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Novel Results for the Multivariate Ricean Distribution With Non-Identical Parameters

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***Abstract*—This paper studies the multivariate Ricean distribution, which is the most important distribution for modeling the magnitude of the received signal when line-of-sight propagation conditions exist. In this paper, assuming different mean values for the Ricean distributed random variables, generic expressions for the multivariate Ricean probability density function, cumulative distribution function, and the joint moments are derived in terms of integral representations. As an application example, these results have been used to analyze the performance of a multi-branch selection diversity receiver, operating in spatially correlated Ricean fading environment. Moreover, an asymptotic closed-form expression has been also provided, which approximates quite well to the exact one, and has been used to study various performance metrics. Finally, a complexity analysis has been performed and simulation results have been also presented.**

***Index Terms*—Bit error rate, multivariate statistics, probability density function, outage probability, Ricean distribution.**

I. Introduction

Multipath fading is responsible for the instantaneous random fluctuations of the received signal-to-noise ratio (SNR). Widely adopted distributions that have been used for modeling the stochastic behavior of the received SNR with well-accepted physical justification, are the Rayleigh and Rice [1]. The first one appears in situations where isotropic scattering occurs due to the non-line-of-sight (NLoS) propagation conditions, while the second one appears in scenarios where an additional specular component exists as a result of the LoS propagation [1]. In practical situations, the received signals tend to be spatially or temporally correlated due to small separations in distance and/or in time. This is why multivariate distributions have attracted the interest in wireless communication studies, e.g., [2]–[5].

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The multivariate Ricean distribution has been investigated many times in the past [6]–[11]. In [6], infinite series expressions for important statistical characteristics of this distribution were provided, including the PDF and the cumulative distribution function (CDF), which are used in [7] to study the performance of various receivers. In [8], integral representations for the bivariate Ricean PDF and CDF were presented, assuming different variances and correlations among the random variables (RVs). In that paper, it was also proved that the integral representation for the CDF requires significantly less computational time as compared to the infinite series derived in [6]. The results derived in [8] were expanded in [9] in the context of multivariate statistics, while some related corrections were given in [11]. In all these studies, equal mean values for the Ricean RVs were assumed. Such an approach has clear benefits regarding the simplification of the developed analytical framework. However, in practical scenarios, it is rather unrealistic to assume the same means for all signals.

In this paper, the previous results are generalized to the scenario where all the parameters of the correlated Ricean RVs are different, including the mean values. More specifically, the contributions of the paper can be summarized as follows:

- Assuming generalized correlation model, generic expressions for the multivariate Ricean PDF and the CDF are derived in terms of integral representations;
- The joint moments are also provided for the first time in scenarios where different parameters are considered among the correlated Ricean RVs;
- The results are used to analyze the performance of selection-diversity (SD) receiver;
- New asymptotic closed-form expressions for the outage probability (OP) and the bit error rate (BER) are obtained;
- The computational complexity of the derived expressions has been also investigated.

II. Correlated Ricean Random Variables

We write $N\left(\mu,\sigma^2\right)$ to denote Gaussian distributed RVs with mean μ and variance σ^2. Moreover, $X_k, Y_k \sim N\left(0,\frac{1}{2}\right)$, $X_{0,k} \sim N\left(m_{1,k},\frac{1}{2}\right)$, $Y_{0,k} \sim N\left(m_{2,k},\frac{1}{2}\right)$, with $k \in \{1,\ldots,K\}$. In this context, assuming independence among $X_{0,k}, X_k, Y_{0,k}, Y_k$, the following generic complex RV is defined

$$\begin{aligned} V_k &= \sigma_k\left(\sqrt{1-\lambda_k^2}X_k + \lambda_k X_{0,k}\right) \\ &\quad + i\sigma_k\left(\sqrt{1-\lambda_k^2}Y_k + \lambda_k Y_{0,k}\right) \\ &= \sigma_k\left[\sqrt{1-\lambda_k^2}X_k + \lambda_k\left(m_{1,k}+\tilde{X}_0\right)\right] \\ &\quad + i\sigma_k\left[\sqrt{1-\lambda_k^2}Y_k + \lambda_k\left(m_{2,k}+\tilde{Y}_0\right)\right]. \end{aligned} \tag{1}$$

In (1), $i = \sqrt{-1}$, $\tilde{X}_0, \tilde{Y}_0 \sim N(0, 1/2)$, and $0 < |\lambda_k| < 1$, with $\lambda_k \neq 0$ taking arbitrary values and being related to the correlation among V_ks. For any $p, q \in \{1, \ldots, K\}$, $\mathbb{E}\langle X_p Y_q \rangle = 0$, $\mathbb{E}\langle X_p X_q \rangle = \mathbb{E}\langle Y_p Y_q \rangle = (1/2)\delta_{p,q}$, with $\delta_{p,q}$ denoting the Kronecker delta symbol [12], and $\mathbb{E}\langle\cdot\rangle$ denoting expectation. The same identities also hold for $\tilde{X}_0, \tilde{Y}_0$. Since V_k are non-zero mean complex Gaussian RVs, i.e., $V_k \sim N_c(\sigma_k \lambda_k (m_{1,k} + i m_{2,k}), \sigma_k^2)$[1], then the magnitude of V_k, i.e., $R_k = |V_k|$, is a Ricean distributed RV. It can be proved that the mean square value of R_k is given by $\mathbb{E}\langle R_k^2 \rangle = \sigma_k^2 (1 + \lambda_k^2 (m_{1,k}^2 + m_{2,k}^2))$, while the Ricean factor is given by $K_k = \lambda_k^2 (m_{1,k}^2 + m_{2,k}^2)$. It is noted that in contrast to similar studies, e.g., [9], [14], [15], in this paper, for the first time different mean values $m_{1,k}$ and $m_{2,k}$ have been considered for each RV, i.e., $m_{1,1} \neq m_{1,2} \neq \cdots \neq m_{1,K}$ and $m_{2,1} \neq m_{2,2} \neq \cdots \neq m_{2,K}$. The correlation coefficient between V_k and V_j, with $j \in \{1, \ldots, K\}$ and $k \neq j$, is given by

$$\rho_{k,j} = \frac{\mathbb{E}\left\langle V_k V_j^* \right\rangle - \mathbb{E}\left\langle V_k \right\rangle \mathbb{E}\left\langle V_j^* \right\rangle}{\sqrt{\mathrm{Var}(V_k)\,\mathrm{Var}(V_j)}} = \lambda_k \lambda_j, \tag{2}$$

where V_j^* denotes the complex conjugate of V_j. The correlation model assumed in this paper is a generalized one, which for $\lambda_k = \lambda_j$ it simplifies to the equal correlation model, assumed in previous studies, e.g., in [14]. The marginal PDF of R_k, conditioned on $\tilde{X}_0$ and $\tilde{Y}_0$, is given by [1, eq. (2.45)]

$$f_{R_k|\tilde{X}_0, \tilde{Y}_0}(r_k | x_0, y_0) = \frac{r_k}{b_k} \exp\left(-\frac{r_k^2 + s_k^2}{2b_k}\right) I_0\left(\frac{r_k s_k}{b_k}\right). \tag{3}$$

In (3), $s_k = \sigma_k \lambda_k [(m_{1,k} + x_0)^2 + (m_{2,k} + y_0)^2]^{1/2}$, $b_k = \sigma_k^2 \frac{1-\lambda_k^2}{2}$, and $I_0(\cdot)$ denotes the zero-th order modified Bessel function of the first kind [12, eq. (8.406/1)]. The corresponding marginal CDF is given by

$$F_{R_k|\tilde{X}_0, \tilde{Y}_0}(r_k | x_0, y_0) = 1 - Q_1\left[\frac{s_k}{\sqrt{b_k}}, \frac{r_k}{\sqrt{b_k}}\right], \tag{4}$$

where $Q_1(\cdot, \cdot)$ denotes the first order Marcum-Q function.

III. Multivariate Ricean Statistics

Let $\mathbf{R} = \{R_1, R_2, \ldots, R_K\}$ denoting a set of generalized correlated and non-identically distributed Ricean RVs. For different mean values of $R_1, R_2, \ldots, R_K$, the multivariate Ricean PDF is given by

$$f_{\mathbf{R}}(r_1, r_2, \ldots, r_K) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-x_0^2 - y_0^2\right) \times \prod_{k=1}^{K} \frac{r_k}{b_k} \exp\left(-\frac{r_k^2 + s_k^2}{2b_k}\right) I_0\left(\frac{r_k s_k}{b_k}\right) dx_0 dy_0. \tag{5}$$

The proof for the derivation of (5) is provided in Appendix A. By assuming equal means, i.e., $m_{1,k} = m_1$ and $m_{2,k} = m_2$, making a change of variables of the form $x_0 + m_1 = \tilde{x}_0$ and $y_0 + m_2 = \tilde{y}_0$, and following the approach provided in [8], (5) simplifies to [9, eq. (18)]. Moreover, under the same assumptions and for $K = 2$, it further simplifies to [8, eq. (12)]. It is noteworthy that an efficient and simple approach to numerically evaluate the integrals in (5) is the application of the Gauss-Hermite integration approximation [16]. Following this approach, two sums of products will be obtained of the form $f_{\mathbf{R}}(r_1, r_2, \ldots, r_K) = \sum_{i_1, i_2 = 0}^{n} w_{i_1} w_{i_2} \prod_{k=1}^{K} A_k \exp(B_k(x_{i_1}, x_{i_2})) I_0(C_k(x_{i_1}, x_{i_2}))$, where $A_k \in \mathbb{R}$, $B_k(x, y), C_k(x, y)$ are functions of x, y, whose accuracy improve as the number of sample points n increases. Therefore, the evaluation complexity of the derived PDF expression, in terms of arithmetic operations, i.e., additions, multiplications, is not so important. The same observation holds for any K-dimensional distribution, as well as for all expressions that have been derived and require numerical integration. In order to examine the applicability of the derived expressions, in Fig. 1, assuming $K = 2$, a three-dimensional (3D) plot of the bivariate PDF given in (5) is depicted. To obtain this figure, numerical integration of (5) has been performed using the techniques that are available in the mathematical software package Mathematica. For obtaining this figure, the following assumptions have been made: $m_{1,1} = 2$, $m_{1,2} = 3$, $m_{2,1} = 2.5$, $m_{2,2} = 3.5$, $\sigma_1 = 4$, $\sigma_2 = 5$, $\lambda_1 = \lambda_2 = 0.92$. It is shown that smaller values for the mean and variance result to narrower range for the values of the bivariate PDF. Moreover, it should be noted that the required computational time for obtaining the results depicted in Fig. 1 (using the two fold integration) is quite small. It is also interesting to note that using the Gauss-Hermite integration approach, only a small number of sample points are required, i.e., <40, in order to achieve a satisfactory accuracy. The corresponding multivariate CDF expression of $\mathbf{R}$ is given by

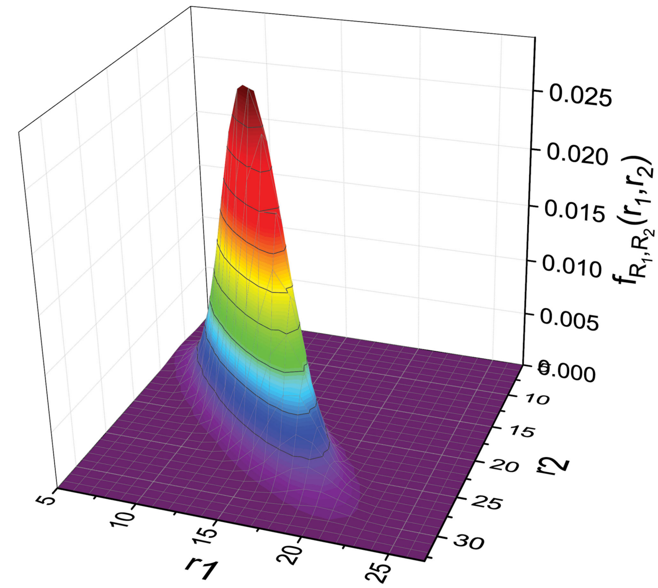

Fig. 1. 3-D plot of the bivariate Ricean PDF.

$$F_{\mathbf{R}}(r_1, r_2, \ldots, r_K) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-x_0^2 - y_0^2\right) \times \prod_{k=1}^{K} \left[1 - Q_1\left(\frac{s_k}{\sqrt{b_k}}, \frac{r_k}{\sqrt{b_k}}\right)\right] dx_0 dy_0. \tag{6}$$

The proof for the derivation of (6) is also given in Appendix A. For equal mean values, (6) simplifies to [9, eq. (19)] and [11, eq. (8)], while for $K = 2$, it further simplifies to [8, eq. (16)]. The joint moments of

[1]The variance of a complex RV V_k can be evaluated using (1) as $\mathrm{Var}(V_k) = \mathbb{E}\langle |V_k|^2 \rangle - |\mathbb{E}\langle V_k \rangle|^2$ as shown in [13, p. 249].

$\mathbf{R}$ are defined as $\mu_{\mathbf{R}}(n_1,\ldots,n_K) \triangleq \mathbb{E}\langle R_1^{n_1}\cdots R_K^{n_k}\rangle$ [13, eq. (7.18)]. Substituting (5) in this definition, using [17, eq. (2.15.5/4)], and after a mathematical procedure, the following expression for the joint moments is deduced

$$\mu_{\mathbf{R}}(n_1,n_2,\ldots,n_K) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-x_0^2-y_0^2\right) \times \prod_{k=1}^{K}(2b_k)^{n_k/2}\exp\left(-\frac{s_k^2}{2b_k}\right)\Gamma(1+n_k/2) \times {}_1F_1\left(\frac{2+n_k}{2};1;\frac{s_k^2}{2b_k}\right)dx_0dy_0, \tag{7}$$

where $\Gamma(\cdot)$ denotes the Gamma function [12, eq. (8.310)] and ${}_1F_1(\cdot;\cdot;\cdot)$ represents the confluent hypergeometric function [12, eq. (9.210)].

IV. Application Example

In this section, analytical expressions for the OP and BER of multi-branch SD receiver are presented. Moreover, simplified asymptotic closed-form expressions for the same performance criteria are also obtained. Let us consider a K-branch SD receiver operating in a Ricean fading channel. The instantaneous received SNR at the kth input branch, γ_k, and the corresponding average SNR, $\overline{\gamma}_k$, are, respectively, given by

$$\gamma_k = R_k^2\frac{E_s}{N_0}, \overline{\gamma}_k = \mathbb{E}\langle R_k^2\rangle\frac{E_s}{N_0} = \mathbb{E}\langle R_k^2\rangle R, \tag{8}$$

where E_s is the average energy per symbol and is N_0 the noise power spectral density. Based on the mean square value of R_k derived previously, the average SNR can be also expressed as $\overline{\gamma}_k = \sigma_k^2(1+K_k)R$. The instantaneous output SNR of SD is given by $\gamma_{\text{out}} = \max\{\gamma_1,\ldots,\gamma_K\}$ with the CDF given by [13, eq. (6.54)]

$$F_{\gamma_{\text{out}}}(\gamma) = F_{\gamma_1,\gamma_2,\ldots,\gamma_K}(\gamma,\gamma,\ldots,\gamma). \tag{9}$$

Using a change of variables of the form (8) in (6) and (9) yields to

$$F_{\gamma_{\text{out}}}(\gamma) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-x_0^2-y_0^2\right) \times \prod_{k=1}^{K}\left[1-Q_1\left[\frac{s_k}{\sqrt{b_k}},\left(\frac{\gamma}{b_kR}\right)^{1/2}\right]\right]dx_0dy_0. \tag{10}$$

Assuming $m_{1,k}=m_1, m_{2,k}=m_2, \lambda_k=\lambda=\sqrt{\rho}, \sigma_k^2=1/2$, making a change of variables of the form $x_0+m_1=\tilde{x}_0$ and $y_0+m_2=\tilde{y}_0$, (10) simplifies to [14, eq. (21)], i.e., the CDF of the K-branch SD assuming equal correlation.

A. Asymptotic Analysis

An alternative, to the numerical integration, and quite efficient approach for obtaining numerically evaluated results is to employ asymptotic expressions and tight bounds. Such an approach is widely adopted in the open technical literature in scenarios where the performance of diversity schemes in correlated environments is required, e.g., [18]. To this aim, in Appendix B, it has been proved that for large values of $\overline{\gamma}_k$, a simplified asymptotic closed-form expression for the CDF of γ_{out} can been obtained as

$$F_{\gamma_{\text{out}}}(\gamma) \simeq \sum_{q_1=0}^{1}\cdots\sum_{q_K=0}^{1}(-1)^{\sum_{j=1}^{K}q_j}\left[\prod_{j=1}^{K}\exp\left(-\frac{q_j\gamma}{2b_jR}\right)\right]\mathcal{S}, \tag{11}$$

where

$$\mathcal{S} = \left[\prod_{k=1}^{K}\exp\left(-\frac{\sigma_k^2\lambda_k^2\left(m_{1,k}^2+m_{2,k}^2\right)}{2b_k}\right)\right] \times \left[\prod_{i=1}^{2}\exp\left(\left(\sum_{j=1}^{K}\frac{\sigma_j^2\lambda_j^2m_{j,i}}{b_j}\right)^2(4P)^{-1}\right)\right]P^{-1},$$

while $P = 1+\sum_{k=1}^{K}\frac{\sigma_k^2\lambda_k^2}{2b_k}$.

B. Performance Analysis

1) Outage Probability: The OP is defined as $P_{\text{out}} \triangleq F_{\gamma_{\text{out}}}(\gamma_{\text{th}})$, where γ_{th} is the outage threshold. Using (10) and (11) in this definition, exact and asymptotic results for the OP are obtained, respectively.

2) Average Bit Error Rate: By adopting the CDF-based approach [14] and using the corresponding expressions provided above, i.e., (10) and (11), the BER performance of SD can be readily evaluated for various modulation schemes. Next, some representative results for differential binary phase shift keying (DBPSK) and binary phase shift keying (BPSK), are provided.

a) Differential binary phase shift keying: Assuming DBPSK, the BER can be analytically evaluated as

$$P_b = \frac{1}{2}\int_0^{\infty}F_{\gamma_{\text{out}}}(\gamma)\exp(-\gamma)\,d\gamma. \tag{12}$$

The exact solution can be obtained by substituting (10) in (12), while an approximated one can be obtained by using (11) in (12). Following the latter approach and using [12, eq. (8.310/1)], yields the following closed-form approximation

$$P_b^{DBPSK} \simeq \sum_{q_1=0}^{1}\cdots\sum_{q_K=0}^{1}\frac{(-1)^{\sum_{j=1}^{K}q_j}}{1+\sum_{j=1}^{K}\frac{q_j}{2b_jR}}\frac{\mathcal{S}}{2}. \tag{13}$$

b) Binary phase shift keying: Assuming BPSK, the BER can be analytically evaluated as

$$P_b = \frac{1}{2\sqrt{\pi}}\int_0^{\infty}F_{\gamma_{\text{out}}}(\gamma)\gamma^{-1/2}\exp(-\gamma)\,d\gamma. \tag{14}$$

The exact solution can be obtained by substituting (10) in (14). An approximated one can be obtained by substituting (11) in (14), and using [12, eq. (8.310/1)], yielding to the following closed-form approximated expression

$$P_b^{BPSK} \simeq \sum_{q_1=0}^{1}\cdots\sum_{q_K=0}^{1}\frac{(-1)^{\sum_{j=1}^{K}q_j}}{\sqrt{\sum_{j=1}^{K}\frac{q_j}{2b_j^2R}+1}}\frac{\mathcal{S}}{2}. \tag{15}$$

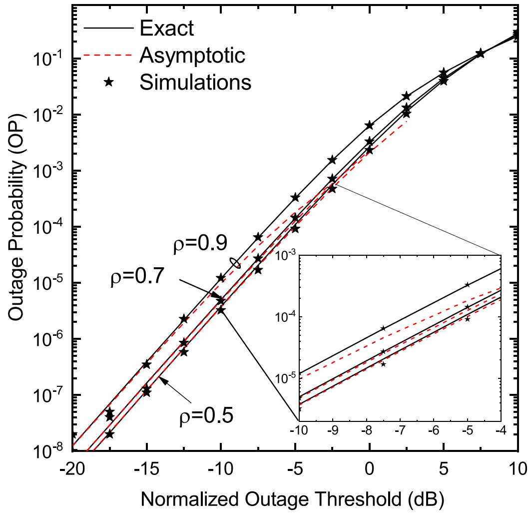

Fig. 2. OP vs $\gamma_{\rm th}/R$ for different values of ρ.

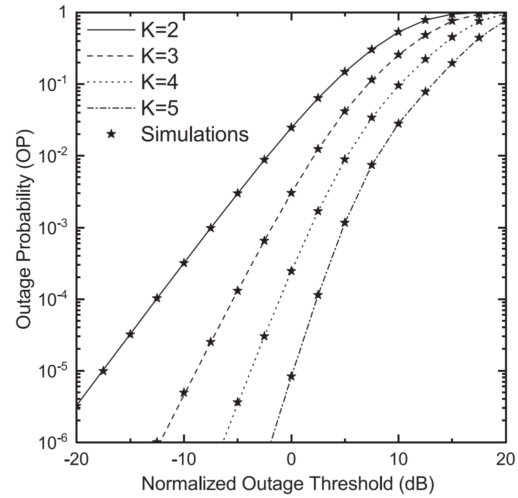

Fig. 3. OP vs $\gamma_{\rm th}/R$ for different values of K.

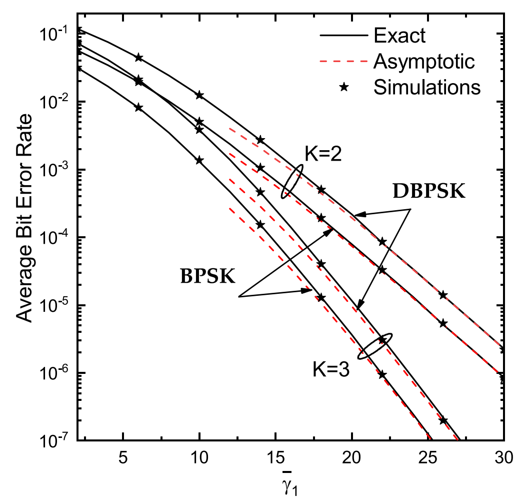

Fig. 4. Average BER of DBPSK and BPSK modulation schemes.

V. Numerical Results

In this section, representative numerical examples have been prepared using numerical integrations of the expressions derived in previous sections. It is noteworthy that the computational time for obtaining the results is not significant. Therefore, as it is already mentioned in [8], numerical integration solutions offer clear benefits including the simple implementation and the fast evaluation, as compared to the infinite series representations. Moreover, using the definition for the correlated Ricean RVs given in (1), various simulation results have been also included by producing 2×10^8 samples.

In Fig. 2, using (10) and assuming $K = 3$, the OP of a triple-branch SD receiver is plotted as a function of the normalized outage threshold $\gamma_{\rm th}/R$, for different values of the correlation coefficients $\rho = \rho_{k,j}$. To obtain this figure, the following parameters were considered: $m_{1,1} = 0.5$, $m_{1,2} = 0.6$, $m_{2,1} = 0.7$, $m_{2,2} = 0.8$, $m_{3,1} = 0.9$, $m_{3,2} = 1$, $\sigma_1 = 2$, $\sigma_2 = 2.5$, $\sigma_3 = 3$. It is shown that the performance improves, i.e., the OP decreases, as ρ decreases. Moreover, the rate of improvement is higher for increased values of ρ. In the same figure, the corresponding results based on the asymptotic expression given in (11) are also provided. It can be clearly observed the close approximation of the asymptotic results, especially for lower values of the $\gamma_{\rm th}/R$ and/or ρ.

In Fig. 3, the OP is plotted as a function of $\gamma_{\rm th}/R$ for different values of K. To obtain this figure the following parameters were considered: $m_{1,1} = 0.5$, $m_{1,2} = 0.6$, $m_{i,j} = m_{i-1,j} + 0.2$, with $i \in \{2, 3, 4\}$, $j = 1, 2$, $\sigma_k = 1.5 + 0.5k$ and $\lambda_k = 0.76 + 0.04k$, with $k \in \{1, \ldots, 5\}$. In all cases it is shown that the performance improves as K increases, with a decreased rate. It is interesting to note that based on the proposed approach, it is possible to obtain numerical results in a relatively simple computational manner for any values of the Ricean's parameters and K. In Fig. 4, assuming different modulation schemes and number of branches, the BER of SD is plotted as a function of $\overline{\gamma}_1$. To obtain this figure the following parameters have been considered: $m_{1,1} = 0.1$, $m_{1,2} = 0.2$, $m_{2,1} = 0.3$, $m_{2,2} = 0.4$, $m_{3,1} = 0.5$, $m_{3,2} = 0.6$, $\sigma_2 = \sigma_1 + 0.1$, $\sigma_3 = \sigma_1 + 0.2$. As it is shown in Fig. 4, the performance improves with an increase of K and with BPSK. What is interesting to note is the close agreement of the analytical results, obtained using numerical integration, with the simulated ones, while the asymptotic curves approximate quite well to the exact ones, especially for $\overline{\gamma}_1 > 15$.

VI. Conclusions

This paper presented novel results for the generalized multivariate Ricean distribution, in which the mean values of each RV is different to the corresponding ones of all the others. Easy-to-evaluate integral representations for the PDF, CDF, and the joint moments were provided. Based on them, the CDF of the output SNR of a multibranch SD receiver operating in correlated Ricean fading channels was investigated. Moreover, a new asymptotic closed-form expression for this CDF was also provided, which was used to provide tight bounds for the OP and the BER. The generic nature of the derived results can be exploited to analytical investigations where the mean values of the specular components are individual RVs.

Appendix A
Proof for Equations (5) and (6)

It can be observed from (1) that conditioned on $\tilde{X}_0$ and $\tilde{Y}_0$, $\{R_k\}$ are independent so that the joint conditional PDF can be written as [9, eq. (44)]. Using this expression and the total probability theorem [13, eq. (7.44)], results to the following unconditional joint PDF

$$f_{\mathbf{R}}\left(r_1, r_2, \ldots, r_K\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{k=1}^{K} f_{R_k\left|\tilde{X}_0,\tilde{Y}_0\right.}\left(r_k\left|x_0, y_0\right.\right) \times f_{\tilde{X}_0}(x_0) f_{\tilde{Y}_0}(y_0) dx_0 dy_0, \tag{A-1}$$

where $f_{R_k\left|\tilde{X}_0,\tilde{Y}_0\right.}\left(r_k\left|x_0, y_0\right.\right)$ is given in (3). By substituting the PDFs of $\tilde{X}_0, \tilde{Y}_0$, i.e., $f_{\tilde{X}_0}(x_0) = \frac{1}{\sqrt{\pi}}\exp\left(-x_0^2\right)$ and $f_{\tilde{Y}_0}(y_0) = \frac{1}{\sqrt{\pi}}\exp\left(-y_0^2\right)$, in (A-1), (5) is obtained. The corresponding CDF expression for $\mathbf{R}$ is given by

$$f_{\mathbf{R}}\left(r_1, r_2, \ldots, r_K\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{k=1}^{K} F_{R_k\left|\tilde{X}_0,\tilde{Y}_0\right.}\left(r_k\left|x_0, y_0\right.\right) \times f_{\tilde{X}_0}(x_0) f_{\tilde{Y}_0}(y_0) dx_0 dy_0, \tag{A-2}$$

where $F_{R_k\left|\tilde{X}_0,\tilde{Y}_0\right.}\left(r_k\left|x_0, y_0\right.\right)$ is given in (4). Using the PDFs of $\tilde{X}_0, \tilde{Y}_0$ in (A-2), yields (6) and also completes this proof.

Appendix B
Proof for Equation (11)

Starting with (10), by employing an infinite series representation for the Marcum Q-function, i.e., [19, eq. (4)], the following expression is obtained

$$F_{\gamma_{\rm out}}\left(\gamma\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp\left(-x_0^2 - y_0^2\right)}{\pi}\prod_{k=1}^{K}\left[\exp\left(-\frac{s_k^2}{2b_k} - \frac{\gamma/R}{2b_k}\right) \times \sum_{i_k=1}^{\infty}\left(\frac{\gamma/R}{s_k^2}\right)^{i_k/2} I_{i_k}\left(\left(\frac{s_k^2\gamma}{b_k^2 R}\right)^{1/2}\right)\right] dx_0 dy_0. \tag{B-1}$$

Moreover, assuming higher values for $\overline{\gamma}_k$ and using the asymptotic expansion $I_{i_k}\left(\left(\frac{s_k^2\gamma}{b_k^2 R}\right)^{1/2}\right) \simeq \frac{1}{\Gamma(i_k+1)}\left(\frac{s_k^2\gamma}{4b_k^2 R}\right)^{i_k/2}$ [16, eq. (9.6.7)], (B-1) simplifies to

$$F_{\gamma_{\rm out}}\left(\gamma\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp\left(-x_0^2 - y_0^2\right)}{\pi} \times \prod_{k=1}^{K}\left[\exp\left(-\frac{s_k^2}{2b_k} - \frac{\gamma/R}{2b_k}\right)\sum_{i_k=1}^{\infty}\left(\frac{\gamma/R}{s_k^2}\right)^{i_k/2} \times \frac{1}{\Gamma(i_k+1)}\left(\frac{s_k^2\gamma}{4b_k^2 R}\right)^{i_k/2}\right] dx_0 dy_0. \tag{B-2}$$

After some mathematical manipulations, the infinite series appearing in (B-2) can be expressed as $\sum_{j=0}^{\infty} x^j/j! = \exp(x)$, [12, eq. (1.211)]. Based on this approach, integrals of the following form appear $\mathcal{I} = \int_{-\infty}^{\infty}\exp\left(-Ax\right)\exp\left(-Bx^2\right)dx \overset{(1)}{=} \left(\frac{\pi}{A}\right)^{1/2}\exp\left(\frac{B^2}{4A}\right)$, with $A, B \in \mathbb{R}$. The solution to this integral is given in [12, eq. (3.462/2)] and after some mathematical manipulations yields (11), which completes this proof.

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