Recap:

$$\hat{\beta}_1 \sim N\left(\beta_1, \sigma^2/SSX\right)$$

$$\hat{\beta}_0 \sim N\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{SSX}\right)\right)$$

$$e_i \overset{\cancel{iid}}{\sim} N\left(0, \sigma^2\left(1 - \frac{1}{n} - \frac{(x_i - \bar{x})^2}{SSX}\right)\right)$$

$$\hat{y}_i \sim N\left(\beta_0 + \beta_1 x_i, \underline{\qquad\qquad}\right)$$

?

Ex: $Var(\hat{y}_i)$

Properties of $\hat{y}_i$ and e_i:

① $\sum_{i=1}^{n} e_i = 0$ (why?) $\rightarrow$ $e_1 + \sum_{i=2}^{n} e_i = 0$

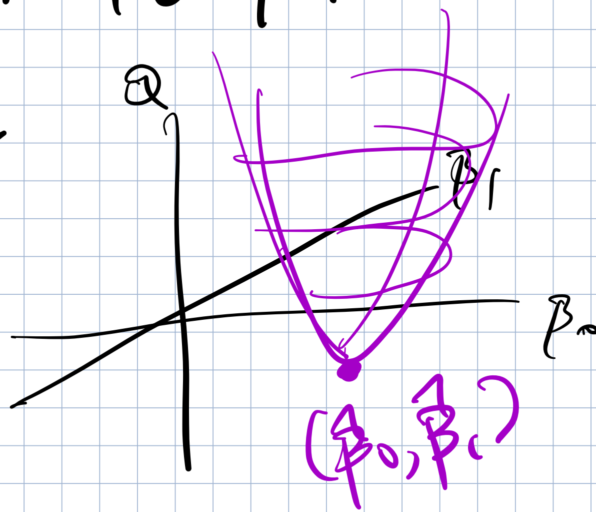
$e_1 = -\sum_{i=2}^{n} e_i$

② $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \hat{y}_i$

③ $\sum_{i=1}^{n} e_i^2$ is the global minimum of

$$Q(\beta_0,\beta_1)=\sum_{i=1}^{n}(y_i-\beta_0-\beta_1x_i)^2.$$

$$Q(\hat\beta_0,\hat\beta_1)=\sum_{i=1}^{n}e_i^2$$



④ $$\sum_{i=1}^{n}e_ix_i=0$$

⑤ $$\sum_{i=1}^{n}e_i\hat y_i=0$$

$$\sum_{i=1}^{n}e_i(\hat\beta_0+\hat\beta_1x_i)$$

$$=\sum_{i=1}^{n}e_i\hat\beta_0+\sum_{i=1}^{n}e_i\hat\beta_1x_i$$

$$=\hat\beta_0\underbrace{\cancelto{0}{\sum_i e_i}}_{(1)}+\hat\beta_1\underbrace{\cancelto{0}{\sum_i e_ix_i}}_{(4)}=0$$

SLR

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \; ; \; \varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$$

To get an idea of prediction error magnitude we would like an est of σ^2.

Let's say I really want an unbiased estimator $\hat{\sigma}^2$. $\Leftrightarrow E(\hat{\sigma}^2) = \sigma^2$.

Remark:

$$\boxed{\frac{\sum_{i=1}^{n} e_i^2}{\sigma^2} \sim \chi^2_{n-p}}$$

$p = \#$ of betas

In SLR, $p = 2$

Aside: what is a χ^2_{df} dist?

Constructing a χ^2_{df} RV:

If I had n std. normal RVs:

$$Z_1, Z_2, \ldots Z_n \overset{iid}{\sim} N(0,1).$$

Then

$$V = Z_1^2 + Z_2^2 + \cdots + Z_n^2 \sim \chi^2_n$$

"V is a chi-squared RV w/ n degrees of freedom."

$E(V) = n$

$Var(V) = 2n$

If $$\frac{\sum_i e_i^2}{\sigma^2} \sim \chi^2_{n-2}$$

$$\Rightarrow E\left(\frac{\sum_i e_i^2}{\sigma^2}\right) = n-2$$

$$\Rightarrow \frac{1}{\sigma^2} E\left(\sum_i e_i^2\right) = n-2$$

$$\Rightarrow \frac{1}{n-2} E\left(\sum_i e_i^2\right) = \sigma^2$$

$$\Rightarrow E\left(\frac{\sum_i e_i^2}{n-2}\right) = \sigma^2$$

$$\hat{\sigma}^2 = \frac{\sum_i e_i^2}{n-2}$$ mean squared error MSE

Hypothesis Testing & Confidence Intervals for SLR

Suppose I want to test some claim about the slope.

We know

$$\hat{\beta}_1 \sim N(\beta_1, \sigma^2/SSX)$$

$\boxed{H_0: \beta_1 = 2}$

$$z = \frac{\hat{\beta}_1 - \beta_1}{SE(\hat{\beta}_1)} = \frac{\hat{\beta}_1 - \beta_1}{\sigma/\sqrt{SSX}}$$

plug in $\hat{\sigma}$ for σ

$$t = \frac{\hat{\beta}_1 - \beta_1}{\widehat{SE}(\hat{\beta}_1)} = \frac{\hat{\beta}_1 - \beta_1}{\hat{\sigma}/\sqrt{SSX}}$$

We can construct a t-statistic to test the hypotheses:

$$H_0: \beta_1 = \beta_{10}$$ ← some #, commonly 0.

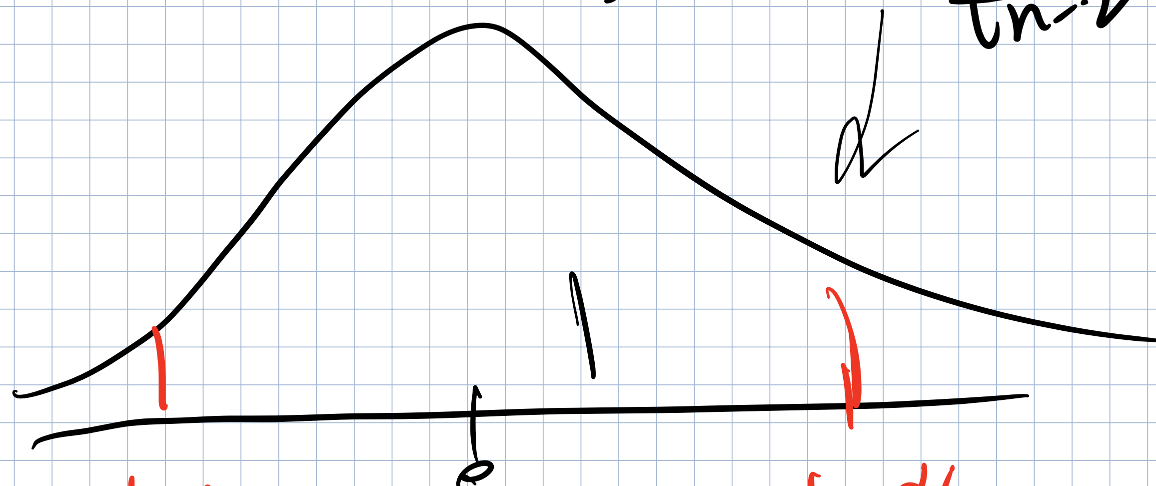
$$H_1: \beta_1 \neq \beta_{10}$$

The t-stat is then:

$$t = \frac{\hat{\beta}_1 - \beta_{10}}{\widehat{SE}(\hat{\beta}_1)} = \frac{\hat{\beta}_1 - \beta_{10}}{\hat{\sigma} / \sqrt{SSX}}$$

If $|t| > t^*_{1-\frac{\alpha}{2}, n-2} \Rightarrow$ reject H_0

t_{n-2}



$t_{\alpha/2}$ $t_{1-\alpha/2}$

$\hat{\mu}$ $\mu^{(0)}$

Ex:

Test the hyp.

$$H_0: \beta_1 = 2$$

$$H_1: \beta_1 \neq 2$$

$$t = \frac{\hat{\beta}_1 - 2}{\hat{\sigma}/\sqrt{SS_x}}$$

If we find $|t| > t^*_{1-\alpha/2,\, n-2}$ then we reject H_0. We have evidence that shows that the true slope is not actually β_{10}.

In the specific case where we let $H_0: \beta_1 = 0$ vs $H_1: \beta_1 \neq 0$,

if we reject H_0, we are claiming we have evidence that the slope is not 0.

$\Leftrightarrow$ "X is a sig. predictor of Y."

$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$

If $\beta_1 = 0$ $\Downarrow$

$y_i = \beta_0 + \varepsilon_i$ $\Rightarrow$ X is not predictive of Y

Same test for β_0:

α sig level

$H_0: \beta_0 = \beta_{00} = \beta_0^{(0)}$

$H_1: \beta_0 \neq \beta_{00} = \beta_0^{(0)}$

T-test:

$$t = \frac{\hat{\beta}_0 - \beta_{00}}{\widehat{SE}(\hat{\beta}_0)} = \frac{\hat{\beta}_0 - \beta_{00}}{\hat{\sigma}\sqrt{\frac{1}{n} + \frac{\bar{x}^2}{SSX}}}$$

Decision Rule:

if $|t| > t^{*}_{1-\frac{\alpha}{2}, n-2}$ then reject H_0.

CIs for β_0 & β_1

For the slope, a $100(1-\alpha)\%$ CI takes the form:

$$\hat{\beta}_1 \pm t^*_{1-\frac{\alpha}{2}, n-2} \widehat{SE}(\hat{\beta}_1)$$

$$\Rightarrow \boxed{\hat{\beta}_1 \pm t^*_{1-\frac{\alpha}{2}, n-2} \hat{\sigma}/\sqrt{SSX}}$$

Similarly for the intercept, the $100(1-\alpha)\%$ CI is

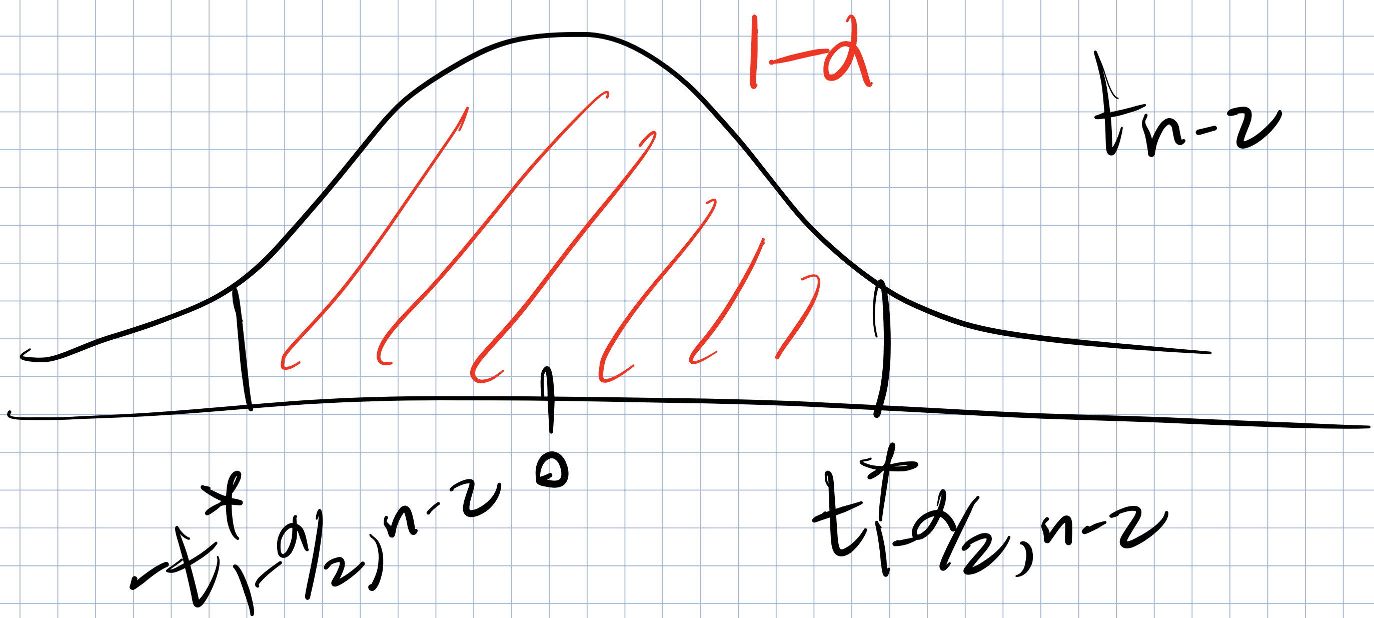
$$\hat{\beta}_0 \pm t^*_{1-\frac{\alpha}{2}, n-2} \widehat{SE}(\hat{\beta}_0)$$

$$\boxed{\hat{\beta}_0 \pm t^*_{1-\frac{\alpha}{2}, n-2} \hat{\sigma}\sqrt{\frac{1}{n} + \frac{\bar{x}^2}{SSX}}}$$

Idea for why

We know

$$t = \frac{\hat{\beta}_1 - \beta_1}{\widehat{SE}(\hat{\beta}_1)} \sim t_{n-2}$$



$$P\left(-t^*_{\dots} \le \frac{\hat{\beta}_1 - \beta_1}{\widehat{SE}(\hat{\beta}_1)} \le t^*_{\dots}\right) = 1-\alpha$$

$\Rightarrow$

$$P\left(-t^* \widehat{SE}(\hat{\beta}_1) \le \hat{\beta}_1 - \beta_1 \le t^* \widehat{SE}(\hat{\beta}_1)\right) = 1-\alpha$$

$\Rightarrow$

$$P\left(-\hat{\beta}_1 - t^* \widehat{SE}(\hat{\beta}_1) \le -\beta_1 \le -\hat{\beta}_1 + t^* \widehat{SE}(\hat{\beta}_1)\right) = 1-\alpha$$

$\Rightarrow$

$$P\left(\hat{\beta}_1 + t^* \widehat{SE}(\hat{\beta}_1) \ge \beta_1 \ge \hat{\beta}_1 - t^* \widehat{SE}(\hat{\beta}_1)\right) = 1-\alpha$$

$\Downarrow$

Take the bounds:

$$\hat{\beta}_1 \pm t^*_{1-\frac{\alpha}{2}, n-2} \widehat{SE}(\hat{\beta}_1)$$

This is the $100(1-\alpha)\%$ CI for β_1