1. Simple random walk in one dimension

Remark. First, let us properly introduce what a random walk is. After stating some of the basic definitions we will move to the core of this thesis which is to explore properties of occupation of a set times.

Definition 1 (Simple random walk in $\mathbb{Z}$)**.** Let $\{X_n\}_{n=0}^{+\infty}$ be a sequence of independent and identically distributed $\{-1, 1\}$-valued random variables, that for some $p \in (0, 1)$ and for $n \in \mathbb{N}$ satisfy $\mathsf{P}(X_n = 1) = p$ and $\mathsf{P}(X_n = -1) = 1 - p =: q$.

Let $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$. We call the pair $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *Simple random walk in* $\mathbb{Z}$.

If $p = q = \frac{1}{2}$, $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ is called *Symmetric simple random walk in* $\mathbb{Z}$.

Remark. Very often we refer to n as time , X_i as the i-th step and S_n as position of the walk in time n. While referring to simple random walk in $\mathbb{Z}$ we refer to $X_i = +1$ as i-th step was rightwards and to $X_i = -1$ as i-th step was leftwards. If it is not stated otherwise, we assume that $S_0 = 0$.

Remark. The most important element in random walks is the probability of being in position x in time n. In order to calculate such probability we have to firstly define what are even possible positions. For example it is impossible for the random walk to be in position x at time n if $x > n$ simply because there have not been enough steps to make it up to x. It is also impossible (given the preposition that $S_0 = 0$) that after even number of steps the random walk is in odd-numbered position and vice versa. Therefore we define the set of possible positions.

Definition 2 (Set of possible positions)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$be a random walk. We call the set $A_n = \{z \in \mathbb{Z}; |z| \leq n, \frac{z+n}{2} \in \mathbb{Z}\}$ *set of all possible positions* of random walk $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$in time n.

Theorem 1 (Probability of position x in time n)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$*be a random walkand* A_n *its set of possible positions.*

$$\mathsf{P}(S_n = x) = \begin{cases} \binom{n}{\frac{n+x}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}} & \textit{for } x \in A_n \ , \\ 0, & \textit{for } x \notin A_n. \end{cases}$$

Remark. While having the definition of set of possible positions it is easy to prove the theorem by finding random variable with alternative distribution in each step. By summing them we get a variable with binomial distribution and then we simply modify the result to get desired probability.

Proof. Consider random variables $\mathbf{1}_{[X_i=1]}$, and $\mathbf{1}_{[X_i=-1]}$, and define new random variables $R_n = \sum_{i=1}^{n} \mathbf{1}_{[X_i=1]}, L_n = \sum_{i=1}^{n} \mathbf{1}_{[X_i=-1]}$. The random variable $\mathbf{1}_{[X_i=1]}$ can be interpreted as indicator whether i-th step was rightwards. Then, R_n is number of rightwards steps and L_n is number of leftwards steps. We can easily see that

$R_n + L_n = n$ and $R_n - L_n = S_n$. Therefore we get by adding these two equations $R_n = \frac{S_n + n}{2}$.

Clearly, $\mathbf{1}_{[X_i=1]}$ has alternative distribution with parameter p (*Alt(p)*). Hence, R_n as a sum of independent and identically distributed random variables with distribution *Alt(p)* has binomial distribution with parameters n and p (*Bi(n,p)*). Therefore we get $\mathsf{P}\left(R_n = x\right) = \binom{n}{x} p^x q^{n-x}$, where we define binomial coefficients as statet in preface. Finally, for $a \in A_n$ we get

$$\mathsf{P}\left(S_n = x\right) = \mathsf{P}\left(R_n = \frac{x+n}{2}\right) = \binom{n}{\frac{x+n}{2}} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}.$$

□

Remark. Following are three simple lemmata that simplify many calculations in the rest of thesis. After proving them we can ask

Lemma 2 (Spatial homogeneity)
Let $n \in \mathbb{N}, a, b, j \in \mathbb{Z}$. Then for all $b \in \mathbb{Z}$

$$\mathsf{P}\left(S_n = j \mid S_0 = a\right) = \mathsf{P}\left(S_n = j + b \mid S_0 = a + b\right)$$

Proof. For any $j, a, b \in \mathbb{Z}$ holds

$$\begin{aligned}\mathsf{P}\left(S_n = j \mid S_0 = a\right) &= \mathsf{P}\left(\sum_{i=1}^{n} X_i = j - a\right) = \mathsf{P}\left(\sum_{i=1}^{n} X_i = (j+b) - (a+b)\right) \\ &= \mathsf{P}\left(S_n = j + b \mid S_0 = a + b\right).\end{aligned}$$

□

Lemma 3 (Temporal homogeneity)
Let $n, m \in \mathbb{N}, a, j \in \mathbb{Z}$. Then for all $m \in \mathbb{N}$

$$\mathsf{P}\left(S_n = j \mid S_0 = a\right) = \mathsf{P}\left(S_{n+m} = j \mid S_m = a\right)$$

Proof. For any $j, a \in \mathbb{Z}$ and $m \in \mathbb{N}$

$$\begin{aligned}\mathsf{P}\left(S_n = j \mid S_0 = a\right) &= \mathsf{P}\left(\sum_{i=1}^{n} X_i = j - a\right) = \mathsf{P}\left(\sum_{i=m+1}^{m+n} X_i = j - a\right) \\ &= \mathsf{P}\left(S_{n+m} = j \mid S_m = a\right),\end{aligned}$$

where the second equality follows from identical distribution of $\{X_n\}_{n=1}^{+\infty}$. □

Lemma 4 (Markov property)
Let $n, m \in \mathbb{N}, n \geq m, a_i \in \mathbb{Z}, i \in \mathbb{N}$. Then

$$\mathsf{P}\left(S_n = j \mid S_0 = a_0, S_1 = a_1, \ldots, S_m = a_m\right) = \mathsf{P}\left(S_n = j \mid S_m = a_m\right)$$

Proof. Because $\{X_n\}_{n=1}^{+\infty}$ is a sequence of independent variables, once S_m is known, then distribution of S_n depends only on steps $X_{m+1}, X_{m+2}, \ldots X_n$ and therefore cannot depend on any information concerning values $X_1, X_2, \ldots, X_{m-1}$ and accordingly $S_1, S_2, \ldots, S_{m-1}$. □

Remark. Once having stated basic definitions we may succeed to ask ourselves questions about occupation of a set times. Let $a > 0$. How many steps up to time n does our walk spend above a (in interval $[a, +\infty)$)? Similarly how many steps does the walk spend in interval $[-a, +\infty)$? We are going to answer these questions in following pages.

Remark. While calculating probabilities in symmetric random walks the fact that $p = q = \frac{1}{2}$ simplifies calculating because in stead of $p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}$ we now have 2^{-n}. Therefore the probabilities only depend on $\binom{n}{\frac{x+n}{2}}$ which can be more generalized as it is in the following definition.

Definition 3 (Number of possible paths)**.** Let $N_n(a, b)$ be *number of posssible paths* of random walk $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$from point $(0, a)$ to point (n, b) and $N_n^x(a, b)$ be number of possible paths from point $(0, a)$ to point (n, b) that visit point (z, x) for some $z \in \{1, 2, \ldots, n\}$.

Theorem 5 (Number of possible paths)
Let $a, b \in \mathbb{Z}, n \in \mathbb{N}$ then

$$N_n(a, b) = \binom{n}{\frac{1}{2}(n + b - a)}.$$

Proof. Let us choose a path from the point $(0, a)$ to point (n, b) and let α be number of rightwards steps and β be number of leftwards steps. Then $\alpha + \beta = n$ and $\alpha - \beta = b - a$. By adding these two equations we get that $\alpha = \frac{1}{2}(n + b - a)$. The number of possible paths is the number of ways of picking α rightwards steps from n steps. Therefore we get $N_n(a, b) = \binom{n}{\alpha} = \binom{n}{\frac{1}{2}(n+b-a)}$. □

Theorem 6 (Reflection principle)
Let $a, b > 0$, then

$$N_n^0(a, b) = N_n(-a, b).$$

Proof. Each path from $(0, -a)$ to (n, b) has to intersect $y = 0$-axis at least once at some point. Let k be the time of earliest intersection with x-axis. By reflecting the segment from $(0, -a)$ to $(k, 0)$ in the x-axis and letting the segment from $(k, 0)$ to (n, b) be the same, we get a path from point $(0, a)$ to (n, b) which visits 0 at point k. Because reflection is a bijective operation on sets of paths, we get the correspondence between the collections of such paths. □

Remark. Following is the definition of return to origin which is a curucial term for this thesis. Let us come back to our question. While calculating the number of steps the walk spends in interval $[a, +\infty)$ we calculate our first passage through a and then set a as a new origin and use achieved results in the following subchapter.

Definition 4 (Return to origin)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$be a random walk. Let $k \in \mathbb{N}$. We say a *return to origin* occurred in time $2k$ if $S_{2k} = 0$. The probability that in time $2k$ occurred a return to origin shall be denoted by u_{2k}. We say that in time $2k$ occurred *first return to origin* if $S_1, S_2, \ldots S_{2k-1} \neq 0$ and $S_{2k} = 0$. The probability that in time $2k$ occurred first return to origin shall be denoted by f_{2k}. We define $f_0 := 0$. Let $\alpha 2n\,(2k)$ denote $u_{2k} u_{2(n-k)}$

Theorem 7 (Ballot theorem)
Let $n, b \in N$ Number of paths from point $(0,0)$ to point (n,b) which do not return to origin is equal to $\frac{b}{n} N_n(0,b)$

Proof. Let us call N the number of paths we are referring to. Because the path ends at point (n,b), the first step has to be rightwards. Therefore we now have

$$N = N_{n-1}(1,b) - N^0_{n-1}(1,b) \stackrel{\text{T6}}{=} N_{n-1}(1,b) - N_{n-1}(-1,b).$$

Hence we get that:

$$\begin{aligned}
N = N_{n-1}(1,b) - N_{n-1}(-1,b) &= \binom{n-1}{\frac{n}{2}+\frac{b}{2}-1} - \binom{n-1}{\frac{n}{2}+\frac{b}{2}} \\
&= \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!} - \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!} \\
&= \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}\right)\left(\frac{n}{2}-\frac{b}{2}-1\right)!} - \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}\right)\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!} \\
&= \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!}\left(\frac{1}{\frac{n}{2}-\frac{b}{2}} - \frac{1}{\frac{n}{2}+\frac{b}{2}}\right) \\
&= \frac{1}{n}\frac{n!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!}\left(\frac{\left(\frac{n}{2}+\frac{b}{2}-\frac{n}{2}+\frac{b}{2}\right)}{\left(\frac{n}{2}-\frac{b}{2}\right)\left(\frac{n}{2}+\frac{b}{2}\right)}\right) \\
&= \frac{b}{n}\frac{n!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!} = \frac{b}{n}\binom{n}{\left(\frac{n}{2}+\frac{b}{2}\right)} = \frac{b}{n} N_n(0,b)
\end{aligned}$$

□

Remark. The name *Ballot theorem* comes from the question: In a ballot where candidate A receives p votes and candidate B receives q votes with $p > q$, what is the probability that A had been strictly ahead of B throughout the whole count?

Definition 5 (Maximum and minimum)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$be a random walk. $M_n^+ := \max\{S_i, i \in \{1,2,\dots,n\}\}$ is called *maximum of random walk* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *up to time* n and $M_n^- := \min\{S_i, i \in \{1,2,\dots,n\}\}$ is called *minimum of random walk* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$*up to time* n. $M_n = \max M_n^+, -M_n^-$ is called *absolute maximum of random walk* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$*up to time* n.

Theorem 8 (Probability of maximum up to time n)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$*be a random walk.*

$$\mathsf{P}\left(M_n^+ \geq r, S_n = b\right) = \begin{cases} \mathsf{P}(S_n = b) & \textit{for } b \geq r\ , \\ \mathsf{P}(S_n = 2r-b)\left(\frac{q}{p}\right)^{r-b}, & \textit{for otherwise.} \end{cases}$$

Proof. Let us firstly consider the easier case in which $b \geq r$. Because we defined M_n^+ as $\max\{S_i, i \in \{1,2,\dots,n\}\}$ we get that $M_n^+ \geq b \geq r$ therefore $[M_n^+ \geq r] \subset [S_n = b]$ hence we get $\mathsf{P}(M_n^+ \geq r, S_n = b) = \mathsf{P}(S_n = b)$.

Now let $r \geq 1, b < r$. $N_n^r(0,b)$ stands for number of paths from point $(0,0)$ to point (n,b) which reach up to r. Let $k \in \{1,2,\ldots,n\}$ denote the first time we reach r. By reflection principle (6), we can reflect the segment from (k,r) to (n,b) in the axis:$y=r$. Therefore we now have path from $(0,0)$ to $(n,2r-b)$ and we get that

$$N_n^r(0,b) = N_n(0,2r-b) \text{ hence } \mathsf{P}\left(S_n = b, M_n^+ \geq r\right) = N_n^r(0,b)\, p^{\frac{n+b}{2}} q^{\frac{n-b}{2}} =$$
$$N_n(0,2r-b)\, p^{\frac{n+(2r-b)}{2}} q^{\frac{n-(2r-b)}{2}} p^{b-r} q^{r-b} = \left(\frac{q}{p}\right)^{r-b} \mathsf{P}\left(S_n = 2r-b\right).$$

□

Definition 6 (Walk reaching new maximum at particular time)**.** Let $n,b \in \mathbb{N}, b > 0$. We say that *the walk reached new maximum b in time n* if $M_{n-1}^+ = S_{n-1} = b-1, S_n = b$. We denote such probability by $f_b(n)$.

Theorem 9 (Probability of reaching new maximum b in time n)
Let $n,b \in \mathbb{N}, b > 0$ then

$$f_b(n) = \frac{b}{n} \mathsf{P}\left(S_n = b\right).$$

Proof.

$$\begin{aligned}
f_b &= \mathsf{P}\left(M_{n-1} = S_{n-1} = b-1, S_n = b\right) = \mathsf{P}\left(M_{n-1} = S_{n-1} = b-1, X_n = +1\right) \\
&= p\,\mathsf{P}\left(M_{n-1} = S_{n-1} = b-1\right) \\
&\overset{*}{=} p\left(\mathsf{P}\left(M_{n-1} \geq b-1, S_{n-1} = b-1\right) - \mathsf{P}\left(M_{n-1} \geq b, S_{n-1} = b-1\right)\right) \\
&\overset{\text{T8}}{=} p\left(\mathsf{P}\left(S_{n-1} = b-1\right) - \frac{q}{p}\mathsf{P}\left(S_{n-1} = b+1\right)\right) \\
&= p\,\mathsf{P}\left(S_{n-1} = b-1\right) - q\,\mathsf{P}\left(S_{n-1} = b+1\right) \\
&= \binom{n-1}{\frac{n}{2}+\frac{b}{2}-1} p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} - \binom{n-1}{\frac{n}{2}+\frac{b}{2}} p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \\
&= p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \left(\frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!} - \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!}\right) \\
&= p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \left(\frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!}\right)\left(\frac{1}{\frac{n}{2}-\frac{b}{2}} - \frac{1}{\frac{n}{2}+\frac{b}{2}}\right) \\
&= p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \frac{b}{n}\left(\frac{n!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!}\right) = \frac{b}{n} p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \binom{n}{\frac{n}{2}+\frac{b}{2}} = \frac{b}{n}\mathsf{P}\left(S_n = b\right).
\end{aligned}$$

Where $*$ comes from the fact that the event $[M_{n-1} \geq b-1]$ can be split into two disjoint events: $[M_{n-1} \geq b-1] = [M_{n-1} \geq b] \cup [M_{n-1} = b-1]$. Therefore $\mathsf{P}\left(M_{n-1} \geq b-1\right) = \mathsf{P}\left(M_{n-1} \geq b\right) + \mathsf{P}\left(M_{n-1} = b-1\right)$.

Hence: $\mathsf{P}\left(M_{n-1} = b-1\right) = \mathsf{P}\left(M_{n-1} \geq b-1\right) - \mathsf{P}\left(M_{n-1} \geq b\right)$. The same applies for the probability $\mathsf{P}\left(M_{n-1} = b-1, S_{n-1} = b-1\right)$ □

Lemma 10 (Binomial identity)
Let $n,k \in \mathbb{N}, n > k$ then following equation holds

$$\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{n}\binom{n}{k}$$

Proof.

$$\binom{n-1}{k}-\binom{n-1}{k-1}=\frac{(n-1)!}{k!\,(n-k-1)!}-\frac{(n-1)!}{(k-1)!\,(n-k)!}$$
$$=\frac{(n-1)!}{(k-1)!\,(n-k-1)!}\left(\frac{1}{k}-\frac{1}{n-k}\right)$$
$$=\frac{1}{n}\frac{n!}{(k-1)!\,(n-k-1)!}\frac{n-2k}{k\,(n-k)}=\frac{n-2k}{n}\frac{n!}{k!\,(n-k)!}=\frac{n-2k}{n}\binom{n}{k}$$

□

Lemma 11 (Main lemma)
Let $\left(\{S_n\}_{n=0}^{+\infty},p\right)$ *be a symmetrical random walk, then*

$$\mathsf{P}\left(S_1,S_2,\ldots,S_{2n}\neq 0\right)=\mathsf{P}\left(S_{2n}=0\right).$$

Proof.

$$\mathsf{P}\left(S_1,S_2,\ldots,S_{2n}\neq 0\right)\overset{LTP}{=}\sum_{i=-\infty}^{+\infty}\mathsf{P}\left(S_1,S_2,\ldots,S_{2n}\neq 0,S_{2n}=2i\right)$$
$$=\sum_{i=-n}^{n}\mathsf{P}\left(S_1,S_2,\ldots,S_{2n}\neq 0,S_{2n}=2i\right)\overset{*}{=}2\sum_{i=1}^{n}\mathsf{P}\left(S_1,S_2,\ldots,S_{2n}\neq 0,S_{2n}=2i\right)$$
$$\overset{\text{T7}}{=}2\sum_{i=1}^{n}\frac{2i}{2n}\mathsf{P}\left(S_{2n}=2i\right)=2\sum_{i=1}^{n}\frac{2i}{2n}\binom{2n}{n-i}2^{-2n}$$
$$\overset{\text{L10}}{=}2\cdot 2^{-2n}\sum_{i=1}^{n}\left(\binom{2n-1}{n-i}-\binom{2n-1}{n-i-1}\right)\overset{**}{=}2\cdot 2^{-2n}\binom{2n-1}{n}$$
$$=2^{-2n}\frac{2n}{n}\binom{2n-1}{n-1}=2^{-2n}\binom{2n}{n}=\mathsf{P}\left(S_{2n}=0\right).$$

Where $*$ comes from the fact that the random walk is symmetric and $**$ comes from the fact that the positive part of i-th term cancels against the negative part of $i+1$-st term. □

Theorem 12
Let $\left(\{S_n\}_{n=0}^{+\infty},p\right)$ *be a symmetric random walk. The probability that the last return to origin up to time* $2n$ *occurred in time* $2k$ *is* $\mathsf{P}\left(S_{2k}=0\right)\mathsf{P}\left(S_{2(n-k)}=0\right)$.

Proof.

$$\alpha_{2n}\left(2k\right)=u_{2k}u_{2(n-k)}=\mathsf{P}\left(S_{2k}=0\right)\mathsf{P}\left(S_{2k+1},S_{2k+2},\ldots,S_{2n}\neq 0\mid S_{2k}=0\right)$$
$$\overset{\text{L3}}{=}\mathsf{P}\left(S_{2k}=0\right)\mathsf{P}\left(S_1,S_2,\ldots,S_{2(n-k)}\neq 0\right)\overset{\text{T11}}{=}\mathsf{P}\left(S_{2k}=0\right)\mathsf{P}\left(S_{2(n-k)}=0\right).$$

□

Theorem 13
Let $\left(\{S_n\}_{n=0}^{+\infty},p\right)$ *be a symmetric random walk and* $b\in\mathbb{Z}$*, then the following equation holds.*

$$\mathsf{P}\left(S_1,S_2,\ldots,S_n\neq 0,S_n=b\right)=\frac{|b|}{n}\mathsf{P}\left(S_n=b\right).$$

Proof. Let us without loss of generality assume that $b > 0$. In that case, first step has to be rightwards $(X_1 = +1)$. Now we have path from point $(1, 1)$ to point (n, b) that does not return to origin. By Ballot theorem 7 there are $\frac{b}{n} N_n(0, b)$ such paths. Each path consists of $\frac{n+b}{2}$ rightwards steps and $\frac{n-b}{2}$ leftwards steps. Therefore $\mathsf{P}(S_1 \cdot S_2 \cdot, \dots, S_n \neq 0, S_n = b) = \frac{b}{n} N_n(0, b) p^{\frac{n+b}{2}} q^{\frac{n-b}{2}} = \frac{b}{n} \mathsf{P}(S_n = b)$. Case $b < 0$ is identical. $\square$

Lemma 14
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a symmetric random walk.*

$$\mathsf{P}(S_1, S_2, \dots, S_{2n} > 0) = \frac{1}{2} \mathsf{P}(S_{2n} = 0).$$

Proof.

$$\mathsf{P}(S_1, S_2, \dots, S_{2n} > 0) \stackrel{LTP}{=} \sum_{r=1}^{n} \mathsf{P}(S_1, S_2, \dots, S_{2n} > 0, S_{2n} = 2r).$$

The r-th term follows equation:

$$\begin{aligned}
&\mathsf{P}(S_1, S_2, \dots, S_{2n} > 0, S_{2n} = 2r) = \mathsf{P}(X_1 = 1, S_2, \dots, S_{2n} > 0, S_{2n} = 2r) \\
&= \frac{1}{2} \mathsf{P}(S_2, S_3, \dots, S_{2n} > 0, S_{2n} = 2r \mid S_1 = 1) \\
&\stackrel{*}{=} \frac{1}{2} \left(\mathsf{P}(S_{2n} = 2r \mid S_1 = 1) - \mathsf{P}(S_2, S_3, \dots, S_{2n} = 0, S_{2n} = 2r \mid S_1 = 1)\right) \\
&= \frac{1}{2} \left(2^{-(2n-1)} N_{2n-1}(1, 2r) - 2^{-(2n-1)} N_{2n-1}^{0}(1, 2r)\right) \\
&= \frac{1}{2} 2^{-(2n-1)} \left(N_{2n-1}(1, 2r) - N_{2n-1}^{0}(1, 2r)\right) \\
&\stackrel{\text{T6}}{=} \frac{1}{2} 2^{-(2n-1} \left(N_{2n-1}(1, 2r) - N_{2n-1}(-1, 2r)\right) \\
&= \frac{1}{2} 2^{-(2n-1)} \left(\binom{2n-1}{n+r-1} - \binom{2n-1}{n+r}\right).
\end{aligned}$$

Where $*$ comes from the disjoint decomposition: $[S_{2n} = 2r] = [S_{2n} = 2r, S_1 \cdot S_2 \cdot \dots \cdot S_{2n} \neq 0] \cup [S_{2n} = 2r, S_1 \cdot S_2 \cdot \dots \cdot S_{2n} = 0]$.

Because of the fact that the negative parts of r-th terms cancel against the positive parts of $(r+1)$-st terms and the sum reduces to just

$$\begin{aligned}
&\frac{1}{2} 2^{-(2n-1)} \binom{2n-1}{n} = \frac{1}{2} 2 2^{-2n} \binom{2n-1}{n} = \frac{1}{2} 2^{-2n} \frac{2n}{n} \binom{2n-1}{n} \\
&= \frac{1}{2} 2^{-2n} \binom{2n}{n} = \frac{1}{2} \mathsf{P}(S_{2n} = 0).
\end{aligned}$$

$\square$

Theorem 15 (Probability of no return and return)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a symmetric random walk. The following equation holds:*

$$\mathsf{P}(S_1, S_2, \dots, S_{2n} \neq 0) = \mathsf{P}(S_{2n} = 0).$$

Proof. The event $[S_1, S_2, \ldots, S_{2n} \neq 0]$ can be split into two disjoint events: $[S_1, S_2, \ldots, S_{2n} \neq 0] = [S_1, S_2, \ldots, S_{2n} < 0] \cup [S_1, S_2, \ldots, S_{2n} > 0]$.

By previous theorem (L15) we get that the probability of both terms is $\frac{1}{2}u_{2n}$. Because the the events are disjoint we can sum their probabilities and we get the desired result. $\square$

Lemma 16
$\mathsf{P}\left(S_1, S_2, \ldots, S_{2n} \geq 0\right) = \mathsf{P}\left(S_{2n} = 0\right) = u_{2n}$

Proof. $\frac{1}{2}u_{2n} = \mathsf{P}\left(S_1, S_2, \ldots, S_{2n} > 0\right) = \mathsf{P}\left(X_1 = 1, S_2, S_3 \ldots, S_{2n} \geq 1\right)$
$\overset{nasobeni}{=} \mathsf{P}\left(S_1 = 1\right) \mathsf{P}\left(S_2, S_3 \ldots, S_{2n} \geq 1 \mid S_1 = 1\right)$
$= \frac{1}{2}\mathsf{P}\left(S_2, S_3 \ldots, S_{2n} \geq 1 \mid S_1 = 1\right)$
$\overset{L3}{=} \frac{1}{2}\mathsf{P}\left(S_1, S_2 \ldots, S_{2n-1} \geq 1 \mid S_0 = 1\right)$
$\overset{L2}{=} \frac{1}{2}\mathsf{P}\left(S_1, S_2 \ldots, S_{2n-1} \geq 0\right)$
$= \frac{1}{2}\mathsf{P}\left(S_1, S_2 \ldots, S_{2n} \geq 0\right)$. Because $[S_{2n-1} \geq 0] \Rightarrow [S_{2n-1} \geq 1] \Rightarrow [S_{2n} \geq 0]$
Therefore $\mathsf{P}\left(S_1, S_2 \ldots, S_{2n} \geq 0\right) = u_{2n}$. $\square$

Theorem 17
$f_{2n} = u_{2n-2} - u_{2n}$

Proof. The event $[S_1, S_2, \ldots S_{2n-1} \neq 0]$ can be split into two disjoint events: $[S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} = 0]$ and $[S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} \neq 0]$. Therefore $\mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0\right) = \mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} = 0\right) + \mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} \neq 0\right)$. Therefore we get $f_{2n} = \mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0, S_{2n} = 0\right) = \mathsf{P}\left(S_1, S_2, \ldots S_{2n-1} \neq 0\right) - \mathsf{P}\left(S_1, S_2, \ldots, S_{2n} \neq 0\right)$. Because $2n-1$ is odd. $\mathsf{P}\left(S_{2n-1} = 0\right) = 0$. Therefore the first term is equal to $\mathsf{P}\left(S_1, S_2, \ldots S_{2n-2} \neq 0\right)$ which is by 15 equal to u_{2n-2}. Second term is by 15 equal to u_{2n}. Therefore we get the result. $\square$

Lemma 18
$f_{2n} = \frac{1}{2n-1}u_{2n}$

Proof. $u_{2n-2} = 2^{-(2n-2)}\binom{2n-2}{n-1} = 4 \cdot 2^{-2n}\frac{(2n-2)!}{(n-1)!(n-1)!} = \frac{4n^2}{(2n)(2n-1)}2^{-2n}\binom{2n}{n} = \frac{2n}{2n-1}u_{2n}$.
Therefore $u_{2n-2} - u_{2n} = u_{2n}\left(\frac{2n}{2n-1} - 1\right) = u_{2n}\frac{1}{2n-1}$. $\square$

Lemma 19 (Decomposition of f_n)
$u_{2n} = \sum\limits_{r=1}^{n} f_{2r}u_{2n-2r}$

Proof. $u_{2n} \overset{D4}{=} \mathsf{P}\left(S_{2n} = 0\right) \overset{LTP}{=} \sum\limits_{r=1}^{n} \mathsf{P}\left(S_{2n} = 0, S_1, S, 2, \ldots, S_{2r-1} \neq 0 S_{2r} = 0\right) \overset{nasobeni}{=}$
$\sum\limits_{r=1}^{n} \mathsf{P}\left(S_{2n} = 0 \mid S_1, S, 2, \ldots, S_{2r-1} \neq 0 S_{2r} = 0\right) \mathsf{P}\left(S_1, S, 2, \ldots, S_{2r-1} \neq 0 S_{2r} = 0\right) =$
$\sum\limits_{r=1}^{n} \mathsf{P}\left(S_{2n} = 0 \mid S_{2r} = 0\right) f_{2r} \overset{L3}{=} \sum\limits_{r=1}^{n} u_{2n-2r}f_{2r}$. $\square$

Theorem 20 (Arcsine law for last visits)
Let $k, n \in \mathbb{N}, k \leq n$. The probability that up to time $2n$ the last return to origin occurred in time $2k$ is given by $\alpha_{2n}(2k) = u_{2n}u_{2(n-k)}$.

Proof. The probability involved can be rewritten as:

$\mathsf{P}(S_{2k+1}, S_{2k+2}, \dots, S_{2n} \neq 0, S_{2k} = 0)$
$\overset{nasobeni}{=} \mathsf{P}(S_{2k+1}, S_{2k+2}, \dots, S_{2n} \neq 0 \mid S_{2k} = 0)\,\mathsf{P}(S_{2k} = 0)$
$\overset{\text{L3}}{=} \mathsf{P}\left(S_1, S_2, \dots, S_{2(n-k)} \neq 0\right)\mathsf{P}(S_{2k} = 0)$
$\overset{\text{T15}}{=} u_{2(n-k)} u_{2k}$

□

Definition 7 (Time spend on the positive and negative sides)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk. We say that the walk spent τ time units of n on the positive side if $\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = \tau$. Let $\beta_n(\tau)$ denote the probability of such an event. We say that the walk spent ζ time units of n on the negative side if $\sum_{i=1}^{n} \mathbf{1}_{[S_i<0 \vee S_{i-1}<0]} = \zeta$.

Theorem 21 (Arcsine law for sojourn times-OWN PROOF)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a symmetric random walk. Then* $\beta_{2n}(2k) = \alpha_{2n}(2k)$.

Proof. Firstly let us start with degenerate cases. $\beta_{2n}(2n) = \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2n\right) \overset{\text{L16}}{=} \mathsf{P}(S_1, S_2, \dots, S_{2n} \geq 0) = u_{2n}$. By symmetry $\beta_{2n}(0) = \beta_{2n}(2n) = u_{2n}$.

Let $1 \leq k \leq v - 1$, where $0 \leq v \leq n$. For such k stands equation: $\beta_{2n}(2k) \overset{\text{D7}}{=} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k\right)$
$\overset{LTP}{=} \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \dots, S_{2r-1} \neq 0, S_{2r} = 0\right)$
$\overset{*}{=} \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \dots, S_{2r-1} < 0, S_{2r} = 0\right)$
$+ \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \dots, S_{2r-1} > 0, S_{2r} = 0\right)$
$\overset{nasobeni}{=} \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k \,\middle|\, S_1, S_2, \dots, S_{2r-1} < 0, S_{2r} = 0\right)$
$\mathsf{P}(S_1, S_2, \dots, S_{2r-1} < 0, S_{2r} = 0)$
$+ \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k \,\middle|\, S_1, S_2, \dots, S_{2r-1} > 0, S_{2r} = 0\right)$
$\mathsf{P}(S_1, S_2, \dots, S_{2r-1} > 0, S_{2r} = 0)$
$\overset{**}{=} \sum_{r=1}^{n} \frac{1}{2} f_{2r}\,\mathsf{P}\left(\sum_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k \,\middle|\, S_{2r} = 0\right)$
$+ \sum_{r=1}^{n} \frac{1}{2} f_{2r}\,\mathsf{P}\left(\sum_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k - 2r \,\middle|\, S_{2r} = 0\right)$
$\overset{\text{L3}}{=} \sum_{r=1}^{n} \frac{1}{2} f_{2r}\,\mathsf{P}\left(\sum_{i=1}^{2n-2r} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k\right) + \sum_{r=1}^{n} \frac{1}{2} f_{2r}\,\mathsf{P}\left(\sum_{i=1}^{2n-2r} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k - 2r\right) =$
$\sum_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k) + \sum_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k - 2r)$. Where $*$ comes from the disjoint decomposition of $[S_1, S_2, \dots, S_{2r-1} \neq 0] = [S_1, S_2, \dots, S_{2r-1} > 0] \cup [S_1, S_2, \dots, S_{2r-1} < 0]$ and $**$ comes from using the condition that up to time $2r$ the steps were on the positive/negative sides.

Now let us proceed by induction. Case for $v = 1$ is trivial because it implies the degenerate case. Let the statement be true for $v \leq n-1$, then $\sum_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k) + \sum_{r=1}^{n} \frac{1}{2} f_{2r} \beta_{2n-2r}(2k - 2r)$

$\overset{IA}{=} \sum_{r=1}^{n} \frac{1}{2} f_{2r} \alpha_{2n-2r} (2k) + \sum_{r=1}^{n} \frac{1}{2} f_{2r} \alpha_{2n-2r} (2k-2r)$
$\overset{D4}{=} \sum_{r=1}^{n} \frac{1}{2} f_{2r} u_{2k} u_{2n-2r-2k} + \sum_{r=1}^{n} \frac{1}{2} f_{2r} u_{2k-2r} u_{2n-2k}$
$= \frac{1}{2} u_{2k} \sum_{r=1}^{n} f_{2r} u_{2n-2r-2k} + \frac{1}{2} u_{2n-2k} \sum_{r=1}^{n} f_{2r} u_{2k-2r} u_{2n-2k}$
L19
$\frac{1}{2}\, u_{2n-2k} u_{2k} + \frac{1}{2} u_{2n-2k} u_{2k} = u_{2n-2k} u_{2k}$
$\overset{D4}{=} \alpha_{2n} (2k)\,.$ □

Definition 8 (Change of a sign)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$be a random walk. We say that in time n occurred a change of sign if if $S_{n-1} \cdot S_{n+1} = -1$ in other words if $(S_{n-1} = +1 \wedge S_{n+1} = -1) \vee (S_{n-1} = -1 \wedge S_{n+1} = +1)$. We shall denote the probability that up to time n occurred r changes of sign by $\xi_{r,n}$.

Theorem 22 (Change of a sign)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$*be a symmetric random walk. The probability* $\xi_{r,2n+1} = 2\,\mathsf{P}\,(S_{2n+1} = 2r+1)$

Proof. Feller □

1.1 Problem chapter 9 Feller-není dokončeno ani zkotrolováno

Definition 9 $(\delta_n, \varepsilon_n^{r,\pm})$**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$be a symmetric random walk. $\delta_n (k)$ shall denote $\mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_n = 0\right)$, $\varepsilon_n^{r} (k)$ shall denote $\mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_1, S\right.$
$\varepsilon_n^{r,+} (k)$ shall denote $\mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_1, S_2, \ldots, S_{r-1} > 0, S_r = 0, S_n = 0\right)$,
$\varepsilon_n^{r,-} (k)$ shall denote $\mathsf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_1, S_2, \ldots, S_{r-1} < 0, S_r = 0, S_n = 0\right)$.

Lemma 23 (Factorization of $\delta_{2n} (2k)$)
$\delta_{2n} (2k) = \frac{1}{2} \sum_{r=1}^{n} (f_{2r} \delta_{2n-2r} (2k-2r) + f_{2r} \delta_{2n-2r} (2r))$.

Proof. Because $S_{2n} = 0$ a return to origin must have happened. Let $2r$ the time of first return to origin, where $r \in \{1, 2, \ldots, n\}$. By the law of total probability:
$\delta_{2n} (2k) \overset{D9}{=} \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = k, S_{2n} = 0\right)$
$\overset{LTP}{=} \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} \neq 0, S_{2r} = 0, S_{2n} = 0\right)$
$\overset{D9}{=} \sum_{r=1}^{n} \varepsilon_{2n}^{2k} \overset{*}{=} \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0, S_{2n} = 0\right)$
$+ \sum_{r=1}^{n} \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0, S_{2n} = 0\right)$
$= \sum_{r=1}^{n} \varepsilon_{2n}^{2r,+} (2k) + \sum_{r=1}^{n} \varepsilon_{2n}^{2r,-} (2k)\,.$

Where $*$ comes from the disjoint decomposition $[S_1, S_2, \ldots, S_{2r-1} \neq 0] = [S_1, S_2, \ldots, S_{2r-1} > 0] \cup [S_1, S_2, \ldots, S_{2r-1} < 0]$.

Now let us calculate $\varepsilon_{2n}^{2r,+} (2k)$
$= \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0, S_{2n} = 0\right)$

$\overset{nasobeni}{=} \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n} = 0 \mid S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right)$
$\mathsf{P}\left(S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right)$
$\overset{*}{=} \mathsf{P}\left(\sum_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k - 2r, S_{2n} = 0 \,\middle|\, S_{2r} = 0\right) \mathsf{P}\left(S_1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right)$
$\overset{**}{=} \mathsf{P}\left(\sum_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k - 2r, S_{2n} = 0 \,\middle|\, S_{2r} = 0\right) \frac{1}{2} f_2 r$
$\overset{\text{L3}}{=} \mathsf{P}\left(\sum_{i=1}^{2n-2r} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k - 2r, S_{2n-2r} = 0\right) \frac{1}{2} f_{2r}$
$\overset{\text{D9}}{=} \delta_{2n-2r}\left(2k - 2r\right) \frac{1}{2} f_{2r}.$

Where $*$ comes from Lemma (4) and using the condition.

Where $**$ comes from the fact that $f_{2r} \overset{\text{D4}}{=} \mathsf{P}\left(S_1, S_2, \ldots, S_{2r-1} \neq 0, S_{2r} = 0\right) = \mathsf{P}\left(S_1 = 1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right) + \mathsf{P}\left(S_1 = -1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right)$ and $\mathsf{P}\left(S_1 = -1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right) = \mathsf{P}\left(S_1 = 1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right)$ because of symmetry. Hence $\mathsf{P}\left(S_1 = 1, S_2, \ldots, S_{2r-1} > 0, S_{2r} = 0\right) = \frac{1}{2} f_{2r}$.

Similarly $\varepsilon_{2n}^{2r,+}\left(2k\right)$
$= \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0, S_{2n} = 0\right)$
$= \mathsf{P}\left(\sum_{i=1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n} = 0 \mid S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right)$
$\mathsf{P}\left(S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right)$
$= \mathsf{P}\left(\sum_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n} = 0 \,\middle|\, S_{2r} = 0\right) \mathsf{P}\left(S_1, S_2, \ldots, S_{2r-1} < 0, S_{2r} = 0\right)$
$= \mathsf{P}\left(\sum_{i=2r+1}^{2n} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n} = 0 \,\middle|\, S_{2r} = 0\right) \frac{1}{2} f_2 r$
$= \mathsf{P}\left(\sum_{i=1}^{2n-2r} \mathbf{1}_{[S_i>0 \vee S_{i-1}>0]} = 2k, S_{2n-2r} = 0\right) \frac{1}{2} f_{2r}$
$= \delta_{2n-2r}\left(2k\right) \frac{1}{2} f_{2r}.$

Therefore $\delta_{2n}\left(2k\right) = \frac{1}{2} \sum_{r=1}^{n} f_{2r} \delta_{2n-2r}\left(2k - 2r\right) + \frac{1}{2} \sum_{r=1}^{n} f_{2r} \delta_{2n-2r}\left(2k\right)$
$= \frac{1}{2} \sum_{r=1}^{n} \left(f_{2r} \delta_{2n-2r}\left(2k - 2r\right) + f_{2r} \delta_{2n-2r}\left(2r\right)\right)$ □

Theorem 24 (Equidistributional theorem-ALMOST COMPLETE OWN PROOF) *Let* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *be a symmetric random walkand* $n \in \mathbb{N}$, *then* $\forall k, l \in \{0, 1, 2, \ldots, n\}$: $\delta_{2n}\left(2k\right) = \delta_{2n}\left(2l\right) = \frac{u_{2n}}{n+1}$.

Proof. Let us prove this statement by induction in n. In case that $n = 1$ we have two options for k. Either $k = 0$ or $k = 1$. $\delta_2\left(0\right) = \mathsf{P}\left(S_1 = -1, S_2 = 0\right) = \frac{1}{2} u_2 = \mathsf{P}\left(S_1 = +1, S_2 = 0\right) = \delta_2\left(2\right).$

Let the statement be true for all $l \leq n - 1$. In that case $\delta_{2(n-l)}\left(2k\right) = \frac{u_{2(n-l)}}{n-l+1} \forall k \in \{1, 2, \ldots, n - l\}$. We want to show that $\delta_{2n} = \frac{u_{2n}}{n+1}$.

Let us calculate $\delta_{2n} \overset{\text{L23}}{=} \frac{1}{2} \sum_{r=1}^{n} \left(f_{2r} \delta_{2n-2r}\left(2k - 2r\right) + f_{2r} \delta_{2n-2r}\left(2r\right)\right)$
$\overset{IA}{=} \frac{1}{2} \sum_{r=1}^{n} \left(f_{2r} u_{2n-2r} \frac{1}{n-r+1} + f_{2r} u_{2n-2r} \frac{1}{n-r+1}\right) = \sum_{r=1}^{n} \frac{f_{2r} u_{2n-2r}}{n-r+1} \overset{\text{SNAD TO DOKAZU L25}}{=} \frac{u_{2n}}{n+1}$
□

Lemma 25 (Sum of binomials-POTŘEBUJU DOKÁZAT)
$\sum_{r=1}^{n} \frac{f_{2r} u_{2n-2r}}{n-r+1} = \frac{u_{2n}}{n+1}$

Proof. $f_{2r}u_{2n-2r} \stackrel{\text{L18}}{=} \frac{1}{2r-1}u_{2r}u_{2n-2r} \stackrel{\text{D4}}{=} \frac{1}{2r-1}2^{-2r}\binom{2r}{r}2^{-(2n-2r)}\binom{2n-2r}{n-r}$.

Therefore $\sum_{r=1}^{n} \frac{f_{2r}u_{2n-2r}}{n-r+1} = \sum_{r=1}^{n} \frac{1}{2r-1}\frac{1}{n-r+1}2^{-2n}\binom{2r}{r}\binom{2n-2r}{n-r} \stackrel{???}{=} \frac{1}{n+1}2^{-2n}\binom{2n}{n}$ □