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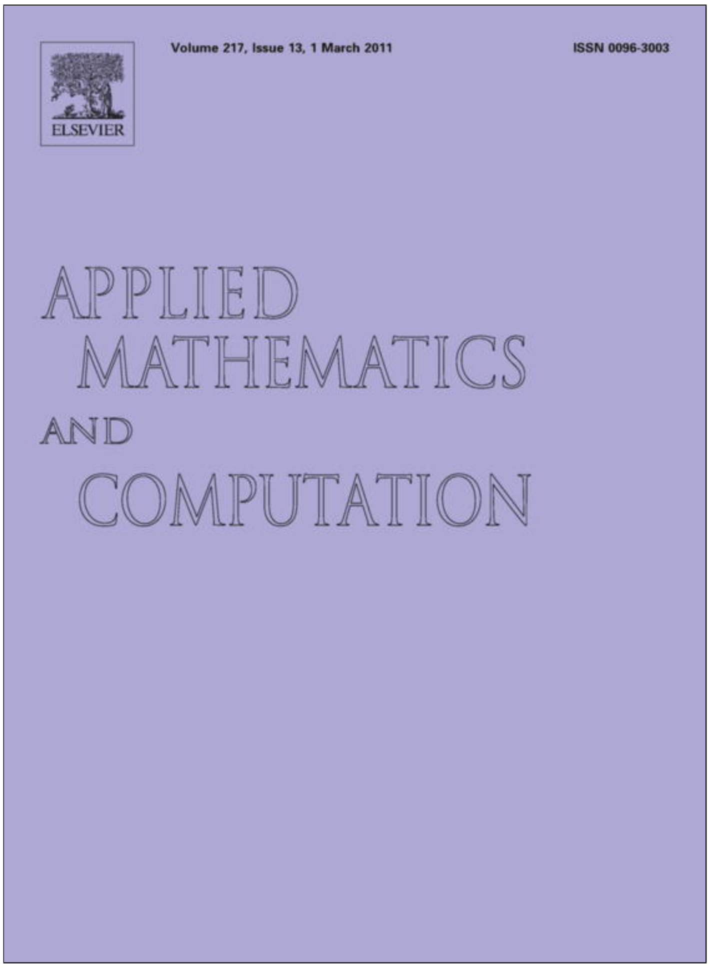
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The bivariate noncentral chi-square distribution – A compound distribution approach

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ABSTRACT

This paper proposes the bivariate noncentral chi-square (BNC) distribution by compounding the Poisson probabilities with the bivariate central chi-square distribution. The probability density and cumulative distribution functions of the joint distribution of the two noncentral chi-square variables are derived for arbitrary values of the correlation coefficient, degrees of freedom(s), and noncentrality parameters. Computational procedures to calculate the upper tail probabilities as well as the percentile points for selected values of the parameters, for both equal and unequal degrees of freedom, are discussed. The graphical representation of the distribution for different values of the parameters are provided. Some applications of the distribution are outlined.

1. Introduction

Despite many studies on the bivariate central chi-square distribution (cf [1–5]), a very limited number of work is available on its noncentral counterpart. The form and representation of the available bivariate noncentral chi-square (BNC) distribution are complicated, and are not suitable for the computation of the cumulative distribution and percentile points [6,7]. Moreover, some of them could only be used under strict conditions or constraints [8]. The BNC distribution is involved in the expression of the power function of various tests in the area of testing after pretest [6,9–11]. None of the studies provided any table of critical values or graphical comparison of power functions, presumably not having any computational method available to find the upper tail area under the BNC distribution due to the absence of any suitable form of the distribution and appropriate computational tools. This paper derives the probability density and cumulative distribution functions for the BNC distribution using compounding method, and provides computational procedure to find the upper tail area as well as the critical values of the BNC distribution under different conditions.

A theoretical paper on BNC distribution is proposed by [8] under some restrictions [12] approximate the distribution of the BNC using some transformation on the variables of the bivariate central chi-square distribution. A latest work by [13] derives the joint density of the noncentral bivariate and trivariate chi-square distribution corresponding to the diagonal elements of a complex noncentral Wishart matrix. Readers may refer [13] for some motivations on the uses of this distribution in some studies of signal processing applications. So far none of the proposed BNC distribution found in the literature is derived based on the compounding method.

The compounding method to derive distributions by mixing more than one distribution is well known in the literature (see [14–17]). However, most of the papers deal with developing theoretical distributions rather than providing suitable form for computational advantage. Although there is a suggestion to derive the BNC distribution by compounding the

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Poisson probabilities with the bivariate central chi-square distributions in [4, p. 475], the density function of the BNC distribution is not provided in the text. Yunus and Khan [11] used the bivariate non-central chi-square distribution in the definition of the power function of testing the intercept after pretesting on the slope of a multivariate simple regression model.

The distribution of the noncentral bivariate chi-square as a mixture of bivariate central chi-square distributions with Poisson probabilities has been proposed by Marshall and Olkin [7]. For some reasons (see Section 2), their proposed BNC distribution however is not suitable for computation. By choosing an appropriate central bivariate chi-square distribution to be compounded with the Poisson probabilities, a different density function of the BNC distribution is defined in this paper. The form of the proposed distribution is more suitable than the previous ones from the computational view point.

This paper is motivated by the work on a project of computing size and power of statistical tests for testing parameters of a regression model such as the multivariate multiple regression. It may be noted that there is no program codes in R in the CRAN project to compute the upper tail area under this distribution.

The idea of generating BNC distribution is discussed in Section 2. The proposed BNC distribution is presented in Section 3. In Section 4, the procedure to compute the cumulative distribution is given and some upper tail percentile points are tabulated. Some concluding remarks are included in Section 5. The Appendix contains all the tables and graphs.

2. Compounding method

The procedure of generating probability distribution from a mixture of two probability distributions is considered in this Section.

Suppose $F(\cdot,\lambda)$ is a distribution of any given parametric family, where the parameter λ is a value of a random variable Λ with a distribution G. Thus, $F(\cdot,\lambda)$ is a conditional distribution given $\Lambda = \lambda$ and the corresponding unconditional distribution is given by (see [7])

$$H(x) = \int F(x|\lambda)dG(\lambda). \tag{2.1}$$

The unconditional distribution is obtained by integrating the mixture distribution of F and G. Note that Eq. (2.1) is the same as Definition 2.1 of compounding distribution found in [17] except that the compounding distribution depends on an arbitrary constant that is multiplied to the random variable Λ. The distribution H in Eq. (2.1) is known as a compound as well as a mixture distribution.

For the univariate case, the cumulative distribution of a noncentral chi-square distribution with υ degrees of freedom and noncentrality parameter θ is represented as a weighted sum of univariate central chi-square probabilities with weights equal to the probabilities of a Poisson distribution with expected value $\theta/2$ (see [18, p. 435]). [16] used the mixture distribution technique to derive the likelihood function of a multivariate Student-t distribution. From [19], the distribution of the univariate noncentral chi-square variable is

$$P[X \leqslant x] = \begin{cases} \sum_{\lambda=0}^{\infty} \frac{e^{-\theta/2}(\theta/2)^{\lambda}}{\lambda!} \int_0^x \frac{y^{(\upsilon/2)+\lambda-1}e^{-y/2}}{2^{(\upsilon/2)+\lambda}\Gamma(\upsilon/2+\lambda)}dy, & \text{for} \quad x > 0, \\ 0 & \text{for} \quad x \leqslant 0. \end{cases} \tag{2.2}$$

Now let G be a bivariate distribution function with $\Lambda_i = \lambda_i$, and each $F_i(x_i)$, $i = 1, 2$ is a univariate distribution function. Let $F(x_1,x_2|\lambda_1,\lambda_2)$ be the joint conditional distribution function of X_1 and X_2, and $G(\lambda_1,\lambda_2)$ be that of Λ_1 and Λ_2. Then the unconditional joint distribution of X_1 and X_2 is given by

$$H(x_1,x_2) = \int\int F_1(x_1|\lambda_1)F_2(x_2|\lambda_2)dG(\lambda_1,\lambda_2). \tag{2.3}$$

Ref. [7] deal with properties and examples of mixtures of some parametric families $\{F(\cdot|\lambda) : \lambda \in \Lambda\}$. Two BNC distributions were proposed in the paper and formally obtained using (2.3). One of them has the probability density function (pdf)

$$h_1(x_1,x_2) = \sum_{\lambda=0}^{\infty} \frac{x_1^{\upsilon_1/2+\lambda-1}e^{-x_1/2}}{2^{\upsilon_1/2+\lambda}\Gamma(\upsilon_1/2+\lambda)} \frac{x_2^{\upsilon_1/2+\lambda-1}e^{-x_2/2}}{2^{\upsilon_1/2+\lambda}\Gamma(\upsilon_1/2+\lambda)} \frac{\theta^{\lambda}e^{-\theta}}{\lambda!}, \quad x_1,x_2 \geqslant 0, \tag{2.4}$$

which is a Poisson mixture of central chi-square distributions with only one noncentrality parameter, θ and degrees of freedom υ_1. According to them, the more meaningful BNC distribution is obtained from the representations

$$X_1 = U_1^2 + V_1, \quad X_2 = U_2^2 + V_2, \tag{2.5}$$

where (U_1,U_2) and (T_1,T_2) a re independently distributed with $U_i \sim N(\mu_i, 1/\beta_{ii})$, $i = 1, 2$ and (V_1,V_2) has the bivariate central chi-square distribution. The corresponding density function is given by

$$h_2(x_1,x_2) = \sum_{j=0}^{\infty}\sum_{\ell_1=0}^{\infty}\sum_{\ell_2=0}^{\infty} q_1(x_1|j,\ell_1)q(x_2|j,\ell_2)p(j|\rho^2)\frac{t_1^{\ell_1}e^{t_1}}{\ell_1!}\frac{t_2^{\ell_2}e^{t_2}}{\ell_2!}, \tag{2.6}$$

where $t_i = \beta_{ii}\mu_i^2/2,\ i = 1,2,\ p(j|\rho^2) = \Gamma(v_2+j)\rho^{2j}(1-\rho^2)^{v_2}/[\Gamma(v_2)j!]$ is the density function of the negative binomial distribution, and

$$q_i(x_i|j,\ell_i) = \frac{\beta_{ii}^{(v_2+2j+2\ell_i+1)/2} x_i^{(v_2+2j+2\ell_i+1)/2} e^{-\beta_{ii}x_i/2}}{2^{(v_2+2j+2\ell_i)/2}\Gamma[(v_2+2j+2\ell_i)/2]}, \quad i = 1,2. \tag{2.7}$$

Practically, it is not easy to represent the interested variables X_1 and X_2 in terms of U_i and V_i, respectively with the aforementioned distributions. As a result, β_{ii} could not be obtained and the density $h_2(x_1,x_2)$ is not useful for compounding.

The univariate noncentral chi-square distribution given in (2.2) has been derived in several different ways such as the direct derivation by a limiting process, the geometrical derivation and by the process of induction [18, p. 436] [19,20]. However, the BNC distribution proposed in this paper is derived by the compounding method. Following a suggestion by Kotz et al. [4, p. 475] of compounding distributions, the BNC distribution is proposed in this paper by compounding appropriate bivariate central chi-square distribution with the Poisson probabilities.

3. The proposed BNC distribution

In this section, the BNC distribution is proposed for two distinct cases. The two cases vary for equal and unequal degrees of freedom of the two cental chi-square random variables.

3.1. Case I (equal degrees of freedom)

Suppose $\{(Z_{1j},Z_{2j}), j = 1,\ldots,m\}$ are independent random variables with $Z_{ij} \sim N(0,1)$ for all j and $i = 1, 2$. Let ρ be the canonical correlation between Z_{1j} and Z_{2j}. Then $W_i = \sum_{j=1}^m Z_{ij}^2 (i = 1,2)$ is a chi-square random variable, with m degrees of freedom, and non-zero canonical correlations ρ between W_1 and W_2. The density function of the bivariate central chi-square distribution [1] with random variables W_1 and W_2 is

$$f_w(w_1,w_2,\rho) = (1-\rho^2)^{\frac{m}{2}} \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{m}{2}+j\right)\rho^{2j}(w_1w_2)^{\frac{m}{2}+j-1} e^{-\frac{w_1+w_2}{2(1-\rho^2)}}}{\Gamma\left(\frac{m}{2}\right)j!\left[2^{\frac{m}{2}+j}\Gamma\left(\frac{m}{2}+j\right)(1-\rho^2)^{\frac{m}{2}+j}\right]^2}. \tag{3.8}$$

The corresponding cumulative distribution is given by

$$P(W_1 < d_1, W_2 < d_2) = \sum_{j=0}^{\infty} \frac{(1-\rho^2)^{\frac{m}{2}}\Gamma\left(\frac{m}{2}+j\right)\rho^{2j}}{\Gamma\left(\frac{m}{2}\right)j!} \gamma\left(\frac{m}{2}+j, \frac{d_1}{2(1-\rho^2)}\right)\gamma\left(\frac{m}{2}+j, \frac{d_2}{2(1-\rho^2)}\right), \tag{3.9}$$

where $\gamma(v,d)$ is the incomplete gamma function, that is, $\gamma(v,d) = \int_0^d x^{v-1}e^{-x}/\Gamma(v)dx$.

Instead of (2.6), another pdf is obtained by compounding the bivariate central chi-square distributions given in Eq. (3.8) with probabilities from Poisson distributions. Thus the pdf of the BNC distribution with random variables Y_1 and Y_2 is given by

$$h(y_1,y_2) = \sum_{j=0}^{\infty}\sum_{\lambda_1=0}^{\infty}\sum_{\lambda_2=0}^{\infty} f_w(y_1,y_2|\lambda_1,\lambda_2,\rho)\frac{e^{-\theta_1/2}(\theta_1/2)^{\lambda_1}}{\lambda_1!}\frac{e^{-\theta_2/2}(\theta_2/2)^{\lambda_2}}{\lambda_2!} \tag{3.10}$$

with noncentrality parameters θ_1 and θ_2 and m degrees of freedom, and correlation coefficient ρ.

Then (3.10) takes the form

$$\begin{aligned} h(y_1,y_2) = {} & \sum_{j=0}^{\infty}\sum_{\lambda_1=0}^{\infty}\sum_{\lambda_2=0}^{\infty} \frac{(1-\rho^2)^{\frac{m}{2}}\Gamma\left(\frac{m}{2}+j\right)\rho^{2j}}{\Gamma\left(\frac{m}{2}\right)j!} \times \frac{y_1^{\frac{m}{2}+j+\lambda_1-1}e^{-\frac{y_1}{2(1-\rho^2)}}}{[2(1-\rho^2)]^{\frac{m}{2}+j+\lambda_1}\Gamma\left(\frac{m}{2}+j+\lambda_1\right)} \frac{e^{-\theta_1/2}(\theta_1/2)^{\lambda_1}}{\lambda_1!} \\ & \times \frac{y_2^{\frac{m}{2}+j+\lambda_2-1}e^{-\frac{y_2}{2(1-\rho^2)}}}{[2(1-\rho^2)]^{\frac{m}{2}+j+\lambda_2}\Gamma\left(\frac{m}{2}+j+\lambda_2\right)} \frac{e^{-\theta_2/2}(\theta_2/2)^{\lambda_2}}{\lambda_2!}. \end{aligned} \tag{3.11}$$

So, the cumulative distribution function can be written as

$$\begin{aligned} P(Y_1 < d_1, Y_2 < d_2) = {} & \sum_{j=0}^{\infty}\sum_{\lambda_1=0}^{\infty}\sum_{\lambda_2=0}^{\infty} \frac{(1-\rho^2)^{\frac{m}{2}}\Gamma\left(\frac{m}{2}+j\right)\rho^{2j}}{\Gamma\left(\frac{m}{2}\right)j!} \times \gamma\left(\frac{m}{2}+j+\lambda_1, \frac{d_1}{2(1-\rho^2)}\right)\gamma\left(\frac{m}{2}+j+\lambda_2, \frac{d_2}{2(1-\rho^2)}\right) \\ & \times \frac{e^{-\theta_1/2}(\theta_1/2)^{\lambda_1}}{\lambda_1!}\frac{e^{-\theta_2/2}(\theta_2/2)^{\lambda_2}}{\lambda_2!}. \end{aligned} \tag{3.12}$$

It is possible to express the cumulative distribution as a weighted sum of bivariate central chi-square with weights equal to the probabilities of a Poisson distribution. Note, as the values of d_1 and d_2 increase, $P(Y_1 < d_1, Y_2 < d_2)$ approaches 1.

The infinite series is truncated after $t_1 + 1$, $t_2 + 1$ and $t_3 + 1$ terms and thus giving a bound on the truncation error, say r_t. Since for any finite d_1 and $d_2, \gamma(\cdot,\cdot) \leqslant 1$,

$$k_j = (1-\rho^2)^{\frac{m}{2}} \frac{\Gamma\left(\frac{m}{2}+j\right)\rho^{2j}}{j!\Gamma\left(\frac{m}{2}\right)}, \quad j = 1,\ldots,m,$$

the mass of a negative binomial distribution and

$$s_{\lambda_i} = \frac{e^{-\theta_i/2}(\theta_i/2)^{\lambda_i}}{\lambda_i!}, \quad i = 1,2,$$

the mass of a Poisson distribution, and

$$r_{t_1,t_2,t_3} = \sum_{j=t_1+1}^{\infty}\sum_{\lambda_1=t_2+1}^{\infty}\sum_{\lambda_2=t_3+1}^{\infty} k_j s_{\lambda_1} s_{\lambda_2} = 1 - \sum_{j=0}^{t_1}\sum_{\lambda_1=0}^{t_2}\sum_{\lambda_2=0}^{t_3} k_j s_{\lambda_1} s_{\lambda_2}.$$

The stopping values of t_1, t_2, and t_3 depend on the machine and software precision. In practice, r_t is computed successively, and iteration stops when $r_{t-1} = r_t$ or when the width of r_t is less than a specified tolerance.

3.2. Case II (unequal degrees of freedom)

Let Y_1 and Y_2 follow bivariate central chi-square distributions with m and n degrees of freedom, respectively and have m non-zero canonical correlation ρ. The joint distribution of Y_1 and Y_2 proposed by Wright and Kennedy [5], Gunst and Webster [22] is given by

$$P(Y_1 < d_1, Y_2 < d_2) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(1-\rho^2)^{(m+n)/2}\frac{\Gamma\left(\frac{m}{2}+j\right)}{\Gamma\left(\frac{m}{2}\right)j!}\frac{\Gamma\left(\frac{n}{2}+k\right)}{\Gamma\left(\frac{n}{2}\right)k!}\rho^{2(j+k)} \times \gamma\left(\frac{m}{2}+j, \frac{d_1}{2(1-\rho^2)}\right)\gamma\left(\frac{n}{2}+k, \frac{d_2}{2(1-\rho^2)}\right). \tag{3.13}$$

Now, the cumulative distribution for the BNC variables can be defined as a Poisson mixture of bivariate central chi-square distribution, and is given by

$$\begin{aligned} P(Y_1 < d_1, Y_2 < d_2) = {} & \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\lambda_1=0}^{\infty}\sum_{\lambda_2=0}^{\infty}(1-\rho^2)^{(m+n)/2}\frac{\Gamma\left(\frac{m}{2}+j\right)}{\Gamma\left(\frac{m}{2}\right)j!}\frac{\Gamma\left(\frac{n}{2}+k\right)}{\Gamma\left(\frac{n}{2}\right)k!} \times \rho^{2(j+k)}\gamma\left(\frac{m}{2}+j+\lambda_1, \frac{d_1}{2(1-\rho^2)}\right) \\ & \times \gamma\left(\frac{n}{2}+k+\lambda_2, \frac{d_2}{2(1-\rho^2)}\right) \times \frac{e^{-\theta_1/2}(\theta_1/2)^{\lambda_1}}{\lambda_1!}\frac{e^{-\theta_2/2}(\theta_2/2)^{\lambda_2}}{\lambda_2!}. \end{aligned} \tag{3.14}$$

By the same manner as in Case I, the truncation error for this Case II is

$$r_{t_1,t_2,t_3,t_4} = \sum_{j_1=t_1+1}^{\infty}\sum_{j_2=t_2+1}^{\infty}\sum_{\lambda_1=t_3+1}^{\infty}\sum_{\lambda_2=t_4+1}^{\infty} q_{j_1}q_{j_2}s_{\lambda_1}s_{\lambda_2} = 1 - \sum_{j_1=0}^{t_1}\sum_{j_2=0}^{t_2}\sum_{\lambda_1=0}^{t_3}\sum_{\lambda_2=0}^{t_4} q_{j_1}q_{j_2}s_{\lambda_1}s_{\lambda_2},$$

where $q_{j_1} = (1-\rho^2)^{\frac{m}{2}}\frac{\Gamma\left(\frac{m}{2}+j_1\right)\rho^{2j_1}}{j_1!\Gamma\left(\frac{m}{2}\right)}$, $j_1 = 1,\ldots,m$ and $q_{j_2} = (1-\rho^2)^{\frac{n}{2}}\frac{\Gamma\left(\frac{n}{2}+j_2\right)\rho^{2j_2}}{j_2!\Gamma\left(\frac{n}{2}\right)}$, $j_2 = 1,\ldots,n$.

4. Computation of cumulative distribution and critical points

In each of the two Cases, the cumulative distribution involves multiple infinite sums. The cumulative distribution is only meaningful if the multiple infinite sums can be replaced by finite sums, and the number of terms needed for a good approximation is not too large. In this paper, the selection of t_1 and t_2 depends on the machine and software precision.

Figs. 1–3 display the density function and the cumulative distribution function of the BNC variables for selected values of the noncentrality parameters, θ_1 and θ_2, correlation coefficient, ρ and degrees of freedom, m for Case I. These 3-dimensional graphs are produced using a code called `wireframe` in *R*. It is found that when the correlation between Y_1 and Y_2 is stronger (i.e. ρ gets larger), the volume under the density curve is getting larger for any two fixed intervals $(0,a_1)$ and $(0,a_2)$, where a_1 is a point on Y_1 axis while a_2 is a point on Y_2 axis. Figs. 1A, C and E, 2G, I and K are for probability density function with $\rho = 0.25$ while Figs. 1B, D and F, 2H, J and L are the density curves with $\rho = 0.75$. We observe that for a larger value of ρ, the density curves become sharper with location parameters (median, mean and mode) shifting toward the origin.

The increase in the mean, median and mode is associated with the increase in the values of the degrees of freedom. As degrees of freedom, m increases, the volume under the density curve for intervals $(0,a_1)$ and $(0,a_2)$ decreases or the curves become flatter with median, mean and mode shifting away from the origin. See Figs. 1C, E and, 2G or 1D, F and 2H.

As the noncentrality parameters tend to 0, the distribution, of course, tends to the central chi-square. We obtain smaller volume under the density curve for intervals $(0,a_1)$ and $(0,a_2)$ when we choose larger values of noncentrality parameters (see Figs. 1A, C and 2I or 1B, D and 2J). The density curves become flatter and the location parameters tend to shift away from the origin as the values of the noncentrality parameters increase.

Fig. 3 shows the cumulative distribution function of the BNC variables for selected values of θ_1, θ_2, m and ρ. The value of the cumulative distribution function becomes 1 quicker for smaller values of the noncentrality parameters, smaller number of degrees of freedom, and larger values of the correlation coefficient.

Let c_1 and c_2 be the critical points for respective marginal distributions of Y_1 and Y_2 with fixed values of θ_i, $i = 1, 2$, ρ and m. Additionally, let c_1 and c_2 to be identical and equal to c, such that

$$H(c,c) = \int_0^c \int_0^c h(y_1,y_2)dy_1dy_2 = 1-\alpha, \tag{4.15}$$

where α is the level of significance, $h(y_1,y_2)$ and $H(y_1,y_2)$ are the pdf and cdf of the BNC variables.

Fig. 4 shows that the upper 5th percentile for selected values of noncentrality parameters, correlation coefficient and degrees of freedom. The upper 5th percentile is the value of c in Eq. (4.15) with $\alpha = 0.05$. To compute the value of c, the interval of $(y_{[1]},y_{[2]})$ that contains c, that is, that satisfies $H(y_{[1]},y_{[2]}) - (1-\alpha) < 0$ and $H(y_{[1]},y_{[2]}) - (1-\alpha) > 0$ is obtained first. Then, a `uniroot` code in R is used to refine the search for this critical point, c. Fig. 4 shows that the upper 5th percentile increases as the degrees of freedom increases, m (see Fig. 4E and F), the noncentrality parameter, $\theta = \theta_1 = \theta_2$ increases (see Fig. 4C and D) and correlation coefficient, ρ decreases (see Fig. 4A and B).

Table 1 presents a single critical point for Case I for selected values of the noncentrality parameters, θ_i, $i = 1, 2$, correlation coefficient, ρ and degrees of freedom m. The upper 5th percentile for Case II is presented in Table 2. Here, again the presented values of the upper 5th percentile are limited to $c = c_1 = c_2$ and $\theta = \theta_1 = \theta_2$. Other schemes are possible, for example one may calculate c_2 at any fixed values of c_1, m, ρ, θ_1 and θ_2.

The upper percentile points for the bivariate central chi-square distribution are given by a number of authors in literature [3,5,21,22]. However, there is no tables for the critical points of the BNC distribution in the literature. Note, we cannot include all possible tables as there could be any selection of the parameters of ρ, m and θ_i, $i = 1, 2$. However, a straightforward technique to produce the critical points, for any selection of the parameters, is given in this paper.

The multiplication of probabilities for marginal Y_1 and Y_2 give exactly the same values as the probabilities calculated from the proposed BNC distribution when $\rho = 0$. Therefore, there is no need to tabulated them separately.

5. An application

Consider a multivariate simple regression model, $\boldsymbol{X}_i = \boldsymbol{a} + \boldsymbol{b}c_i + \boldsymbol{e}_i$, $i = 1,\ldots,n$, where $\boldsymbol{X}_i$ is the p dimensional response vector, c_i is a non-zero scalar value of the explanatory variable, $\boldsymbol{a}$ and $\boldsymbol{b}$ are unknown intercept and slope vectors, and $\boldsymbol{e}_i$ is the p dimensional vector of errors.

Yunus and Khan [11] consider statistical tests, (i) the unrestricted test (UT) for testing $H_0^{(1)} : \boldsymbol{a} = \boldsymbol{a}_0$ when $\boldsymbol{b}$ is unspecified, (ii) the restricted test (RT) for testing $H_0^{(2)} : \boldsymbol{a} = \boldsymbol{a}_0$ (when $\boldsymbol{b} = \boldsymbol{b}_0$), (iii) the pre-test (PT) for testing $H_0^{(3)} : \boldsymbol{b} = \boldsymbol{b}_0$, and (iv) the pre-test test (PTT), a choice between the UT and the RT. If the null hypothesis $H_0^{(3)}$ is rejected in the pre-test (PT), then the UT is used to test $H_0^{(1)}$, otherwise the RT is used.

Under a sequence of local alternative hypothesis, $\{K_n\}$, where $K_n : (\boldsymbol{a},\boldsymbol{b}) = \left(\boldsymbol{a}_0 + n^{-\frac{1}{2}}\boldsymbol{\varrho}_1, \boldsymbol{b}_0 + n^{-\frac{1}{2}}\boldsymbol{\varrho}_2\right)$, $\boldsymbol{\varrho}_1 \geqslant \boldsymbol{0}$, $\boldsymbol{\varrho}_2 \geqslant \boldsymbol{0}$, the asymptotic power function for the PTT is given by

$$\Pi^{PTT}(\boldsymbol{\varrho}_1,\boldsymbol{\varrho}_2) = G_p\left(\chi^2_{p,\alpha};\Delta^{PT}\right)\left\{1 - G_p\left(\chi^2_{p,\alpha};\Delta^{RT}\right)\right\} + \lim_{n\to\infty} P\left(L_n^{PT} > \tau_{n,\alpha}^{PT}, L_n^{UT} > \tau_{n,\alpha}^{UT} | K_n\right), \tag{5.16}$$

where $G_p(\chi^2_{p,\alpha};\theta)$ is the cdf of a noncentral chi-square distribution with p degrees of freedom and noncentrality parameter Δ, $\chi^2_{p,\alpha}$ is the upper $100\alpha\%$ critical value of a central chi-square distribution, and $\tau_{n,\alpha}^{UT}$ and $\tau_{n,\alpha}^{PT}$ are the critical values with level of significance α for the UT and the PT.

Here, the test statistics for the UT and PT, (L_n^{UT}, L_n^{PT}) are distributed as correlated BNC with p degrees of freedom and noncentrality parameters, $\Delta^{UT} = (\boldsymbol{\varrho}_1'\boldsymbol{T}^{-1}\boldsymbol{\varrho}_1)(C^{\star 2}/(C^{\star 2} + \bar{c}^2))$, $\Delta^{RT} = (\boldsymbol{\varrho}_1 + \bar{c}\boldsymbol{\varrho}_2)'\boldsymbol{T}^{-1}(\boldsymbol{\varrho}_1 + \bar{c}\boldsymbol{\varrho}_2)$, $\Delta^{PT} = (\boldsymbol{\varrho}_2'\boldsymbol{T}^{-1}\boldsymbol{\varrho}_2)C^{\star 2}$, correlation coefficient, $\rho = -\bar{c}/\sqrt{C^{\star 2} + \bar{c}^2}$, where $C^{\star 2} = \lim_{n\to\infty} n^{-1}\sum_{i=1}^n c_i^2 - n\bar{c}_n^2$, $\bar{c} = \lim_{n\to\infty} n^{-1}\sum_{i=1}^n c_i$ and $\boldsymbol{T}$ is estimated using $\hat{\boldsymbol{T}} = \left(\left(\frac{\hat{\zeta}_{jk}}{\hat{\eta}_j\hat{\eta}_k}\right)\right)_{p\times p}$, $\hat{\eta}_j = \frac{1}{n}\sum_{i=1}^n \psi'(e_{ij})$, $\hat{\zeta}_{jk} = \frac{1}{n}\sum_{i=1}^n \psi(e_{ij})\psi(e_{ik})$, $j,k = 1,\ldots,p$, with $\psi(\cdot)$ is the score function in the M-estimation methodology.

Let $p = 2$, $\psi(u) = u$ for any $u \in \Re$ and the regressor values, c_i, $i = 1,2,\ldots,n$ are 0 and 1 with 50% for each. The error term, $\boldsymbol{e}_i$, $i = 1,2,\ldots,n$ of size $n = 60$ are generated randomly from the standard normal distribution. In Fig. 5, the power of the PTT is plotted with respect to v_1 for two selected values of v_2. Here, v_2 is a function of the difference between the true slope and its suspected value while v_1 is a function of the difference between the true intercept and its suspected value. We find that the power of the PTT when $\alpha = 0.01$ is lower than that of when $\alpha = 0.05$ for smaller values of v_1. However, for larger values of v_1, the power of the PTT when $\alpha = 0.01$ is higher than that of when $\alpha = 0.05$.

The complicated form of the power function of the PTT causes a difficulty to explain the behaviour of the power of the PTT theoretically. So, the computation or graphical representation of the power function of the PTT is more useful to study the properties of the PTT. Several versions of the bivariate central and BNC distributions found in the literature [6,23,24] are computationally expensive and complicated in form, because Laguerre polynomials and convolution summations are

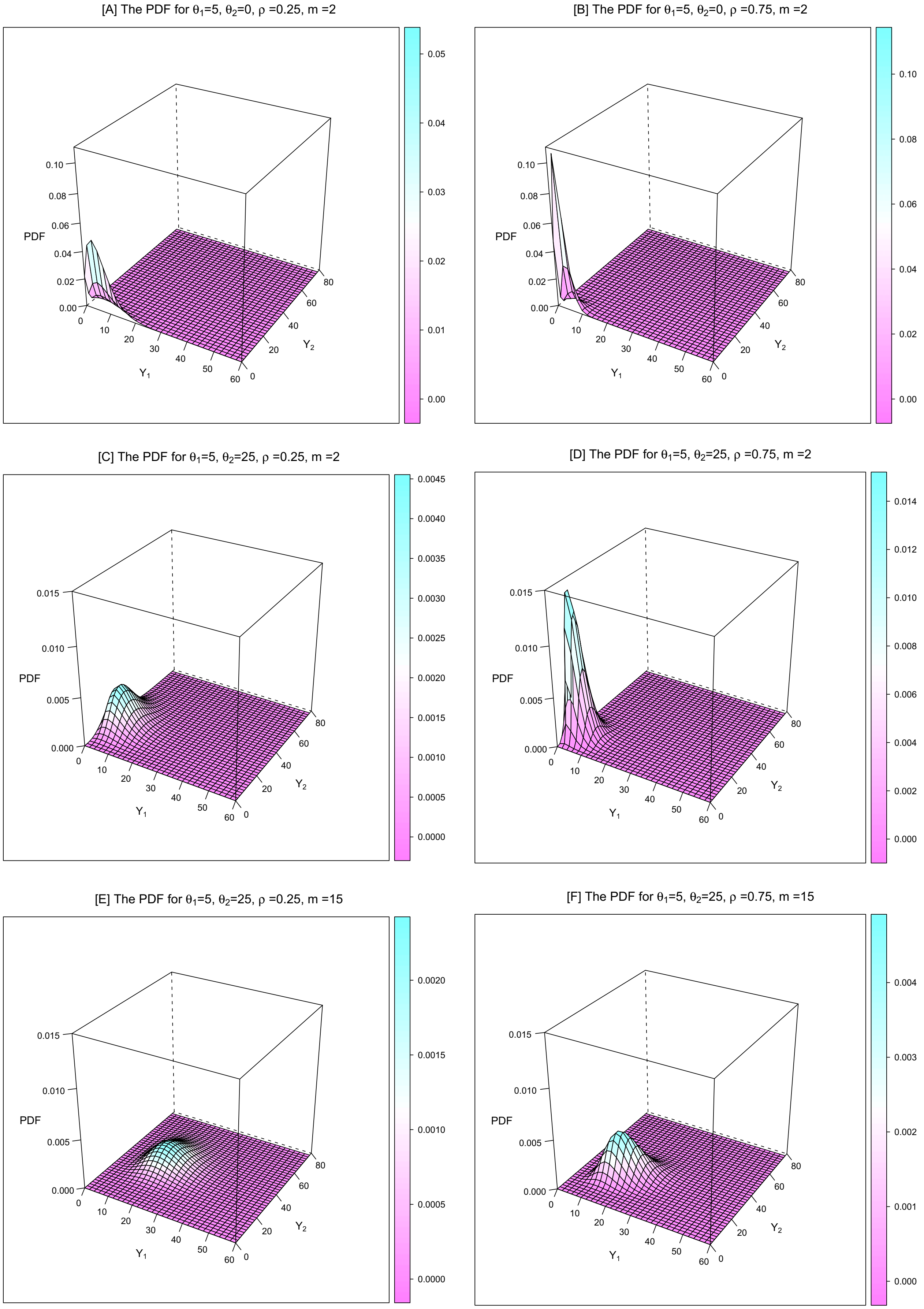


Fig. 1. Graphs of density function of the BNC distribution for selected values of θ_1, θ_2, m and ρ.

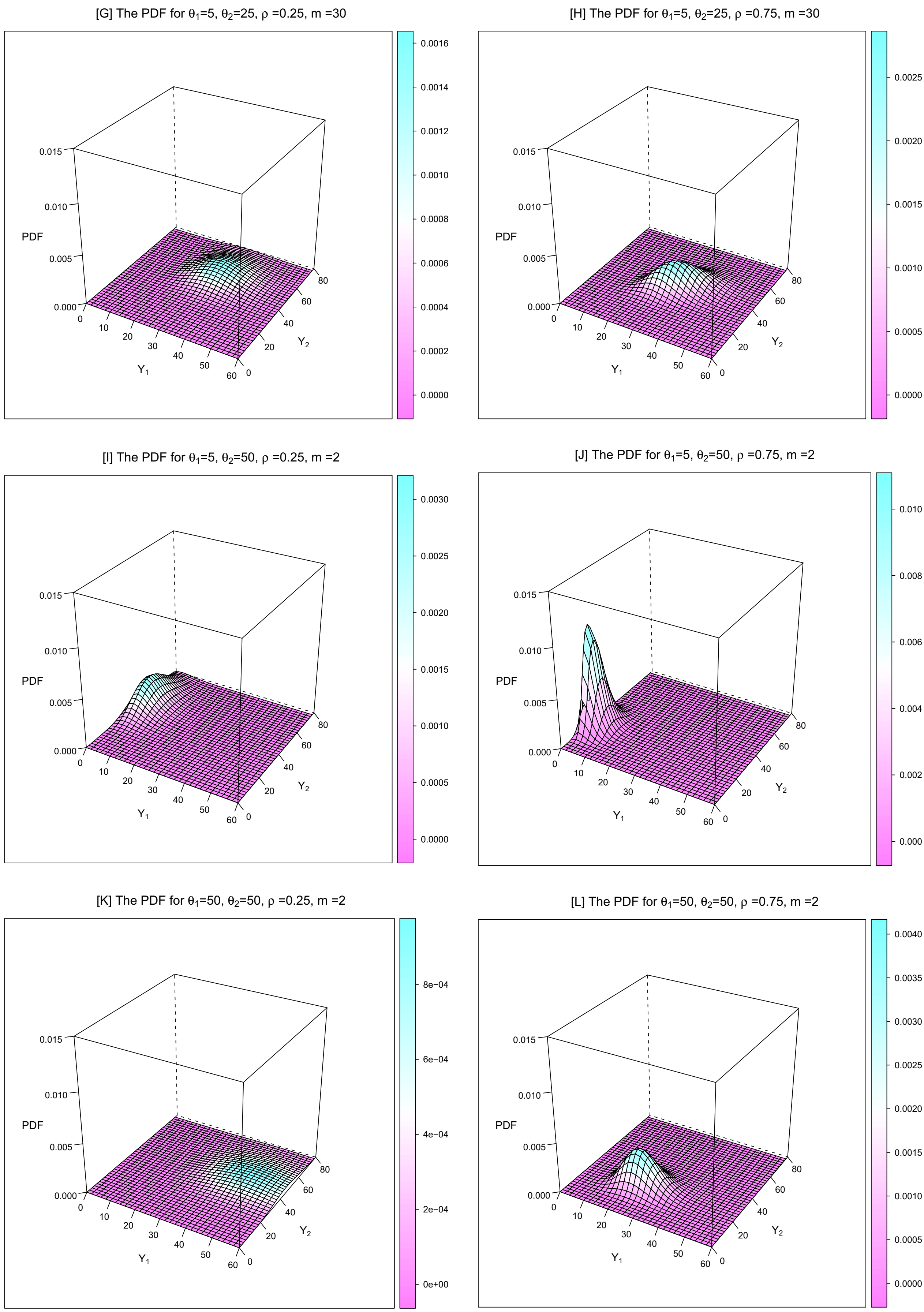


Fig. 2. Graphs of density function of the BNC distribution for selected values of θ_1, θ_2, m and ρ.

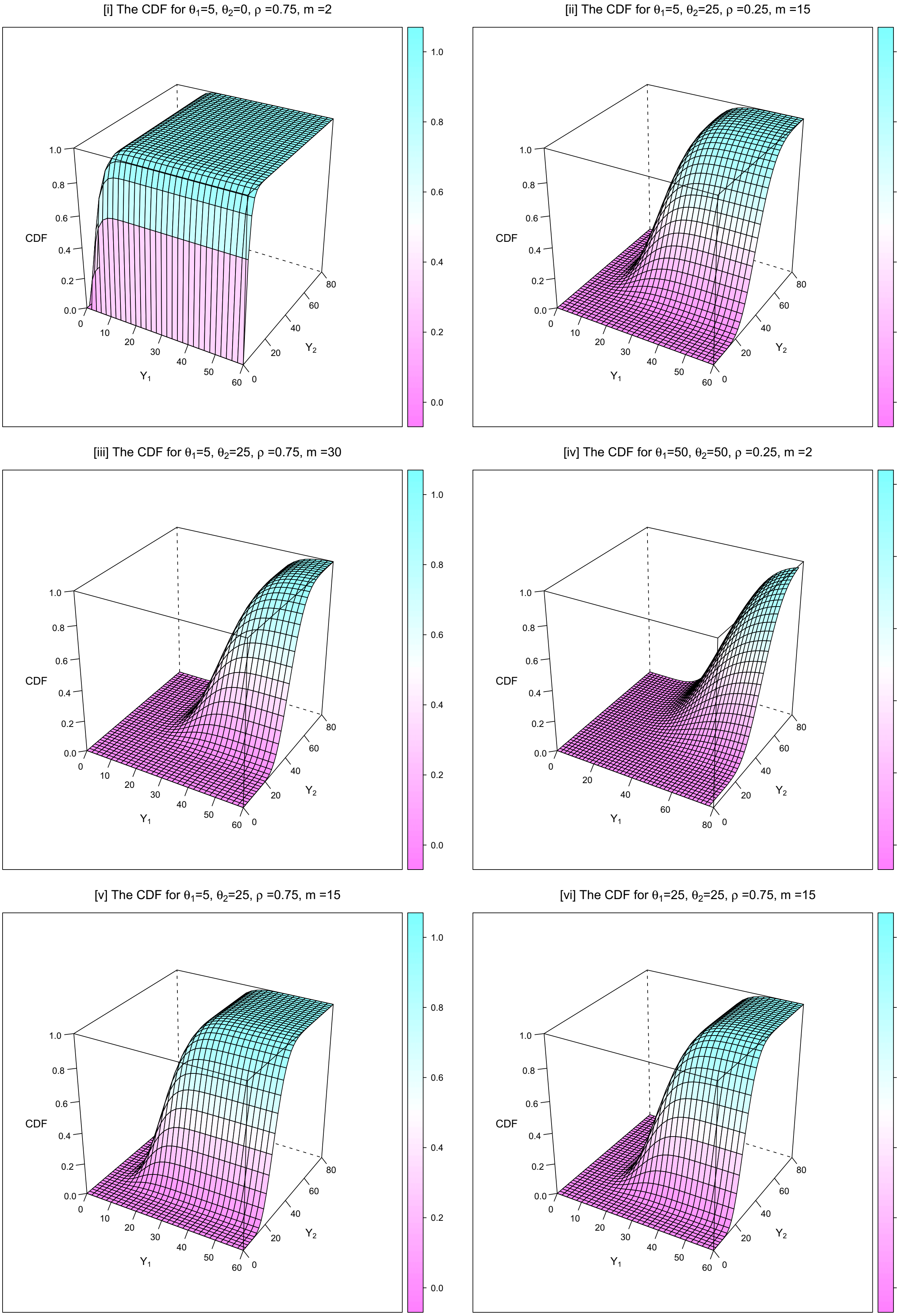


Fig. 3. Graphs of cumulative distribution of the BNC variables at selected values of θ_1, θ_2, m and ρ.

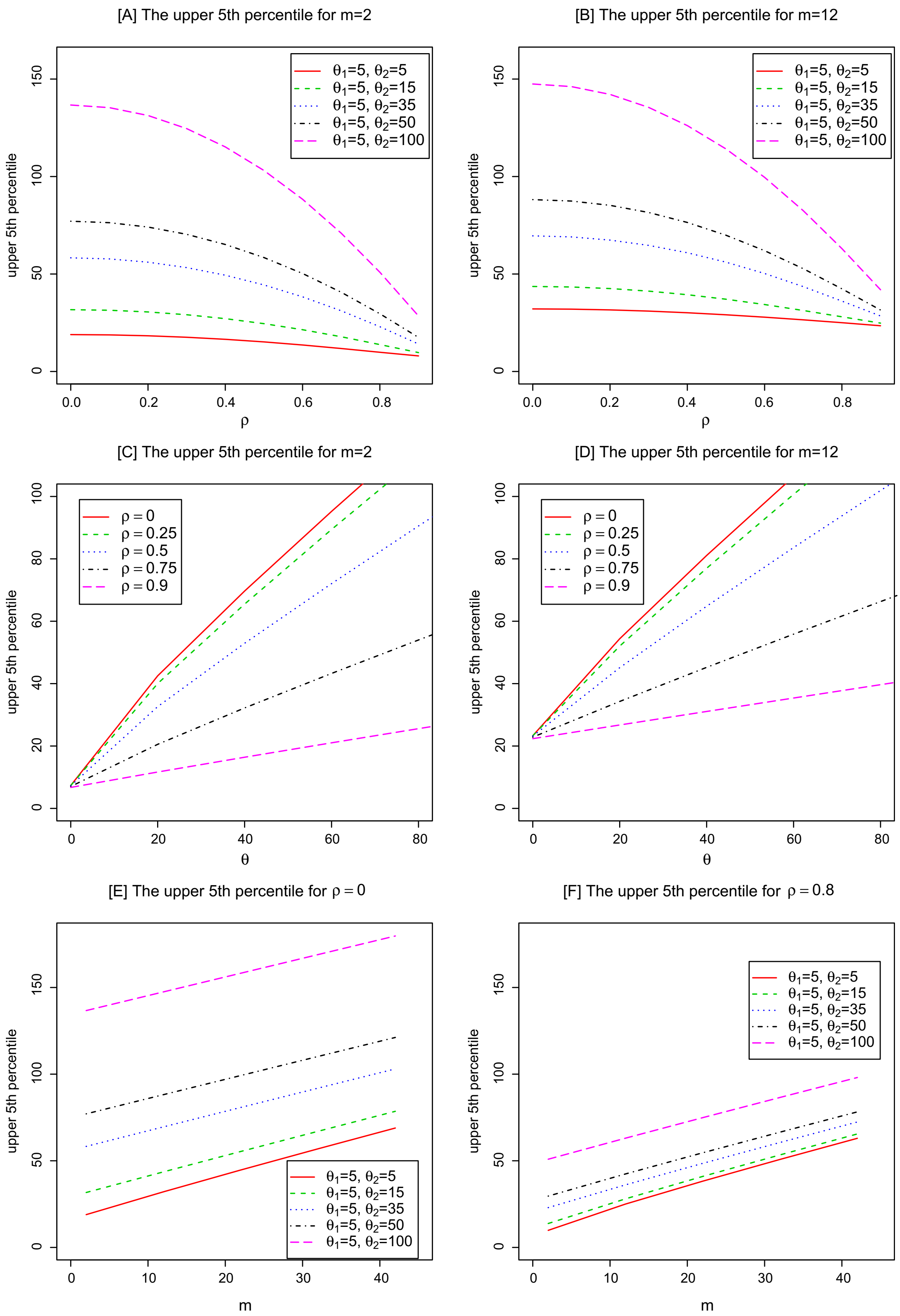
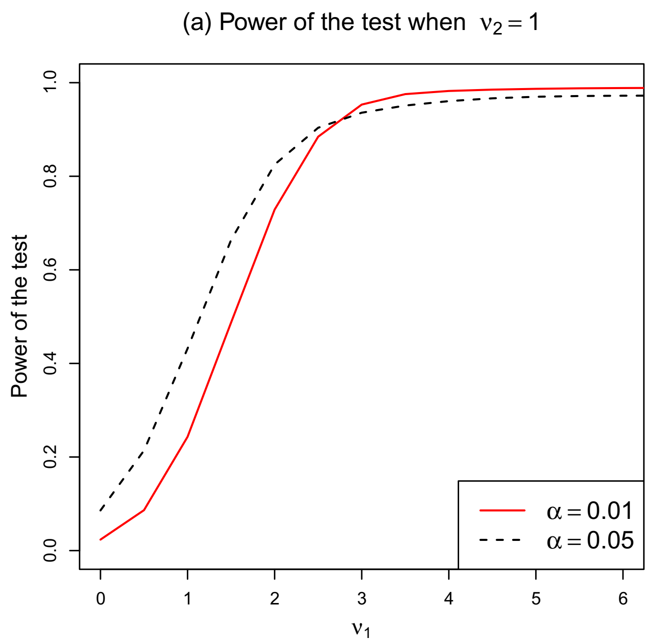


Fig. 4. Graphs of upper 5th percentile of the BNC distribution at selected values of θ_1, θ_2, m and ρ where $\theta = \theta_1 = \theta_2$.

required in the computation of the distribution (cf [3, p. 19]). The proposed from of the BNC distribution is simple in form and quite easy to compute, compared to its existing forms.



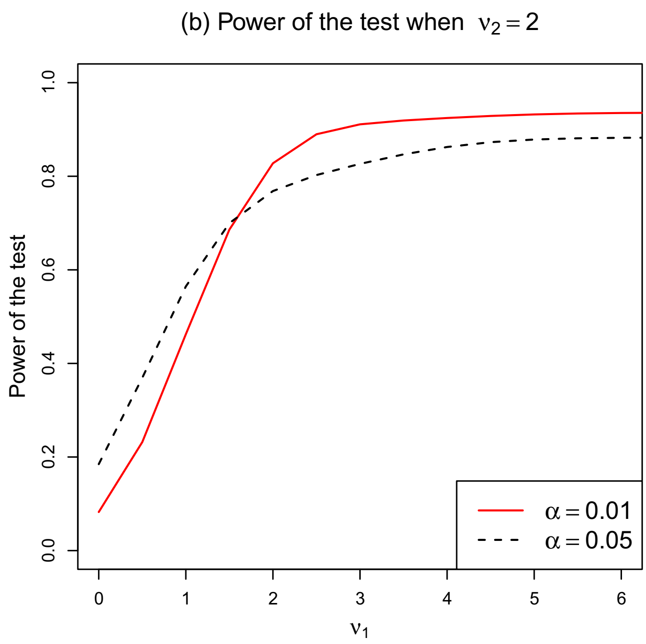


Fig. 5. Graphs of power of the test.

Table 1
Upper 5th percentile of the BNC distribution at selected values of ρ, θ_i and m for Case I.

m	θ_i, $i = 1, 2$, $\theta_1 = \theta_2$				θ_i, $i = 1, 2$, $\theta_1 = \theta_2$			
	0	10	20	30	0	10	20	30
	$\rho = 0.00$				$\rho = 0.25$			
2	7.3523	27.4524	42.5001	56.3953	7.3291	25.9384	40.0237	53.0406
10	20.4442	37.4697	52.0235	65.6748	20.4198	36.0840	49.6497	62.4074
30	46.9225	61.6925	75.4362	88.6263	46.8926	60.4810	73.2386	85.5250
	$\rho = 0.50$				$\rho = 0.75$			
2	7.2434	21.4416	32.6279	43.0036	7.0273	14.2392	20.5400	26.4786
10	20.3171	32.0546	42.6472	52.7123	20.0161	25.9350	31.6423	37.2093
30	46.7605	56.9924	66.8344	76.4196	46.3447	51.6782	56.9513	62.1752

Table 2
Upper 5th percentile of the BNC distribution at selected values of ρ, θ_i, m and n for Case II.

m	n	θ_i, $i = 1, 2$, $\theta_1 = \theta_2$				θ_i, $i = 1, 2$, $\theta_1 = \theta_2$			
		0	10	20	30	0	10	20	30
		$\rho = 0.00$				$\rho = 0.25$			
2	4	9.8491	28.8703	43.8062	57.6490	9.8491	27.3909	41.3549	54.3148
2	6	12.6847	30.5623	45.3197	59.0765	12.6847	29.1328	42.9067	55.7743
2	8	15.5316	32.4790	47.0189	60.6647	15.5316	31.1071	44.6542	57.4042
		$\rho = 0.50$				$\rho = 0.75$			
2	4	9.8491	23.0353	34.0625	44.3625	9.8491	16.3101	22.3582	28.1613
2	6	12.6847	24.9660	35.7663	45.9507	12.6847	18.7174	24.5247	30.1740
2	8	15.5316	27.7447	37.6975	47.7436	15.5316	21.3097	26.9294	32.4382

6. Concluding remarks

The paper provides the density and distribution functions of the BNC distribution. The representation of the density function in this paper is handy for computation of the upper tail area and the critical points of the BNC distribution. The distribution has been displayed for some arbitrary values of the parameters. A technique to compute the critical values is included, and some sample tables are provided. The computational method of this paper can be used to compute the power function of tests stated earlier.

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Appendix A

Figs. 1–5, Tables 1 and 2.

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