1. Number of paths+maxima

Remark. Once having stated basic definitions we may ask questions about occupation times of a set. Let $a > 0$. How many steps up to time n does our walk spend above a (in interval $[a, +\infty)$)? Similarly how many steps does the walk spend in interval $[-a, +\infty)$? We are going to answer these questions in following XXX.

Remark. The assumption $p = \frac{1}{2}$ simlifies the calculation of probabilities because we have 2^{-n} instead of $p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}$. Therefore each path of length n hes equal probability of 2^{-n}. Therefore the probabilities only depend on $\binom{n}{\frac{x+n}{2}}$ which can be more generalized like in the following definition.

Definition 1 (Number of possible paths)**.** Let $n \in \mathbb{N}, a, b \in \mathbb{Z}$. Let $N_n(a, b)$ be *number of posssible paths* of random walk $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ from point $(0, a)$ to point (n, b) and $N_n^x(a, b)$ be number of possible paths from point $(0, a)$ to point (n, b) that visit point (z, x) for some $z \in \{1, 2, \ldots, n\}$.

Theorem 1 (Number of possible paths)
Let $a, b \in \mathbb{Z}, n \in \mathbb{N}$ then

$$N_n(a, b) = \binom{n}{\frac{1}{2}(n + b - a)}.$$

Proof. Let us choose a path from the point $(0, a)$ to point (n, b) and let α be number of the rightwards steps and β be number of the leftwards steps. Then $\alpha + \beta = n$ and $\alpha - \beta = b - a$. By adding these two equations we get that $\alpha = \frac{1}{2}(n + b - a)$. The number of possible paths is the number of ways of picking α rightwards steps from n steps. Therefore we get $N_n(a, b) = \binom{n}{\alpha} = \binom{n}{\frac{1}{2}(n+b-a)}$. □

Theorem 2 (Reflection principle)
Let $a, b \in \mathbb{N}$, then

$$N_n^0(a, b) = N_n(-a, b).$$

Proof. Each path from $(0, -a)$ to (n, b) has to intersect $y = 0$-axis at least once at some point. Let k be the time of the first intersection with x-axis. By reflecting the segment from $(0, -a)$ to $(k, 0)$ in the x-axis and letting the segment from $(k, 0)$ to (n, b) stay the same as it was, we get a path from point $(0, a)$ to (n, b) which visits 0 in time k. Because reflection is a bijective operation on sets of paths, we get the correspondence between the collections of such paths. □

Remark. Following is the definition of return to origin which is a crucial term for this thesis. Let us come back to our question. While calculating the number of steps the walk spends in interval $[a, +\infty)$ we find the first passage through a and then set a as a new origin. However, our achievements concerning return to origin are in the latter subchapter.

Definition 2 (Return to origin)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk, $k \in \mathbb{N}$. We say a *return to origin* occurred in time $2k$ if $S_{2k} = 0$. The probability that in time $2k$ occurred a return to origin shall be denoted by u_{2k}. We say that in time $2k$

occurred *first return to origin* if $S_1, S_2, \dots S_{2k-1} \neq 0$ and $S_{2k} = 0$. The probability that in time $2k$ occurred first return to origin shall be denoted by f_{2k}. We define $f_0 := 0$. Let $\alpha_{2n}(2k)$ denote $u_{2k}u_{2n-2k}$.

Theorem 3 (Ballot theorem)
Let $n, b \in N$ Number of paths from point $(0,0)$ to point (n,b) which do not return to origin is equal to $\frac{b}{n} N_n(0,b)$

Proof. Let us call N the number of paths we are referring to. Because the path ends at point (n,b) and it does not return to origin, the first step has to be rightwards. Therefore we now have

$$N = N_{n-1}(1,b) - N^0_{n-1}(1,b) \overset{\text{T2}}{=} N_{n-1}(1,b) - N_{n-1}(-1,b).$$

Hence we get that:

$$\begin{aligned}
N &= N_{n-1}(1,b) - N_{n-1}(-1,b) \overset{\text{T1}}{=} \binom{n-1}{\frac{n}{2}+\frac{b}{2}-1} - \binom{n-1}{\frac{n}{2}+\frac{b}{2}} \\
&= \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!} - \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!} \\
&= \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}\right)\left(\frac{n}{2}-\frac{b}{2}-1\right)!} - \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}\right)\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!} \\
&= \frac{(n-1)!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!}\left(\frac{1}{\frac{n}{2}-\frac{b}{2}} - \frac{1}{\frac{n}{2}+\frac{b}{2}}\right) \\
&= \frac{1}{n}\frac{n!}{\left(\frac{n}{2}+\frac{b}{2}-1\right)!\left(\frac{n}{2}-\frac{b}{2}-1\right)!}\left(\frac{\left(\frac{n}{2}+\frac{b}{2}-\frac{n}{2}+\frac{b}{2}\right)}{\left(\frac{n}{2}-\frac{b}{2}\right)\left(\frac{n}{2}+\frac{b}{2}\right)}\right) \\
&= \frac{b}{n}\frac{n!}{\left(\frac{n}{2}+\frac{b}{2}\right)!\left(\frac{n}{2}-\frac{b}{2}\right)!} = \frac{b}{n}\binom{n}{\left(\frac{n}{2}+\frac{b}{2}\right)} \overset{\text{T1}}{=} \frac{b}{n}N_n(0,b)
\end{aligned}$$

□

Remark. The name *Ballot theorem* comes from the question: In a ballot where candidate A receives p votes and candidate B receives q votes with $p > q$, what is the probability that A had been strictly ahead of B throughout the whole count?

Answer to this question can be derived from the previous theorem. In our case $b = p - q$ and $n = p + q$.

Definition 3 (Maximum and minimum)**.** Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk. $M_n^+ := \max\{S_i, i \in \{1,2,\dots,n\}\}$ is called *maximum of random walk* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *up to time* n and $M_n^- := \min\{S_i, i \in \{1,2,\dots,n\}\}$ is called *minimum of random walk* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *up to time* n. $M_n = \max\{M_n^+, -M_n^-\}$ is called *absolute maximum of random walk* $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ *up to time* n.

Theorem 4 (Probability of maximum up to time n)
Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk.

$$\mathsf{P}\left(M_n^+ \geq r, S_n = b\right) = \begin{cases} \mathsf{P}(S_n = b) & \text{for } b \geq r\ , \\ \mathsf{P}(S_n = 2r - b)\left(\frac{q}{p}\right)^{r-b}, & \text{for otherwise.} \end{cases}$$

Proof. Let us firstly consider the easier case in which $b \geq r$. Because we defined M_n^+ as $\max\{S_i, i \in \{1, 2, \dots, n\}\}$ we get that $M_n^+ \geq b \geq r$ therefore $[M_n^+ \geq r] \subset [S_n = b]$ hence we get $\mathsf{P}(M_n^+ \geq r, S_n = b) = \mathsf{P}(S_n = b)$.

Now let $r \geq 1, b < r$. $N_n^r(0, b)$ stands for number of paths from point $(0, 0)$ to point (n, b) which reach up to r. Let $k \in \{1, 2, \dots, n\}$ denote the first time tha walk reaches r. By reflection principle (2), we can reflect the segment from (k, r) to (n, b) in the axis:$y = r$. Therefore we now have path from $(0, 0)$ to $(n, 2r - b)$ and we get that

$$N_n^r(0, b) = N_n(0, 2r - b) \text{ hence } \mathsf{P}\left(S_n = b, M_n^+ \geq r\right) = N_n^r(0, b) p^{\frac{n+b}{2}} q^{\frac{n-b}{2}} = N_n(0, 2r - b) p^{\frac{n+(2r-b)}{2}} q^{\frac{n-(2r-b)}{2}} p^{b-r} q^{r-b} = \left(\frac{q}{p}\right)^{r-b} \mathsf{P}(S_n = 2r - b).$$

□

Definition 4 (Walk reaching new maximum at particular time). Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk $n, b \in \mathbb{N}$. We say that *the walk reached new maximum b in time n* if $M_{n-1}^+ = S_{n-1} = b - 1, S_n = b$. We denote such probability by $f_b(n)$.

Theorem 5 (Probability of reaching new maximum b in time n) *Let $\left(\{S_n\}_{n=0}^{+\infty}, p\right)$ be a random walk $n, b \in \mathbb{N}$ then*

$$f_b(n) = \frac{b}{n} \mathsf{P}(S_n = b).$$

Proof.

$$\begin{aligned}
f_b &= \mathsf{P}(M_{n-1} = S_{n-1} = b - 1, S_n = b) = \mathsf{P}(M_{n-1} = S_{n-1} = b - 1, X_n = +1) \\
&= p\,\mathsf{P}(M_{n-1} = S_{n-1} = b - 1) \\
&\overset{*}{=} p\left(\mathsf{P}(M_{n-1} \geq b - 1, S_{n-1} = b - 1) - \mathsf{P}(M_{n-1} \geq b, S_{n-1} = b - 1)\right) \\
&\overset{\text{T4}}{=} p\left(\mathsf{P}(S_{n-1} = b - 1) - \frac{q}{p}\mathsf{P}(S_{n-1} = b + 1)\right) \\
&= p\,\mathsf{P}(S_{n-1} = b - 1) - q\,\mathsf{P}(S_{n-1} = b + 1) \\
&= \binom{n-1}{\frac{n}{2} + \frac{b}{2} - 1} p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} - \binom{n-1}{\frac{n}{2} + \frac{b}{2}} p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \\
&= p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \left(\frac{(n-1)!}{\left(\frac{n}{2} + \frac{b}{2} - 1\right)!\left(\frac{n}{2} - \frac{b}{2}\right)!} - \frac{(n-1)!}{\left(\frac{n}{2} + \frac{b}{2}\right)!\left(\frac{n}{2} - \frac{b}{2} - 1\right)!}\right) \\
&= p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \left(\frac{(n-1)!}{\left(\frac{n}{2} + \frac{b}{2} - 1\right)!\left(\frac{n}{2} - \frac{b}{2} - 1\right)!}\right)\left(\frac{1}{\frac{n}{2} - \frac{b}{2}} - \frac{1}{\frac{n}{2} + \frac{b}{2}}\right) \\
&= p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \frac{b}{n}\left(\frac{n!}{\left(\frac{n}{2} + \frac{b}{2}\right)!\left(\frac{n}{2} - \frac{b}{2}\right)!}\right) = \frac{b}{n} p^{\frac{n}{2}+\frac{b}{2}} q^{\frac{n}{2}-\frac{b}{2}} \binom{n}{\frac{n}{2} + \frac{b}{2}} = \frac{b}{n}\mathsf{P}(S_n = b).
\end{aligned}$$

Where $*$ comes from the fact that the event $[M_{n-1} \geq b - 1]$ can be split into two disjoint events: $[M_{n-1} \geq b - 1] = [M_{n-1} \geq b] \cup [M_{n-1} = b - 1]$.

Hence: $\mathsf{P}(M_{n-1} = b - 1) = \mathsf{P}(M_{n-1} \geq b - 1) - \mathsf{P}(M_{n-1} \geq b)$. The same applies for the probability $\mathsf{P}(M_{n-1} = b - 1, S_{n-1} = b - 1)$ □