clear pf that

$$\mathrm{Var}(e_i) = \sigma^2\left(1 - \frac{1}{n} - \frac{(x_i-\bar{x})^2}{SSX}\right)$$

Pf:

$$\mathrm{Var}(e_i) = \mathrm{Var}(y_i - \hat{y}_i)$$

$$= \mathrm{Var}(y_i) + \mathrm{Var}(\hat{y}_i) - 2\mathrm{Cov}(y_i, \hat{y}_i)$$

$$= \overset{A}{\sigma^2} + \overset{B}{\left[\mathrm{Var}(\hat{\beta}_0 + \hat{\beta}_1 x_i)\right]} - 2\overset{C}{\left[\mathrm{Cov}(y_i, \hat{\beta}_0 + \hat{\beta}_1 x_i)\right]}$$

Notice:

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i = \sum_{j=1}^{n} c_j y_j + \left(\sum_{j=1}^{n} k_j y_j\right) x_i$$

$$= \sum_{j=1}^{n} (c_j + x_i k_j)\, y_j$$

$$= \sum_{j=1}^{n} \left(\frac{1}{n} - \bar{x} k_j + x_i k_j\right) y_j$$

$$= \sum_{j=1}^{n} \left(\frac{1}{n} + (x_i - \bar{x}) k_j\right) y_j$$

B

$$Var(\hat{\beta}_0+\hat{\beta}_1 x_i) = Var\left(\sum_{j=1}^{n}\left(\frac{1}{n}+(x_i-\bar{x})k_j\right)y_j\right)$$

$$= \sum_{j=1}^{n} Var\left(\left(\frac{1}{n}+(x_i-\bar{x})k_j\right)y_j\right)$$

$$= \sum_{j=1}^{n}\left(\frac{1}{n}+(x_i-\bar{x})k_j\right)^2\sigma^2$$

$$= \sigma^2\sum_{j=1}^{n}\left(\frac{1}{n^2}+\frac{2(x_i-\bar{x})k_j}{n}+\left[(x_i-\bar{x})k_j\right]^2\right)$$

$$= \sigma^2\left(\frac{1}{n}+\frac{2(x_i-\bar{x})}{n}\overset{0}{\cancel{\sum_{j=1}^{n}k_j}}+(x_i-\bar{x})^2\overset{1/SSX}{\cancel{\sum_{j=1}^{n}k_j^2}}\right)$$

$$= \sigma^2\left(\frac{1}{n}+\frac{(x_i-\bar{x})^2}{SSX}\right)$$

(c)

$$\mathrm{Cov}(y_i, \hat{y}_i)$$

$$= \mathrm{Cov}\left(y_i, \sum_{j=1}^{n} \left(\frac{1}{n} + (x_i - \bar{x})k_j\right) y_j\right)$$

$$= \sum_{j=1}^{n} \mathrm{Cov}\left(y_i, \left(\frac{1}{n} + (x_i - \bar{x})k_j\right) y_j\right)$$

$\downarrow$ only $\neq 0$ if $i = j$

$$= \mathrm{Cov}\left(y_i, \left(\frac{1}{n} + (x_i - \bar{x})k_i\right) y_i\right)$$

$$= \left(\frac{1}{n} + (x_i - \bar{x})k_i\right) \mathrm{Cov}(y_i, y_i)$$

$$= \left(\frac{1}{n} + (x_i - \bar{x})k_i\right) \sigma^2$$

$$= \left(\frac{1}{n} + \frac{(x_i - \bar{x})(x_i - \bar{x})}{SSX}\right) \sigma^2$$

$$= \left(\frac{1}{n} + \frac{(x_i - \bar{x})^2}{SSX} \right) \sigma^2$$

$\Rightarrow$

$$Var(e_i) = \overset{A}{\sigma^2} + \overset{B}{Var(\hat{\beta}_0 + \hat{\beta}_1 x_i)} - \overset{C}{2Cov(y_i, \hat{\beta}_0 + \hat{\beta}_1 x_i)}$$

$$= \sigma^2 + \sigma^2 \left(\frac{1}{n} + \frac{(x_i - \bar{x})^2}{SSX} \right) - 2\sigma^2 \left(\frac{1}{n} + \frac{(x_i - \bar{x})^2}{SSX} \right)$$

$$= \sigma^2 \left(1 - \frac{1}{n} - \frac{(x_i - \bar{x})^2}{SSX} \right)$$