Occupation of a set time of random walks

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Overview

Motivation

- How long can you profit if you buy n independent stocks

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- Arcsine laws

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- Probability of last return to origin at time k: $\beta_n\left(k\right)$

Definition of a random walk

Definition

Let $\{X_n\}_{n=0}^{+\infty}$ be a sequence of independent and identically distributed $\{-1, 1\}$-valued random variables, that for some $p \in (0, 1)$ and for $n \in \mathbb{N}$ satisfy $\mathrm{P}(X_n = 1) = p$ and $\mathrm{P}(X_n = -1) = 1 - p =: q$.

Let $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$. We call the pair $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *Simple random walk in* $\mathbb{Z}$.

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  1. $p = q$ the walk stays approximately around origin (symmetric)
  2. $p > q$ the walk shifts rightwards/upwards for $n \to +\infty$
  3. $p < q$ the walk shifts leftwards/downwards for $n \to +\infty$

Probability of position at time

Theorem

Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk.

$$\mathrm{P}\left(S_n = x\right) = \begin{cases} \binom{n}{\frac{n+x}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}} & \text{for } x \in A_n \ , \\ 0, & \text{for } x \notin A_n, \end{cases}$$

where $A_n = \{z \in \mathbb{Z}; |z| \leq n, \frac{z+n}{2} \in \mathbb{Z}\}$

Probability of return to origin

Theorem

Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk.

$$\mathrm{P}\left(S_n = 0\right) = \begin{cases} \binom{n}{\frac{n}{2}} p^{\frac{n}{2}} q^{\frac{n}{2}} & \text{for } n \text{ even}\ , \\ 0, & \text{for } n \text{ odd}, \end{cases}$$

Probability of no return to origin

Theorem

Let $\{S_n\}_{n=0}^{+\infty}$ be a symmetric random walk.

$$\begin{aligned} \mathrm{P}\left(S_n = 0\right) &= \mathrm{P}\left(S_1, S_2, \ldots, S_{2n} \neq 0\right) = \\ &= 2\,\mathrm{P}\left(S_1, S_2, \ldots, S_{2n} > 0\right) = \mathrm{P}\left(S_1, S_2, \ldots, S_{2n} \geq 0\right) \end{aligned}$$

Arcsine law for last visits

Theorem

Let $\{S_n\}_{n=0}^{+\infty}$ be a symmetric random walk. Then the probability that the last return to the origin up to time $2n$ happened at time $2k$ is

$$\alpha_{2n}(2k) = \mathrm{P}(S_{2k}=0)\,\mathrm{P}(S_{2n-2k}=0).$$

- $$\mathrm{P}(S_{2k}=0) = \binom{2k}{k}2^{-2k} = \frac{(2k)!}{(k!)^2}2^{-2k} \sim \frac{\left(\frac{2k}{e}\right)^{2k}\sqrt{4\pi k}}{\left(\left(\frac{k}{e}\right)^{k}\sqrt{2\pi k}\right)^2}2^{-2k}$$

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- $$= \frac{\left(\frac{k}{e}\right)^{2k} 2^{2k} 2\sqrt{\pi k}}{\left(\frac{k}{e}\right)^{2k} 2\pi k} 2^{-2k} = \frac{1}{\sqrt{\pi k}}$$

Arcsine law for last visits-continuing

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- therefore we get the distribution function:

$$F(nx)=\mathrm{P}(X\leq nx)=\int_0^{nx}\frac{1}{\pi\sqrt{y(n-y)}}dy=\frac{2}{\pi}\arcsin\sqrt{x}$$

Arcsine law for sojourn times

Theorem

Let $\{S_n\}_{n=0}^{+\infty}$ be a symmetric random walk. Then the probability that up to time $2n$ the walk spends $2k$ steps on the positive side is

$$\alpha_{2n}(2k) = \mathrm{P}(S_{2n} = 0)\,\mathrm{P}(S_{2n-2k} = 0).$$

Type I random walk-definition

Definition

Let $m \in \mathbb{N}$. Let e_i denote i-th vector of standard basis in $\mathbb{R}^m$. Let $\{X_n\}_{n=0}^{+\infty}$ be a sequence of independent and identically distributed random variables with values in $\{-e_1, +e_1, -e_2, +e_2, \ldots, -e_m, +e_m\}$. That $\forall n \in \mathbb{N}$ satisfy the conditions $\mathrm{P}\left(X_n = e_i\right) = p_i \in (0,1)$, $\mathrm{P}\left(X_n = -e_i\right) = q_i \in (0,1)$ and $\sum\limits_{i=1}^{m} p_i + \sum\limits_{i=1}^{m} q_i = 1$. Let $\boldsymbol{p} = \left(p_1, p_2, \ldots, p_m\right)^T$, $\boldsymbol{q} = \left(q_1, q_2, \ldots, q_m\right)^T$ and $0 = \left(0, 0, \ldots, 0\right)^T$. Let $S_0 = 0$ and $S_n = \sum\limits_{i=1}^{n} X_i$. Then the triplet $\left(\{\boldsymbol{S_n}\}_{n=0}^{+\infty}, \boldsymbol{p}, \boldsymbol{q}\right)$ is called *Type I random walk in* $\mathbb{Z}^m$.

Type II random walk-definition

Definition

Let $m \in \mathbb{N}$. $\forall n \in \mathbb{N}$, let $\boldsymbol{X_n} = \begin{pmatrix} x_n^1 & x_n^2 & \dots, x_n^m \end{pmatrix}^T$, where $\{x_n^i\}_{i=1}^m$ are $\forall n \in \mathbb{N}$ independent.
Let $\forall i \in \{1, 2, \dots, m\}$ x_n^i be a $\{-1, +1\}$-valued random variable with probabilities $\mathrm{P}\left(x_n^i = +1\right) = p_i \in (0, 1)$ and $\mathrm{P}\left(x_n^i = -1\right) = 1 - p_i =: q_i \in (0, 1)\, \forall n \in \mathbb{N}$.
Let $\{X_n\}_{n=0}^{+\infty}$ be a sequence of independent and identically distributed random vectors. Let $\boldsymbol{S}_0 = 0$ and $\forall n \in \mathbb{N} : \boldsymbol{S_n} = \sum_{i=1}^{n} \boldsymbol{X_i}$ and $\boldsymbol{p} = \left(p_1, p_2, \dots, p_m\right)^T$. Then the pair $\left(\{\mathrm{S_n}\}_{n=0}^{+\infty}, \boldsymbol{p}\right)$ is called *Type II random walk in* $\mathbb{Z}^m$.

Type II random walk probability of position at time

Theorem

Let $m \in \mathbb{N}$ and $\left(\{\boldsymbol{S_n}\}_{n=0}^{+\infty}, \boldsymbol{p}\right)$be a Type II random walk in $\mathbb{Z}^m$. Let $\boldsymbol{y} = \left(y_1, y_2, \ldots, y_m\right)^T \in \mathbb{Z}^m$. Then following equation stands:

$$\mathrm{P}\left(S_n = \boldsymbol{y}\right) = \prod_{i=1}^{m} \binom{n}{\frac{y_i+n}{2}} {p_i}^{\frac{n+y_i}{2}} {q_i}^{\frac{n-y_i}{2}}$$

if $\forall i \in \{1, 2, \ldots, m\} : y_i \in A_n, 0$ otherwise.

Type II random walk probability of position at time

Theorem

Let $\{\mathrm{S_n}\}_{n=0}^{+\infty}$ be a Symmetric type II random walk in $\mathbb{Z}^m$. Let C be a closed orthant in Z^m.

$$\mathrm{P}\left(\boldsymbol{S_n} \in C\right) = \left(\mathrm{P}\left(S_{2n} = 0\right)\right)^m = \left(2^{-2n}\binom{2n}{n}\right)^m .$$

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My goals

- Find probabilites of occupation times of given sets in more dimensions
- Implement in R/Python for simulation study
- Statistical tests

Thank for your attention