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Person running in the rain with an umbrella: invariance of distances and angles under a Galilean transformation

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Abstract

It is a common experience that raindrops falling vertically appear to fall at an angle to a moving person. When trying to describe the sheltered area provided by an umbrella from the viewpoints of both moving and stationary observers, at first glance it may appear that the descriptions for the sheltered area in these two frames are not identical. It is shown as well as illustrated by simulation that the sheltered areas are indeed identical in both these frames as expected. The physics behind this is explained and in the process confirming distances, angles, and consequently areas, remain invariant under a Galilean transformation. The detailed treatment of the problem given in this paper may help undergraduate students gain a robust understanding of the important concepts of frame of reference and Galilean transformation.

Keywords: rain, umbrella, trajectory, Galilean transformation, simulation

1. Introduction

Frame of reference and Galilean transformation are fundamental concepts of mechanics and a robust understanding of these are needed to grasp classical and relativistic dynamics by undergraduate students. The well-known Galilean principle of relativity states that there are no experiments which can distinguish between frames at rest or that are moving with constant velocities. In other words, the laws of physics are equally valid in every one of these frames and what is observed by a moving observer is as valid as what is observed by a stationary observer [1–3]. A fundamental property of the Galilean transformation is that distances and angles are invariant under any number of such transformations. One may encounter situations, where at first glance it would appear that angles and distances may not be invariant under a

Galilean transformation. One such situation is a person running in the rain with an umbrella. It is a common experience that raindrops falling vertically down appear to fall at an angle to a moving person. However, as per the principle of relativity, irrespective of the observed raindrop trajectories, both observers must report the same sheltered area. This is not readily apparent with the observed raindrop trajectories.

Over the years, there have been several papers published in physics [4–10] and mathematics [11, 12] teaching journals as well as in meteorological [13, 14] journals, regarding running or walking in the rain *without* holding an umbrella. These works were mainly concerned with the amount of rain falling on the running person. Fairly recent articles by Ehrmann and Blachowicz [9] as well as by Bocci [10] in this journal summarises the previous work and further expands the research in this area.

Physics textbooks normally consider only transformation of the position of a point when discussing Galilean transformations, but transformations of other physical quantities are rarely dealt with. The papers by Tefft and Tefft [15] and by Diaz *et al* [16] regarding the transformation of work and energy as well as on the invariance of Lagrangian by Mohallem [17] are interesting in this regard.

2. Theory

In general, raindrops are acted upon by two forces, gravity pulling it down and wind pushing it sideways. The force of gravity is counterbalanced by buoyancy and velocity dependent drag forces, making the raindrops fall at a steady velocity, called the terminal velocity. Let us now define two frames of reference, one associated with the Earth's surface, called the 'stationary frame' and the other associated with the moving person, called the 'moving frame'. A point P may be represented as (x, y) in the stationary frame and by (x', y') in the moving frame. The coordinates in the two frames are related by Galilean transformation and are

$$x' = x - v_p t, \tag{1}$$

$$y' = y, \tag{2}$$

where v_p is the person's velocity in the positive X-direction, and t is the elapsed time. This assumes that the person is running on level ground. Should the person be running on a gradient, the equations are $x' = x - v_{ph} t$ and $y' = y \pm v_{pv} t$, where v_{ph} and v_{pv} are the horizontal and vertical components of the person's velocity. To keep things simple, only motion on level ground is considered.

Let us assume that the stationary and moving frames were coincident at time $t = 0$ and at that instant the raindrop under consideration was at $P_0(x_0, y_0)$. The position of the raindrop in the stationary frame at a future time is then given by

$$x = x_0 + v_w t, \tag{3}$$

$$y = y_0 - v_{\mathrm{rd}} t, \tag{4}$$

where v_w is the velocity of the wind and v_{rd} is the terminal velocity of the raindrops. The direction of v_{rd} is always vertically down.

Using the Galilean transformation, the position of the raindrop in the moving frame is given by

$$x' = (x_0 + v_w t) - v_p t, \tag{5}$$

$$y' = y_0 - v_{\mathrm{rd}} t. \tag{6}$$

2.1. Trajectory of the raindrop

An elegant way to obtain the trajectory of the raindrops is to treat equations (3) and (4) as parametric equations and eliminating the parameter t. This gives us an equation connecting the coordinates x and y of the raindrop:

$$y = y_0 - \frac{v_{\mathrm{rd}}}{v_w}(x - x_0). \tag{7}$$

The trajectory in the stationary frame is an inclined line with a slope $-v_{\mathrm{rd}}/v_w$ and passing through $P(x_0, y_0)$. Slope is positive for headwind and negative for tailwind. Note that the trajectory of the raindrop is not affected by the person's movement, as the movement of the person and falling of the raindrops are causally independent.

To obtain the trajectory of the raindrops in the moving frame, as earlier, we eliminate the parameter t from equations (5) and (6). This gives us an equation connecting the coordinates x' and y':

$$y' = y_0 - \frac{v_{\mathrm{rd}}}{v_w - v_p}(x' - x_0). \tag{8}$$

The trajectory is an inclined line with a slope $v_{\mathrm{rd}}/(v_w + v_p)$ for headwind and $-v_{\mathrm{rd}}/(v_w - vp)$ for tailwind. In the case of tailwind, if the person can run as fast as the wind, the denominator of the slope term in equation (8) becomes zero and the rain will appear to fall vertical in the moving frame. It is important to note that the slope terms are different in equations (7) and in (8), as it should be.

It is straightforward to find the sheltered area in the moving frame. The trajectories given by equation (8) at the front and back of the umbrella define two sides of the sheltered area. Rain is unable to penetrate the area between these parallel trajectories because of the shielding provided by the umbrella. The other two sides of the sheltered area are defined by the ground and the umbrella canopy. Note that for simplicity, we are ignoring the curvature of the umbrella canopy and consider the canopy as a flat surface. It is then possible to model the canopy as a straight line connecting the front and the rear tips of the umbrella. Hence, the sheltered area in the moving frame has the shape of a trapezoid.

To find the sheltered area in the stationary frame, we need to introduce the independent motion of the person running. One possible way to do this is by taking into consideration the details of the umbrella's interaction with the streams of raindrops. We may note that the moving umbrella continuously cut the streams of falling raindrops in the front and the lower portion of any cut continue to fall without its path being modified. At the rear the umbrella allows, one by one, raindrop streams which were previously blocked to continue on their trajectories. This information may be utilised to find the sheltered area in the stationary frame.

Let us define the coordinates of the tip of the umbrella, which in its forward motion will always be just cutting a stream, as (x_1, y_1). If we consider a stream which was cut t seconds ago, the coordinates of the cut end of the stream may be written as

$$x = x_1 + (v_w - v_p)t \tag{9}$$

and

$$y = y_1 - v_{\mathrm{rd}} t. \tag{10}$$

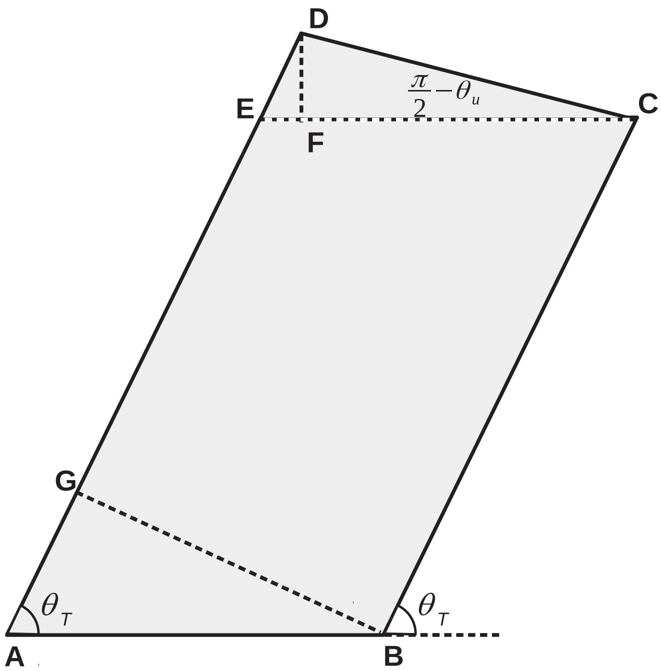

Figure 1. Calculation of trapezoid angles and base height. DC = umbrella diameter(d_u) and GB = base height(h).

Note that we need to consider the relative velocity of the wind and the person. Eliminating the parameter t, we get

$$y = y_1 - \frac{v_{\mathrm{rd}}}{v_w - v_p}(x - x_1). \tag{11}$$

This equation gives the line joining the rear ends of the streams which were previously cut by the forward movement of the umbrella. Equation (11) is similar to equation (8) obtained previously for the trajectory of raindrops in the moving frame. Further, an identical equation may be obtained for the forward tips of all streams which are being released by the umbrella at the rear.

Hence the trapezoidal sheltered area obtained in the stationary frame is identical to that obtained from the moving frame of reference. This proves that the sheltered area is transforming without modification from the stationary frame to the moving frame or vice versa.

2.2. Calculation of sheltered area parameters

The trapezoid shaped sheltered area is shown in figure 1. The trapezoid may be completely defined by the length of it's base, base angles and base height. AD is the base, $\angle BAG$ and $\angle ADC$ are the base angles and GB the base height. $\angle BAG$ is henceforth referred to as the trapezoid angle θ_T and is given by $\arctan(v_{\mathrm{rd}}/(v_w + v_p))$ for headwind and $\arctan(-v_{\mathrm{rd}}/(v_w - v_p))$ for tailwind. $\angle ADC$ depends on the umbrella angle θ_u and is equal to $(\theta_u + \pi/2) - \theta_T$. The base height h may easily be derived referring to figure 1 and is given by

$$h = d_u \cos(\theta_u - \theta_T). \tag{12}$$

Base height h may be used as a parameter to get a measure of the sheltered area. Equation (12) shows that one can maximise the sheltered area by having a larger umbrella (large d_u) and/or by tilting the umbrella such that $\theta_u = \theta_T$, i.e. the umbrella pole is parallel to the apparent velocity of the raindrops. The optimal tilt angle is given by

$$\theta_{\text{opt}} = \arctan\left(\frac{\pm v_{\text{rd}}}{v_w \pm v_p}\right) \tag{13}$$

with positive sign for headwind and negative sign for tailwind.

3. Simulation

In these simulations, the person's velocity is varied from 1 m $\sec^{-1}$ (3.6 km h^{-1}, slow-walking) to 4 m $\sec^{-1}$ (14.4 km h^{-1}, running). The terminal velocity of raindrops is taken as 10 m s^{-1} (36 km h^{-1}) [18]. The wind velocity 10 m s^{-1} (Fresh Breeze, Beaufort number 5) to 20 m s^{-1} (Fresh Gale, Beaufort number 8) and is in the same (or opposed) direction as that of the person's velocity. The umbrella diameter is taken as 1200 mm. In these simulations, the person is always running from left to right. All angles are measured from the horizontal, counter clockwise.

In the absence of any wind, raindrops are assumed to fall continuously and vertically down in a straight line. When wind is present, the raindrops fall at an angle determined by the addition of the velocities.

All simulations are drawn as a snap-shot in time and as observed in the stationary frame or in the moving frame. For convenience, simulation parameters and results are given with the figures as well.

3.1. Vertically falling rain

First we simulate the case of vertically falling raindrops and a person running at 4 m s^{-1} with an umbrella and as observed in the moving frame. From the moving person's view, the raindrops are not only falling down but also moving backwards at 4 m s^{-1}. In other words, in the moving person's frame, she or he is not moving but the stationary frame and the associated rainfall is going backwards. Using equation (8) the raindrops should appear to be falling at an angle $\arctan(-4/(-10)) = 68.2°$. The umbrella is held vertical. Figure 2 shows what is observed in the moving frame.

The sheltered area is a trapezoid bound by the raindrops streams immediately adjacent to the umbrella canopy on the right and on the left of the figure (parallel sides), and with umbrella canopy and the ground being the non-parallel side pair. The base height of the trapezoid h is 1.11 m. Ignore the dimension lines shown in figure 2 for the time being, but later compare it with the dimension lines drawn in figure 3.

Now let us simulate the view from the stationary frame. In this frame the raindrops are falling vertically down. The person's movement does not influence the direction of the fall of raindrops. For ease of drawing, and more importantly to relate to the view from the moving frame, the motion direction is divided into 0.15 m-long segments. At the person's velocity of 4 m s^{-1}, to cover 0.15 m, it takes 0.0375 s. Referring to figure 3, the rightmost stream of raindrops is just being cut by the tip of the umbrella. After it is cut, the lower portion of the stream should continue falling as no new external force has acted or is acting on the stream. The second stream from right is one which was cut 0.0375 s previously and during that

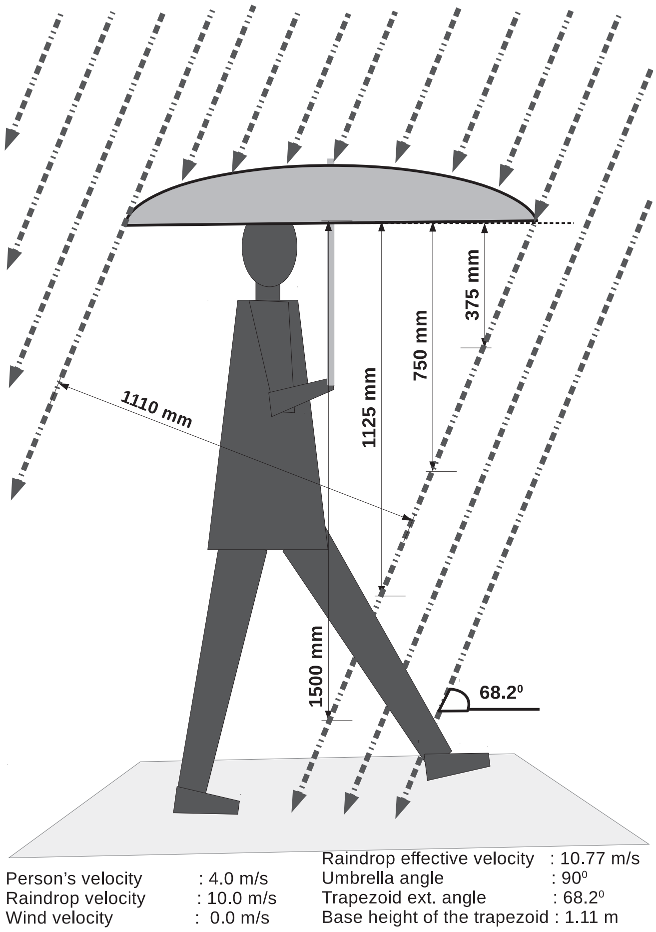

Figure 2. Person running in the rain at $4\ \mathrm{m\ s^{-1}}$ as observed from the moving frame.

period, the rear end of the stream should travel 0.375 m down (time required to travel 0.15 m times raindrop downward velocity of $10\ \mathrm{m\ s^{-1}}$). In a similar vein, the third stream depicted was cut 0.075 s previously (corresponding to 0.3 m umbrella movement) and must have travelled 0.75 m and so on. When a line is drawn connecting the rear ends of all the four cut streams depicted on the right, we get an inclined line at an angle 68.2°, which is identical to the angle of tilt for raindrops in the moving frame. If we let the segmentation go to very small sizes, this line exactly represents the trajectory of the raindrops in the moving frame.

On the left-hand side of the figure, the fourth stream from the left shows a stream which is just being allowed to fall. The third stream from the left was allowed to fall 0.0375 s earlier and the front end has travelled down 0.375 m. In a similar fashion, the front end of the second stream from left has travelled 0.75 m and the first one 1.125 m down. If we draw a line connecting the front ends of all the streams shown on the left, we again get a line inclined at 68.2°. A trapezoidal sheltered area is evident. The base height of the trapezoid is 1.11 m which is exactly the same for the moving frame. At this point we can compare the distance markings shown in figure 2 and verify that in the forward direction, for a given horizontal

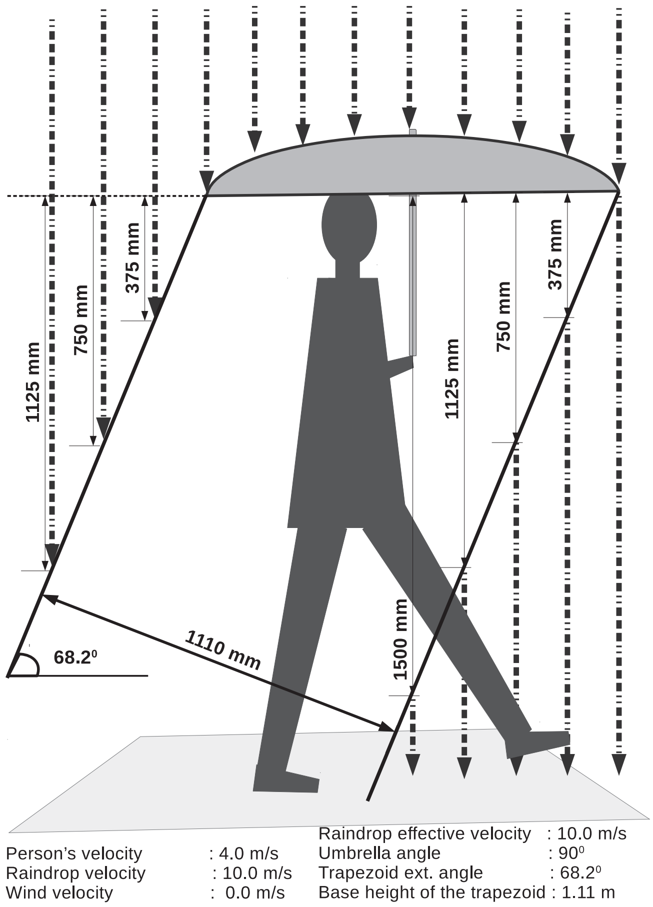

Figure 3. Person running in the rain at 4 m s^{-1} as observed from the stationary frame; rain falling vertically down. The trapezoidal sheltered area is clearly identifiable and is identical to that in the moving frame of reference.

distance, the raindrop streams are at the same positions. Similar is the case for raindrop streams being released at the rear of the umbrella.

Let us now investigate the effect of tilting the umbrella on the sheltered area. Figure 4 shows the effect of introducing an umbrella tilt and as observed in the moving frame. The umbrella is tilted 70° from the horizontal. Physically, the case of the tilted umbrella is not much different from the previous case of the umbrella being held vertical, the difference being that the rear edge of the umbrella is now lifted up by 0.2 m. The tilt angle of the raindrops are the same, but the base height of the trapezoid h increases to 1.19 m.

Let us now explore how this appears in the stationary frame. Referring to figure 5, the rightmost stream of raindrops is just being cut by the umbrella and the lower portion of the stream should continue falling. The second stream, cut 0.0375 s earlier, falls 0.375 m and so on. On the left side of the figure, the fifth stream from the left is just being allowed to fall. The fourth from the left stream, which was released 0.0375 s earlier, has fallen a distance of 0.375 m and so on. As previously, when a line connecting the rear ends (on the right) and another one connecting the forward ends (on the left) of the streams are drawn, we get the

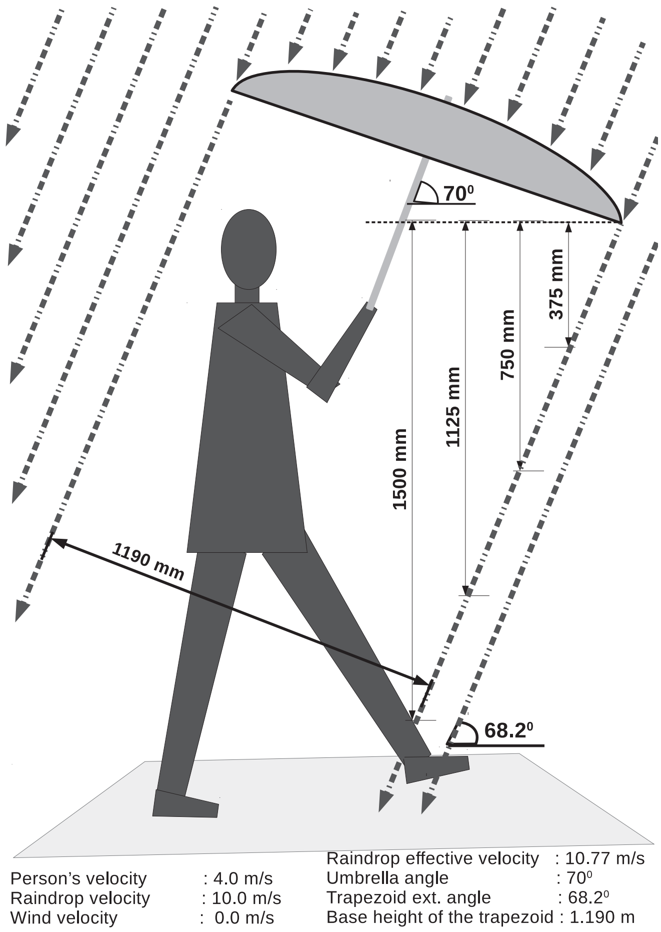

Figure 4. View from the moving frame, person running at 4 m s^{-1}; umbrella tilted.

same slope angle as previously, 68.2°. The base height of the trapezoid is 1.19 m and is equal to that for the corresponding case in the moving frame. It is important to note that the trapezoid exterior angle, which is determined by the velocities, has not changed because of the tilting of the umbrella; only the base height has changed, providing more sheltered area.

3.2. Rain affected by fresh breeze

Rain is almost always accompanied by wind and this is when people tend to tilt their umbrellas to gain better protection. In these simulations, wind is assumed to be blowing horizontally and in the same direction or opposite to the direction the person is running (tailwind or headwind respectively). Figure 6 shows the result of a simulation with headwind blowing at 10 m s^{-1} and umbrella tilted at 60° and as observed in the moving frame. The trapezoid exterior angle has become shallower, $\arctan(-10/(-10-4)) = 35.5°$, and the base height of the trapezoid is 1.09 m.

Figure 7 shows the view from the stationary frame. The base height of the trapezoid is 1.09 m and the trapezoid exterior angle 35.5°, which matches the numbers for the sheltered area in the moving frame.

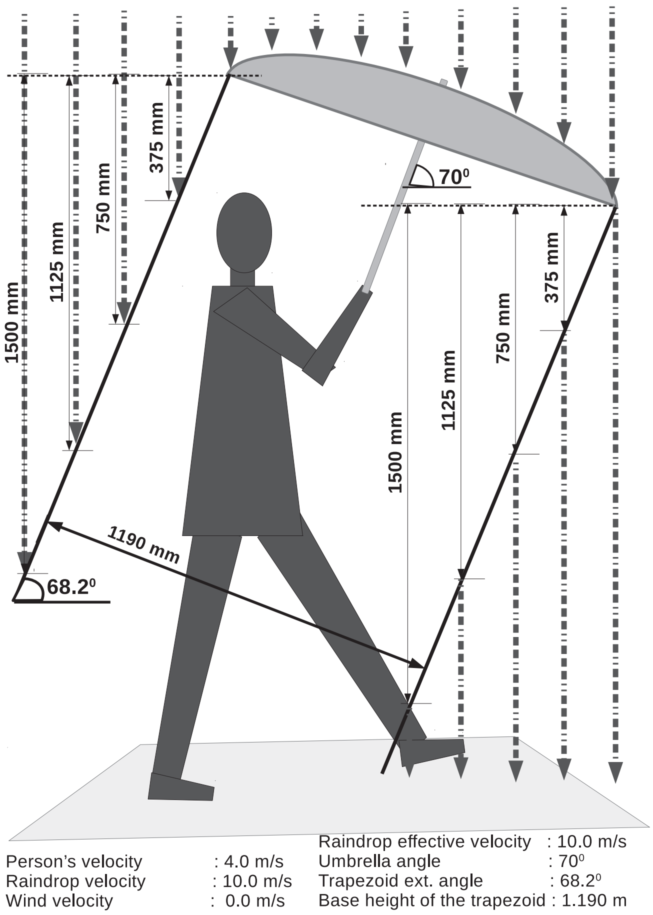

Figure 5. View from the stationary frame; person running at $4\ \mathrm{m\ s^{-1}}$ umbrella tilted.

Figure 8 shows the situation with a tailwind of $10\ \mathrm{m\ s^{-1}}$, umbrella tilted and as observed in the stationary frame. The tilt angle of the trapezoid edge line when calculated gives $-59°$ which is the result of the way arctan() function is implemented in computers. The actual angle is $121°$. When the velocity of the tailwind equals the person's forward velocity, in the moving frame, the raindrops will fall vertically down and there is no need to tilt the umbrella. This conclusion has previously been reached by other researchers [11, 19].

Tilting the umbrella can make the sheltered area larger, as may be seen from figures 4 and 5. When the umbrella is tilted, it is not necessary that the person's body be directly under the umbrella. This is especially true when running with a headwind, see figures 6 and 7.

4. Discussion

Let us discuss the following 'thought experiment' which may shed more light on the reality of the *slanted* raindrop trajectory for the observer in the moving frame. Consider a single drop of water released from a stationary water nozzle, say, at a height of 2 m. Let us picture a person running towards the water drop holding a device capable of measuring the horizontal distance

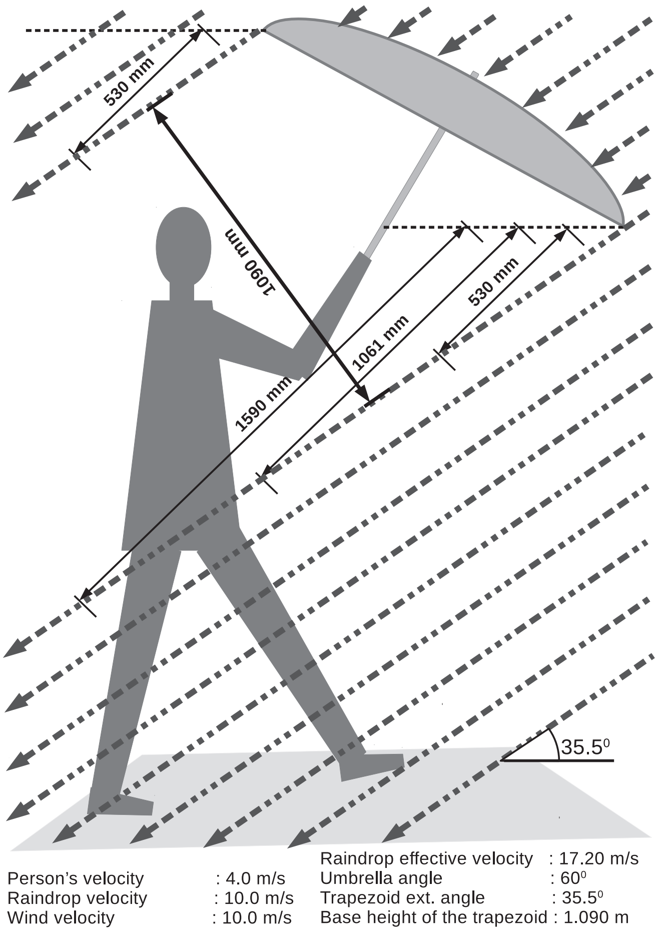

Figure 6. Headwind, making the rainfall at an angle; umbrella tilted at 60°. As observed in the moving frame.

to the water drop continuously. As time progresses, the horizontal distance to the water drop *decreases* due to the person's forward movement. This creates the impression that the trajectory of the water drop is tilted towards the running person. It should be noted that the water drop, which was released in the stationary frame, is unaffected by the person's motion and continue to fall vertically down. However, the experience of the running person is real, in the sense that, if she/he plots the readings of the horizontal distance measuring device against time, a slanted trajectory is obtained.

An umbrella cutting a raindrop stream and the cut end falling uninterrupted or an umbrella just releasing raindrop stream to fall, is something reminiscent of water dot-matrix printers used for creating artwork [20]. Instead of computer-generated images where pixels are turned on and off as required, these displays use groups of falling water droplets, released from several hundred nozzles at precisely timed bursts, to create moving images.

Using the theoretical trajectories of raindrops and simulations, we were able to demonstrate that sheltered areas are the same when described in a moving or in a stationary frame of reference. Physically, where a person holding an umbrella gets wet is predicted to be

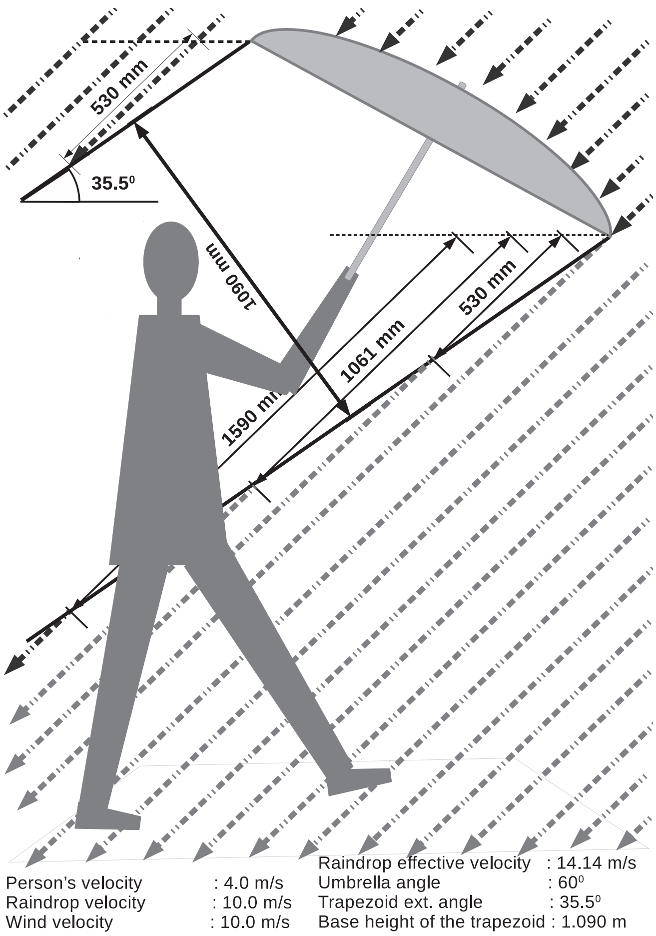

Figure 7. Headwind and umbrella tilted. As observed in the stationary frame. Note that the drawing segments are for 100 mm movement of the person.

the same, irrespective of the frame of reference. This is a property of *proper rigid transformations*, a transformation that does not alter the size or shape of a figure. In our particular case, during the transformation, only translations come into play as given by equation (1). An interested reader may read more about such transformations in any textbook on theoretical kinematics [21].

5. Conclusion

It is shown that a scalene trapezoidal sheltered area is provided by a moving umbrella. This area transforms under a Galilean transformation without violating the principle of Galilean relativity. Expressions are derived for the trapezoid base height, as well as interior and exterior angles in terms of the velocities involved.

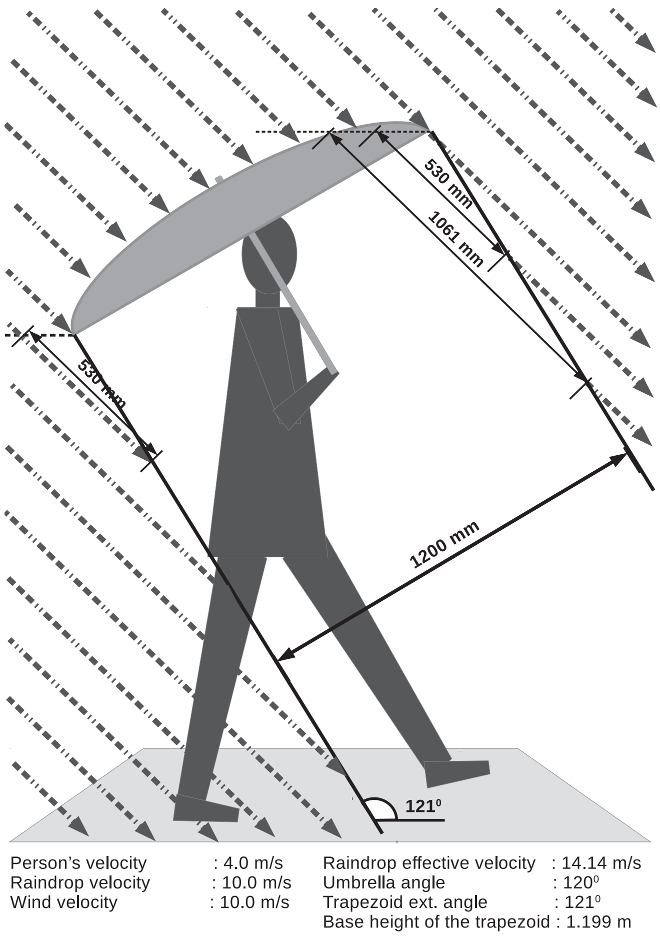

Figure 8. Tailwind of 10 m s^{-1} and umbrella tilted. As observed in the stationary frame.

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