a, b \in \mathbb{Z}, n \in \mathbb{N}$ then $N_n(a, b) = \binom{n}{\frac{1}{2}(n+b-a)}$.*

Proof. Let us choose a path from $(0, a)$ to (n, b) and let α be number of rightwards steps and β be number of leftwards steps. Then $\alpha + \beta = n$ and $\alpha - \beta = b - a$. By adding these two equations we get that $\alpha = \frac{1}{2}(n + b - a)$. The number of possible paths is the number of ways of picking α rightwards steps from n steps. Therefore we get $N_n(a, b) = \binom{n}{\alpha} = \binom{n}{\frac{1}{2}(n+b-a)}$. $\square$

Theorem 6 (Reflection principle). *Let $a, b > 0$, then $N_n^0(a, b) = N_n(-a, b)$.*

Proof. Each path from $(0, -a)$ to (n, b) has to intersect x-axis at least once at some point. Let k be the time of earliest intersection with x-axis. By reflexing the segment from $(0, -a)$ to $(k, 0)$ in the x-axis, we get a path from point $(0, a)$ to (n, b) which visits 0 at point k. Because reflection is bijective operation on sets of paths, we get the correspondence between the collections of such paths. $\square$

Definition 4 (Return to origin). *Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk. Then if $\exists k \in \mathbb{N}$ such that $S_k = 0$ then we say that in k-th step occured return to origin. Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk. Then if $S_1, S_2, \ldots, S_{2n-1} \neq 0$ and $S_{2n} = 0$*

Theorem 7 (Ballot theorem). *Let $n, b \in N$ Number of paths from point $(0,0)$ to point (n, b) which do not return to origin is equal to $\frac{b}{n} N_n(0, b)$*

Proof. Let us call N the number of paths we are refering to. Because the path ends at point (n, b), the first step has to be rightwards. Therefore we now have $N = N_{n-1}(1, b) - N_{n-1}^0(1, b) = N_{n-1}(1, b) - N_{n-1}(-1, b)$. The last equation was aquired using Ballot theorem (6). We now have:

$$N_{n-1}(1,b) - N_{n-1}(-1, b) = \binom{n-1}{\frac{n}{2} + \frac{b}{2} - 1} - \binom{n-1}{\frac{n}{2} + \frac{b}{2}} = \frac{(n-1)!}{(\frac{n}{2} + \frac{b}{2} - 1)!(\frac{n}{2} - \frac{b}{2})!} - \frac{(n-1)!}{(\frac{n}{2} + \frac{b}{2})!(\frac{n}{2} - \frac{b}{2} - 1)!}$$

$\square$

Definition 5. $M_n = \max\{S_i, i \in \{0, 1, \ldots, n\}\}$

Theorem 8 (Probability of maximum up to time n). *Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk.*

$$P(M_n \geq r, S_n = b) = \begin{cases} P(S_n = b) & \text{for } b \geq r, \\ P(S_n = 2r - b)\left(\frac{q}{p}\right)^{r-b}, & \text{for otherwise.} \end{cases}$$

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Proof. Let us firstly consider the easier case in which $b \geq r$. Because we defined M_n as $\max\{S_i, i \in \{0, 1, \ldots, n\}\}$ we get that $M_n \geq b \geq r$ therefore $[M_n \geq r] \subset [S_n = b]$ therefore we get $\mathsf{P}(M_n \geq r, S_n = b) = \mathsf{P}(S_n = b)$. Let $r \geq 1, b < r$. $N_n^r(0, b)$ stands for number of paths from point $(0, 0)$ to point (n, b) which reach up to r. Let $k \in \{0, 1, \ldots, n\}$ denote the first time we reach r. By reflection principle (6), we can reflex the segment from (k, r) to (n, b) in the axis:$y = r$. Therefore we now have path from $(0, 0)$ to $(n, 2r - b)$ and we get that $N_n^r(0, b) = N_n(0, 2r - b)$. $\mathsf{P}(S_n = b, M_n \geq r) = N_n^r(0,b)p^{\frac{n+b}{2}}q^{\frac{n-b}{2}} = N_n(0, 2r - b)p^{\frac{n+(2r-b)}{2}}q^{\frac{n-(2r-b)}{2}}p^{\frac{n+b}{2}}q^{\frac{n-b}{2}}p^{b-r}q^{r-b} = \left(\frac{q}{p}\right)^{r-b}\mathsf{P}(S_n = 2r - b).$ $\qquad \square$

Definition 6 (Walk reaching new maximum at particular time). *Let $b > 0$. $f_b(n)$ denotes the probability that we reach new maximum b in time n. $f_b(n) = \mathsf{P}(M_{n-1} = S_{n-1} = b - 1, S_n = b)$*

Theorem 9 (Probability of reaching new maximum b in time n). *Let $b > 0$ then $f_b(n) = \frac{b}{n}\mathsf{P}(S_n = b)$.*

Proof. $f_b = \mathsf{P}(M_{n-1} = S_{n-1} = b - 1, S_n = b) = p\,\mathsf{P}(M_{n-1} = S_{n-1} = b - 1) = p\left(\mathsf{P}(M_{n-1} \geq b - 1, S_{n-1} = b - 1) - \mathsf{P}(M_{n-1} \geq b, S_{n-1} = b - 1)\right) = p\left(\mathsf{P}(S_{n-1} = b - 1) - \frac{q}{p}\mathsf{P}(S_{n-1} = b - 1) \right.$
$p\,\mathsf{P}(S_{n-1} = b - 1) - q\,\mathsf{P}(S_{n-1} = b + 1) = \binom{n-1}{\frac{n}{2}+\frac{b}{2}-1}p^{\frac{n}{2}+\frac{b}{2}}q^{\frac{n}{2}-\frac{b}{2}} - \binom{n-1}{\frac{n}{2}+\frac{b}{2}}p^{\frac{n}{2}+\frac{b}{2}}q^{\frac{n}{2}-\frac{b}{2}} =$
$p^{\frac{n}{2}+\frac{b}{2}}q^{\frac{n}{2}-\frac{b}{2}}\left(\frac{(n-1)!}{(\frac{n}{2}+\frac{b}{2}-1)!(\frac{n}{2}-\frac{b}{2})!} - \frac{(n-1)!}{(\frac{n}{2}+\frac{b}{2})!(\frac{n}{2}-\frac{b}{2}-1)!}\right) = p^{\frac{n}{2}+\frac{b}{2}}q^{\frac{n}{2}-\frac{b}{2}}\left(\frac{(n-1)!}{(\frac{n}{2}+\frac{b}{2})!(\frac{n}{2}-\frac{b}{2})!}\right)\left(\frac{1}{\frac{n}{2}-\frac{b}{2}} - \frac{1}{\frac{n}{2}+\frac{b}{2}}\right) =\Big|$
$p^{\frac{n}{2}+\frac{b}{2}}q^{\frac{n}{2}-\frac{b}{2}}\frac{b}{n}\left(\frac{n!}{(\frac{n}{2}+\frac{b}{2})!(\frac{n}{2}-\frac{b}{2})!}\right) = \frac{b}{n}p^{\frac{n}{2}+\frac{b}{2}}q^{\frac{n}{2}-\frac{b}{2}}\binom{n}{\frac{n}{2}+\frac{b}{2}} = \frac{b}{n}\mathsf{P}(S_n = b).$ $\qquad \square$

Theorem 10 (XXXMean number of visits to b before returning to origin in symmetric random walk). *Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a symmetric random walk. Mean number μ_b of visits of the walk to point b before returning to origin is equal to 1.*

Proof. aa $\qquad \square$

Definition 7 (Return to origin). *Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk. Let $k \in \mathbb{N}$. We say a* return to origin *occurred in time $2k$ if $S_{2k} = 0$. The probability that in time $2k$ occured a return to origin shall be denoted by u_{2k}. We say that in time $2k$ occurred* first return to origin *if $S_1, S_2, \ldots S_{2k-1} \neq 0$ and $S_{2k} = 0$. The probability that in time $2k$ occured first return to origin shall be denoted by f_{2k}. By definition $f_0 = 0$.*

Lemma 11 (Binomial identity). *Let $n, k \in \mathbb{N}, n > k : \binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{n}\binom{n}{k}$*

Proof. $\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-k-1)!} - \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{1}{k} - \frac{1}{n-k}\right) = \frac{1}{n}\frac{n!}{(k-1)!(n-k-1)!}\frac{n-2k}{k(n-k)} =$
$\frac{n-2k}{n}\frac{n!}{k!(n-k)!} = \frac{n-2k}{n}\binom{n}{k}$ $\qquad \square$

Lemma 12 (Main lemma). *Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a symmetrical random walk. Then $\mathsf{P}(S_1 \cdot S_2, \cdot, \ldots, S_{2n} \neq 0) = \mathsf{P}(S_{2n} = 0)$*

Proof. $\mathsf{P}(S_1 \cdot S_2, \cdot, \ldots, S_{2n} \neq 0) = \sum\limits_{i=-\infty}^{+\infty} \mathsf{P}(S_1 \cdot S_2, \cdot, \ldots, S_{2n-1} \neq 0, S_{2n} = 2i) =$
$\sum\limits_{i=-n}^{n} \mathsf{P}(S_1 \cdot S_2, \cdot, \ldots, S_{2n-1} \neq 0, S_{2n} = 2i) = 2 \cdot \sum\limits_{i=1}^{n} \mathsf{P}(S_1 \cdot S_2, \cdot, \ldots, S_{2n-1} \neq 0, S_{2n} =$
$2i) = 2\sum\limits_{i=1}^{n}\frac{2i}{2n}\mathsf{P}(S_{2n} = 2k) = 2\sum\limits_{i=1}^{n}\frac{2i}{2n}\binom{2n}{n+i}2^{-2n} = 2 \cdot 2^{-2n}\sum\limits_{i=1}^{n}\left(\binom{2n-1}{m+k-1} - \binom{2n-1}{m+k}\right) =$
$2 \cdot 2^{-2n}\binom{2n-1}{n} = 2^{-2n}\frac{2n}{n}\binom{2n-1}{n} = 2^{-2n}\frac{2n(2n-1)!}{m(m-1)!m!} = 2^{-2n}\frac{(2n)!}{m!m!} = 2^{-2n}\binom{2n}{n} = \mathsf{P}(S_2n = 0)$ $\qquad \square$

1.1 Title of the first subchapter of the first chapter

1.2 Title of the second subchapter of the first chapter

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A. Attachments

A.1 First Attachment