ELASTIC NOTES

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1 Anisotropic Derivation

1.1 Notation

Recall the deformation gradient

$$\mathbb{J} := \frac{\partial \boldsymbol{\Phi}(\boldsymbol{\xi}, t)}{\partial \boldsymbol{\xi}} \tag{1}$$

where $\boldsymbol{\Phi}$ is the mapping from reference to current configuration. The right Cauchy-Green deformation tensor is defined as

$$\boldsymbol{C} := \mathbb{J}^{\mathsf{T}}\mathbb{J}. \tag{2}$$

The reduced right Cauchy-Green deformation tensor is defined as

$$\overline{\boldsymbol{c}} := \frac{\boldsymbol{C}}{|\boldsymbol{C}|^{1/3}} \tag{3}$$

1.2 Hyperelasticity

In order to guarantee the existence of solutions, the strain energy function must be sequentially weakly lower semicontinuous (s.w.l.s.) and must meet a coercivity condition [3].

[[MS: Define s.w.l.s. and coercivity condition.]]

Adopting Ball's polyconvexity framework [1] guarantees sequential weak lower semicontinuity and coercivity and, in particular, implies ellipticity. For this reason, we formulate the strain-energy function to be polyconvex.

Definition 1.1 (Polyconvexity)**.** A function $W \in C^2\left(\mathbb{M}^{3\times3}, \mathbb{R}\right)$ is polyconvex if there exists a convex function $P : \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \times \mathbb{R} \to \mathbb{R}$ such that

$$W(\mathbb{J}) = P(\mathbb{J}, \operatorname{Adj}\mathbb{J}, \det\mathbb{J}) \tag{4}$$

for all $\mathbb{J} \in \mathbb{R}^{3\times3}$ with $\det\mathbb{J} > 0$.

1.3 Uniaxial Tension Test

In [2, Sec. 4.1] for the Uniaxial tension tests on coronary arteries the authors describe their fiber based stress to account for the anisotropic case. They define the strain energy function in Eq. (39) as

$$\begin{aligned}\boldsymbol{\sigma} = \left(1 - 2w_1^1\right)\left[\frac{A_1}{2}(I_1 - 3) + \frac{B_1}{2}\left(\frac{I_2}{I_3} - 3\right)\right] \\ + 2w_1^1 D_1\left[\frac{1}{A_1}\left(e^{A_1(J_{4i}-1)} - 1\right) + \frac{1}{B_1}\left(e^{B_1\left(K_{5i}^{\mathrm{inc}}-1\right)} - 1\right)\right]\end{aligned} \tag{5}$$

where

$$\begin{aligned} I_1 &= \operatorname{tr}[\boldsymbol{C}] & J_{4i} &= \operatorname{tr}[\boldsymbol{C}\boldsymbol{G}_i] \\ I_2 &= \frac{1}{2}\left(\operatorname{tr}[\boldsymbol{C}]^2 - \operatorname{tr}\left[\boldsymbol{C}^2\right]\right) & J_{5i} &= \operatorname{tr}\left[\boldsymbol{C}^2\boldsymbol{G}_i\right] \\ I_3 &= \det[\boldsymbol{C}] & K_{5i} &= J_{5i} - I_1 J_{4i} + I_2 \operatorname{tr}[\boldsymbol{G}_i]\,. \end{aligned}$$

Remark 1.2. Notice that in [2] they work in the Cauchy-Green Strain tensor $\boldsymbol{C}$, not the reduced tensor $\overline{\boldsymbol{c}}$. This will mean slight modifications to our representation.

Remark 1.3. Notice also that the strain energy function given in (5) is in terms of K_{5i}. This is because the invariant J_{5i} is not convex with respect to $\mathbb{J}$, as shown in [4, Lem. C.7].

1.4 Reduced Invariant Implementation

We adopt the following reduced invariants

$$
\begin{aligned}
j_1 &= \operatorname{tr}\left[\overline{\boldsymbol{c}}\right] & j_{4i} &= \operatorname{tr}\left[\overline{\boldsymbol{c}}\boldsymbol{G}_i\right] \\
j_2 &= \operatorname{tr}\left[\overline{\boldsymbol{c}}^2\right] & j_{5i} &= \operatorname{tr}\left[\overline{\boldsymbol{c}}^2\boldsymbol{G}_i\right] \\
j_3 &= \det\left[\overline{\boldsymbol{c}}\right] = 1 & k_{5i} &= j_{5i} - j_1 j_{4i} + j_2 \operatorname{tr}\left[\boldsymbol{G}_i\right].
\end{aligned}
$$

Then we have the following relationships between the invariants:

$$
\begin{aligned}
I_1 &= |\boldsymbol{C}|^{1/3}\, j_1 & J_{4i} &= |\boldsymbol{C}|^{1/3}\, j_{4i} \\
I_2 &= |\boldsymbol{C}|^{2/3}\, \frac{j_1^2 - j_2}{2} & J_{5i} &= |\boldsymbol{C}|^{2/3}\, j_{5i} \\
 & & K_{5i} &= |\boldsymbol{C}|^{2/3}\left(j_{5i} - j_1 j_{4i} + \frac{j_1^2 - j_2}{2}\right).
\end{aligned}
$$

Then the strain energy function (5) becomes

$$
\begin{aligned}
e^s = \left(1 - 2w_1^1\right)&\left[\frac{A_1}{2}\left(|\boldsymbol{C}|^{1/3}\, j_1 - 3\right) + \frac{B_1}{2}\left(\frac{1}{2\,|\boldsymbol{C}|^{1/3}}\left(j_1^2 - j_2\right) - 3\right)\right] \\
+2w_1^1 D_1 &\left[\frac{1}{A_1}\left(\exp\left\{A_1\left(|\boldsymbol{C}|^{1/3}\, j_{4i} - 1\right)\right\} - 1\right) + \frac{1}{B_1}\left(\exp\left\{\left(B_1\left(|\boldsymbol{C}|^{2/3}\left(j_{5i} - j_1 j_4 + \frac{j_1^2 - j_2}{2}\right) - 1\right)\right)\right\} - 1\right)\right]
\end{aligned}
\tag{6}
$$

References

[1] John M Ball. Convexity conditions and existence theorems in nonlinear elasticity. Archive for rational mechanics and Analysis, 63(4):337–403, 1976.

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[3] J. Schröder, P. Neff, and V. Ebbing. Anisotropic polyconvex energies on the basis of crystallographic motivated structural tensors. Journal of the Mechanics and Physics of Solids, 56(12):3486–3506, 2008. ISSN 0022-5096. doi: https://doi.org/10.1016/j.jmps.2008.08.008. URL https://www.sciencedirect.com/science/article/pii/S0022509608001373.

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