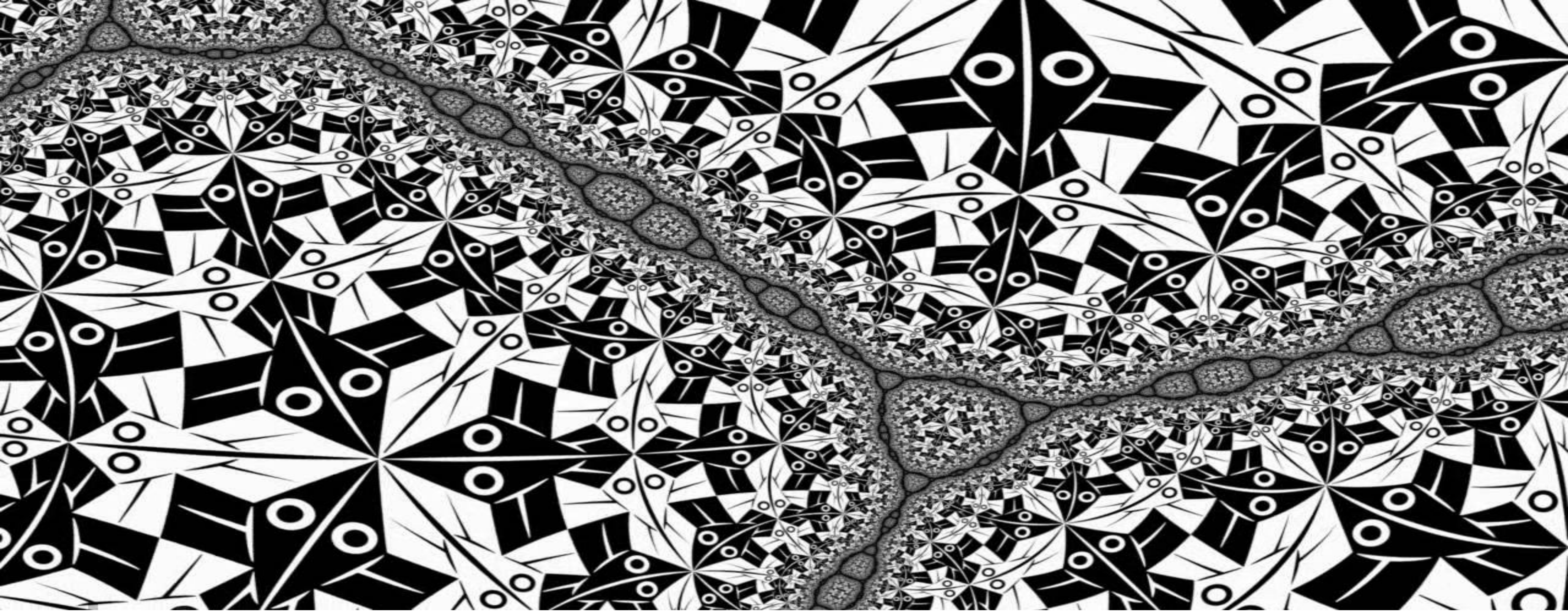


A STRANGE NEW UNIVERSE

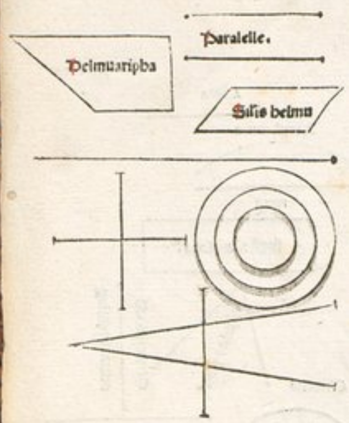
From Euclidean to Hyperbolic geometry

EUCLID AND THE *ELEMENTS*

- Euclid of Alexandria (c. 300 BC)
- Studied at Plato's Academy (or with Plato's students)
- Published a number of works, all but 5 lost
- The *Elements* extremely influential

Preclarissimũ opus elementoꝝ Euclidis megarẽsis vna cũ cõmentis Campani pspicacissimi in artẽ geometriã incipit feliciť.

Punctus est cuius ps non est. ¶ Linea est longitudo sine latitudĩe cuiũs quidem extremitates sunt duo puncta. ¶ Linea recta ẽ ab vno pũcto ad aliũm breuissima extensio in extremitates suas vtrũq; eoꝝ recipiens. ¶ Supficies ẽ q̃ lõgitudinẽ ⁊ latitudinẽ tm̃ habet: cui⁹ termini quidẽ sũt linee ¶ Superficies plana ẽ ab vna linea ad aliam extensio in extremitates suas recipiẽs ¶ Angulus planus ẽ duarum lineaꝝ alterius cõtactus: quaꝝ expansio est super supficiẽ applicatioq; nõ directa. ¶ Quãdo autẽ angulũ cõtinent due linee recte rectiline⁹ angulus noĩatur. ¶ Qñ recta linea sup rectã steterit duoq; anguli vtrobiq; fuerint eq̃les eoꝝ vterq; rect⁹ erit. ¶ Lineaq; linee superstans ei cui supstat ppẽdicularis vocať. ¶ Angulus vero qui recto maior est obtusus diciť. ¶ Angul⁹ vero mĩor recto acut⁹ appellať. ¶ Termin⁹ ẽ qd vniuscuiusq; finis ẽ. ¶ Figura ẽ q̃ termino vel terminis ꝯtineť. ¶ Circul⁹ ẽ figura plana vna q̃dẽ linea cõtenta: q̃ circũferẽtia noĩaťã cui⁹ medio pũct⁹ ẽ a quo oẽs linee recte ad circũferentiã exeũtes sibiĩnicẽ sunt equales. Et hic quidẽ punct⁹ cẽtꝝ circuli diciť. ¶ Diameter circuli ẽ linea recta q̃ sup ei⁹ centꝝ transiẽs extremitatesq; suas circũferentie applicans circulũ in duo media diuidit. ¶ Semicirculus ẽ figura plana diametro circuli ⁊ medietate circũferentie cõtenta. ¶ Portio circuli est figura plana recta linea ⁊ parte circũferentie cõtenta: semicirculo quidem aut maior aut minor. ¶ Rectilinee figure sũt que rectis lineis cõtinenť quaꝝ quedã trilatere q̃ trib⁹ rectis lineis: quedam quadrilatere q̃ quatuor rectis lineis: q̃dã multilatere q̃ pluribus q̃; quatuor rectis lineis continentur. ¶ Figuraꝝ trilaterarum: alia est triangulus habens tria latera equalia. Alia triangulus duo habẽs equalia latera. Alia triangulus triũ inequaliũ laterũ. Harũ iteru; alia est orthogoniũ: vnũ .s. rectũ angulum habens. Alia est ambligonium aliquem obtusum angulum habens. Alia est oxigonium: in qua tres anguli sunt acuti. ¶ Figurarum autem quadrilaterarũ. Alia est quadratũ quod ẽ equilateru; atq; rectangulũ. Alia est tetragonus longus: que est figura rectangula: sed equilatera non est Alia est helmuaym: que est equilatera: sed rectangula non est.

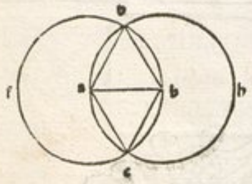


¶ Alia est similis helmuaym que opposita latera habet equalia atq; oppositos angulos equales: idem tamẽ nec rectis angulis nec equis lateribus cõtinetur. Preter has aũt omnes quadrilatere figure helmuariphe nominantur. ¶ Equidistantes linee sunt q̃ in eadem superficie collocate atq; in alterutram partem protracte nõ conueniũt etiam si in infinitum protrahantur.

Petitiones sunt quinq;. ¶ A quolibet puncto i quemlib; punctũ rectã lineam ducere atq; lineam definitãm in continuum rectũq; quãtũlibet protrahere. ¶ Super centrũ quodlibet quãtũlibet occupando spacium: circulum designare: ¶ Omnes rectos angulos sibi inuicem esse equales. ¶ Si linea recta super duas lineas rectas ceciderit duoq; anguli ex vna parte duobus rectis angulis minores fuerint istas duas lineas in eandem partem protractas proculdubio coniuncti ire. ¶ Duas lineas rectas superficiem nullam concludere.

Communes animi cõceptiões sunt hec ¶ Que vni ⁊ eide; sunt equalia ⁊ sibiĩuicẽ sunt equalia. ¶ Et si equalib⁹ eq̃lia addanť tota quoq; fient equalia. ¶ Et si ab eq̃libus eq̃lia auferantur que relinquũť erũt equalia. ¶ Et si ab ineq̃libus eq̃lia demas q̃ relinquunť erũt ineq̃lia. ¶ Et si ineq̃libus equalia addas ipsa quoq; fient ineq̃lia. ¶ Si fuerint due res vni equales ipse sibiĩuicẽ erunt equales. ¶ Si fuerint due res quaꝝ vtraq; vnius eiusdẽ fuerit dimidium vtraq; erit equalis alteri. ¶ Si aliqua res alicui superponatur appliceťq; ei nec excedat altera alteram: ille sibiĩuicem erunt equales. ¶ Omne totum est maius sua pte

Sciendũ est aũt ꝙ preter has animi ꝯceptiones: siue cões scias multas alias que numero sunt incõprehensibiles pretermisit Euclides: quarũ hec ẽ vna. ¶ Si due quantitates equales ad quãlibet tertia; eiusdem generis comparentur simul erunt ambe illa tertia aut eque maiores: aut eque minores: aut simul eq̃les. ¶ Item alia Quanta est aliqua quantitas ad quãlibet aliam eiusdem generis tantam esse quãlibet tertiam ad aliquam quartam eiusdem generis in quantitatibus continuis: hoc vniuersaliter verum est siue antecedentes maiores fuerint consequentibus siue minores. magnitudo enim decrescit in infinitum. in numeris autem non sic: sed si fuerit primus submultiplex secundi: erit quilibet tertius eque submultiplex alicui⁹ quarti: quoniã numerus crescit in infinitũ: sicut magnitudo in infinitũ minuitur.



Propositio .1.

Triangulum equilaterum supra datam lineam rectam collocare.

¶ Esto data linea recta .a.b. volo super ipsam triangulũ equilaterũ cõstituere super alterã eius extremitatẽ .s. in puncto .a. ponam pedẽ circini immobilem: ⁊ alterũ pedem mobilem extendam vsq; ad .b. ⁊ describam sm quantitatẽ ipsius linee date per secundã petitionẽ circulu;. c.b.d.f.

POSTULATES I-IV

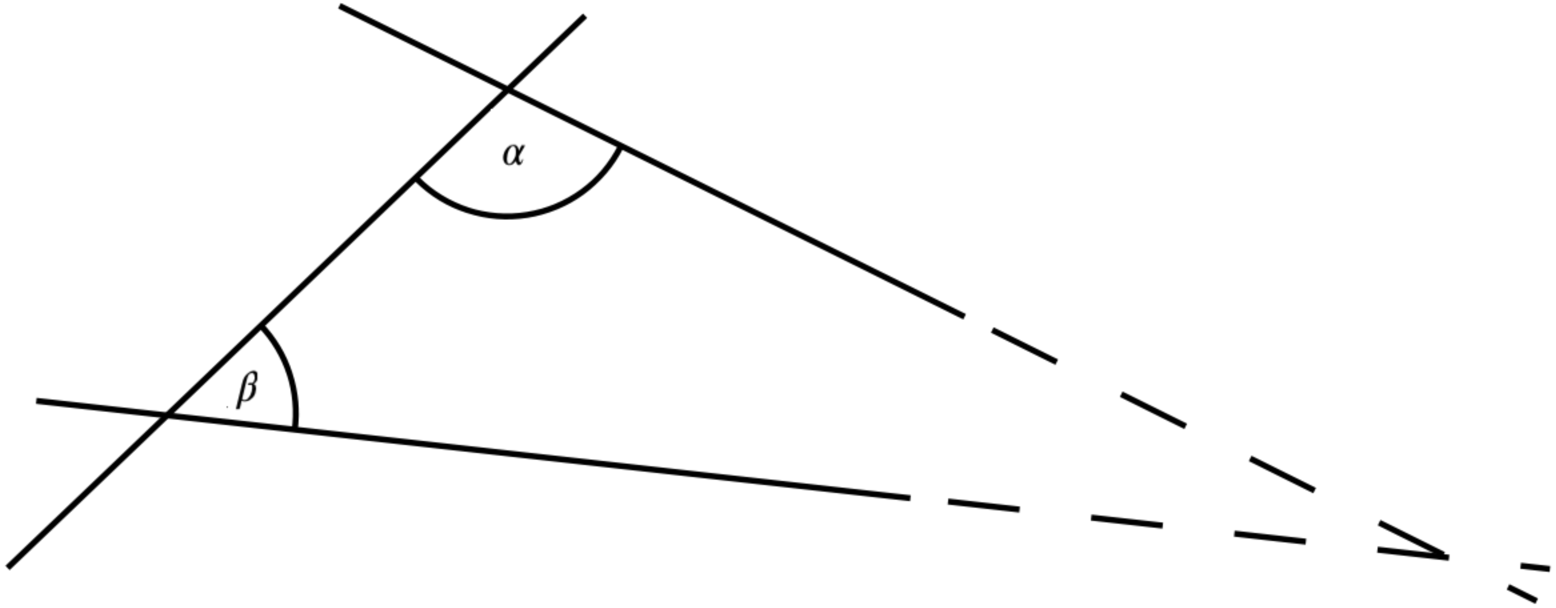
I. [It is possible …] to draw a straight line from any point to any point.
II. And to produce a finite straight line continuously in a straight line.
III. And to describe a circle with any centre and distance.
IV. And that all right angles are equal to one another.

(Heath’s translation)

POSTULATE V

V. And that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

i.e., with reference to the image, if $\alpha + \beta < \pi$, then the lines intersect on the side of α and β.



POSTULATE V

- Nowadays most commonly stated in the form of Playfair’s axiom:
  - *In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point.*

- Highly controversial, even to Euclid himself
- Many failed attempts to prove

ATTEMPTED PROOFS OF V

- Usually followed this schema:
  - Replace V with a more ‘acceptable’ assumption (whether explicitly or implicitly)
  - Retain postulates I-IV (‘neutral geometry’)
  - Prove V
- These assumptions logically equivalent to V *in neutral geometry*, i.e.,
  - Neutral geometry + A $\Rightarrow$ V and
  - Neutral geometry + V $\Rightarrow$ A
- E.g. Proclus, Ptolemy, Wallis, Gauss…

EQUIVALENT STATEMENTS (IN NEUTRAL GEOMETRY)

There is at most one line that can be drawn parallel to another given one through an external point (Playfair's axiom).

The sum of the angles in every triangle is 180° (triangle postulate).

There exists a triangle whose angles add up to 180°.

The sum of the angles is the same for every triangle.

There exists a pair of similar, but not congruent, triangles.

Every triangle can be circumscribed.

If three angles of a quadrilateral are right angles, then the fourth angle is also a right angle.

There exists a quadrilateral in which all angles are right angles, that is, a rectangle.

There exists a pair of straight lines that are at constant distance from each other.

Two lines that are parallel to the same line are also parallel to each other (transitivity of parallelism).

In a right-angled triangle, the square of the hypotenuse equals the sum of the squares of the other two sides (Pythagoras' Theorem).

The Law of cosines, a generalization of Pythagoras' Theorem.

There is no upper limit to the area of a triangle.

The summit angles of the Saccheri quadrilateral are 90°.

If a line intersects one of two parallel lines, both of which are coplanar with the original line, then it also intersects the other (Proclus' axiom).

THE PROBLEM OF PARALLELS

"You must not attempt this approach to parallels. I know this way to its very end. I have traversed this bottomless night, which extinguished all light and joy of my life. ***I entreat you, leave the science of parallels alone****.... I thought I would sacrifice myself for the sake of the truth. I was ready to become a martyr who would remove the flaw from geometry and return it purified to mankind. [But] I turned back when I saw that* ***no man can reach the bottom of the night****. I turned back unconsoled, pitying myself and all mankind."*

– Farkas Bolyai to his son

"I have discovered such wonderful things that I was amazed, and it would be an everlasting piece of bad fortune if they were lost. When you, my dear Father, see them, you will understand; at present I can say nothing except this: that out of nothing ***I have created a strange new universe****."*

– Janos Bolyai to his father

HYPERBOLIC GEOMETRY

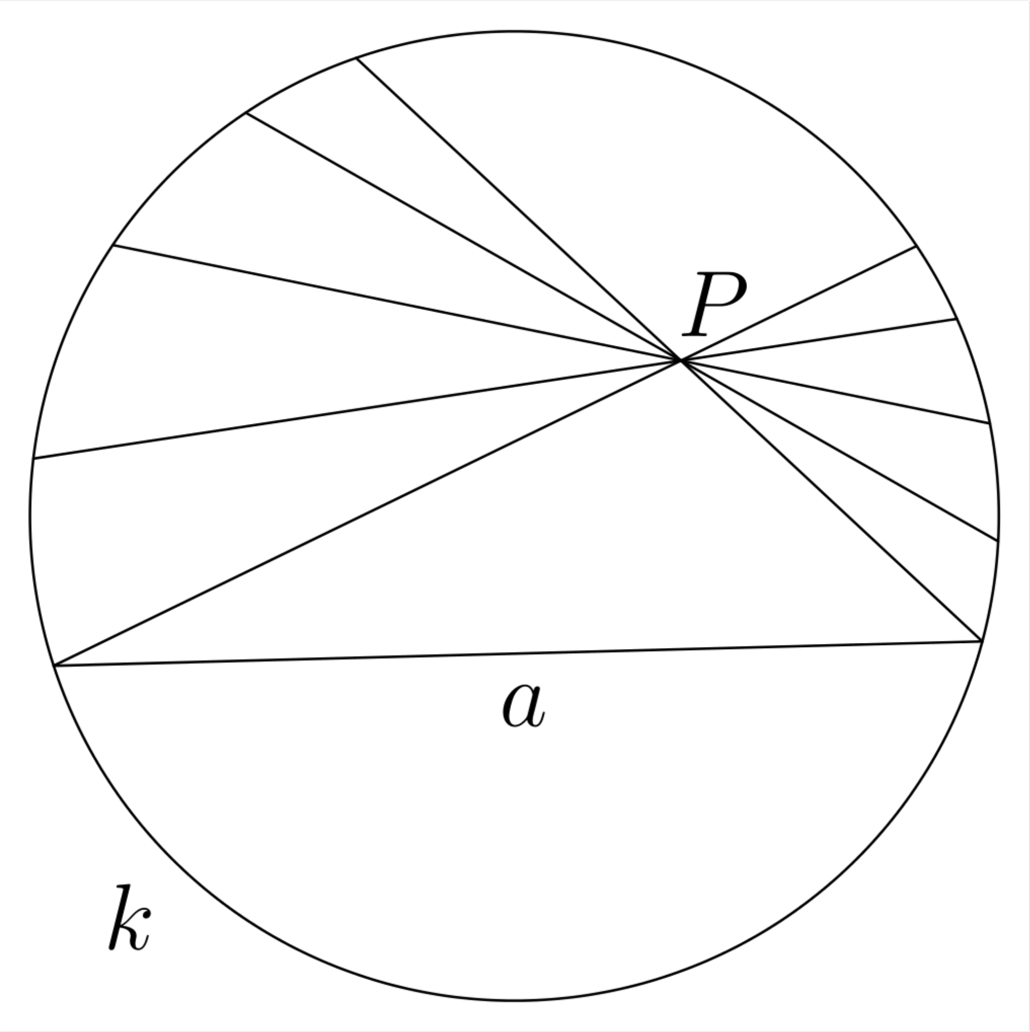
- Discovered by Janos Bolyai and Lobachevsky independently (and almost simulatenously)
- Shares the neutral axioms with Euclidean geometry, but postulates the converse of V:
  - ~V (using Playfair): *For any given line and point not on said line, in the plane containing both the line and the point there are at least two distinct lines through the point that do not intersect the line.*
- All ‘equivalent statements’ no longer true

HYPERBOLIC GEOMETRY

- Beltrami, 1868: Models for the Euclidean and hyperbolic plane can be constructed from one another, reveals *relative consistency* of Euclidean and hyperbolic geometry.
- Implies *independence of V from axioms I-IV!*
  - Proof: Assume there exists a proof of V from I-IV. Then hyperbolic geometry is inconsistent (axiom ~V contradicts proved result). But hyperbolic geometry is consistent relative to Euclidean geometry, hence there can be no neutral proof of V. Under the same hypothesis that Euclidean geometry is consistent, there can also be no disproof of V.
- Had the endeavour to prove V in the pursuit of elegance succeeded, Euclidean geometry would have been shown to be inconsistent!

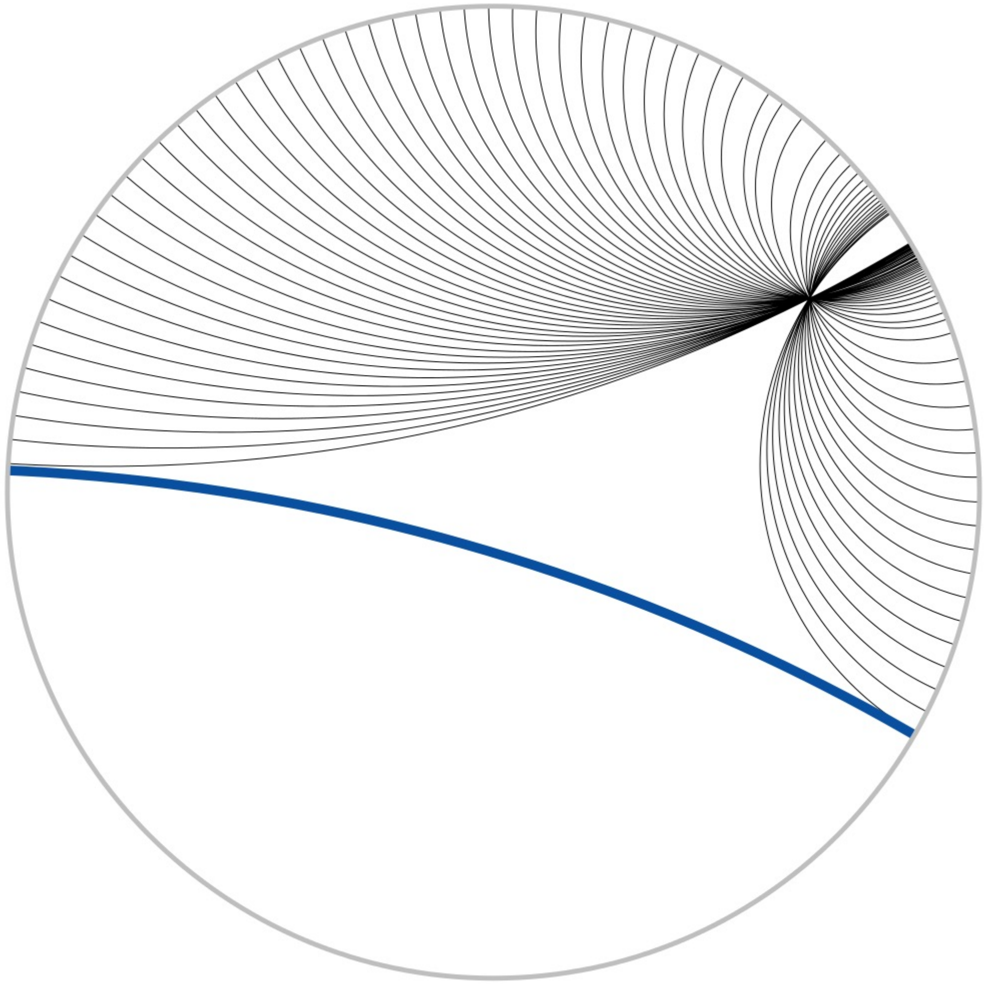
THE KLEIN DISK

- Open unit disk represents entire hyperbolic plane
- Boundary circle the 'limit'
- Points in model: Euclidean points in circle
- Lines in model: (open) chords of limit circle
- Not conformal (angles and circles distorted)
- V clearly doesn't hold



THE POINCARÉ DISK

- Unit disk represents entire hyperbolic plane
- Points in model: Euclidean points in circle
- Lines in model: all circular arcs orthogonal to boundary circle + all diameters
- Conformal (angles given by Euclidean measure between rays)
- V clearly doesn't hold



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