Deriving System Equations & the State-Space Model for Cessna 172

State-Space Representation and Transfer Function Derivation

$$\begin{aligned} \text{State Equation:} \quad & x' = Ax + B\mu \\ \text{Output Equation:} \quad & y = Cx + D\mu \end{aligned}$$

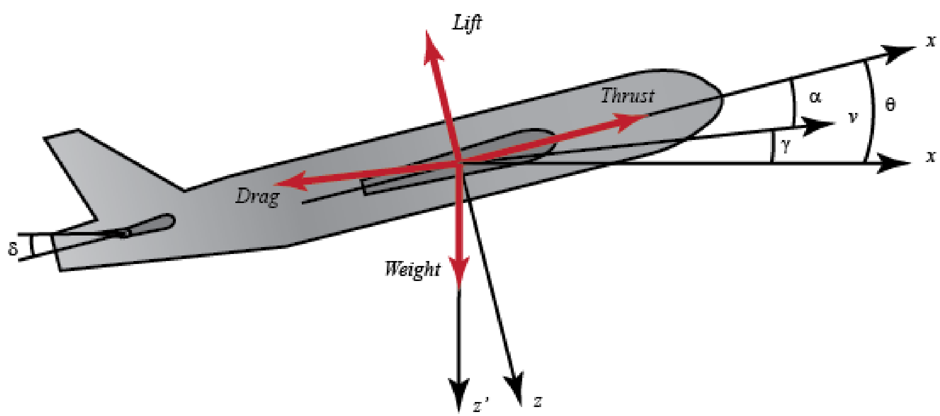

Figure 1: Basic coordinate axes and forces acting on an aircraft

Source: `https://ctms.engin.umich.edu/CTMS/?example=AircraftPitch&section=SystemModeling`

From the basic model shown in Figure 1, I was able to derive the following equations.

$$\dot{\alpha} = \mu\Omega\sigma\left[-(C_L + C_D)\alpha + \frac{1}{\mu - C_L}q - (C_W \sin\lambda)\delta + C_L\right]$$

$$\dot{q} = \frac{\mu\Omega}{2I_{yy}}[[m - \eta(C_L + C_D)]\alpha + [C_M + \sigma C_m(1 - \mu C_L)]q + (\eta C_w \sin\lambda)\delta]$$

$$\dot{\theta} = \Omega q$$

Symbol	Description/Equation	Value	Units
α	Angle of attack	N/A	rad
q	Pitch rate	N/A	rad/s
δ	Elevator deflection angle	N/A	rad
μ	$\frac{\rho S \bar{c}}{4m}$	0.00658	Constant
ρ	Density of air	1.204	kg/m^3
S	Platform area of the wing	16.165	m^2
$\bar{c}$	Average chord length	1.472	m
m	Mass of the aircraft	1088.622	kg
Ω	$\frac{2U}{\bar{c}}$	83.860	Constant
U	Equilibrium flight speed	62.733	m/s
σ	Normalization constant	0.04	Constant
C_L	Lift coefficient	1.60	Constant
C_D	Drag coefficient	0.0292	Constant
C_M	Pitch moment coefficient	-0.3	Constant
C_W	$\frac{2m}{\rho U^2 S}$	2.88	Constant
λ	Roll angle	0.05	rad
σ	$\frac{1}{1+\mu C_L}$	0.990	Constant
I_{yy}	$\frac{i_{yy}}{m\bar{c}^2}$	0.773	Constant
i_{yy}	Moment of inertia	1824.93	$kg \cdot m^2$
η	$\mu \sigma C_M$	0.00195	Constant

Table 1: Cessna 172 model variables

After plugging in the values from Table 1 into the equations I was able to simplify the modeling equations.

$$\dot{\alpha} = -0.890\alpha - 0.343q - 1.57\delta$$

$$\dot{q} = -0.106\alpha - 0.117q - 0.002\delta$$

$$\dot{\theta} = 83.86q$$

Next, to find the transfer function I took the Laplace transform of the modeling equations seen above and performed basic algebra to rearrange them into $P(s) = \frac{\Theta(s)}{\Delta(s)}$.

$$sA(s) = -0.890A(s) - 0.343Q(s) - 1.57\Delta(s)$$

$$Q(s) = -0.106A(s) - 0.117Q(s) - 0.002\Delta(s)$$

$$\Theta(s) = 83.86Q(s)$$

$$P(s) = \frac{\Theta(s)}{\Delta(s)} = \frac{-0.168s + 14.0886}{s^3 + 1.007s^2 + 0.0682s}$$

As the equations above are already in their state-variable form, they can easily be rewritten into the necessary matrices.

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.890 & -0.343 & 0 \\ -0.106 & -0.117 & 0 \\ 0 & 83.86 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} -1.57 \\ -0.002 \\ 0 \end{bmatrix} [\delta]$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix}$$

Closed-Loop System Response and PID Tuning

I then designed a pitch angle PID controller, considering actuator dynamics, and tuned it to a stable, responsive behavior.

$$Actuator(s) = \frac{1}{0.5s + 1}$$

$$P_{combined}(s) = P(s) \cdot Actuator(s)$$

By performing the PID tuning via adjusting the target bandwidth I was able to create a controller with stable closed-loop poles and a moderate rise time, settling time, and overshoot.

Table 2: Step Response Characteristics

Characteristic	**Value**
Rise Time	1.4478 s
Transient Time	14.5328 s
Settling Time	14.5328 s
Minimum Value (Settling)	0.9197
Maximum Value (Settling)	1.1966
Overshoot	19.66 %
Undershoot	0
Peak Value	1.1966
Peak Time	3.2599 s

$$C(s) = K_p + \frac{K_i}{s} + K_d s$$

$$K_p = 0.00248$$
$$K_i = 1.91 \times 10^{-5}$$
$$K_d = 0.0805$$

Table 3: Closed-Loop Poles

Closed-Loop Poles	**Value**
Pole 1	-20.0733
Pole 2	$-0.4521 + 0.9819i$
Pole 3	$-0.4521 - 0.9819i$
Pole 4	$-0.0148 + 0.0032i$
Pole 5	$-0.0148 - 0.0032i$

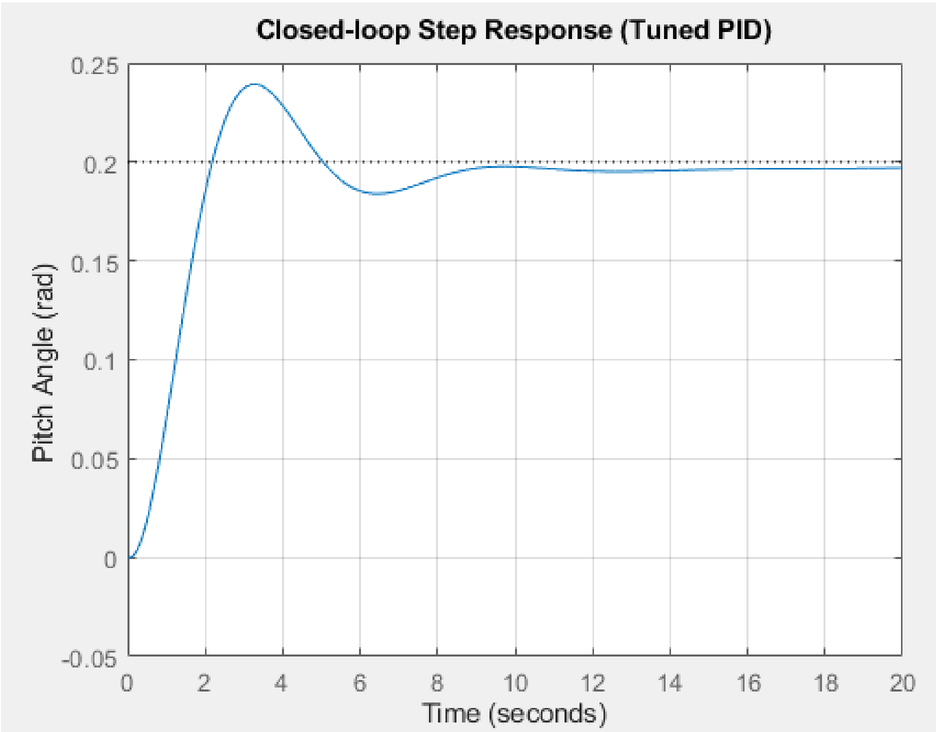

Figure 2: Closed-Loop Step Response for the Tuned PID Controller