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Uniting Control Lyapunov and Control Barrier Functions

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Abstract— **In this paper, we propose a nonlinear control design for solving the problem of stabilization with guaranteed safety. The design is based on the merging of a Control Lyapunov Function and a Control Barrier Function. The proposed control method allows us to combine the design of a stabilizer based on CLF and the design of safety control based on CBF. The efficacy of the proposed approach is shown in the simulation results.**

Keywords: Control Lyapunov Function, Control Barrier Function, Uniting Control Lyapunov and Barrier Function

I. INTRODUCTION

One of the modern control design tools for the stabilization of affine nonlinear systems is to use the so-called Control Lyapunov Function (CLF). Artstein in [1] has given a necessary and sufficient condition for the existence of such CLF and it has been used to design a universal control law for affine nonlinear systems in [27]. Various recent Lyapunov-based control designs, such as, the forwarding [24], backstepping [14], Passivity Based Control [20], [21], have been derived using the same principle as CLF.

Since CLFs can be designed to meet a number of specific performance criteria, such as, optimality, transient behaviour or robustness properties, the question on how to combine several CLFs for combining the performances has been addressed, to name a few, in [2], [3], [7], [12], [25], [26]. With the exception of combined/merging/uniting CLF proposed in [7] that results in a non-smooth CLF, the *merged* CLF is designed based on a convex combination of two CLFs where the weights can be state-dependent.

With the recent surge of research interests in cyber-physical systems and in networked control systems, the safety of the process has become part of the control design. The feedback control must comply to the state constraint, avoid unsafe states and adhere to the input constraint. Akin to the CLF, Wieland and Allgöwer in [30] have proposed the construction of Control Barrier Function (CBF) where the Lyapunov function is interchanged with the Barrier certificate studied in [23], [22]. Using CBF as in [30], one can design a universal controller for steering the states from the set of initial conditions to the set of terminal conditions without violating the set of unsafe states.

There have been a number of control design methods proposed in literature that deal with (non-)linear constraints for (non-)linear systems. For example, Model Predictive Control-based approach has been proposed in [18], [17], [6] and reference governor has been proposed in [4], [5], [11]. Both approaches lead to a high-level controller that generates admissible reference signals for the low-level controller, in order to avoid the constraints. Another control design approach for dealing with constraint is the invariance control principle proposed in [10], [32].

In order to combine the stabilization property of CLF with the safety aspect from the CBF, we will study in this paper a simple control design procedure where CLF can be merged with CBF. Some previous relevant works, where a barrier function is incorporated explicitly in the CLF control design method, have been proposed in [19] and [29]. In these papers, a stabilization control problem with state saturation is considered which is solved by incorporating explicitly a "barrier function" in the design of a CLF. The resulting CLF has a strong property of being unbounded in the boundary of the state's domain. While in this paper, we consider a more general problem where the unsafe set can be any form of compact set in the domain of the state. It is solved by combining a CLF and CBF that results in a Control Lyapunov-Barrier Function (CLBF) control design method which does not impose unboundedness condition on the boundary of the unsafe set. Hence we admit a larger class of functions than the former approaches.

Note that important features of the CLF for stabilizing the origin are the (local-)convexity and globally minimum at the origin. Hence the merged CLF has these properties and they are inherited from the original CLFs. On the other hand, the important characteristic of the CBF is that it is (locally-)concave with the level-set of zero belonging to the safe domain. Moreover, CBF may not have global minimum at all. As a result, CBF and CLF cannot be merged using the same principle of merging two CLFs as before. It may shift the desired equilibrium point (away from the origin) and the merged CLF-CBF may not be proper (i.e., the level-set may not be compact).

Another related control problem in the literature is the obstacle avoidance control problem [8] where the systems are described by single integrator and the proposed control law is based on a gradient of a particular potential function. Other relevant works in the context of avoidance control problem for multi-agent systems are [28], [9].

One important characteristic of the potential function in such method is that it grows unbounded as it reaches the boundary of the obstacle (or the set of unsafe state). The construction of such function can be complicated and is not easy to construct.

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In Section II, we will review briefly on the concept of Control Lyapunov Function, of Control Barrier Function and of universal control law which are based on [27], [30]. In Section III, we will propose methods to merge CBF and CLF. In Section IV, we will provide numerical simulations and the conclusions will be given in Section V.

II. Preliminaries

In this section, we consider a nonlinear affine system in the form of

$$\dot{x} = f(x) + g(x)u, \qquad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^p$ denote the state and the control input of the system, respectively. We assume also that the functions $f(x)$ and $g(x)$ are smooth, $f(0) = 0$, and $g(x)$ is full rank for all x.

As usual, we denote $L_f V(x)$ as the Lie derivative of $V(x)$ along the vector field $f(x)$, i.e. $L_f V(x) := \frac{\partial V(x)}{\partial x} f(x)$. A function $V : \mathbb{R}^n \to \mathbb{R}$ is called *proper* if the set $\{x | V(x) \leq c\}$ is compact for all constant $c \in \mathbb{R}$, or equivalently, V is radially unbounded. For a given set $\mathcal{D} \subset \mathbb{R}^n$, we denote the boundary of $\mathcal{D}$ by $\partial\mathcal{D}$. The set $\mathbb{R}_+ := [0, \infty)$ and $\overline{\mathbb{R}}_+$ is the compactification of $\mathbb{R}_+$, i.e., $\overline{\mathbb{R}}_+ := [0, \infty]$.

In the following, let us recall some basic results relating to Control Lyapunov Function and its universal control law (see also [27]).

A proper, $\mathcal{C}^1$, positive-definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ that satisfies

$$L_f V(x) < 0 \qquad \forall x \in \{z \in \mathbb{R}^n \backslash \{0\} \mid L_g V(z) = 0\} \tag{2}$$

is called a *Control Lyapunov Function* (CLF).

Given a CLF $V : \mathbb{R}^n \to \mathbb{R}_+$, the system (1) has the *Small Control Property* (SCP) w.r.t. V if for every $\varepsilon > 0$ there exists a $\delta > 0$ s.t.

$$0 < \|x\| < \delta \Rightarrow \exists u \in \mathbb{R}^p \qquad \text{s.t.} \qquad \|u\| < \varepsilon \qquad \text{and} \qquad L_f V(x) + L_g V(x) u < 0.$$

We define a function $k : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^p$ by

$$k(\gamma, a, b) = \begin{cases} -\frac{a + \sqrt{a^2 + \gamma\|b\|^4}}{b^T b} b & \text{if} \quad b \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Using the notion of CLF and small-control property, Sontag in [27] has proposed a universal control law as summarized in the following theorem.

Theorem 1: Assume that the system (1) has a CLF $V(x)$ and satisfies the small-control property w.r.t. V. Then the feedback law

$$u = k(\gamma, L_f V(x), (L_g V(x))^T) \qquad \gamma > 0, \tag{4}$$

is continuous and ensures that the closed-loop system is globally-asymptotically stable.

In order to incorporate the safety aspect into the control design, we adopt the following definition as used in [30].

Let us denote $\mathcal{X}_0 \subset \mathbb{R}^n$ be the set of initial conditions, and an open set $\mathcal{D} \subset \mathbb{R}^n$ be the set of unsafe state. Throughout this paper, for simplicity of presentation, we always assume that $\mathcal{D} \cap \mathcal{X}_0 = \emptyset$. Of course, we assume that $0 \in \mathcal{X}_0$.

Definition 1 (Safety): Given an autonomous system

$$\dot{x} = f(x) \qquad x(0) = x_0 \in \mathcal{X}_0 \tag{5}$$

where $x(t) \in \mathbb{R}^n$, the system is called *safe* if for all $x_0 \in \mathcal{X}_0$ and for all $t \in \overline{\mathbb{R}}_+$, $x(t) \notin \mathcal{D}$.

Note that in our definition above, compare to the one in [30], it has a slight modification in the asymptotic behaviour of the state trajectory x in order to remove the possibility of x converging to the boundary of $\mathcal{D}$.

Using this safety definition, the control problem that is considered in [30] is given as follows (see also *Problem 5* in [30]).

Safety control problem: Given the system (1) with a given initial condition $\mathcal{X}_0$ and a given set of unsafe state $\mathcal{D} \subset \mathbb{R}^n$, design a feedback law $u = \alpha(x)$ s.t. the closed loop system $\dot{x} = f(x) + g(x)\alpha(x)$, $x(0) = x_0 \in \mathcal{X}_0$ is safe.

In order to solve the above problem and motivated by universal control law using CLF, Wieland and Allgöwer have recently proposed the concept of control barrier function in [30].

Let us recall the basic definition of a Control Barrier Function as in [30].

Given a set of unsafe state $\mathcal{D} \subseteq \mathbb{R}^n$, a $\mathcal{C}^1$ function $B : \mathbb{R}^n \to \mathbb{R}$ that satisfies the following conditions

$$B(x) > 0 \qquad \forall x \in \mathcal{D} \tag{6}$$

$$L_f B(x) \leq 0 \qquad \forall x \in \{z \in \mathbb{R}^n \setminus \mathcal{D} | L_g B(z) = 0\} \tag{7}$$

$$\mathcal{U} := \{x \in \mathbb{R}^n |\ B(x) \leq 0\} \neq \emptyset \tag{8}$$

is called a *Control Barrier Function* (CBF).

In the following theorem, we present the safety control design which generalizes the result in [30].

Theorem 2: Assume that the system (1) has a CBF $B : \mathbb{R}^n \to \mathbb{R}$ with a given a set of unsafe state $\mathcal{D} \subseteq \mathbb{R}^n$, then the feedback law

$$u = k(\gamma, L_f B(x), (L_g B(x))^T) \qquad \gamma > 0, \tag{9}$$

solves the safety control problem, i.e. the closed-loop system is safe with the set of initial states $\mathcal{X}_0 = \mathcal{U}$ with $\mathcal{U}$ be as in (8).

Let us consider now the incorporation of safety aspect in the standard stabilization problem.

Stabilization with guaranteed safety control problem: Given the system (1) with a given set of initial condition $\mathcal{X}_0$ and a given set of unsafe state $\mathcal{D}$, design a feedback law $u = \alpha(x)$ s.t. the closed loop system is safe and asymptotically stable, i.e. $\lim_{t\to\infty} \|x(t)\| = 0$. Moreover, when $\mathcal{X}_0 = \mathbb{R}^n \setminus \mathcal{D}$ we call it the *global stabilization with guaranteed safety control problem*.

For solving the stabilization with guaranteed safety control problem, one can intuitively consider to merge or to unite the CLF and CBF by a convex combination a'la Andreu and Prieur [2] or Grammatico et al [12]. However, such approach may not immediately guarantee the solvability of the problem. Firstly, the convex combination can lead to the shifting of the global minimum of the combined function which can result in the shifting of the equilibrium point away from the origin. This does not happen in the uniting/merging CLFs since each CLF has minimum at the origin. In the extreme case, when the function of $B(x)$ is not lower-bounded, the combined function may not even admit a global minima. Secondly, we need a theoretical framework to combine the stability analysis via Lyapunov method and the safety analysis via Barrier Certificate. Motivated by the safety analysis using Barrier Certificate (see, for example [23], [31]), we provide below a proposition on the stability with safety.

Proposition 1: Consider an autonomous system

$$\dot{x} = f(x) \qquad x(0) = x_0 \tag{10}$$

with a set of unsafe state $\mathcal{D}$ which is open. Suppose that there exists a proper and lower-bounded $\mathcal{C}^1$ function $W : \mathbb{R}^n \to \mathbb{R}$ such that

$$W(x) > 0 \qquad \forall x \in \mathcal{D} \tag{11}$$

$$L_f W(x) < 0 \qquad \forall x \in \mathbb{R}^n \setminus (\mathcal{D} \cup \{0\}) \tag{12}$$

$$\mathcal{U} := \{x \in \mathbb{R}^n | W(x) \leq 0\} \neq \emptyset \tag{13}$$

$$\overline{\mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{U})} \cap \overline{\mathcal{D}} = \emptyset \tag{14}$$

then the system is safe with $\mathcal{X}_0 = \mathbb{R}^n \setminus \mathcal{D}$ and asymptotically stable, i.e. $\lim_{t\to\infty} \|x(t)\| = 0$.

Proof : We firstly prove that if $x_0 \in \mathcal{X}_0$, then the state trajectory x never enters $\mathcal{D}$, i.e., for all $t \geq 0$, $x(t) \notin \mathcal{D}$.

If $x_0 \in \mathcal{U}$ (i.e. $W(x(0)) \leq 0$ by definition) then it follows from (12), $\dot{W} < 0$ that $W(x(t)) - W(x(0)) < 0$ for all $t \in R_+$. Hence, it implies that $W(x(t)) < 0$ for all $t \in R_+$. In other words, the set $\mathcal{U}$ is forward invariant and $x(t) \notin \mathcal{D}$ for all $t \in R_+$ by (11). Moreover, by the properness of W, the set $\mathcal{U}$ is compact. Note that by the compactness of $\mathcal{U}$, it holds that $\lim_{t\to\infty} x(t) \notin \mathcal{D}$. Now consider the other case when $x \in \mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{U})$. By using the same argument as in the proof of the second claim of Theorem 2, the trajectory x will remain in $\mathcal{U}$ and will never enter $\mathcal{D}$.

We will now prove that if $x_0 \in \mathbb{R}^n \setminus \mathcal{D}$ then $x(t) \to 0$ as $t \to \infty$.

Let $x_0 \in \mathbb{R}^n \setminus \mathcal{D}$ which (according to the previous arguments) implies that the trajectory $x(t) \notin \mathcal{D}$ for all $t \geq 0$. Correspondingly, it follows from (12) that

$$\frac{d}{dt} W(x(t)) < 0 \qquad \forall x(t) \notin (\mathcal{D} \cup \{0\}) \tag{15}$$

$$\Rightarrow W(x(t)) < W(x(0)) < \infty \qquad \forall t \geq 0.$$

By the properness of W, the last inequality implies that the trajectory x is bounded, and thus it is pre-compact [1]. i.e., the closure of $\{x(t)|t \in [0, \infty)\}$ is compact. This implies that the ω-limit set $\Omega(x_0)$ is non-empty, compact, connected and $\lim\limits_{t\to\infty} d(x(t), \Omega(x_0)) = 0$ where d defines the distance[2].

[1]The trajectory x in $\mathcal{X}$ is *pre-compact* if it is bounded for all $t \in [0, \infty)$ and for any sequences (t_n) in $[0, \infty)$, the limit $\lim_{n\to\infty} x(t_n)$ exists and is in $\mathcal{X}$ [15].

[2]For the concept of ω-limit set, we refer interested readers to [13], [15], [16].

Additionally, since the function $\mathcal{W} := W \circ x$ is an absolutely continuous function of t and bounded from below, (15) implies that $\mathcal{W}(t)$ is monotonically decreasing and it has a limit h as $t \to \infty$. On the other hand, for any point ξ in the ω-limit set $\Omega(x_0)$, there is a sequence (t_n) in $\mathbb{R}_+$ such that $t_n \to \infty$ and $x(t_n) \to \xi$. By the continuity of W, $W(\xi) = \lim_n \mathcal{W}(t_n) = h$. Therefore, in the invariant set $\Omega(x_0)$, W is constant and is given by h. Using (12), and the fact that $\mathcal{D} \not\subset \Omega(x(0))$, we have that W is constant only at $x = 0$ and thus $\Omega(x_0) = \{0\}$. Hence,

$$\lim_{t\to\infty} \|x(t)\| = 0.$$

□

Few remarks on the assumptions in Proposition 1. When we restrict the state space to $\mathcal{D} \cup \mathcal{U}$, the conditions in (12)-(13) are reminiscent of the conditions in Barrier Certificate theorem (c.f. [23, Prop. 2.18]).

On the other hand, the properness of W together with (12) resembles the standard Lyapunov stability theorem (albeit, in this proposition, we do not impose positive-definiteness of W). The addition of condition (14) is to ensure that the first entry point to the set of $\mathcal{D} \cup \mathcal{U}$ is the boundary of $\mathcal{D} \cup \mathcal{U}$, and not that of $\mathcal{D}$.

Obviously, one can observe from the condition (12) and (13) that the origin lies inside the set of $\mathcal{U}$. Indeed, we can prove this by contradiction. Suppose that $0 \notin \mathcal{U}$. Let $x_0 \in \mathcal{U}$ which implies that $x(t) \in \mathcal{U}$ for all t (following the same argument as in the proof of Proposition 1). By (12), $W(x(t))$ is decreasing and converge to a constant. Thus, in the ω-limit set, $W(x(t))$ is constant, and by (12), the ω-limit set is 0 which is a contradiction since $0 \notin \mathcal{U}$.

III. Main results

In this section, we will present a control design framework for solving the stabilization with guaranteed safety control problem. For this, we introduce the notion of Control Lyapunov-Barrier Function as follows.

Definition 2 (CLBF): Given a set of unsafe state $\mathcal{D}$, a proper and lower-bounded $\mathcal{C}^1$ function $W : \mathbb{R}^n \to \mathbb{R}$ that satisfies the following conditions

$$W(x) > 0 \ \forall x \in \mathcal{D} \tag{16}$$

$$L_f W(x) < 0 \ \forall x \in \{z \in \mathbb{R}^n \setminus (\mathcal{D} \cup \{0\}) | L_g W(z) = 0\} \tag{17}$$

$$\mathcal{U} := \{x \in \mathbb{R}^n | W(x) \leq 0\} \neq \emptyset \tag{18}$$

$$\overline{\mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{U})} \cap \overline{\mathcal{D}} = \emptyset \tag{19}$$

is called a Control Lyapunov Barrier Function (CLBF).

Using this notion and Proposition 1, we can solve the problem in the following theorem.

Theorem 3: Assume that the system (1) has a CLBF $W : \mathbb{R}^n \to \mathbb{R}$ with a given set of unsafe state $\mathcal{D}$ and satisfy the small-control property w.r.t. W, then the feedback law

$$u = k(\gamma, L_f W(x), (L_g W(x))^T) \qquad \gamma > 0, \tag{20}$$

solves the global stabilization with guaranteed safety control problem.

Proof : We prove the theorem by showing that the conditions (11)-(14) in Proposition 1 hold for the closed-loop autonomous system

$$\dot{x} = F(x)$$

where $F(x) = f(x) + g(x)k(\gamma, L_f W(x), (L_g W(x))^T)$.

The conditions (11), (13) and (14) follow trivially from (16), (18) and (19), respectively. Now, for all $x \in \{z \in \mathbb{R}^n \setminus (\mathcal{D} \cup \{0\}) \mid L_g W(z) \neq 0\}$, we have that

$$\begin{aligned} L_F W(x) &= L_f W(x) + L_g W(x) k(\gamma, L_f W(x), (L_g W(x))^T) \\ &= -\sqrt{\|L_f W(x)\|^2 + \gamma \|L_g W(x)\|^4} < 0 \end{aligned}$$

holds. On the other hand, for all $x \in \{z \in \mathbb{R}^n \setminus (\mathcal{D} \cup \{0\}) \mid L_g W(z) = 0\}$, the condition (17) implies that

$$L_F W(x) < 0.$$

These two inequalities show that (12) also holds. □

Using the same argument as in the proof of Proposition 1, it can be checked that the condition (17) can be weakened by

$$L_f W(x) \leq 0 \qquad \forall x \in \mathcal{M}$$

where the CLBF function W is still assumed to be C^1,

$$\mathcal{M} := \{z \in \mathbb{R}^n \setminus \mathcal{D} | L_g W(z) = 0\}$$

and the largest invariant set in $\mathcal{M}$ is 0. This condition will be useful later in the simulation result where we combine CLF and CBF. This is formalized in the following proposition

Proposition 2: Let $\mathcal{D}$ be a given set of unsafe state. Assume that the system (1) has a proper and lower-bounded C^1 function $W : \mathbb{R}^n \to \mathbb{R}$ that satisfies the following conditions

$$W(x) > 0 \qquad \forall x \in \mathcal{D} \tag{21}$$

$$L_f W(x) \leq 0 \qquad \forall x \in \mathcal{M} := \{z \in \mathbb{R}^n \setminus \mathcal{D} \mid L_g W(z) = 0\} \tag{22}$$

$$\mathcal{U} := \{x \in \mathbb{R}^n \mid W(x) \leq 0\} \neq \emptyset \tag{23}$$

$$\overline{\mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{U})} \cap \overline{\mathcal{D}} = \emptyset. \tag{24}$$

Assume also that the system is zero-state detectable with respect to $L_g W(x)$, i.e., $L_g W(x(t)) = 0\ \forall t \geq 0 \Rightarrow x(t) \to 0$. Suppose that the system (1) has the small-control property w.r.t. W. Then the feedback law

$$u = k\left(\gamma, L_f W(x), (L_g W(x))^T\right) \qquad \gamma > 0, \tag{25}$$

solves the global stabilization with guaranteed safety control problem.

Proof : The proof is akin to the proof of Theorem 3 and Proposition 1. Similar to the proof of Proposition 1, if $x_0 \in \mathbb{R}^n \setminus \mathcal{D}$ then the trajectory x will never enter $\mathcal{D}$, i.e., $x(t) \in \mathbb{R}^n \setminus \mathcal{D}$ for all $t \geq 0$ and $\mathcal{D} \not\subseteq \Omega(x_0)$.

It remains to show that in the closed-loop system, for every $x_0 \in \mathbb{R}^n \setminus \mathcal{D}$ we have $\Omega(x_0) = \{0\}$. As in the proof of Theorem 3, the time-derivative of W satisfies

$$\begin{aligned} L_F W(x) &= -\sqrt{\|L_f W(x)\|^2 + \gamma \|L_g W(x)\|^4} \\ &\leq -\gamma \|L_g W(x)\|^2 \qquad \forall x \in \mathbb{R}^n \setminus (\mathcal{D} \cup \mathcal{M}). \end{aligned}$$

On the other hand, for all $x \in \mathcal{M}$, the assumption (22) implies that $L_F W(x) \leq 0$. Hence, combining these two inequalities, we have that for all $x(t) \in \mathbb{R}^n \setminus \mathcal{D}$,

$$\dot{W}(x(t)) \leq -\gamma \|L_g W(x(t))\|^2.$$

This inequality implies that W converges to a constant and the trajectory x converges to the largest invariant set $\mathcal{C}$ contained in $\mathcal{M}$, i.e., $\Omega(x_0) \subset \mathcal{C} \subset \mathcal{M}$. By the zero-state detectability assumption with respect to $L_g W$, we have that the largest invariant set $\mathcal{C} = \{0\}$. Hence, $\Omega(x_0) = \mathcal{C} = \{0\}$, i.e., $\lim_{t \to \infty} \|x(t)\| = 0$. □

Equipped with this theorem we can now present results on a possible method to combine CLF and CBF. This will potentially allow us to separate the control design for achieving the asymptotic stability and safety by designing the CLF and CBF, independently, and then combine them together. In the following proposition, we assume first that B is lower-bounded.

Proposition 3: Suppose that for system (1), with a given set of unsafe state $\mathcal{D}$, there exist a CLF $V(x)$ and a CBF $B(x)$ which satisfies

$$c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2 \qquad \forall x \in \mathbb{R}^n \qquad c_1, c_2 > 0, \tag{26}$$

and a compact and connected set $\mathcal{X}$ s.t.

$$\mathcal{D} \subset \mathcal{X},\ 0 \notin \mathcal{X} \text{ and } B(x) = -\varepsilon,\ \varepsilon > 0 \qquad \forall x \in \mathbb{R}^n \setminus \mathcal{X}. \tag{27}$$

If

$$L_f W(x) < 0 \qquad \forall x \in \{z \in \mathbb{R}^n \setminus (\mathcal{D} \cup \{0\}) | L_g W(z) = 0\} \tag{28}$$

where

$$W(x) = V(x) + \lambda B(x) + \kappa$$

with $\lambda > \frac{c_2 c_3 - c_1 c_4}{\varepsilon}$, $\kappa = -c_1 c_4$, $c_3 := \max_{x \in \partial \mathcal{X}} \|x\|^2$, $c_4 := \min_{x \in \partial \mathcal{D}} \|x\|^2$, then the feedback law (20) solves the stabilization with guaranteed safety control problem with the set of initial states $\mathcal{X}_0 = \mathbb{R}^n \setminus \mathcal{D}_{relaxed}$ where $\mathcal{D}_{relaxed} := \{x \in \mathcal{X} | W(x) > 0\} \supset \mathcal{D}$.

Proof : The proof of the proposition will be based on proving that $\mathcal{D} \subset \mathcal{D}_{relaxed}$ and (16)-(19) hold with $\mathcal{D}$ being replaced by $\mathcal{D}_{relaxed}$. Note that (16) holds by the definition of $\mathcal{D}_{relaxed}$. A routine computation shows that for all $x \in \mathcal{D}$,

$$W(x) = V(x) + \lambda B(x) - c_1 c_4 \geq c_1 \|x\|^2 - c_1 c_4 > 0 \tag{29}$$

since c_4 is evaluated in the boundary of $\mathcal{D}$ and $\mathcal{D}$ is open set by assumption.

Also for all $x \in \partial \mathcal{X}$,

$$\begin{aligned} W(x) &= V(x) + \lambda B(x) - c_1 c_4 \\ &= V(x) - \lambda \varepsilon - c_1 c_4 \\ &\leq c_2 \|x\|^2 - \lambda \varepsilon - c_1 c_4 \\ &< c_2 c_3 - (c_2 c_3 - c_1 c_4) - c_1 c_4 = 0 \end{aligned} \tag{30}$$

where the strict inequality is due to the hypotheses of $\lambda > \frac{c_2 c_3 - c_1 c_4}{\varepsilon}$. Hence we have that (18) holds. By the continuity of $W(x)$, the inequality (29) and (30) implies that the open set $\mathcal{D}_{relaxed}$ is the interior of $\mathcal{X}$ and moreover $\mathcal{D} \subset \mathcal{D}_{relaxed}$. Hence $\partial\mathcal{X} \cap \partial\mathcal{D}_{relaxed} = \emptyset$ and we have

$$\mathcal{D} \subset \mathcal{D}_{relaxed} \subset \mathcal{X} \subset \mathcal{D}_{relaxed} \cup \mathcal{U}. \tag{31}$$

The last relation is due to the decomposition of $\mathcal{X} = \mathcal{D}_{relaxed} \cup \mathcal{X}_{-}$ where $\mathcal{X}_{-} := \{x \in \mathcal{X} | W(x) \leq 0\} \subset \mathcal{U}$. Since $\mathcal{D} \subset \mathcal{D}_{relaxed}$, we have that (28) $\implies$ (16) (with $\mathcal{D}$ being replaced by $\mathcal{D}_{relaxed}$). Finally, since the boundary of $\mathcal{X}$ does not intersect with the boundary of $\mathcal{D}_{relaxed}$, (31) implies that $\overline{\mathbb{R}^n \setminus (\mathcal{D}_{relaxed} \cup \mathcal{U})} \cap \overline{\mathcal{D}_{relaxed}} = \emptyset$, i.e. (19) holds. $\square$

We note that the condition (28) implies that the function W has a global minimum in $\mathbb{R}^n \setminus \mathcal{D}$ at 0. This can be shown by contradiction. Suppose that W admits another minimum $x^* \neq 0$ in $\mathbb{R}^n \setminus \mathcal{D}$ such that (28) holds. The point x^* being minimum implies that $\frac{\partial W(x^*)}{\partial x} = 0$ so that $L_f W(x^*) = 0$ which contradicts (28).

In Proposition 3, it is assumed that B is lower-bounded. In general, when the CBF $B(x)$ is not lower-bounded, we can always construct another CBF $\widetilde{B}(x)$ satisfying (27) based on $B(x)$ which satisfies (6)-(8). Hence Proposition 2 can still be applicable using this new CBF $\widetilde{B}(x)$.

Proposition 4: Suppose that $B : \mathbb{R}^n \to \mathbb{R}$ is a CBF which satisfies (6)-(8), the function $-B$ is proper and the set of unsafe state $\mathcal{D}$ is connected. Then for all non-decreasing $\mathcal{C}^1$ function $\rho : \mathbb{R} \to [0, 1]$ s.t. $\rho(z) = 0$ for all $z \leq -\delta$ with $\delta > 0$ and $\rho(z) = 1$ for all $z \geq 0$, the function $\widetilde{B}(x) : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\widetilde{B}(x) = B(\omega) + \oint_{\Gamma} \rho(B(\sigma)) \frac{\partial B(\sigma)}{\partial x} d\sigma \tag{32}$$

where the integration is taken along any path from any point $\omega \in \partial\mathcal{D}$ to x, is also a CBF, satisfies the conditions (6)-(8), and moreover, (27) holds with $\mathcal{X}$ be compact and connected.

Proof : Firstly, we prove the proposition by showing (6)-(8) holds with the same $\mathcal{D}$. Notice that the integration in (32) is proper and $\widetilde{B}$ is a potential function. Indeed, it is trivial to check that the Hessian matrix of (32) is symmetric and hence, it defines the potential function.

Now, for every $x \in \mathcal{D}$, there exists a path Γ from ω to x since $\mathcal{D}$ is connected and it follow that

$$\widetilde{B}(x) = B(\omega) + \oint_{\Gamma} \frac{\partial B(\sigma)}{\partial x} d\sigma = B(x) > 0$$

where we have used the fact that $\rho(B(\sigma)) = 1$ for all $\sigma \in \Gamma$.

In order to show that (7) holds with the new CBF $\widetilde{B}$, we first note that for all $x \in \mathcal{M} := \{z \in \mathbb{R}^n | \rho(B(z)) > 0\}$, we have that $\frac{\partial \widetilde{B}}{\partial x} g(x) = 0 \Leftrightarrow \frac{\partial B}{\partial x} g(x) = 0$. Hence, for all $x \in \{z \in \mathcal{M} \setminus \mathcal{D} | L_g \widetilde{B}(z) = 0\}$ we have

$$\frac{\partial \widetilde{B}}{\partial x} f(x) = \rho(B(x)) \frac{\partial B}{\partial x} f(x) \leq 0. \tag{33}$$

On the other hand, for all $x \in \mathbb{R}^n \setminus \mathcal{M}$, we have $\frac{\partial \widetilde{B}}{\partial x} = 0$ which implies that $L_f \widetilde{B}(x) = 0$, $\forall x \in \{z \in \mathbb{R}^n \setminus \mathcal{M} | L_g \widetilde{B}(x) = 0\}$. Together with (33), we have that (7) holds.

Now we will proof that $\widetilde{B}(x)$ is constant in $\mathbb{R}^n \setminus \mathcal{M}$. Recall that by assumption, the set $\mathcal{A} := \{B(x) \leq -\delta\}$ is connected, i.e., for all $x \in \mathcal{A}$, $\rho(B(x)) = 0$. Therefore, if we consider two points z_1 and z_2 in $\mathcal{A}$, then using a direct path Γ from z_1 to z_2,

$$\begin{aligned} \widetilde{B}(z_1) &= \widetilde{B}(z_2) + \oint_{\Gamma} \frac{\partial \widetilde{B}(\sigma)}{\partial x} d\sigma \\ &= \widetilde{B}(z_2) + \oint_{\Gamma} \rho(B(\sigma)) \frac{\partial B(\sigma)}{\partial x} d\sigma \\ &= \widetilde{B}(z_2) \end{aligned}$$

Now the proof is completed with $\mathcal{X} = \overline{\mathcal{M}}$. $\square$

IV. EXAMPLES

For numerical example, we consider two systems which is described a follows.

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -x_1 + u, \end{aligned} \tag{34}$$

where $x = [x_1 \ x_2]^T \in \mathbb{R}^2$, and $u \in \mathbb{R}$. This example can represent a mechanical system where x_1 describe the displacement, x_2 describe the velocity. In this case, the mass is 1, the damping constant is considered to be zero and spring constant is 1. For this system, $f(x) = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$ and $g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

It can be checked that the system (34) admit $V(x) = x_1^2 + x_1 x_2 + x_2^2$ as a CLF, i.e. (2) holds and it has small control property. Also,the function

$$B(x) = \begin{cases} e^{-\left(\frac{1}{1-(x_1-2)^2} + \frac{1}{1-x_2^2}\right)} - e^{-4} & \forall x \in \mathcal{X} \\ 0 & \text{elsewhere,} \end{cases} \tag{35}$$

where $\mathcal{X} := (1, 3) \times (-1, 1)$, define a CBF for (34) with the set of unsafe state as $\mathcal{D} := \{x \in \mathcal{X} | \frac{1}{1-(x_1-2)^2} + \frac{1}{1-x_2^2} < 4\}$. Note that for all $x \in \mathcal{D}$, $B(x) > 0$.

Indeed, by direct evaluation, we have that for all $x \in \mathcal{X}$

$$\begin{aligned} &\frac{\partial B}{\partial x} g(x) = 0 \Rightarrow x_2 = 0 \\ &\qquad \frac{\partial B}{\partial x} f(x) = \left.\frac{\partial B}{\partial x}\right|_{x_2=0} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0 \end{aligned}$$

hence (7) holds.

Let us now construct a CLBF $W(x)$ according to the construction as in Proposition 2. It is easy to see that the CLF $V(x)$ satisfies $\frac{1}{2}\|x\|^2 \leq V(x) \leq \frac{3}{2}\|x\|^2$, $\forall x \in \mathbb{R}^2$, i.e. (26) holds with $c_1 = \frac{1}{2}$ and $c_2 = \frac{3}{2}$.

On the other hand, it can be checked that $c_3 = \max_{x \in \partial\mathcal{X}} \|x\|^2 = 10$ and $c_4 = \min_{x \in \partial\mathcal{D}} \|x\|^2 = 1.4$. Hence, by taking $\lambda = 1000$, the condition $\lambda > \frac{c_2 c_3 - c_1 c_4}{\varepsilon}$ is satisfied. Also, as defined in Proposition 2, $\kappa = -c_1 c_4 = -0.7$. Using this constant λ and κ, the CLBF $W(x)$ is given by $W(x) = V(x) + \lambda B(x) + \kappa$ and the control law for solving the problem of stabilization with guaranteed safety is given by (20).

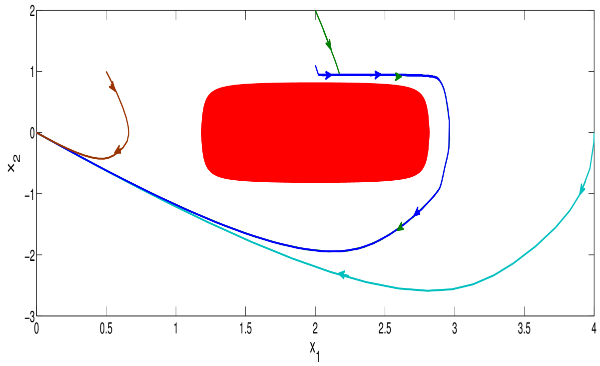

Fig. 1. The numerical simulation result of the closed-loop system using our proposed uniting CLF and CBF method. The set of unsafe state $\mathcal{D}$ is shown in red and the plot of closed-loop trajectories are based on four different initial conditions.

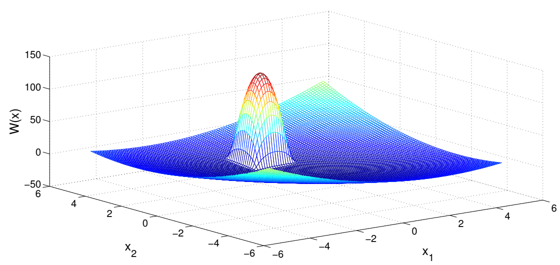

Fig. 2. The plot of the resulting Control Lyapunov-Barrier Function considered in the numerical simulation.

Figure 1 shows the numerical simulation of the closed-loop system with the gain $\gamma = 2$. In this plot, four trajectories are shown with four different initial state (0.5 1.2), (2,1), (2,2) and (4,0). It can be seen from this figure that all trajectories converges to zero and avoid the unsafe state $\mathcal{D}$.

Figure 2 shows the resulting CLBF $W(x)$ where it is shown that for all $x \in \mathcal{D}$, $W(x) > 0$.

V. Conclusions

In this paper, we have proposed a method to combine a CLF and a CBF. Simulation results show the effectiveness of the control law based on the resulting Control Lyapunov-Barrier Function for solving the stabilization with guaranteed safety.

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