

# 1. Title of the first chapter

**Definition 1** (Simple random walk in  $\mathbb{Z}$ ). Let  $\{X_n\}_{n=0}^{+\infty}$  be a sequence of independent and identically distributed random variables with values in  $\{-1, +1\}$ . That  $\forall n \in \mathbb{N}$  satisfy  $P(X_n = 1) = p \in (0, 1)$  and  $P(X_n = -1) = 1 - p = q$ . Let  $S_0 = 0$  and  $S_n = \sum_{i=1}^n X_i$ . We call the pair  $(\{S_n\}_{n=0}^{+\infty}, p)$  Simple random walk in  $\mathbb{Z}$ . In case that  $p = q = \frac{1}{2}$  we call the pair  $(\{S_n\}_{n=0}^{+\infty}, p)$  Symmetric simple random walk in  $\mathbb{Z}$ .

*Remark.* Very often we refer to  $n$  as time,  $X_i$  as  $i$ -th step and  $S_n$  as position in time  $n$ . In simple random walk in  $\mathbb{Z}$  we refer to  $X_i = +1$  as  $i$ -th step was rightwards. If not stated otherwise, we

**Definition 2** (Set of possible positions). Let  $(\{S_n\}_{n=0}^{+\infty}, p)$  be a random walk. We call the set  $A_n = \{z \in \mathbb{Z}; |z| \leq n, \frac{z+n}{2} \in \mathbb{Z}\}$  set of all possible positions of random walk  $(\{S_n\}_{n=0}^{+\infty}, p)$  in time  $n$ .

**Theorem 1** (Probability of position  $x$  in time  $n$ ). Let  $(\{S_n\}_{n=0}^{+\infty}, p)$  be a random walk and  $A_n$  its set of possible positions.

$$P(S_n = x) = \begin{cases} \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}} & \text{for } x \in A_n, \\ 0, & \text{for } x \notin A_n. \end{cases}$$

*Proof.* Let us define new variables  $r_i = \mathbf{1}_{[X_i=1]}, l_i = \mathbf{1}_{[X_i=-1]}, R_n = \sum_{i=1}^n r_i, L_n = \sum_{i=1}^n l_i$ .  $r_i$  can be interpreted as indicator whether  $i$ -th step was rightwards. Then  $R_n$  is number of rightwards steps and  $L_n$  is number of leftwards steps. We can easily see that  $R_n + L_n = n$  and  $R_n - L_n = S_n$ . Therefore we get by adding these two equations  $R_n = \frac{S_n + n}{2}$ .

$r_i$  has alternative distribution with parameter  $p$  ( $Alt(p)$ ). Therefore  $R_n$  as a sum of independent and identically distributed random variables with  $Alt(p)$  has binomial distribution with parameters  $n$  and  $p$  ( $Bi(n, p)$ ). Therefore we get  $P(R_n = x) = \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}$ . Where we define  $\binom{a}{x} := 0$  for  $a \in \mathbb{N}, x \in \mathbb{R} \setminus \mathbb{N}, x < 0, x > n$ . Therefore we get  $P(S_n = x) = P(R_n = \frac{x+n}{2}) = \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}$ .  $\square$

**Lemma 2** (Spatial homogeneity).  $P(S_n = j | S_0 = a) = P(S_n = j + b | S_0 = a + b) \forall b \in \mathbb{Z}$

*Proof.*  $P(S_n = j | S_0 = a) = P\left(\sum_{i=1}^n X_i = j - a\right) = P\left(\sum_{i=1}^n X_i = (j + b) - (a + b)\right) = P(S_n = j + b | S_0 = a + b)$ .  $\square$

**Lemma 3** (Temporal homogeneity).  $P(S_n = j | S_0 = a) = P(S_{n+m} = j | S_m = a) \forall m \in \mathbb{N}$

*Proof.*  $P(S_n = j | S_0 = a) = P\left(\sum_{i=1}^n X_i = j - a\right) = P\left(\sum_{i=m+1}^{m+n} X_i = j - a\right) = P(S_{n+m} = j | S_m = a)$ .  $\square$

**Lemma 4** (Markov property). *Let  $n \geq m$  and  $a_i \in \mathbb{Z}$ . Then  $P(S_n = j | S_0 = a_0, S_1 = a_1, \dots, S_m = a_m) = P(S_n = j | S_m = a_m)$*

*Proof.* Once  $S_m$  is known, then distribution of  $S_n$  depends only on steps  $X_{m+1}, X_{m+2}, \dots, X_n$  and therefore cannot be dependent on any information concerning values  $X_1, X_2, \dots, X_m$  and accordingly  $S_1, S_2, \dots, S_m - 1$ .  $\square$

*Remark.* In symmetric random walk, everything can be counted by number of possible paths from point to point.

**Definition 3** (Number of possible paths). *Let  $N_n(a, b)$  be number of possible paths of random walk  $(\{S_n\}_{n=0}^{+\infty}, p)$  from point  $(0, a)$  to point  $(n, b)$  and  $N_n^x(a, b)$  be number of possible paths from point  $(0, a)$  to point  $(n, b)$  that visit point  $(z, x)$  for some  $z \in \{0, \dots, n\}$ .*

**Theorem 5.** *Let  $a, b \in \mathbb{Z}, n \in \mathbb{N}$  then  $N_n(a, b) = \binom{n}{\frac{1}{2}(n+b-a)}$ .*

*Proof.* Let us choose a path from  $(0, a)$  to  $(n, b)$  and let  $\alpha$  be number of rightwards steps and  $\beta$  be number of leftwards steps. Then  $\alpha + \beta = n$  and  $\alpha - \beta = b - a$ . By adding these two equations we get that  $\alpha = \frac{1}{2}(n + b - a)$ . The number of possible paths is the number of ways of picking  $\alpha$  rightwards steps from  $n$  steps. Therefore we get  $N_n(a, b) = \binom{n}{\alpha} = \binom{n}{\frac{1}{2}(n+b-a)}$ .  $\square$

**Theorem 6** (Reflection principle). *Let  $a, b > 0$ , then  $N_n^0(a, b) = N_n(-a, b)$ .*

*Proof.* Each path from  $(0, -a)$  to  $(n, b)$  has to intersect  $x$ -axis at least once at some point. Let  $k$  be the time of earliest intersection with  $x$ -axis. By reflexing the segment from  $(0, -a)$  to  $(k, 0)$  in the  $x$ -axis, we get a path from point  $(0, a)$  to  $(n, b)$  which visits 0 at point  $k$ . Because reflection is bijective operation on sets of paths, we get the correspondence between the collections of such paths.  $\square$

**Definition 4** (Return to origin). *Let  $(\{S_n\}_{n=0}^{+\infty}, p)$  be a random walk. Then if  $\exists k \in \mathbb{N}$  such that  $S_k = 0$  then we say that in  $k$ -th step occurred return to origin. Let  $(\{S_n\}_{n=0}^{+\infty}, p)$  be a random walk. Then if  $S_1, S_2, \dots, S_{2n-1} \neq 0$  and  $S_{2n} = 0$*

**Theorem 7** (Ballot theorem). *Let  $n, b \in \mathbb{N}$  Number of paths from point  $(0, 0)$  to point  $(n, b)$  which do not return to origin is equal to  $\frac{b}{n} N_n(0, b)$*

*Proof.* Let us call  $N$  the number of paths we are refering to. Because the path ends at point  $(n, b)$ , the first step has to be rightwards. Therefore we now have  $N = N_{n-1}(1, b) - N_{n-1}^0(1, b) = N_{n-1}(1, b) - N_{n-1}(-1, b)$ . The last equation was aquired using Ballot theorem (??). We now have:

$$N_{n-1}(1, b) - N_{n-1}(-1, b) = \binom{n-1}{\frac{n}{2} + \frac{b}{2} - 1} - \binom{n-1}{\frac{n}{2} + \frac{b}{2}} = \frac{(n-1)!}{(\frac{n}{2} + \frac{b}{2} - 1)!(\frac{n}{2} - \frac{b}{2})!} - \frac{(n-1)!}{(\frac{n}{2} + \frac{b}{2})!(\frac{n}{2} - \frac{b}{2} - 1)!}$$

$\square$

**Definition 5.**  $M_n = \max\{S_i, i \in \{0, 1, \dots, n\}\}$

**Theorem 8** (Probability of maximum up to time  $n$ ). *Let  $(\{S_n\}_{n=0}^{+\infty}, p)$  be a random walk.*

$$P(M_n \geq r, S_n = b) = \begin{cases} P(S_n = b) & \text{for } b \geq r, \\ P(S_n = 2r - b) \left(\frac{q}{p}\right)^{r-b}, & \text{for otherwise.} \end{cases}$$

*Proof.* Let us firstly consider the easier case in which  $b \geq r$ . Because we defined  $M_n$  as  $\max\{S_i, i \in \{0, 1, \dots, n\}\}$  we get that  $M_n \geq b \geq r$  therefore  $[M_n \geq r] \subset [S_n = b]$  therefore we get  $P(M_n \geq r, S_n = b) = P(S_n = b)$ . Let  $r \geq 1, b < r$ .  $N_n^r(0, b)$  stands for number of paths from point  $(0, 0)$  to point  $(n, b)$  which reach up to  $r$ . Let  $k \in \{0, 1, \dots, n\}$  denote the first time we reach  $r$ . By reflection principle (??), we can reflex the segment from  $(k, r)$  to  $(n, b)$  in the axis:  $y = r$ . Therefore we now have path from  $(0, 0)$  to  $(n, 2r - b)$  and we get that  $N_n^r(0, b) = N_n(0, 2r - b)$ .  $P(S_n = b, M_n \geq r) = N_n^r(0, b)p^{\frac{n+b}{2}}q^{\frac{n-b}{2}} = N_n(0, 2r - b)p^{\frac{n+(2r-b)}{2}}q^{\frac{n-(2r-b)}{2}}p^{\frac{n+b}{2}}q^{\frac{n-b}{2}}p^{b-r}q^{r-b} = \left(\frac{q}{p}\right)^{r-b}P(S_n = 2r - b)$ .  $\square$

**Definition 6** (Walk reaching new maximum at particular time ). *Let  $b > 0$ .  $f_b(n)$  denotes the probability that we reach new maximum  $b$  in time  $n$ .  $f_b(n) = P(M_{n-1} = S_{n-1} = b - 1, S_n = b)$*

**Theorem 9** (Probability of reaching new maximum  $b$  in time  $n$ ). *Let  $b > 0$  then  $f_b(n) = \frac{b}{n} P(S_n = b)$ .*

*Proof.*  $f_b = P(M_{n-1} = S_{n-1} = b - 1, S_n = b) = pP(M_{n-1} = S_{n-1} = b - 1) = p(P(M_{n-1} \geq b - 1, S_{n-1} = b - 1) - P(M_{n-1} \geq b, S_{n-1} = b - 1)) = p(P(S_{n-1} = b - 1) - \frac{q}{p}P(S_{n-1} = b)) = pP(S_{n-1} = b - 1) - qP(S_{n-1} = b) = \left(\frac{n-1}{\frac{n}{2} + \frac{b}{2} - 1}\right)p^{\frac{n}{2} + \frac{b}{2}}q^{\frac{n}{2} - \frac{b}{2}} - \left(\frac{n-1}{\frac{n}{2} + \frac{b}{2}}\right)p^{\frac{n}{2} + \frac{b}{2}}q^{\frac{n}{2} - \frac{b}{2}} = p^{\frac{n}{2} + \frac{b}{2}}q^{\frac{n}{2} - \frac{b}{2}} \left( \frac{(n-1)!}{(\frac{n}{2} + \frac{b}{2} - 1)!(\frac{n}{2} - \frac{b}{2})!} - \frac{(n-1)!}{(\frac{n}{2} + \frac{b}{2})!(\frac{n}{2} - \frac{b}{2} - 1)!} \right) = p^{\frac{n}{2} + \frac{b}{2}}q^{\frac{n}{2} - \frac{b}{2}} \left( \frac{(n-1)!}{(\frac{n}{2} + \frac{b}{2})!(\frac{n}{2} - \frac{b}{2})!} \right) \left( \frac{1}{\frac{n}{2} - \frac{b}{2}} - \frac{1}{\frac{n}{2} + \frac{b}{2}} \right) = p^{\frac{n}{2} + \frac{b}{2}}q^{\frac{n}{2} - \frac{b}{2}} \frac{b}{n} \left( \frac{n!}{(\frac{n}{2} + \frac{b}{2})!(\frac{n}{2} - \frac{b}{2})!} \right) = \frac{b}{n} p^{\frac{n}{2} + \frac{b}{2}}q^{\frac{n}{2} - \frac{b}{2}} \binom{n}{\frac{n}{2} + \frac{b}{2}} = \frac{b}{n} P(S_n = b)$ .  $\square$

**Theorem 10** (XXXMean number of visits to  $b$  before returning to origin in symmetric random walk). *Let  $(\{S_n\}_{n=0}^{+\infty}, p)$  be a symmetric random walk. Mean number  $\mu_b$  of visits of the walk to point  $b$  before returning to origin is equal to 1.*

*Proof.* aa  $\square$

**Definition 7** (Return to origin). *Let  $(\{S_n\}_{n=0}^{+\infty}, p)$  be a random walk. Let  $k \in \mathbb{N}$ . We say a return to origin occurred in time  $2k$  if  $S_{2k} = 0$ . The probability that in time  $2k$  occurred a return to origin shall be denoted by  $u_{2k}$ . We say that in time  $2k$  occurred first return to origin if  $S_1, S_2, \dots, S_{2k-1} \neq 0$  and  $S_{2k} = 0$ . The probability that in time  $2k$  occurred first return to origin shall be denoted by  $f_{2k}$ . By definition  $f_0 = 0$ . Let  $\alpha_{2n}(2k)$  denote  $u_{2k}u_{2(n-k)}$*

**Lemma 11** (Binomial identity). *Let  $n, k \in \mathbb{N}, n > k : \binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{n} \binom{n}{k}$*

*Proof.*  $\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-k-1)!} - \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k-1)!} \left( \frac{1}{k} - \frac{1}{n-k} \right) = \frac{1}{n} \frac{n!}{(k-1)!(n-k-1)!} \frac{n-2k}{k(n-k)} = \frac{n-2k}{n} \frac{n!}{k!(n-k)!} = \frac{n-2k}{n} \binom{n}{k}$   $\square$

**Lemma 12** (Main lemma). *Let  $(\{S_n\}_{n=0}^{+\infty}, p)$  be a symmetrical random walk. Then  $P(S_1 \cdot S_2, \cdot, \dots, S_{2n} \neq 0) = P(S_{2n} = 0)$*

*Proof.*  $P(S_1 \cdot S_2, \cdot, \dots, S_{2n} \neq 0) = \sum_{i=-\infty}^{+\infty} P(S_1 \cdot S_2, \cdot, \dots, S_{2n-1} \neq 0, S_{2n} = 2i) = \sum_{i=-n}^n P(S_1 \cdot S_2, \cdot, \dots, S_{2n-1} \neq 0, S_{2n} = 2i) = 2 \cdot \sum_{i=1}^n P(S_1 \cdot S_2, \cdot, \dots, S_{2n-1} \neq 0, S_{2n} = 2i)$

$$\begin{aligned}
2i) &= 2 \sum_{i=1}^n \frac{2i}{2n} P(S_{2n} = 2k) = 2 \sum_{i=1}^n \frac{2i}{2n} \binom{2n}{n+i} 2^{-2n} = 2 \cdot 2^{-2n} \sum_{i=1}^n \left( \binom{2n-1}{m+k-1} - \binom{2n-1}{m+k} \right) = \\
&= 2 \cdot 2^{-2n} \binom{2n-1}{n} = 2^{-2n} \frac{2n}{n} \binom{2n-1}{n} = 2^{-2n} \frac{2n(2n-1)!}{m(m-1)!m!} = 2^{-2n} \frac{(2n)!}{m!m!} = 2^{-2n} \binom{2n}{n} = P(S_{2n} = 0)
\end{aligned}$$

□

**Theorem 13.** Let  $(\{S_n\}_{n=0}^{+\infty}, p)$  be a symmetric random walk. The probability that the last return to origin up to time  $2n$  occurred in time  $2k$  is  $P(S_{2k} = 0) P(S_{2(n-k)} = 0)$ .

*Proof.*  $\alpha_{2n}(2k) = u_{2k} u_{2(n-k)} = P(S_{2k} = 0) P(S_{2k+1} \cdot S_{2k+2}, \dots, S_{2n} \neq 0 | S_{2k} = 0) = P(S_{2k} = 0) P(S_1 \cdot S_2, \dots, S_{2(n-k)} \neq 0) = P(S_{2k} = 0) P(S_{2(n-k)} = 0)$  □

**Theorem 14.** Let  $b \in \mathbb{Z}$ .  $P(S_1 \cdot S_2, \dots, S_n \neq 0, S_n = b) = \frac{|b|}{n} P(S_n = b)$ .

*Proof.* Let us Without loss of generality assume that  $b > 0$ . In that case, first step has to be rightwards ( $X_1 = +1$ ). Now we have path from point  $(1, 1)$  to point  $(n, b)$  that does not return to origin. By Ballot theorem 7 there are  $\frac{b}{n} N_n(0, b)$  such paths. Each path consists of  $\frac{n+b}{2}$  rightwards steps and  $\frac{n-b}{2}$  leftwards steps. Therefore  $P(S_1 \cdot S_2, \dots, S_n \neq 0, S_n = b) = \frac{b}{n} N_n(0, b) p^{\frac{n+b}{2}} q^{\frac{n-b}{2}} = \frac{b}{n} P(S_n = b)$ . Case  $b < 0$  is identical. □

**Lemma 15.** Let  $(\{S_n\}_{n=0}^{+\infty}, p)$  be a symmetric random walk.  $P(S_1, S_2, \dots, S_{2n} > 0) = \frac{1}{2} P(S_{2n} = 0) = \frac{1}{2} u_{2n}$ .

*Proof.* Because  $S_i > 0 \forall i \in \mathbb{N}$  the first step has to be rightwards ( $X_1 = S_1 = 1$ ). Therefore we get  $P(S_1, S_2, \dots, S_{2n} > 0) = \sum_{i=1}^n P(S_1, S_2, \dots, S_{2n} > 0, S_{2n} = 2r)$ . The  $r$ -th term follows equation:  $P(S_1, S_2, \dots, S_{2n} > 0, S_{2n} = 2r) = P(X_1 = 1, S_2, \dots, S_{2n} > 0, S_{2n} = 2r) = \frac{1}{2} P(S_2, S_3, \dots, S_{2n} > 0, S_{2n} = 2r) = \frac{1}{2} (P(S_{2n} = 2r) - P(S_2 \cdot S_3 \cdot \dots \cdot S_{2n} = 2r)) = \frac{1}{2} \left( \frac{1}{2} {}^{1/2}_{2n-1} N_{2n-1}(1, 2r) - \frac{1}{2} {}^{1/2}_{2n-1} N_{2n-1}^0(1, 2r) \right) = \frac{1}{2} \frac{1}{2} {}^{1/2}_{2n-1} (N_{2n-1}(1, 2r) - N_{2n-1}^0(1, 2r)) = \frac{1}{2} \frac{1}{2} {}^{1/2}_{2n-1} (N_{2n-1}(1, 2r) - N_{2n-1}(-1, 2r)) = \frac{1}{2} \frac{1}{2} {}^{1/2}_{2n-1} \left( \binom{2n-1}{n+r-1} - \binom{2n-1}{n+r} \right)$ . Therefore the negative parts of  $r$ -th terms cancel against the positive parts of  $(r+1)$ -st terms and the result reduces to just □

## 1.1 Title of the first subchapter of the first chapter

## 1.2 Title of the second subchapter of the first chapter

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