ELASTIC NOTES

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1 Anisotropic Derivation

1.1 Notation

Recall the deformation gradient

$$\mathbb{J} := \frac{\partial \boldsymbol{\Phi}\left(\boldsymbol{\xi}, t\right)}{\partial \boldsymbol{\xi}} \tag{1}$$

where $\boldsymbol{\Phi}$ is the mapping from reference to current configuration. The right Cauchy-Green deformation tensor is defined as

$$\boldsymbol{C} := \mathbb{J}^{\mathsf{T}}\mathbb{J}. \tag{2}$$

The reduced right Cauchy-Green deformation tensor is defined as

$$\boldsymbol{c} := \frac{\boldsymbol{C}}{\left|\boldsymbol{C}\right|^{1/3}} \tag{3}$$

1.2 Hyperelasticity

In order to guarantee the existence of solutions, the strain energy function must be sequentially weakly lower semicontinuous (s.w.l.s.) and must meet a coercivity condition [3].

[[MS: Define s.w.l.s. and coercivity condition.]]

Adopting Ball's polyconvexity framework [1] guarantees sequential weak lower semicontinuity and coercivity and, in particular, implies ellipticity. For this reason, we formulate the strain-energy function to be polyconvex.

> **Definition 1.1** (Polyconvexity)**.** A function $W \in C^2\left(\mathbb{M}^{3\times 3}, \mathbb{R}\right)$ is polyconvex if there exists a convex function $P : \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \times \mathbb{R} \to \mathbb{R}$ such that
>
> $$W(\mathbb{J}) = P(\mathbb{J}, \operatorname{Adj}\mathbb{J}, \det\,\mathbb{J}) \tag{4}$$
>
> for all $\mathbb{J} \in \mathbb{R}^{3\times 3}$ with $\det\,\mathbb{J} > 0$.

1.3 Uniaxial Tension Test

In [2, Sec. 4.1] for the Uniaxial tension tests on coronary arteries the authors describe their fiber based stress to account for the anisotropic case. They define the strain energy function in Eq. (39) as

$$\begin{aligned} e^s = \left(1 - 2w_1^1\right)\left[\frac{A_1}{2}\left(I_1 - 3\right) + \frac{B_1}{2}\left(\frac{I_2}{I_3} - 3\right)\right] \\ + 2w_1^1 D_1\left[\frac{1}{A_1}\left(e^{A_1\left(J_{4i}-1\right)} - 1\right) + \frac{1}{B_1}\left(e^{B_1\left(K_{5i}^{\text{inc}}-1\right)} - 1\right)\right] \end{aligned} \tag{5}$$

where

$$\begin{aligned} I_1 &= \operatorname{tr}\left[\boldsymbol{C}\right] & J_{4i} &= \operatorname{tr}\left[\boldsymbol{C}\boldsymbol{G}_i\right] \\ I_2 &= \frac{1}{2}\left(\operatorname{tr}\left[\boldsymbol{C}\right]^2 - \operatorname{tr}\left[\boldsymbol{C}^2\right]\right) & J_{5i} &= \operatorname{tr}\left[\boldsymbol{C}^2\boldsymbol{G}_i\right] \\ I_3 &= \det\left[\boldsymbol{C}\right] & K_{5i} &= J_{5i} - I_1 J_{4i} + I_2 \operatorname{tr}\left[\boldsymbol{G}_i\right]. \end{aligned}$$

> **Remark 1.2.** Notice that in [2] they work in the Cauchy-Green Strain tensor $\boldsymbol{C}$, not the reduced tensor $\boldsymbol{c}$. To work with the reduced invariants there is no issue since in Chaimoon (2019) the work is done under the assumption of incompressiblity ($\det\left|\boldsymbol{C}\right| = 1$).

Remark 1.3. Notice also that the strain energy function given in (5) is in terms of K_{5i}. This is because the invariant J_{5i} is not convex with respect to $\mathbb{J}$, as shown in [4, Lem. C.7].

1.4 Reduced Invariant Implementation

We adopt the following reduced invariants

$$\begin{aligned} j_1 &= \operatorname{tr}[\boldsymbol{c}] & j_{4i} &= \operatorname{tr}[\boldsymbol{c}\boldsymbol{G}_i] \\ j_2 &= \operatorname{tr}\left[\boldsymbol{c}^2\right] & j_{5i} &= \operatorname{tr}\left[\boldsymbol{c}^2\boldsymbol{G}_i\right] \\ j_3 &= \det[\boldsymbol{c}] = 1 & k_{5i} &= j_{5i} - j_1 j_{4i} + j_2 \operatorname{tr}[\boldsymbol{G}_i]\,. \end{aligned}$$

Then we have the following relationships between the invariants:

$$\begin{aligned} I_1 &= |\boldsymbol{C}|^{1/3}\, j_1 & J_{4i} &= |\boldsymbol{C}|^{1/3}\, j_{4i} \\ I_2 &= |\boldsymbol{C}|^{2/3}\, \frac{j_1^2 - j_2}{2} & J_{5i} &= |\boldsymbol{C}|^{2/3}\, j_{5i} \\ & & K_{5i} &= |\boldsymbol{C}|^{2/3} \left(j_{5i} - j_1 j_{4i} + \frac{j_1^2 - j_2}{2} \right). \end{aligned}$$

Then the strain energy function (5) becomes

$$\begin{aligned} e^s &= \left(1 - 2w_1^1\right) \left[\frac{A_1}{2} \left(j_1 - 3\right) + \frac{B_1}{2} \left(j_1^2 - j_2 - 6\right) \right] \\ &+ 2w_1^1 D_1 \left[\frac{1}{A_1} \left(\exp\left\{A_1 \left(j_{4i} - 1\right)\right\} - 1 \right) + \frac{1}{B_1} \left(\exp\left\{ \left(B_1 \left(j_{5i} - j_1 j_{4i} + \frac{j_1^2 - j_2}{2} \right) - 1 \right) \right\} - 1 \right) \right] \end{aligned} \tag{6}$$

To compute the stress, we need the partial derivatives of the strain energy function with respect to the invariants:

$$\frac{\partial e^s}{\partial j_1} = \left(1 - 2w_1^1\right) \left[\frac{A_1}{2} + \frac{B_1}{2} j_1 \right] + 2w_1^1 D_1 \left(-j_{4i} + j_1\right) \exp\left\{ B_1 \left(j_{5i} - j_1 j_{4i} + \frac{j_1^2 - j_2}{2} \right) - 1 \right\} \tag{7a}$$

$$\frac{\partial e^s}{\partial j_2} = -\left(1 - 2w_1^1\right) \frac{B_1}{4} - w_1^1 D_1 \exp\left\{ B_1 \left(j_{5i} - j_1 j_{4i} + \frac{j_1^2 - j_2}{2} \right) - 1 \right\} \tag{7b}$$

$$\frac{\partial e^s}{\partial j_{4i}} = 2w_1^1 D_1 \left[\exp\left\{A_1 \left(j_{4i} - 1\right)\right\} - j_1 \exp\left\{ B_1 \left(j_{5i} - j_1 j_{4i} + \frac{j_1^2 - j_2}{2} \right) - 1 \right\} \right] \tag{7c}$$

$$\frac{\partial e^s}{\partial j_{5i}} = 2w_1^1 D_1 \exp\left\{ B_1 \left(j_{5i} - j_1 j_{4i} + \frac{j_1^2 - j_2}{2} \right) - 1 \right\} \tag{7d}$$

We must also compute the partial derivatives of the invariants with respect to $\boldsymbol{C}$:

$$
\begin{aligned}
\frac{\partial j_1}{\partial \boldsymbol{C}} &= \frac{\partial}{\partial \boldsymbol{C}} \left(\operatorname{tr}\left(\boldsymbol{c}\right)\right) \\
&= \frac{\partial}{\partial \boldsymbol{C}} \left[\left|\boldsymbol{C}\right|^{-1/3} \operatorname{tr}\left(\boldsymbol{C}\right)\right] \\
&= \left|\boldsymbol{C}\right|^{-1/3} \mathbb{I}_d - \frac{1}{3} \left|\boldsymbol{C}\right|^{-4/3} \operatorname{tr}\left(\boldsymbol{C}\right) \operatorname{adj}\left(\boldsymbol{C}^T\right) \\
&= \left|\boldsymbol{C}\right|^{-1/3} \mathbb{I}_d - \frac{1}{3} \left|\boldsymbol{C}\right|^{-1/3} \operatorname{tr}\left(\boldsymbol{C}\right) \boldsymbol{C}^{-1} \\
&= \left|\boldsymbol{C}\right|^{-1/3} \mathbb{I}_d - \frac{1}{3} j_1 \boldsymbol{C}^{-1} \\
\frac{\partial j_2}{\partial \boldsymbol{C}} &= \frac{\partial}{\partial \boldsymbol{C}} \left(\operatorname{tr}\left(\boldsymbol{c}^2\right)\right) \\
&= \frac{\partial}{\partial \boldsymbol{C}} \left[\left|\boldsymbol{C}\right|^{-2/3} \operatorname{tr}\left(\boldsymbol{C}^2\right)\right] \\
&= 2\left|\boldsymbol{C}\right|^{-2/3} \boldsymbol{C} - \frac{2}{3} \left|\boldsymbol{C}\right|^{-5/3} \operatorname{tr}\left(\boldsymbol{C}^2\right) \operatorname{adj}\left(\boldsymbol{C}^T\right) \\
&= 2\left|\boldsymbol{C}\right|^{-2/3} \boldsymbol{C} - \frac{2}{3} \left|\boldsymbol{C}\right|^{-2/3} \operatorname{tr}\left(\boldsymbol{C}^2\right) \boldsymbol{C}^{-1} \\
&= 2\left|\boldsymbol{C}\right|^{-2/3} \boldsymbol{C} - \frac{2}{3} j_2 \boldsymbol{C}^{-1} \\
\frac{\partial j_{4i}}{\partial \boldsymbol{C}} &= \frac{\partial}{\partial \boldsymbol{C}} \left(\operatorname{tr}\left(\boldsymbol{c}\boldsymbol{G}_i\right)\right) \\
&= \frac{\partial}{\partial \boldsymbol{C}} \left[\left|\boldsymbol{C}\right|^{-1/3} \operatorname{tr}\left(\boldsymbol{C}\boldsymbol{G}_i\right)\right] \\
&= \left|\boldsymbol{C}\right|^{-1/3} \boldsymbol{G}_i^{\mathsf{T}} - \frac{1}{3} \left|\boldsymbol{C}\right|^{-4/3} \operatorname{tr}\left(\boldsymbol{C}\boldsymbol{G}_i\right) \operatorname{adj}\left(\boldsymbol{C}^T\right) \\
&= \left|\boldsymbol{C}\right|^{-1/3} \boldsymbol{G}_i^{\mathsf{T}} - \frac{1}{3} j_{4i} \boldsymbol{C}^{-1} \\
\frac{\partial j_{5i}}{\partial \boldsymbol{C}} &= \frac{\partial}{\partial \boldsymbol{C}} \left(\operatorname{tr}\left(\boldsymbol{c}^2\boldsymbol{G}_i\right)\right) \\
&= \frac{\partial}{\partial \boldsymbol{C}} \left[\left|\boldsymbol{C}\right|^{-2/3} \operatorname{tr}\left(\boldsymbol{C}^2\boldsymbol{G}_i\right)\right] \\
&= \left|\boldsymbol{C}\right|^{-2/3} \left(\boldsymbol{G}_i^{\mathsf{T}}\boldsymbol{C}^{\mathsf{T}} + \boldsymbol{C}^{\mathsf{T}}\boldsymbol{G}_i^{\mathsf{T}}\right) - \frac{2}{3} \left|\boldsymbol{C}\right|^{-5/3} \operatorname{tr}\left(\boldsymbol{C}^2\boldsymbol{G}_i\right) \operatorname{adj}\left(\boldsymbol{C}^T\right) \\
&= \left|\boldsymbol{C}\right|^{-2/3} \left(\boldsymbol{G}_i^{\mathsf{T}}\boldsymbol{C}^{\mathsf{T}} + \boldsymbol{C}^{\mathsf{T}}\boldsymbol{G}_i^{\mathsf{T}}\right) - \frac{2}{3} \left|\boldsymbol{C}\right|^{-2/3} \operatorname{tr}\left(\boldsymbol{C}^2\boldsymbol{G}_i\right) \boldsymbol{C}^{-1} \\
&= \left|\boldsymbol{C}\right|^{-2/3} \left(\boldsymbol{G}_i^{\mathsf{T}}\boldsymbol{C}^{\mathsf{T}} + \boldsymbol{C}^{\mathsf{T}}\boldsymbol{G}_i^{\mathsf{T}}\right) - \frac{2}{3} j_{5i} \boldsymbol{C}^{-1} \\
&= 2\left|\boldsymbol{C}\right|^{-2/3} \boldsymbol{C}\boldsymbol{G}_i - \frac{2}{3} j_{5i} \boldsymbol{C}^{-1}
\end{aligned}
$$

where particularly for the derivation of $\frac{\partial j_{5i}}{\partial \boldsymbol{C}}$ we have used the fact that

$$
\frac{\partial}{\partial \boldsymbol{C}} \operatorname{tr}\left(\boldsymbol{C}^2\boldsymbol{G}_i\right) = \boldsymbol{G}_i^{\mathsf{T}}\boldsymbol{C}^{\mathsf{T}} + \boldsymbol{C}^{\mathsf{T}}\boldsymbol{G}_i^{\mathsf{T}}
$$

and since $\boldsymbol{C}$ and $\boldsymbol{G}_i$ are both symmetric. In particular, this implies for the Cauchy stress tensor that

$$\mathbb{J}\frac{\partial j_1}{\partial \boldsymbol{C}}\mathbb{J}^{\mathsf{T}} = \boldsymbol{b} - \frac{1}{3}j_1\mathbb{I} \tag{8a}$$

$$\mathbb{J}\frac{\partial j_2}{\partial \boldsymbol{C}}\mathbb{J}^{\mathsf{T}} = 2\boldsymbol{b}^2 - \frac{2}{3}j_2\mathbb{I} \tag{8b}$$

$$\mathbb{J}\frac{\partial j_{4i}}{\partial \boldsymbol{C}}\mathbb{J}^{\mathsf{T}} = \boldsymbol{b}\boldsymbol{G}_i - \frac{1}{3}j_{4i}\mathbb{I} \tag{8c}$$

$$\mathbb{J}\frac{\partial j_{5i}}{\partial \boldsymbol{C}}\mathbb{J}^{\mathsf{T}} = 2\boldsymbol{b}^2\boldsymbol{G}_i - \frac{2}{3}j_{5i}\mathbb{I} \tag{8d}$$

With the definition of the strain energy and the above partial derivatives, our stress tensor then takes the form

$$\begin{aligned}
\boldsymbol{\sigma} &= 2\rho\frac{\partial e^s}{\partial \boldsymbol{C}}\boldsymbol{C} - \rho^2\frac{\partial e^h}{\partial \rho}\mathbb{I} \\
&= 2\rho\left[\frac{\partial e^s}{\partial j_1}\frac{\partial j_1}{\partial \boldsymbol{C}}\boldsymbol{C} + \frac{\partial e^s}{\partial j_2}\frac{\partial j_2}{\partial \boldsymbol{C}}\boldsymbol{C}\right] - \rho^2\frac{\partial e^h}{\partial \rho}\mathbb{I} + 2\rho\left[\frac{\partial e^s}{\partial j_{4i}}\frac{\partial j_{4i}}{\partial \boldsymbol{C}}\boldsymbol{C} + \frac{\partial e^s}{\partial j_{5i}}\frac{\partial j_{5i}}{\partial \boldsymbol{C}}\boldsymbol{C}\right] \\
&= 2\rho\left[\frac{\partial e^s}{\partial j_1}\left(\boldsymbol{c} - \frac{1}{3}j_1\mathbb{I}\right) + 2\frac{\partial e^s}{\partial j_2}\left(\boldsymbol{c}^2 - \frac{1}{3}j_2\mathbb{I}\right)\right] - \rho^2\frac{\partial e^h}{\partial \rho}\mathbb{I} \\
&\quad + 2\rho\left[\frac{\partial e^s}{\partial j_{4i}}\left(\boldsymbol{c}\boldsymbol{G}_i - \frac{1}{3}j_{4i}\mathbb{I}\right) + 2\frac{\partial e^s}{\partial j_{5i}}\left(\boldsymbol{c}^2\boldsymbol{G}_i - \frac{1}{3}j_{5i}\mathbb{I}\right)\right]
\end{aligned}$$

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