$$S(N) = \frac{1}{N} \sum_{k=1}^{N} \begin{bmatrix} \bar{z}_\alpha(k) \\ \bar{v}(k) \end{bmatrix} p'(k).$$

Similarly the H estimator may be implemented as

$$\hat{H}(N) = \text{first } n \text{ columns } \{Q(N)[S(N) - P]^{-1}\} \tag{27}$$

where

$$Q(N) = \frac{1}{N} \sum_{k=1}^{N} z(k)p'(k).$$

The following probability limits implying the convergence of the proposed algorithm are easily shown

$$p \cdot \lim_{N\to\infty} \hat{\phi}(N) = \phi; \; p \cdot \lim_{N\to\infty} H(N) = H. \tag{28}$$

An efficient scheme for on-line estimation can be developed from the foregoing as is done in [9].

Finally, G can be calculated from the estimate of ϕ utilizing (14).

IV. Conclusion

A one-shot algorithm has been presented for the estimation of the system matricies F, G, H from noisy measurements of the input and output signals. The output structural indicies are not employed. Moreover, no special difficulty is encountered in the case of dependent outputs.

At present there is no rationale for the choice α and β. In the absence of measurement noise almost any α and β will suffice. The utility of the wide latitude in the choice α and β may lie in being able to iteratively improve the system parameter estimates in the presence of measurement noise.

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An Identification Procedure for Discrete Multivariable Systems

PRABHAKAR KUDVA, STUDENT MEMBER, IEEE, AND KUMPATI S. NARENDRA, SENIOR MEMBER, IEEE

***Abstract*—A new procedure is presented for the identification of discrete linear multivariable systems. Using Lyapunov's direct method the asymptotic stability of the overall system is established when the inputs are sufficiently general. It is shown that the procedure may also be extended for the identification of a certain class of nonlinear systems.**

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The authors are with the Becton Center, Yale University, New Haven, Conn. 06520.

I. Introduction

A stable identification scheme for continuous multivariable systems, using Lyapunov's direct method, was suggested in [1] by Pazdera and Pottinger and later independently by the authors in [2]. In [3] Mendel discussed a deterministic-gradient identification scheme for a discrete single variable system and proved its stability. In this paper, the direct method of Lyapunov is employed to develop a stable procedure for the identification of a discrete multivariable system from the measurements on the inputs and the state variables of the system.

For the procedure to be applicable, it is assumed that all the state variables of the plant are accessible and that there is no measurement noise. The procedure results in the convergence of every parameter of a model to the corresponding one of the system as the number of stages $k \to \infty$. The identification scheme is derived first for a linear system; the extension to the case of a nonlinear system, where the form of the nonlinearity is known, is subsequently indicated.

Computer simulations are provided to demonstrate the feasibility of the proposed scheme for both linear and nonlinear systems.

II. Statement of the Problem

A linear time invariant discrete system is described by the vector difference equation

$$x_p(k+1) = Ax_p(k) + Bu(k) \tag{1}$$

where some or all of the elements of the $(n \times n)$ matrix A and those of the $(n \times m)$ matrix B are unknown but constant. It is desired to set up a model and suitably adjust its parameters so that as the number of stages k increases, the unknown elements of A and B can be estimated.

III. Model for the Identification Scheme

The model used to track the parameters of the given system is described by the difference equation

$$x_m(k+1) = \hat{A}(k+1)x_p(k) + \hat{B}(k+1)u(k). \tag{2}$$

The identification scheme describes a method of adjusting the $n(n+m)$ elements of $\hat{A}$ and $\hat{B}$ in (2) in order to satisfy the following condition:

$$\lim_{k\to\infty} \hat{A}(k) = A; \; \lim_{k\to\infty} \hat{B}(k) = B;$$
$$\lim_{k\to\infty} [x_m(k) - x_p(k)] = \lim_{k\to\infty} e(k) = 0. \tag{3}$$

IV. The Identification Scheme and Proof of Stability

Defining the following error vector and error matrices as:

$$\begin{array}{ll} \text{State error vector} & e(k) \triangleq x_m(k) - x_p(k) \\ \text{Parameter error matrices} & \begin{cases} \Phi(k) \triangleq \hat{A}(k) - A \\ \Psi(k) \triangleq \hat{B}(k) - B \end{cases} \end{array} \tag{4}$$

it follows from (1) and (2) that $e(k)$ satisfies

$$\begin{aligned} e(k+1) &= \Phi(k+1)x_p(k) + \Psi(k+1)u(k) \\ &= \theta(k+1)y(k) \end{aligned} \tag{5}$$

where

$$\theta \triangleq [\Phi \vdots \Psi]; \; y \triangleq \begin{bmatrix} x_p \\ u \end{bmatrix}.$$

Hence

$$e(k) = \theta(k)y(k-1). \tag{6}$$

The rule chosen to adjust the parameters of the model is given by

$$\theta(k+1) = \theta(k) - F(k)$$

where

$$F(k) = \epsilon(k)Pe(k)y^T(k-1), \qquad \epsilon(k) \text{ is a scalar} > 0 \; \forall \; k,$$
$$P = P^T > 0. \tag{7}$$

Consider the following candidate for the Lyapunov function:

$$V(k) = \mathrm{tr}[\theta^T(k)\theta(k)] \tag{8}$$

where "tr" denotes the trace of a matrix.

Hence

$$\begin{aligned}\Delta V(k) \triangleq V(k+1) - V(k) &= \mathrm{tr}[-2\theta^T(k)F(k) + F^T(k)F(k)]\\ &= \mathrm{tr}[-2\theta^T(k)\epsilon(k)Pe(k)y^T(k-1)\\ &\quad + \epsilon^2(k)y(k-1)e^T(k)P^2e(k)y^T(k-1)].\end{aligned} \tag{9}$$

Since given two vectors μ and η

$$\mathrm{tr}\ \mu\eta^T = \eta^T\mu$$

it follows that

$$\begin{aligned}\Delta V(k) &= -2y^T(k-1)\theta^T(k)\epsilon(k)Pe(k)\\ &\quad + \epsilon^2(k)y^T(k-1)y(k-1)e^T(k)P^2e(k)\\ &= -2\epsilon(k)e^T(k)Pe(k)\\ &\quad + \epsilon^2(k)y^T(k-1)y(k-1)e^T(k)P^2e(k) \quad \text{[from (6)]}\\ &= -e^T(k)[2\epsilon(k)P - \epsilon^2(k)y^T(k-1)y(k-1)P^2]e(k).\end{aligned} \tag{10}$$

Lemma 1: If γ is a real positive scalar, and P, a symmetric positive definite matrix then

$$[2\beta P - \gamma\beta^2P^2] > 0$$

if $\beta = (\alpha)/(\lambda_{\max}\gamma)$ where $0 < \alpha < 2$ and $\lambda_{\max}$ is the largest eigen value of P. [For proof, refer to Appendix A.]

If $\epsilon(k)$ is chosen as

$$\hat{\epsilon(k)} = \frac{\alpha}{\lambda_{\max}y^T(k-1)y(k-1)}, \qquad (0 < \alpha < 2) \tag{11}$$

it can be easily shown, using Lemma 1 and (10), that ΔV is negative semidefinite, i.e.,

$$\Delta V(k) \leqslant 0, \ \forall\ k.$$

The identification law is, therefore, chosen as

$$\theta(k+1) = \theta(k) - \frac{\alpha}{\lambda_{\max}y^T(k-1)y(k-1)}Pe(k)y^T(k-1) \tag{12}$$

i.e.,

$$[\hat{A}(k+1) \vdots \hat{B}(k+1)] = [\hat{A}(k) \vdots \hat{B}(k)] - \frac{\alpha Pe(k)y^T(k-1)}{\lambda_{\max}y^T(k-1)y(k-1)}, \quad (\text{with } u(0) \neq 0 \text{ and } 0 < \alpha < 2). \tag{13}$$

Even though $\Delta V(k)$ is only negative semidefinite, the overall scheme has been shown to be asymptotically stable [Appendix B] if the plant is completely controllable and the input u is sufficiently general.

Extension to Nonlinear Systems

The scheme for the identification of linear systems, can be easily extended to the case of nonlinear multivariable systems where the form of the nonlinearity is known exactly.

Consider a plant given by

$$x_p(k+1) = Af[x_p(k),u(k)] \tag{14}$$

and a model described by

$$x_m(k+1) = \hat{A}(k+1)f[x_p(k),u(k)]. \tag{15}$$

Following an approach identical to the one given above, it can be shown that stability is assured with the identification law

$$\hat{A}(k+1) = \hat{A}(k) - \frac{\alpha}{\lambda_{\max}f^T(k-1)f(k-1)}Pe(k)f^T(k-1) \tag{16}$$

where $f(k) \triangleq f[x_p(k),u(k)]$; and $0 < \alpha < 2$.

V. Examples

Using the proposed scheme, computer simulations were carried out to identify the unknown parameters of the following second-order linear and nonlinear plants.

1) Example 1 (Linear plant):

a) *Plant:*

$$x_p(k+1) = \begin{bmatrix} 0.5 & a_{12} \\ a_{21} & 0.5 \end{bmatrix} x_p(k) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(k)$$

where a_{12}, a_{21}, b_1, and b_2 are the unknown parameters.

b) *Model:*

$$x_m(k+1) = \begin{bmatrix} 0.5 & \hat{a}_{12}(k+1) \\ \hat{a}_{21}(k+1) & 0.5 \end{bmatrix} x_p(k) + \begin{bmatrix} \hat{b}_1(k+1) \\ \hat{b}_2(k+1) \end{bmatrix} u(k).$$

c) *Input:* The input signal $u(k)$ was chosen to be sampled rectangular pulses of unit height with fundamental frequency $\omega = 1$ rad/s.

Simulation Results: Simulations were carried out with the following set of values:

$$a_{12} = a_{21} = 0.1;\ b_1 = 0.5;\ b_2 = 1;\ \alpha = 1.5;\ P = I.$$

The results of these simulations are presented in Fig. 1.

2) Example 2 (Nonlinear plant):

a) *Plant:*

$$x_p(k+1) = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.5 \end{bmatrix} x_p(k) + \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} \cos\,[x_{p_2}(k)]u(k) \\ x_{p_1}{}^3(k)u(k) \end{bmatrix}$$

where c_1 and c_2 are the unknown parameters and x_{p_1}, x_{p_2} are the elements of x_p.

b) *Model:*

$$x_m(k+1) = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.5 \end{bmatrix} x_p(k) + \begin{bmatrix} \hat{c}_1(k+1) & 0 \\ 0 & \hat{c}_2(k+1) \end{bmatrix} \begin{bmatrix} \cos\,[x_{p_2}(k)]u(k) \\ x_{p_1}{}^3(k)u(k) \end{bmatrix}.$$

c) *Input:* The input signal $u(k)$ was chosen to be the same as in the previous case.

d) *Simulation Results:* The following set of values were used:

$$c_1 = 0.5;\ c_2 = 0.1;\ \alpha = 1.5;\ P = I$$

and the results are presented in Fig. 2.

Comments

1) The form of the model chosen (2) is seen to be crucial in proving the convergence of the identification procedure. It is worth pointing out that it is not a direct extension of the models used in [1] and [2] for the continuous case, but appears closer in form to the model used in [3]. However, since a state vector representation is used here in the identification procedure, it appears possible to extend it directly to the more important problem of the adaptive observer, where only the outputs of the system are accessible.

2) The convergence of the identification algorithm is seen to be quite rapid in the examples considered above, at the expense of large transient overshoots. This is due to the fact that the choice of α and P in (13) and (16) are critical factors which affect not only the rate of convergence but also the initial overshoot. The overshoot behavior is highly susceptible also to the initial choices of the parameters and, in general, numerous trial and error runs are required before a suitable performance is obtained.

3) In the examples, only partial parameter identification is required. Here, the known parameters were excluded from the algorithm and a reduced parameter matrix $\theta(k)$ was estimated using (13) and (16) with $P = I$.

VI. Conclusions

This paper described a procedure for adjusting the parameters of a model for the identification of linear and nonlinear discrete multi-

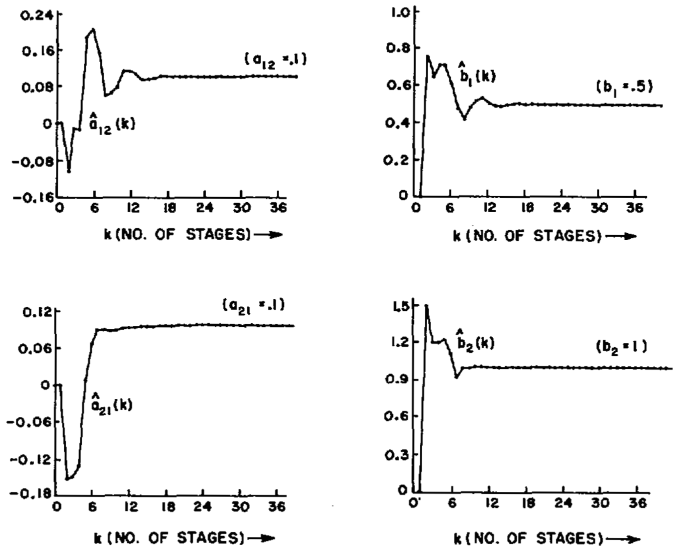

Fig. 1.

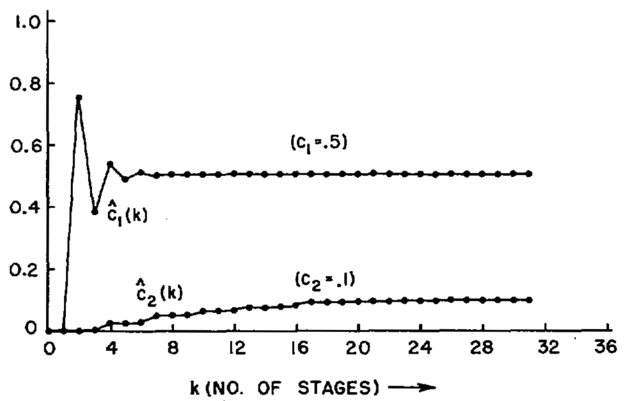

Fig. 2.

variable systems when all the state variables are accessible and no observation noise is present. Using Lyapunov's direct method the scheme is shown to be asymptotically stable, when the plant is completely controllable and the input is sufficiently general.

Extension of these results to the more important cases, namely, 1) when observation noise is present at the output and 2) when not all the state variables of the plant are accessible, is under investigation.

Appendix A

Proof of Lemma 1: Consider the quadratic form

$$q = x^T[2\beta P - \gamma\beta^2P^2]x, \qquad x \neq 0 \tag{A1}$$

$$= \beta x^TPx\left[2 - \gamma\beta\frac{x^TP^2x}{x^TPx}\right]. \tag{A2}$$

Let 1) M be the modal matrix of P so that

a) $M^TPM = \Lambda$ where $\Lambda = \{\lambda_i\}$ is the diagonalized form of P, and

b) $M^T = M^{-1}$ (since P is symmetric)

2) $y = M^{-1}x$

$$\therefore \frac{x^TP^2x}{x^TPx} = \frac{y^T\Lambda^2y}{y^T\Lambda y} = \frac{\sum_{i=1}^{n}\lambda_i^2y_i^2}{\sum_{i=1}^{n}\lambda_iy_i^2}, \qquad \text{where } y_i, i \in \{1,2,\ldots,n\}$$

are the elements of the vector y. (A3)

Since $P > 0$, it follows from (A3) that

$$\frac{x^TP^2x}{x^TPx} \leq \lambda_{\max}, \text{ where } \lambda_{\max} \text{ is the largest eigen value of } P.$$

Hence, from (A2) we have

$$q \geq \beta x^TPx[2 - \gamma\beta\lambda_{\max}] = \beta x^TPx(2 - \alpha), \qquad \text{if } \beta = \frac{\alpha}{\lambda_{\max}\gamma}.$$

If $0 < \alpha < 2$,

$$q > 0, \qquad \forall x(x \neq 0)$$

or, $2\beta P - \gamma\beta^2P^2 > 0$, and the lemma is proved.

Appendix B

Along the trajectories where $\Delta V \equiv 0$ or $e \equiv 0$ we have from (6) that

$$\Phi x_p + \Psi u \equiv 0. \tag{B1}$$

(Note that $e \equiv 0$ implies that Φ and Ψ are constant matrices.) Using (1) and (B1) we can write

$$[\Phi(pI - A)^{-1}B + \Psi]u \equiv 0 \tag{B2}$$

where p is the shift operator such that $p[x(k)] \triangleq x(k+1)$.

Let $\alpha(\lambda)$ be the characteristic polynomial of A where

$$\alpha(\lambda) = \lambda^n + a_n\lambda^{n-1} + a_{n-1}\lambda^{n-2} + \cdots + a_2\lambda + a_1 \tag{B3}$$

then it is well known that

$$(\lambda I - A)^{-1} = \frac{1}{\alpha(\lambda)}\sum_{i=1}^{n}\alpha_i(\lambda)A^{i-1} \tag{B4}$$

where the auxiliary polynomials are defined as

$$\alpha_n(\lambda) = 1, \; \alpha_0(\lambda) = \alpha(\lambda)$$

and

$$\alpha_{i-1}(\lambda) = \lambda\alpha_i(\lambda) + a_i, \qquad i \in \{1,2,\cdots,n\}. \tag{B5}$$

Hence (B2) may be rewritten as

$$\left[\Phi\sum_{i=1}^{n}\alpha_i(p)A^{i-1}B + \alpha(p)\Psi\right]u \equiv 0. \tag{B6}$$

Using the relations given by (B5) in (B6) and rearranging terms, we have

$$\begin{aligned}[\Psi p^n &+ (a_n\Psi + \Phi B)p^{n-1} + (a_{n-1}\Psi + \Phi Ba_n + \Phi AB)p^{n-2} \\ &+ \ldots + (\Psi a_1 + \Phi Ba_2 + \Phi ABa_3 + \cdots \\ &+ \Phi A^{n-2}Ba_n + \Phi A^{n-1}B)]u = 0.\end{aligned} \tag{B7}$$

Let $u_1, u_2, \ldots, u_m$ be the elements of the input vector u. Using the following notation:

$$\Psi = \{b_{ij}\}, \; \Phi B = \{a_{ij}^1\}, \; \Phi A^qB = \{a_{ij}^{q+1}\} \text{ where } q \in \{1,2,\cdots n-1\}$$

we have

$$\begin{aligned}\sum_{q=1}^{m}[b_{iq}p^n &+ (a_nb_{iq} + a_{iq}^1)p^{n-1} + (a_{n-1}b_{iq} + a_na_{iq}^1 + a_{iq}^2)p^{n-2} \\ &+ \cdots + (a_1b_{iq} + a_2a_{iq}^1 + a_3a_{iq}^2 + \cdots + a_{iq}^n)]\,u_q = 0, \\ &\text{where } i \in \{1,2,\ldots,n\}.\end{aligned} \tag{B8}$$

The input u is said to be sufficiently general if there exists a finite $k \geq (n+1)m$, such that the matrix

$$M = [M_1 \vdots M_2 \vdots M_3 \vdots \cdots \vdots M_m] \tag{B9}$$

is of full rank where

$$M_j = \overbrace{\begin{bmatrix} u_j(0) & 0 & 0 & & 0 \\ u_j(1) & u_j(0) & 0 & & 0 \\ u_j(2) & u_j(1) & u_j(0) & \cdot\ \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ u_j(k) & u_j(k-1) & u_j(k-2) & & u_j(k-n) \end{bmatrix}}^{n+1}.$$

It can be easily shown that if u is sufficiently general, then (B8) implies

$$b_{iq} = a_{iq}{}^1 = a_{iq}{}^2 = \cdots = a_{iq}{}^n = 0, \qquad \forall\ q \text{ and } i.$$

Hence $\Psi = 0$, and

$$\Phi B = 0,\ \Phi AB = 0,\cdots,\Phi A^{n-1}B = 0 \tag{B10}$$

that is

$$\Phi[B|AB|A^2B|\cdots|A^{n-1}B] = 0. \tag{B11}$$

If the plant is assumed to be completely controllable, the controllability matrix is of rank n and hence, (B11) implies that $\Phi = 0$. Hence, $e \equiv 0 \Longrightarrow \Phi = 0$, $\Psi = 0$, which proves the asymptotic stability of the identification scheme.

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A Parallel Filtering Algorithm for Linear Systems with Unknown Time Varying Noise Statistics

D. L. ALSPACH, MEMBER, IEEE

Abstract—**The problem of estimating the state of a linear dynamic system driven by additive Gaussian noise with unknown time varying statistics is considered. Estimates of the state of the system are obtained which are based on all past observations of the system. These observations are linear functions of the state contaminated by additive white Gaussian noise. A previously developed algorithm designed for use in the case of stationary noise is modified to allow estimation of an unknown Kalman gain and thus the system state in the presence of unknown time varying noise statistics.**

The algorithm is inherently parallel in nature and if implemented in a computer with parallel processing capability should only be slightly slower than the stationary Kalman filtering algorithm with known noise statistics.

I. Introduction

The Kalman filter gives an optimal minimum mean-square error or maximum *a posteriori* estimate of the state of a linear dynamic system driven by additive white Gaussian noise and observed by a linear function of the state contaminated by additive white Gaussian noise [1]–[3]. However, the filter requires a complete knowledge of system parameters and an accurate description of the first- and second-order statistics of the noise contaminating the measurements and driving the system. When the system parameters are known, but an inaccurate description of second-order moments of the noise is used, the filter may give extremely poor estimates, or even diverge. When these noise statistics are unknown, the problem of estimating the system state is often referred to as the adaptive Kalman filtering problem. A number of people have considered this problem and a variety of methods have been used when it is known that the noise is stationary [4]–[7]. In particular, we have recently reported a new algorithm which gives the *a posteriori* probability density function of the optimal steady-state Kalman gain [8], [9].

In this paper that steady-state algorithm is modified to allow the consideration of time varying noise statistics. The modifications are shown below. Numerical results showing the operation of the algorithm in time varying noise environments are presented.

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The author was with the Department of Applied Mechanics and Engineering Sciences, University of California at San Diego, La Jolla, Calif. 92037. He is now with the Applied Systems Corporation, P.O. Box 22113, San Diego, Calif. 92122.

II. Problem Statement and Autonomous Algorithm

The system is described by the following sets of equations:

$$X_{k+1} = FX_k + \Gamma w_k, \qquad (n \times 1) \tag{1a}$$

$$Z_k = HX_k + v_k, \qquad (m \times 1) \tag{1b}$$

$$E(w_k w_j{}^T) = Q_k \delta_{kj}, \qquad E(w_k) = 0 \tag{2a}$$

$$E(v_k v_j{}^T) = R_k \delta_{kj}, \qquad E(v_k) = 0. \tag{2b}$$

The plant and measurement noises w_k and v_k are assumed to be independent Gaussian white noise sequences with time variable unknown covariances.

To obtain an optimal estimate of the state based on all available data up to time t_k, it is sufficient to obtain the *a posteriori* density of the state $p(X_k/Z^k)$ conditioned on all the data $Z^k = (Z_1,Z_2,\cdots,Z_k)^T$. This can be found by the law of total probability from the density of the state conditioned on the data and the gain, and the probability of the gain conditioned on the data. We have previously considered this problem for the case of time invariant noise statistics and the derivation of the *a posteriori* density of the gain for scalar measurements is presented in [8] and for vector measurements in [9]. It is shown in [8] for scalar measurements that (neglecting certain *a priori* or data independent effects) the *a posteriori* density of the steady-state Kalman gain conditioned on all data up to time N is given by

$$P(K/Z^N) = \begin{cases} C(\hat{\Omega}_N(K))^{(2-N)/2}, & \text{for feasible } K \\ 0, & \text{for infeasible } K. \end{cases} \tag{3}$$

Here feasible gains are those which can be obtained from the steady-state Kalman filter equations for some allowable value of the unknown noise covariances Q and R. The term $\hat{\Omega}_N(K)$ is defined by

$$\hat{\Omega}_N(K) = \frac{1}{N}\sum_{j=1}^{N} \nu_j{}^2(K),\ \nu_j(K) = Z_j - HF\hat{X}_{i-1}(K) \tag{4}$$

where for the scalar measurement case considered here the residual and the covariance of the residual are scalars. The vector measurement case considered in [9] can be extended to the time varying case as a direct extension of the method presented here. Our presentation is restricted to the scalar measurement case ($m = 1$) in order to simplify the notation.

The state estimate $\hat{X}_j(K)$ at time j is found recursively for any given gain by

$$\hat{X}_j(K) = F\hat{X}_{j-1}(K) + K(Z_j - HF\hat{X}_{j-1}(K)),\ \hat{X}_0(K) = \hat{X}_0. \tag{5}$$

This algorithm will give the exact relative magnitude of the *a posteriori* density function of the gain conditioned on all previous measurement data neglecting only *a priori* effects. As discussed in