RANDOM WALK ON PLANE AND SPACE

Technical Report · April 2017

DOI: 10.13140/RG.2.2.14283.69922

Sumish Ajmani
Indian Institute of Technology Kanpur

Indian Institute of Technology Kanpur

IME625 Project Report

on

RANDOM WALK ON PLANE AND SPACE

Authors:
Sumish Ajmani,13817719
Javed Akhtar,13324
Shagun Singh,13639
Akshay raj Singh,13076

Supervisor:
Prof. Avijit Khanra

September 8, 2017

1 Abstract

This report will scrutinize a special Markov Chain known as **Random Walk** on plane and space. We will first elaborate the Working Model , then derive mathematically and theoretically various formulas , Transition Probability Matrices , lemmas to get in-depth understanding of Random Walk in 2D and 3D. We will also analyze the nature of states of this Markov Chain. At the end we will conclude few things related to this model like real life examples related to it, some general formulas for d dimension and Gaussian Random Walk in 1D . Through out the report we have consider every random walk as Simple and Symmetric.

2 Introduction

Firstly we will discuss what's markov chain and it's properties and lemmas that we will use through out the report, after that we will introduce model of 2D and 3D random walk and try to prove / derive carious results associated with it.

2.1 Markov Chain

Consider a markov process with discrete time and states.Let $X_n = \{1, 2, 3 \cdots n\}$ be a Markov Chain with states space S $= 0, 1, 2 \cdots$ then it is a Markov chain if $P(X_{n+1} = j|X_n = i_n, X_{n-1} = i_{n-1}, \cdots X_1 = i_1) = P(X_{n+1} = j|X_n = i_n)$, for all $i_n, i_{n-1}, \cdots i_1, i, j \in$ S.

In words, "the past is conditionally independent of the future given the present state of the process" or "given the present state, the past contains no additional information on the future evolution of the system." Markov chain is considered to be homogeneous if $P(X_{n+1} = j | X_n = i) = P(X_1 = j | X_0 = i)$.

2.1.1 Examples of Markov chain

1. **Speech recognition**: Context can be important for identifying words. This can modeled as a probability distribution for the next word given the most recent k words. This can be modeled as a Markov chain whose state is a vector of k consecutive words.

2. **Epidemics**: Suppose each infected individual has some chance of contacting each susceptible individual in each time interval, before becoming removed (recovered or hospitalized). Then, the number of infected and susceptible individuals may be modeled as a Markov chain.[**Michigan University, dept. of stats, Markov chains**]

2.2 Notations and Equations

1. **Transition probability matrix** Let $P_{ij} = P(X_{n+1} = j | X_n = i)$, denotes the 1 step transition probability matrix. Whence ${P_{ij}}^n$ can be

denoted as n- step transition probability matrix.

2. **Chapman-Kolmogorov equation** ${P_{ij}}^{n+m} = \sum_{k=0}^{\infty} {P_{ij}}^{n} {P_{ij}}^{m}$ and it corresponds to the matrix multiplication identity $P^{n+m} = P^n P^m$.[**Michigan University, dept. of stats, Markov chains**]

2.3 Classification of States and general properties

1. State j is accessible from i if ${P_{ij}}^{k} > 0$ for some k $\geq$ 0.

2. i and j communicate if they are accessible from each other. This is written i $\leftrightarrow$ j, and is an equivalence relation.

3. An equivalence relation divides a set (here, the state space) into disjoint classes of equivalent states (here, called communication classes).

4. A Markov chain is irreducible if all the states communicate with each other, i.e., if there is only one communication class.

5. The communication class containing i is absorbing if P_{jk} =0 whenever i $\leftrightarrow$ j but i !$\leftrightarrow$ k (i.e., when i communicates with j but not with k).

6. State i is recurrent if P(re enter i$|X_0 = i) = 1$. Moreover i is recurrent if and only if $\sum_{n=1}^{\infty} {P_{ii}}^{n} = \infty$

7. Let u_{ii} denotes Expected return time, then i is transient if $u_{ii} = \infty$. If i is recurrent and $u_{ii} < \infty$ then i is said to be positive recurrent.

Otherwise, if $u_{ii} = \infty$, i is null recurrent.[**Michigan University, dept. of stats, Markov chains**]

2.4 Random Walk on Line (Short Demonstration)

A person starts walking along the straight line from an initial point say origin. At each step (s)he takes one step forward with probability p and backward with probability q= (1-p), p $\epsilon (0, 1)$. Define, $Z_i =$ step size at i^{th} step. Clearly, $Z_i's$ are IID Random Variables with P($Z_i = 1$) = p and P($Z_i = -1$) = q, for i ϵ $1, 2 \cdots$, then X_n = position of the person after n steps and is equal to $Z_1 + Z_2 + \cdots Z_n$ for n = 1, 2 so on. Then $X_n = Z_n + X_{n-1}$ is a Homogeneous Markov Chain with S = $0, \pm 1, \pm 2, \cdots$. Probability that starting from origin, person will return to origin in n steps is given by:

$$p_{ii}^n = \begin{cases} \frac{n!}{(n/2!)(n/2!)} p^n q^n, & \text{if n is even} \\ 0, & \text{if n is odd} \end{cases}$$

Clearly markov Chain is irreducible, Now Stirling's approximation states that the approximate value of:

$$n! \approx \sqrt{2\pi n}(\frac{n}{e})^n \tag{1}$$

putting this in above equation we get

$$p_{ii}^{2n} \approx p^n q^n \frac{2\pi e^{-2n} 2n^{2n+0.5}}{(2\pi e^{-n} n^{n+0.5})^2} \tag{2}$$

$$p_{ii}^{2n} \approx \frac{(4pq)^n}{\sqrt{\pi n}} = b_n(say) \tag{3}$$

Thus, i is transient if $\Leftrightarrow \sum_{n=1}^{\infty} b_n \leq \infty$.

- Case : p =1/2 , then $b_n =\sqrt{\pi n}$ implies $\sum_{n=1}^{\infty} b_n = \infty$ then i is recurrent , hence all states are recurrent.
- Case: p $\neq$ 1/2 , then $\frac{b_{n+1}}{b_n}$ = 4pq$\sqrt{\frac{n+1}{n}}$,as 4pq $\leq$ 1 , hence $\sum_{n=1}^{\infty} b_n \leq \infty$. So i is transient , all states are transient. [**Indian Institute of Technology Kanpur, dept. of Mathematics**]

2.4.1 Gaussian Random Walk in 1D

In this each of the next forward or backward steps are independent normal random variable. here we are considering Random variable with Normal Distribution say $N(0, \sigma^2)$. Let Z_i denote $N(0, \sigma^2)$. So position of the particle after n steps is a R.V. given by $X_n = Z_1 + Z_2 \cdots + Z_n$. Clearly some results follows from assumptions that

$$E(X_n) = 0, Var(X_n) = n\sigma^2 \tag{4}$$

Therefore $X_n \approx N(0, n\sigma^2)$ since all Z_i's are IID R.Vs with $N(0, \sigma^2)$. So we see that in this case the exact distribution of the position after n^{th} step of random walk is normally distributed. One of the real life example is the stock price purchasing.[**UCLA University**]

3 Random Walk on Plane

Consider a drunken men walking in the street where at each step he can take one step forward , backward , left or right each with probability 1/4. That is we are assuming that the walk is symmetric .

Let X_n :$n \geq 0$ be a homogeneous markov chain with state space S = { (i,j) : i,j $\in 0, \pm 1, \pm 2, \cdots$ } and Transition probability matrix for this model is given by,

$$p_{(i,j),(k,l)} = \begin{cases} 1/4, & \text{if (k,l)} \in \{\text{(i+1,j) , (i-1,j) , (i,j+1) ,(i,j-1)}\} \\ 0, & \text{otherwise} \end{cases}$$

Clearly , $p^{2n+1}_{(i,i),(i,i)} = 0$, as one can not come back to its initial position in odd number of steps, and

$$p^{2n}_{(i,i),(i,i)} = \sum_{i=0}^{n} \frac{2n!}{i!i!n-i!n-i!} \frac{1}{4^i 4^i 4^{n-i} 4^{n-i}} \tag{5}$$

$$= \frac{1}{4^{2n}} \frac{2n!}{n!n!} \sum_{i=0}^{n} (\frac{n!}{i!n-i!})^2 = \frac{1}{4^{2n}} (\frac{2n!}{n!n!})^2 \tag{6}$$

Now Stirling's approximation states that the approximate value of:

$$n! \approx \sqrt{2\pi n}(\frac{n}{e})^n \tag{7}$$

Hence putting above equation in (2) we get the approximate value of:

$$p^{2n}_{(i,i),(i,i)} \approx \frac{1}{4^{2n}} \frac{\sqrt{2\pi e^{-2n} 2n^{2n+0.5}}}{\sqrt{2\pi e^{-n} n^{n+0.5}}} \tag{8}$$

$$\approx \frac{1}{\pi n} \tag{9}$$

Now to find whether these states are transient of recurrent we will calculate the value of $\sum_{n=1}^{\infty} p^{2n}_{(i,i),(i,i)}$. Clearly the sequence $\frac{1}{n}$ diverges as n goes to infinity , hence $\sum_{n=1}^{\infty} p^{2n}_{(i,i),(i,i)} = \infty$. Therefore all the states in Random walk on plane is recurrent.[**Indian Institute of Technology Kanpur, dept. of Mathematics**]

Before we discuss further, lets prove first that recurrence is equivalent to the infinite number of expected returns.Let u= P{random walk started at 0, returns to 0}, therefore P{state '0' is visited k times} $= u^{k-1}(1-u)$,

$$m = \sum_{k=1}^{\infty} k.u^{k-1}.(1-u) \tag{10}$$

$$= (1-u).\sum_{k=1}^{\infty} \frac{d(u^k)}{du} \tag{11}$$

So, $m = \frac{1}{1-u}$

If m $= \infty$, then u $= 1$ and so the walk will be recurrent and if m $< \infty$ then u < 1 and so the walk will be transient. [**Duke University Dec'4, 1998**]

Alternatively , m = $\sum_{k=1}^{\infty} u_n$, where u_n = {Walk starting at '0' is at '0' on $n^{th} step\}$. Therefore recurrence $\Leftrightarrow \sum_{k=1}^{\infty} u_n$ diverges and transient $\Leftrightarrow$ $\sum_{k=1}^{\infty} u_n$ converges.

3.1 Application

In mathematical ecology, random walks are used to describe individual animal movements, to empirically support processes of bio diffusion, and occasionally to model population dynamics.

4 Random Walk on Space

Considering a similar scenario as in the last case , just image that a fly is discretely flying(walking) and can move up, down, forward, backward , left , right directions all symmetrically with the probability of 1/6.

Let X_n be the trajectory of a random walk in 3 dimensions. So, $X_n = S_1 + S_2 + S_3 + \cdots + S_n$, where S_i's are independent random variables with

$$S_j = \begin{cases} 1/6, & (1,0,0)\ ,\ (0,1,0)\ ,\ (0,0,1),\ (-1,0,0)\ ,\ (0,-1,1)\ ,\ (0,0,-1) \\ 0, & \text{otherwise} \end{cases}$$

Then for each n $= 1, 2, \cdots X_n$ is a random point in Z^3 = Z x Z x Z. Now let us try to figure out about transition probability matrix,

$$P\{X_n = (x,y,z)|X_{n-1} = (x_0,y_0,z_0)\} = \begin{cases} 1/6, & \text{if x} = x_0 \pm 1 \\ 1/6, & \text{if y} = y_0 \pm 1 \\ 1/6, & \text{if z} = z_0 \pm 1 \\ 0, & \text{otherwise} \end{cases}$$

Lets determine transient or recurrent states for this case, but before that we need to calculate the quantity u_n, where u_n = {Walk starting at '0' is at '0' on $n^{th} step\}$. It is similar according to previous cases that, returning to origin, the walker must take same number of steps either in left or right , up or down , forward or backward. Therefore every path that returns in 2n steps has probability $(\frac{1}{6})^{2n}$. Now number of steps, N with k steps left , k steps right , j steps down , j steps up , n - k - j steps steps forward and n - k - j steps backward is :

$$N = \frac{2n!}{k!k!j!j!(n-k-j)!(n-k-j)!} \tag{12}$$

$$u_{2n} = (\frac{1}{6})^{2n} \sum_{j+k \le n} \frac{2n!}{k!k!j!j!(n-k-j)!(n-k-j)!} \tag{13}$$

$$u_{2n} = (\frac{1}{2})^{2n} \frac{2n!}{n!n!} \sum_{j+k \le n} (\frac{1}{3}^n \frac{n!}{k!j!(n-k-j)!})^2 \tag{14}$$

Now, we can analyze this term $(\frac{1}{3}^n \frac{n!}{k!j!(n-k-j)!})^2$ equals to probability of placing n balls in 3 boxes and this is maximized when k = j = n/3 or k , j , n-k-j all three are equal. Therefore,

$$u_{2n} \leq (\frac{1}{2})^{2n} \frac{2n!}{n!n!} \frac{1}{3}^n \frac{n!}{(n/3)!(n/3)!(n/3)!}) \sum_{j+k \leq n} (\frac{1}{3}^n \frac{n!}{k!j!(n-k-j)!}) \tag{15}$$

Clearly $\sum_{j+k \leq n} (\frac{1}{3}^n \frac{n!}{k!j!(n-k-j)!})$ the term is 1 , since it is a distribution of n. Therefore ,

$$u_{2n} \leq (\frac{1}{2})^{2n} \frac{2n!}{n!n!} \frac{1}{3}^n \frac{n!}{(n/3)!(n/3)!(n/3)!} \tag{16}$$

Using Stirling's Approximation in above equation we get

$$u_{2n} \leq \frac{k}{(n)^{3/2}} \tag{17}$$

for some constant k ϵ R^+ . So,

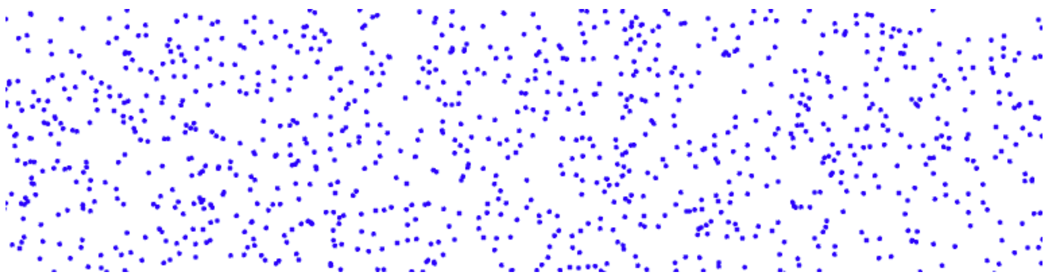
$$m = \sum_n u_{2n} \leq k \sum_n (1/n)^{3/2} \leq \infty \tag{18}$$

Thus simple random Walk in 3 Dimension is transient.[**Duke University Dec'4, 1998**]

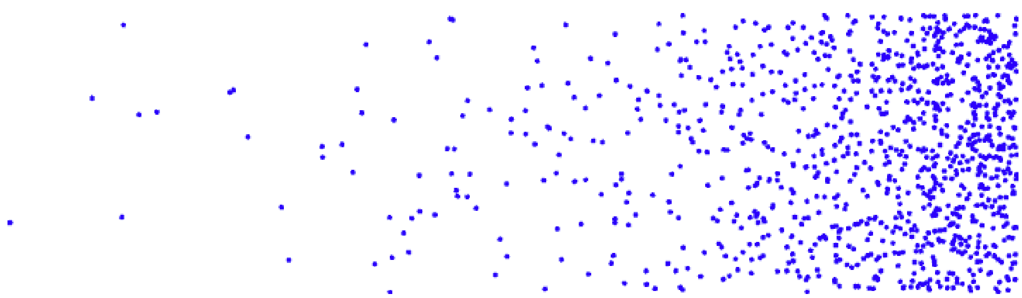
4.1 Applications

- The particles in a drop of food coloring added to a glass of water will spread out, partially due to currents in the water, and partially due to a random walk.

If we picture each individual molecule as a little blue dot, a constant concentration of molecules would look like the picture below:



A concentration gradient, with a higher concentration of molecules on the right than on the left would look like the picture below:



Remember the biased random walk? Well there's always a reason for a bias. Bacteria can bias their walks based on the concentration gradient of a particular chemical. So even though each step is in a random direction, the length of the step is longer if the bacterium is moving towards a higher concentration than it is if the bacterium is moving towards a lower concentration.When a bacterium is looking for a particular chemical signal, it detects this chemical as it moves along its path. If it is moving up the concentration gradient, it will start detecting the chemical's molecules more and more frequently. If it is moving down the concentration gradient, it will start detecting the chemical's molecules less and less frequently. This ultimately determines the direction and strength of the bias in its random walk.[**MIT University, Random Walks**]

5 Simple Random Walk in Higher Dimension

Note, to return to the origin, the walker must take same number of forward and backward steps in each direction. Therefore path that returns in 2n steps has probability $(\frac{1}{2d})^{2n}$ of returning. The number of paths with k_1 steps forward and k_1 steps backward in 1st direction , K_2 steps forward and K_2 steps backward in second direction , $\cdots$ K_{d-1} steps forward and K_d steps backward in d^{th} direction is,

$$\frac{2n!}{k_1!k_1!k_2!k_2!\cdots(n-\sum_{i=1}^{d-1}K_i)!(n-\sum_{i=1}^{d-1}K_i)!} \tag{19}$$

Hence, u_{2n} is equal to

$$= (\frac{1}{2d})^{2n} \sum_{\sum_{i=1}^{d-1} K_i \leq n}^{k_1,k_2,\cdots k_{d-1}} \frac{2n!}{k_1!k_1!k_2!k_2! \cdots (n - \sum_{i=1}^{d-1} K_i)!(n - \sum_{i=1}^{d-1} K_i)!} \tag{20}$$

$$= (\frac{1}{2})^{2n} \frac{2n!}{n!n!} \sum_{\sum_{i=1}^{d-1} K_i \leq n}^{k_1,k_2,\cdots k_{d-1}} (\frac{1}{(d)^n k_1!k_2! \cdots (n - \sum_{i=1}^{d-1} K_i)!})^2 \tag{21}$$

As we did for 3D case, here also the term $\frac{1}{(d)^n k_1!k_2!\cdots(n-\sum_{i=1}^{d-1} K_i)!}$ for d dimension , is the probability of placing n balls in d boxes and due to symmetric behaviour of the model , this term is maximized when $k_1, k_2, \cdots (n - \sum_{i=1}^{d-1} K_i)$ all are equal to n/d. Therefore,

$$u_{2n} \leq (\frac{1}{2})^{2n} \frac{2n!}{n!n!} (\frac{n!}{d^n[n/d]!^d})(\sum^{k_1,k_2,\cdots k_{d-1}} \frac{n!}{(d)^n k_1!k_2! \cdots (n - \sum_{i=1}^{d-1} K_i)!}) \tag{22}$$

Clearly the term $(\sum^{k_1,k_2,\cdots k_{d-1}} \frac{n!}{(d)^n k_1!k_2!\cdots(n-\sum_{i=1}^{d-1} K_i)!})$ is equal to 1 since its a distribution. Hence,

$$u_{2n} \leq (\frac{1}{2})^{2n} \frac{2n!}{n!n!} (\frac{n!}{d^n[n/d]!^d}) \tag{23}$$

Using Stirling's Approximation , we get

$$u_{2n} \leq \frac{k}{(n)^{d/2}} \tag{24}$$

for some constant k ϵ R^+ . So,

$$m = \sum_n u_{2n} \leq k \sum_n (1/n)^{d/2} \leq \infty \tag{25}$$

Since $\sum_n (1/n)^{d/2}$ converges for d $\geq$ 3. Therefore Simple random walk in d dimension is transient for d $\geq$ 3. [**Duke University Dec'4, 1998**]

6 Conclusions

In this report, we investigated a simple symmetric random walk in line, plane and space and also for higher dimensional case. In each of the cases our main interest was to know that when a particle starts at origin, what is the probability that it will return back to origin in n steps? , Would the states be recurrent or transient? Is the walk irreducible?

We concluded that in all the cases the random walk is irreducible, further we concluded that for 1D and 2D , the random walk is recurrent and for higher dimensional cases , it comes out to be transient. We also listed some applications of 1D and 2D random walk in finance and mathematical ecology.

7 Discussion

As evident by the examples of random walk, these results can be used in varying fields " ranging from finance, stock price modeling (Binomial pricing and Black Scholes model) or in mathematical ecology(Modeling movement of animals in 2D space), to Engineering, diffusion can be modeled as Brownian motion". From this discussion we also conclude a famous saying that a "drunken person is more likely to return to his home but a drunken bird will never return to it's home"

8 References

- Michigan University, dept. of stats, Markov chains , lecture slides 4 , 2013
- Indian Institute of Technology kanpur, Neeraj Misra , Stochastic processes, Module 2
- MIT University , Random Walks in 2D and 3D
- UCLA university , Caflisch, lecture 4-5
- An Introduction to Random Walks from Polya to Self-Avoidance Michael Kozdron Duke University December 4, 1998
- Sheldon M Ross, Introduction to probability Models
- Wikipedia , Random Walk