# The Poincaré half plane I

Karina Kniel (Talk 4) 3rd Nov 2021

## 1 Definitions

**Definition 1.1** (hyperbolic half plane)**.**

$$\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\ \} \subseteq \mathbb{C}$$

**Definition 1.2** (hyperbolic metric (line element))**.**

$$d_{\mathbb{H}} = \frac{\sqrt{dx^2 + dy^2}}{y}$$

**Definition 1.3** (hyperbolic length)**.**

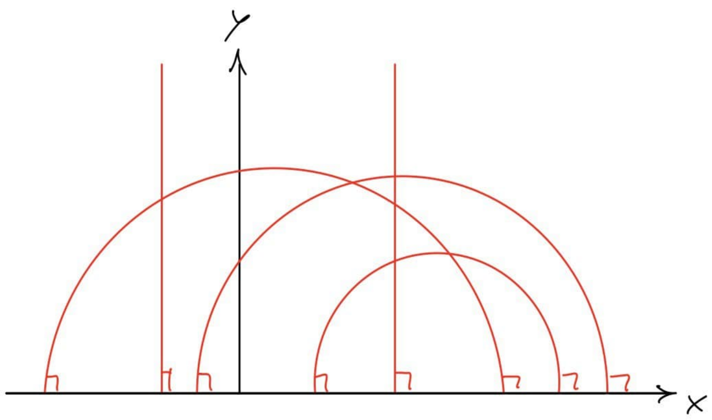
$$l_{\mathbb{H}} = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}$$

**Definition 1.4** (hyperbolic distance)**.** Let P, Q $\in \mathbb{H}$ be two points.

$$d_{\mathbb{H}}(P, Q) = inf\{l_{\mathbb{H}}(\gamma) \mid \gamma \textit{ goes from } P \textit{ to } Q\}$$

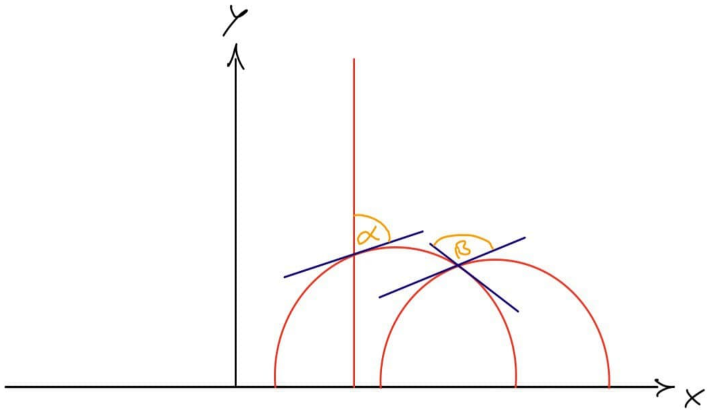
## 2 Geodesics

The geodesics in $\mathbb{H}$ are either vertical lines or half circles with centre at the real axis.



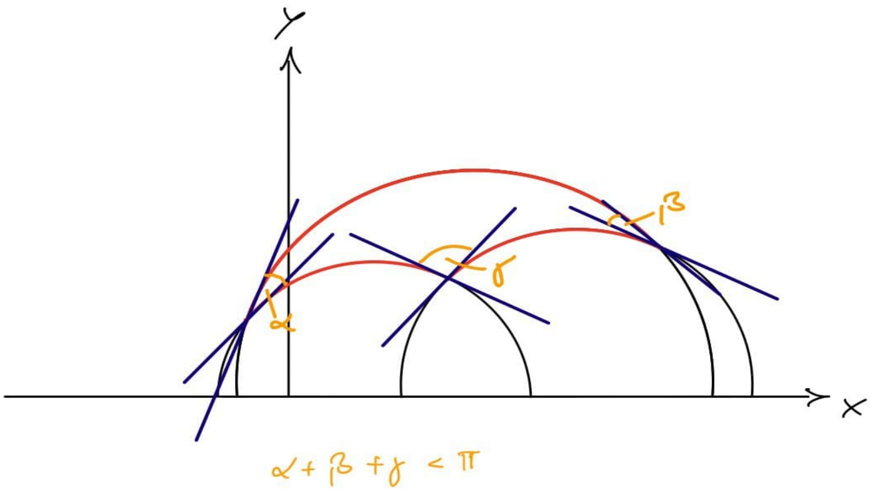
# 3 Angles

Angles in $\mathbb{H}$ are measured by taking the angle between the two tangents of the geodesics at the point where they intersect.




# 4 Curvature

The curvature of $\mathbb{H}$ is -1. You can visualize its constant negatice curvature by drawing a triangle which consists of three geodesics meeting each other in three different points.



# 5 Some examples of isometries

**5.1** Homothety defined by $\phi(x, y) = (\lambda x, \lambda y)$ for $\lambda>0$

**5.2** Reflection defined by $\phi(x, y) = (-x, y)$ (across the y-axis)

**5.3** Horizontal translation defined by $\phi(x, y) = (x + x_0, y)$

**5.4** Standard inversion defined by $\phi(x, y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$

# References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[5] B. Xu. *Introduction to Hyperbolic Geometry, Part I*, 2021.