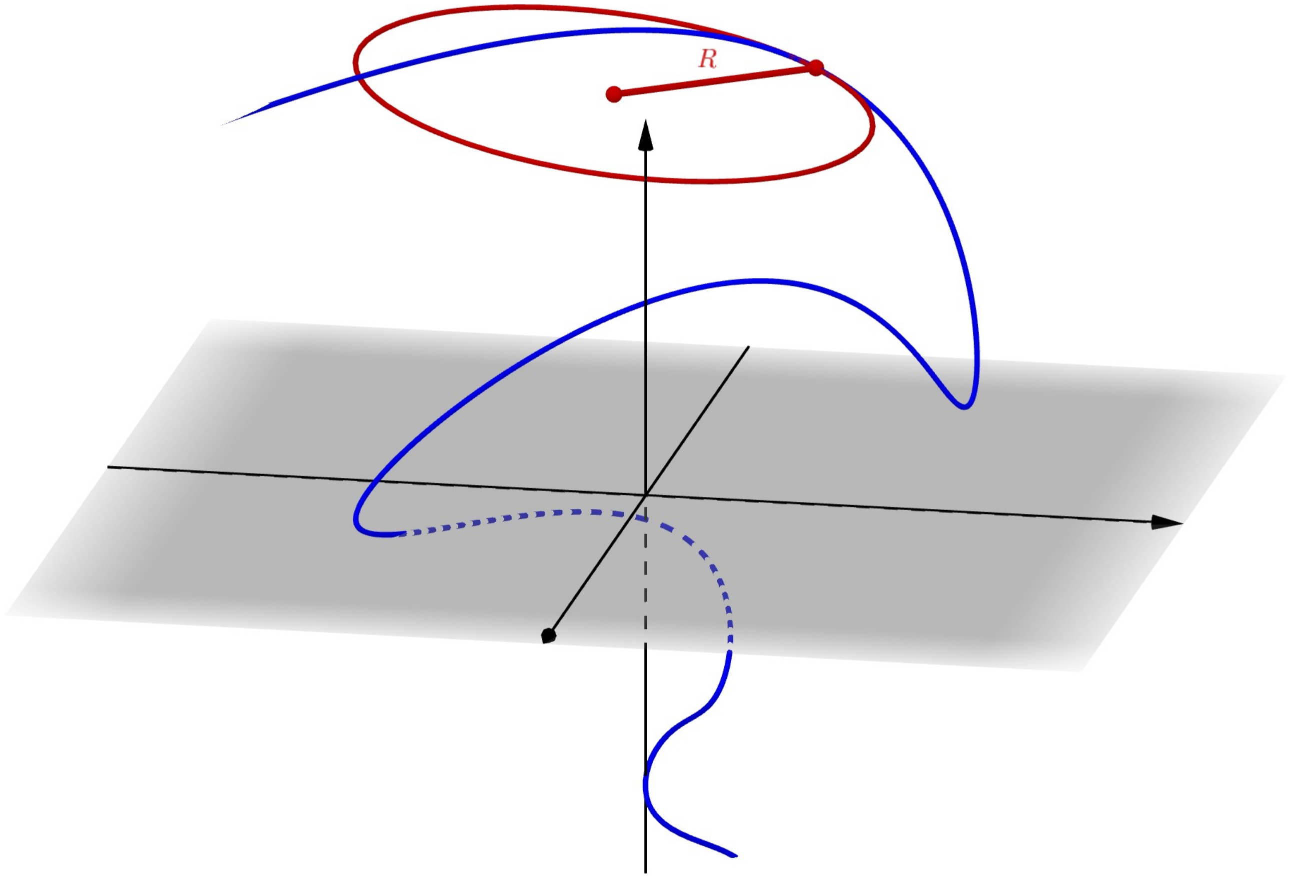
- Curvature

# Curvature of a space curve

- Curvus: bent
- Circle
- Osculating circle
- Curve
- Properties of a curve
- Equivalence of analytical and intuitive definitions


R

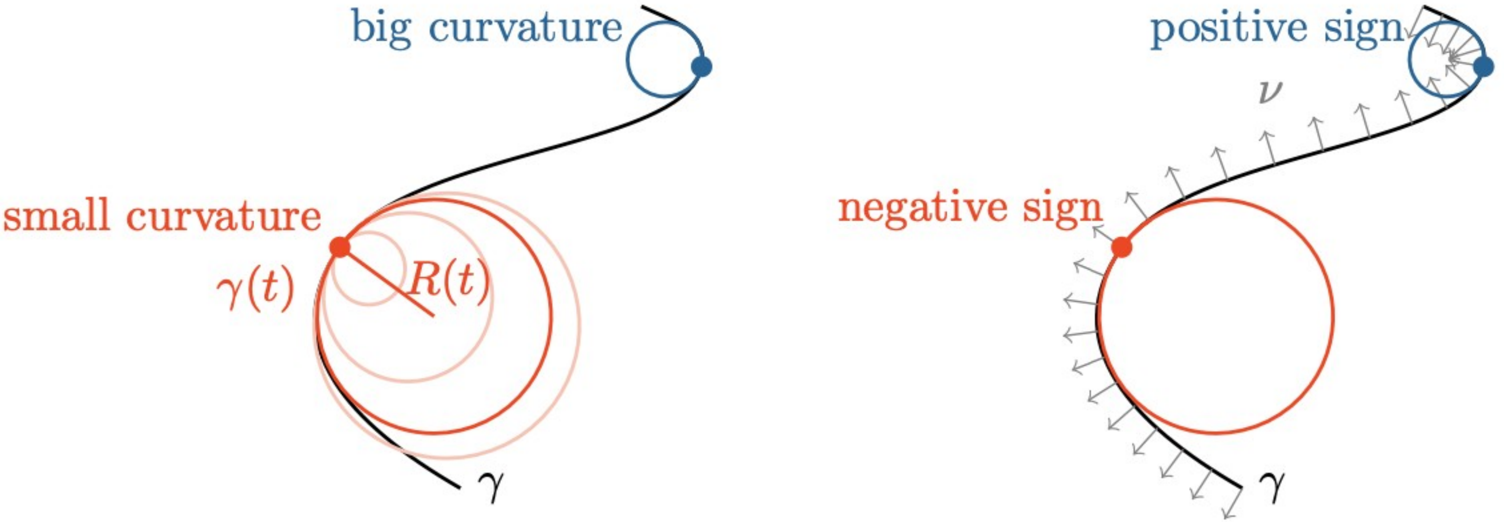


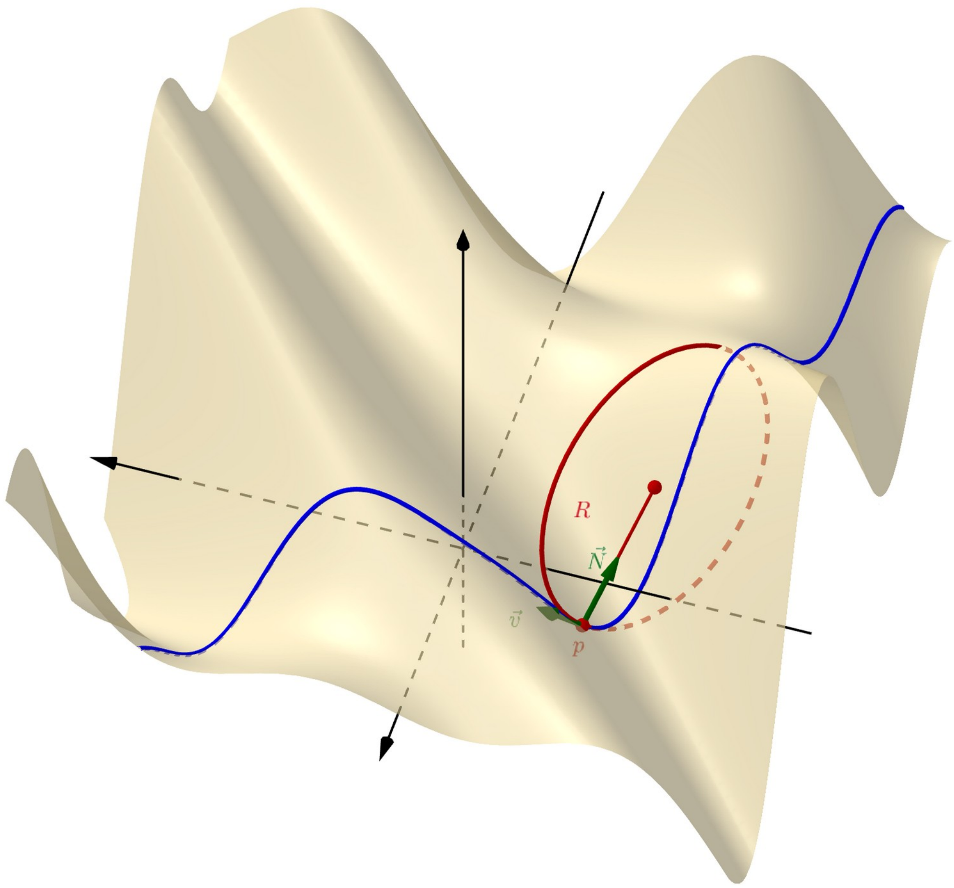

Figure 7.1.: Curvature of a plane curve

# Curvature of surfaces

- Surfaces in the 3-dim euclidian space
- Geodesic
- A problem of choice: we need a criterion
- Negative curvatures
- Gaussian curvature

# Curvature of surfaces

- Theorema Egregium
- Some consequences
- Pseudosphere


R
$\vec{N}$
$\vec{v}$
p

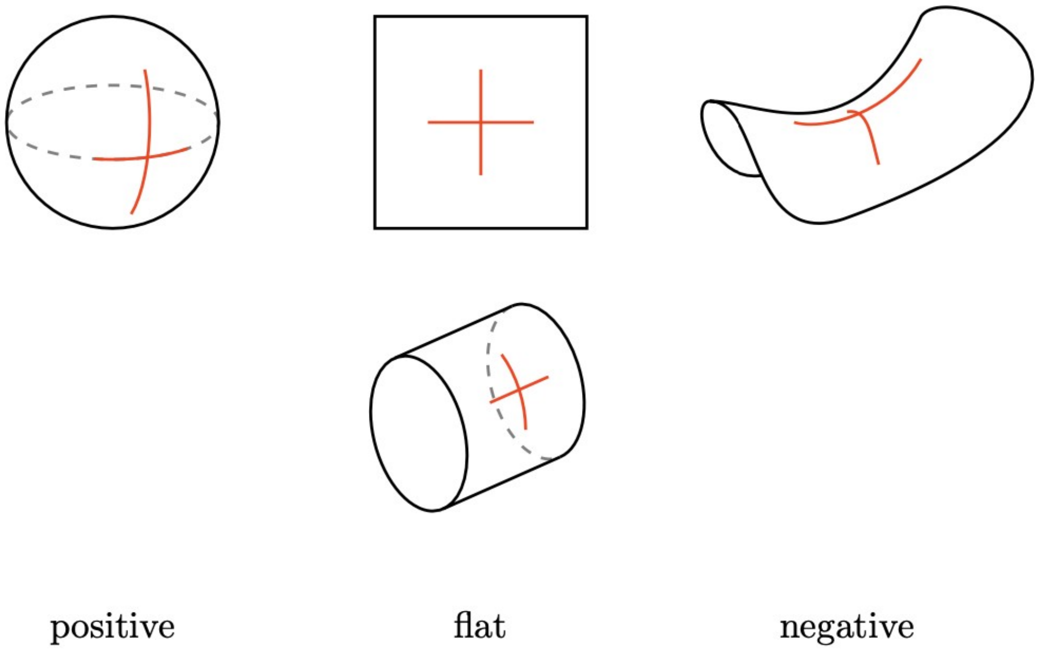
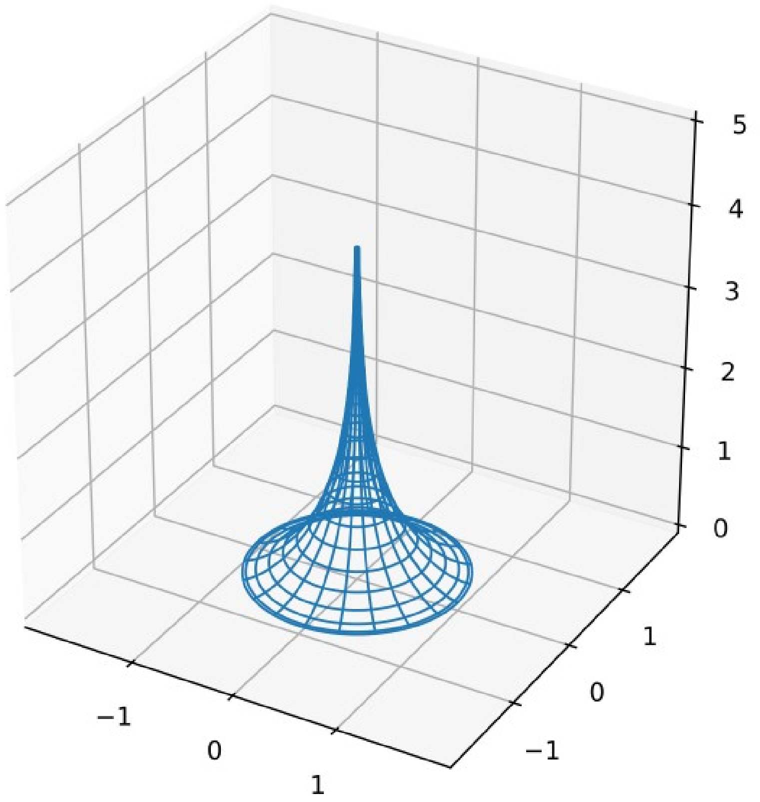


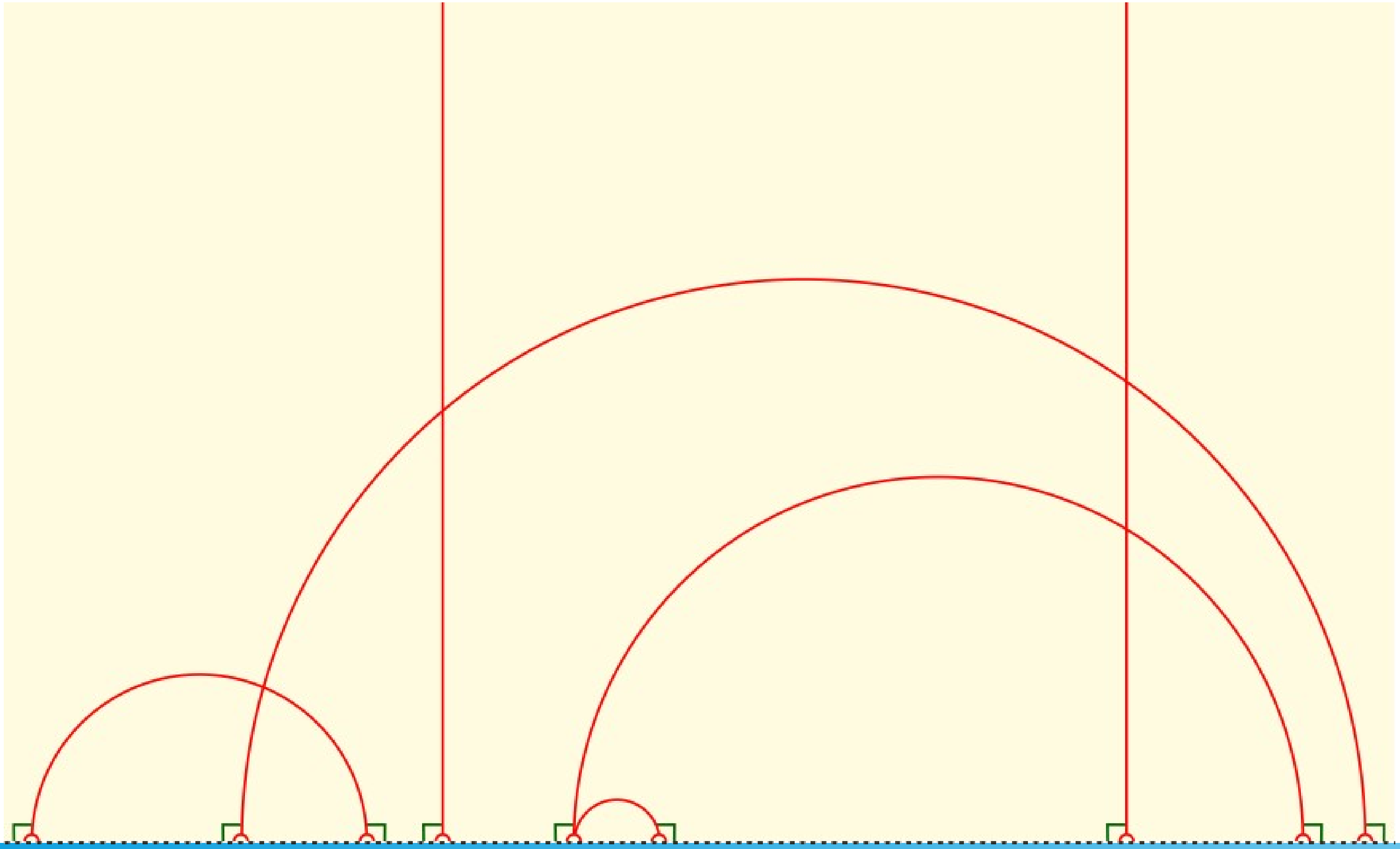

Figure 7.3.: Examples of Gaussian curvatures of surfaces

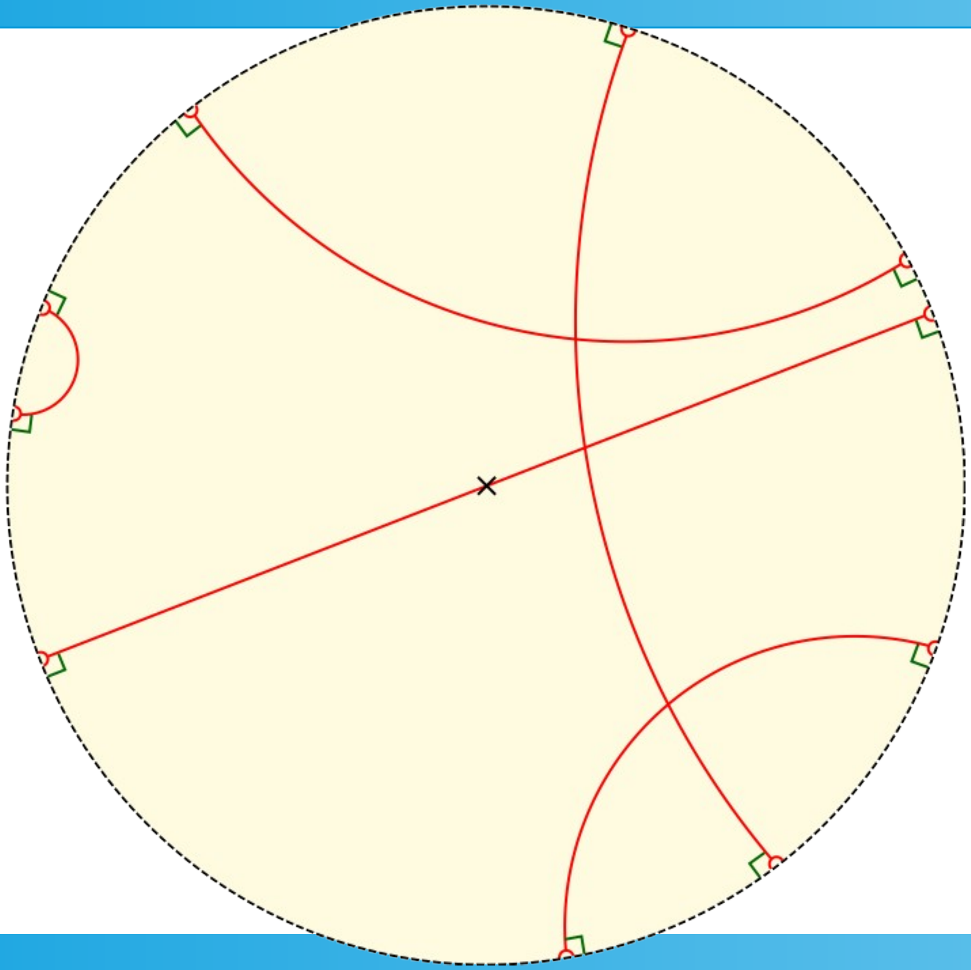


5
4
3
2
1
0
1
0
−1
−1
0
1

# Curvature of Riemannian manifolds

- A new notion of length
- Riemannian line element
- Conformal metric





# Curvature of metric spaces

- Slimmer triangles
- Delta-slim

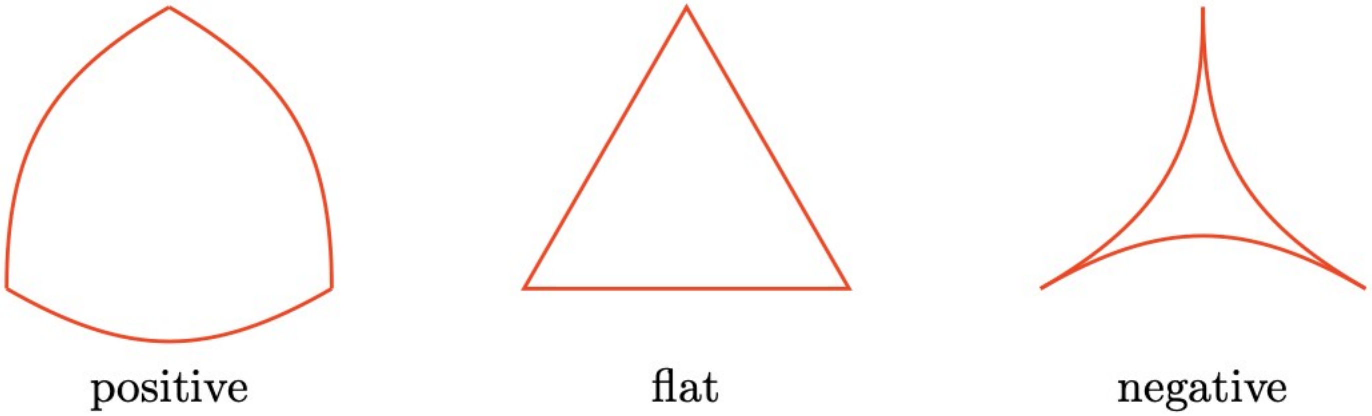



Figure 7.4.: Geodesic triangles in surfaces

**Definition 7.2.1** ($\delta$-Slim geodesic triangle). Let $(X, d)$ be a metric space.

- A *geodesic triangle* in $X$ is a triple $(\gamma_0, \gamma_1, \gamma_2)$ consisting of geodesics $\gamma_j \colon [0, L_j] \longrightarrow X$ in $X$ such that

$$\gamma_0(L_0) = \gamma_1(0), \qquad \gamma_1(L_1) = \gamma_2(0), \qquad \gamma_2(L_2) = \gamma_0(0).$$

- A geodesic triangle $(\gamma_0, \gamma_1, \gamma_2)$ is $\delta$-*slim* if (Figure 7.5)

$$\begin{aligned}
\operatorname{im} \gamma_0 &\subset B_\delta^{X,d}\bigl(\operatorname{im} \gamma_1 \cup \operatorname{im} \gamma_2\bigr), \\
\operatorname{im} \gamma_1 &\subset B_\delta^{X,d}\bigl(\operatorname{im} \gamma_0 \cup \operatorname{im} \gamma_2\bigr), \\
\operatorname{im} \gamma_2 &\subset B_\delta^{X,d}\bigl(\operatorname{im} \gamma_0 \cup \operatorname{im} \gamma_1\bigr).
\end{aligned}$$

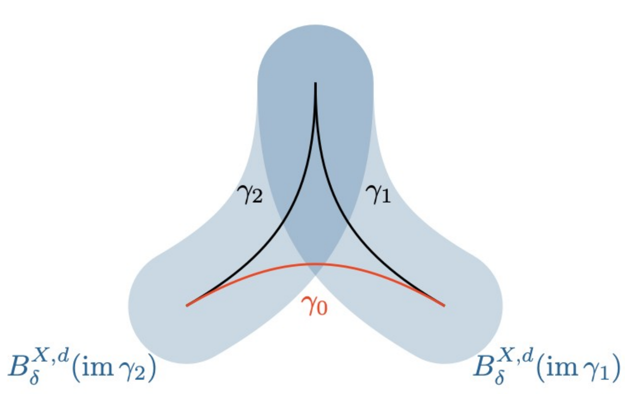



Figure 7.5.: A $\delta$-slim triangle

Here, for $\gamma\colon [0, L] \longrightarrow X$ we use the abbreviation $\operatorname{im}\gamma := \gamma([0, L])$, and for $A \subset X$ we write $B_\delta^{X,d}(A) := \{x \in X \mid \exists_{a\in A}\ d(x, a) \le \delta\}$.

**Definition 7.2.2** ($\delta$-Hyperbolic space)**.** Let $X$ be a metric space.

- Let $\delta \in \mathbb{R}_{\ge 0}$. We say that $X$ is *$\delta$-hyperbolic* if $X$ is geodesic and if all geodesic triangles in $X$ are $\delta$-slim.
- The space $X$ is *hyperbolic* if there exists a $\delta \in \mathbb{R}_{\ge 0}$ such that $X$ is $\delta$-hyperbolic.

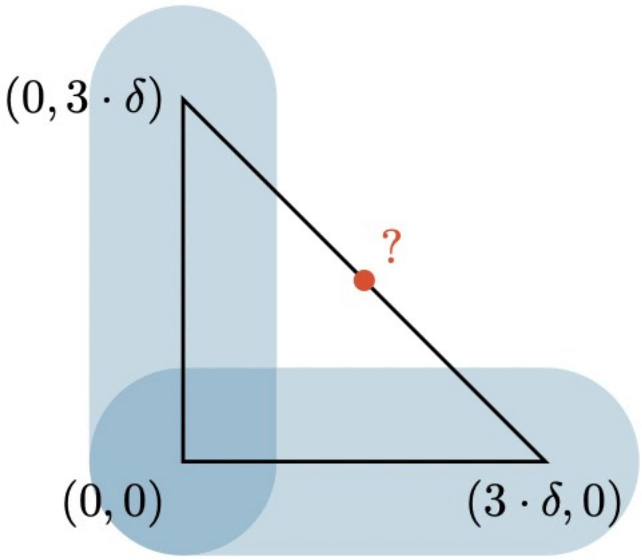



Figure 7.6.: The Euclidean plane $\mathbb{R}^2$ is *not* hyperbolic

# Bibliography


- Do Carmo, M. Differential Geometry of Curves and Surfaces
- Fasano, A; Marmi, S. Analytical Mechanics
- Loustau, B. Hyperbolic Geometry
- Löh, C. Geometric Group Theory

# Image credits

- Loustau, B. Hyperbolic Geometry
- Löh, C. Geometric Group Theory