

# Lecture 8: Camera Calibration

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Stanford Vision Lab

# What we will learn today?

- Review camera parameters
- Affine camera model (**Problem Set 2 (Q4)**)
- Camera calibration
- Vanishing points and lines (**Problem Set 2 (Q1)**)

Reading:

- [FP] Chapter 3
- [HZ] Chapter 7, 8.6

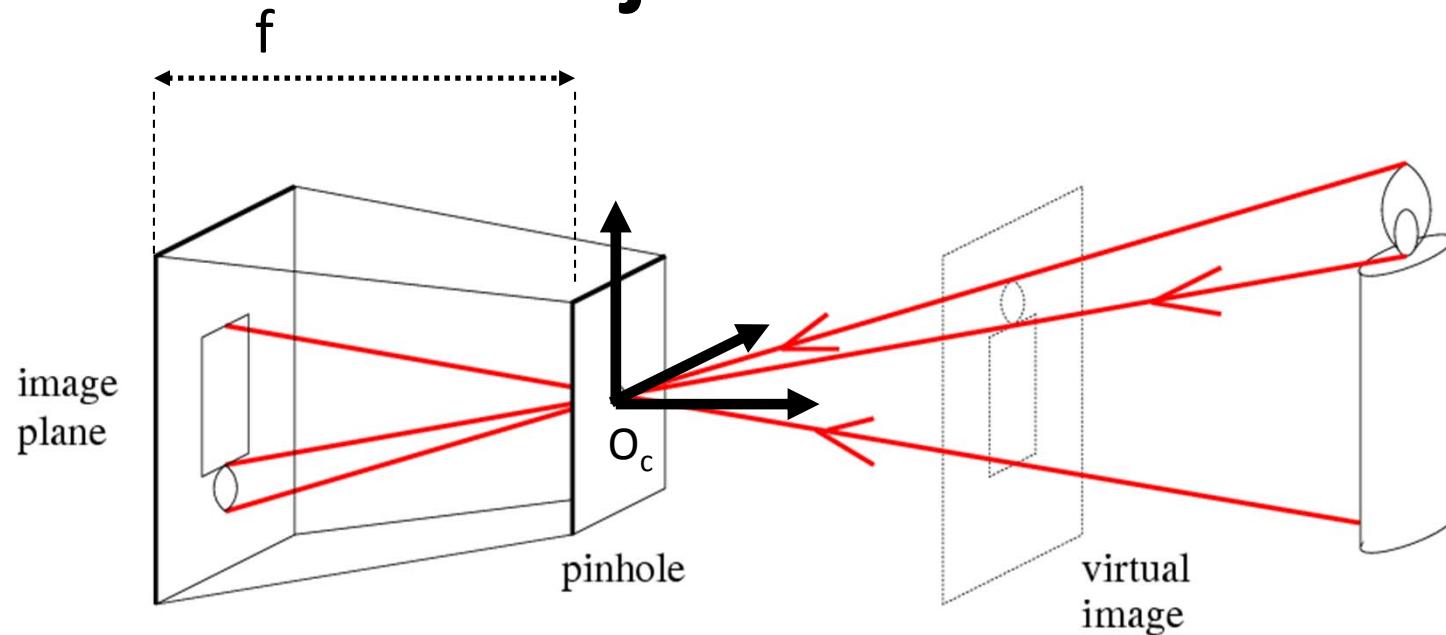
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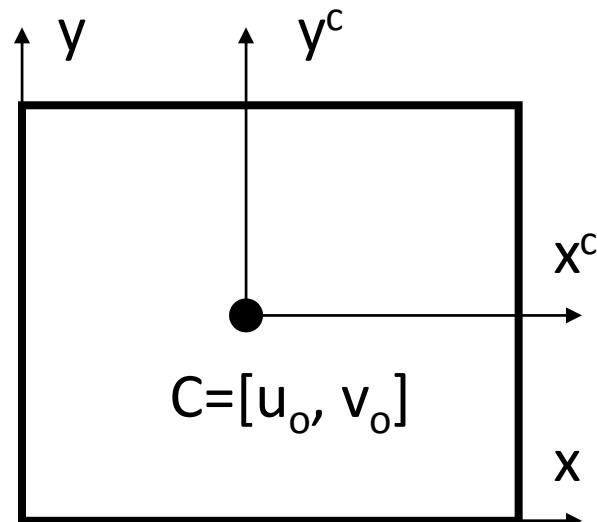
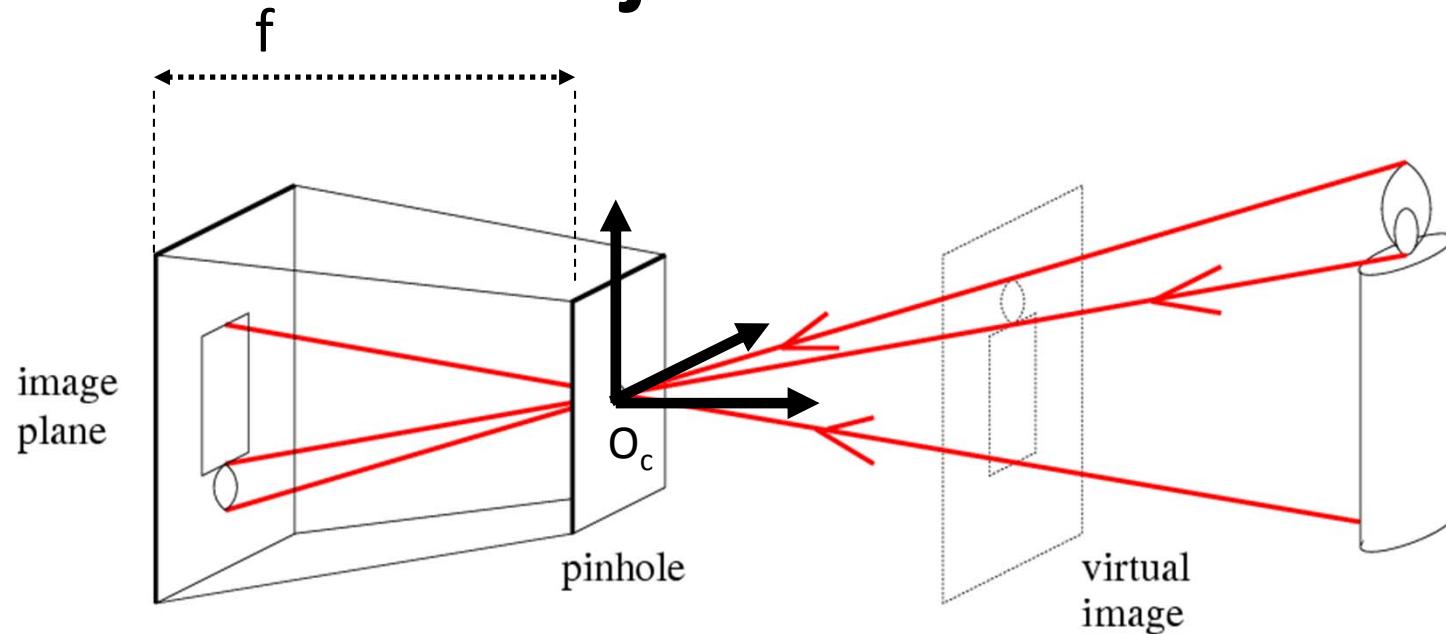
- [FP] Chapter 3
- [HZ] Chapter 7, 8.6

# Projective camera



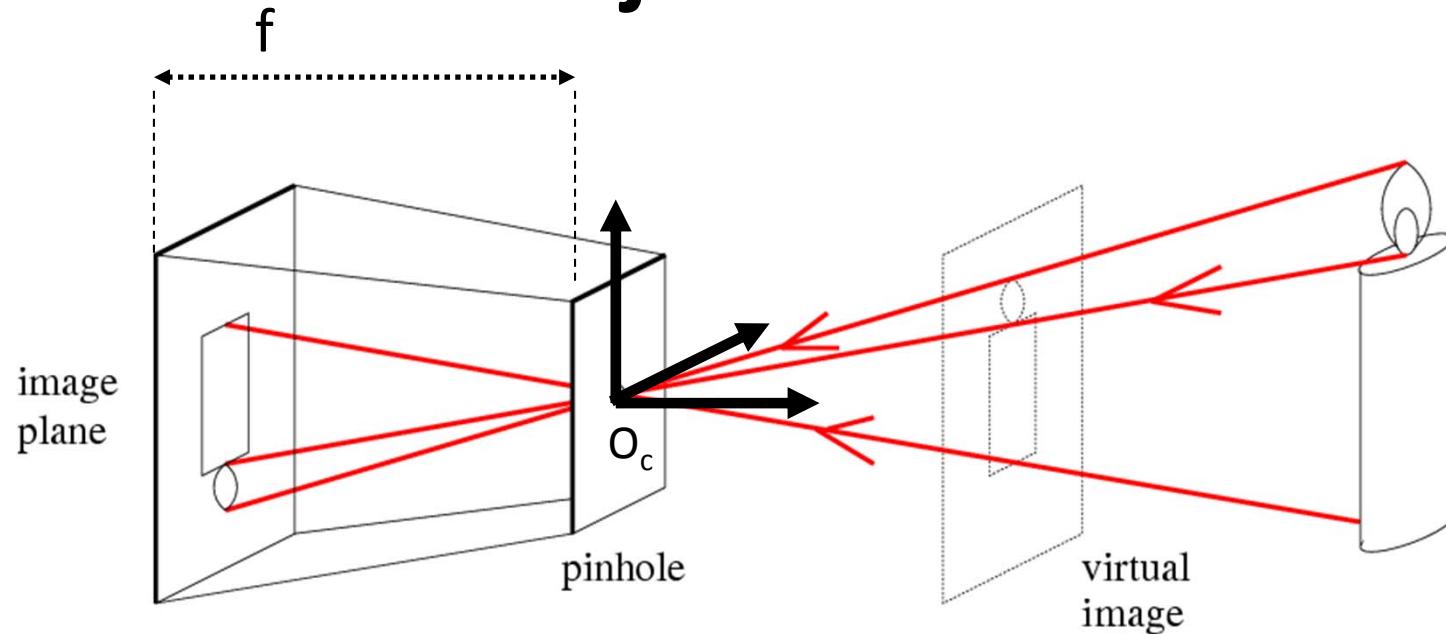
$f$  = focal length

# Projective camera



$f$  = focal length  
 $u_o, v_o$  = offset

# Projective camera



Units:  $k, l$  [pixel/m]

$f$  [m]

Non-square pixels

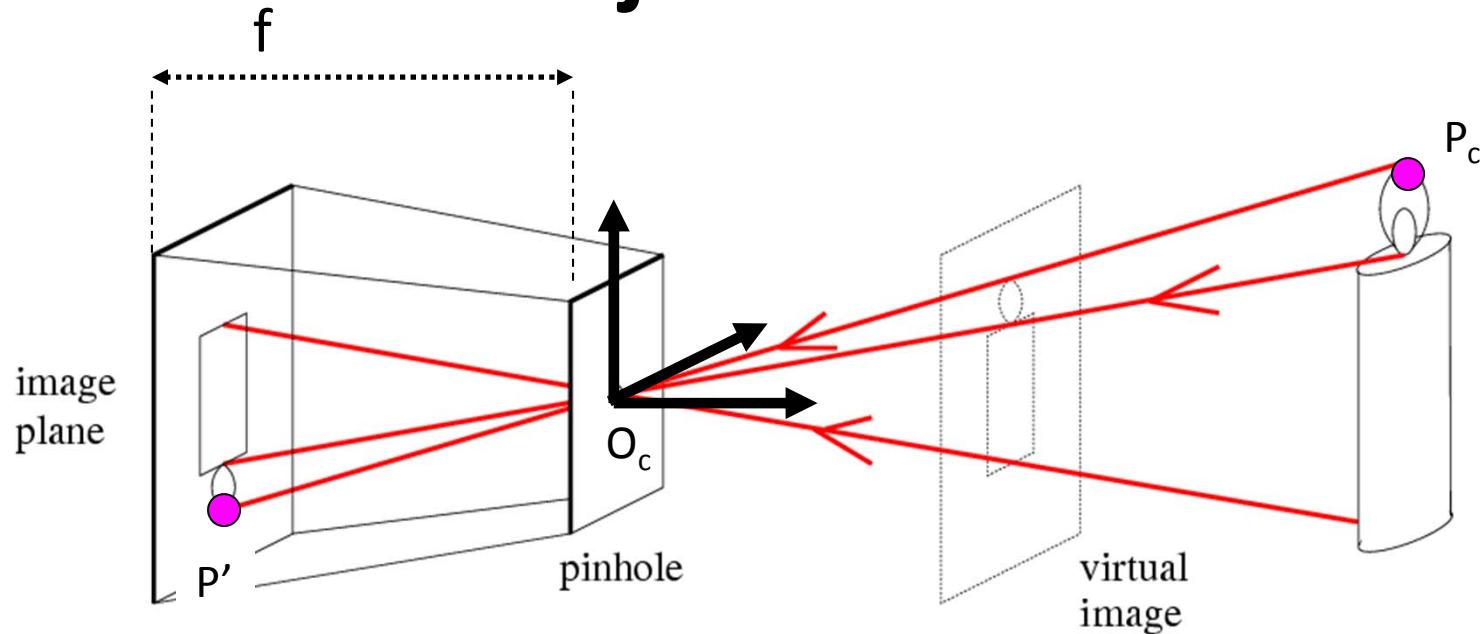
$\alpha, \beta$  [pixel]

$f$  = focal length

$u_o, v_o$  = offset

$\alpha, \beta \rightarrow$  non-square pixels

# Projective camera

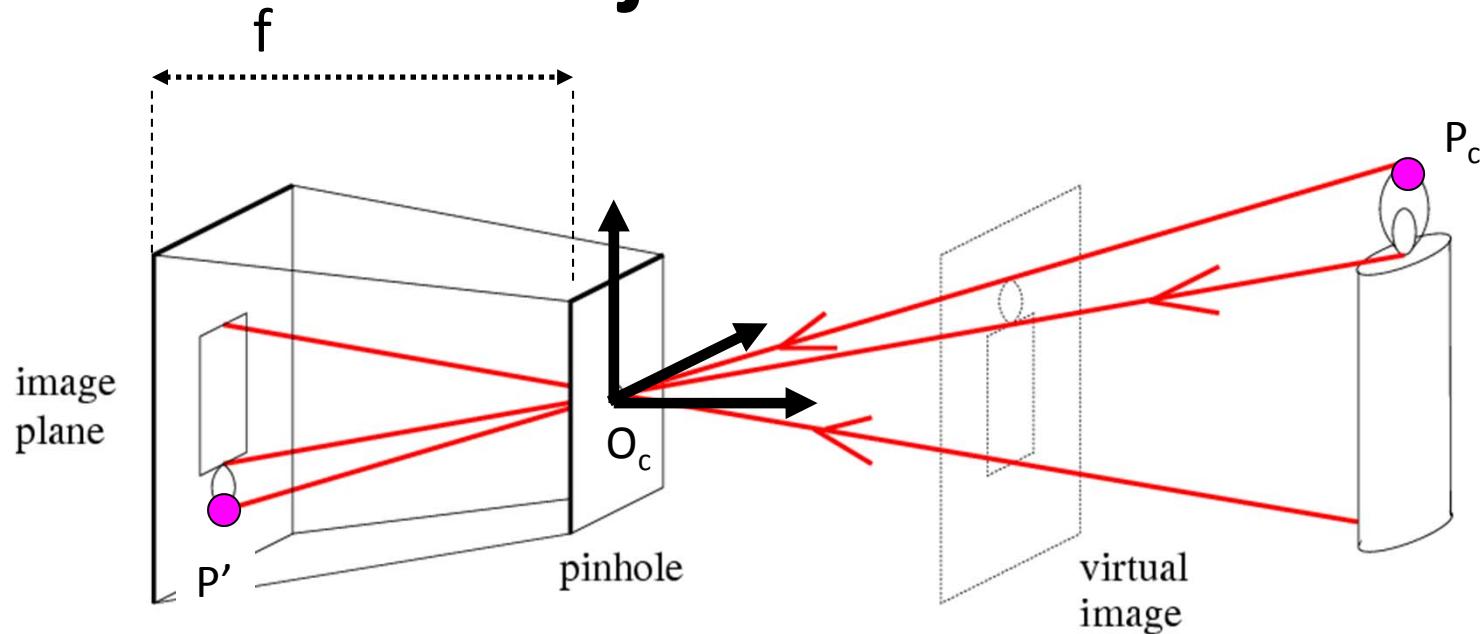


$$P' = \begin{bmatrix} \alpha & s & u_o \\ 0 & \beta & v_o \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$f$  = focal length  
 $u_o, v_o$  = offset  
 $\alpha, \beta$  → non-square pixels  
 $\theta$  = skew angle

$K$  has 5 degrees of freedom!

# Projective camera



$$P' = \begin{bmatrix} \alpha & -\alpha \cot \theta & u_o \\ 0 & \frac{\beta}{\sin \theta} & v_o \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$f$  = focal length

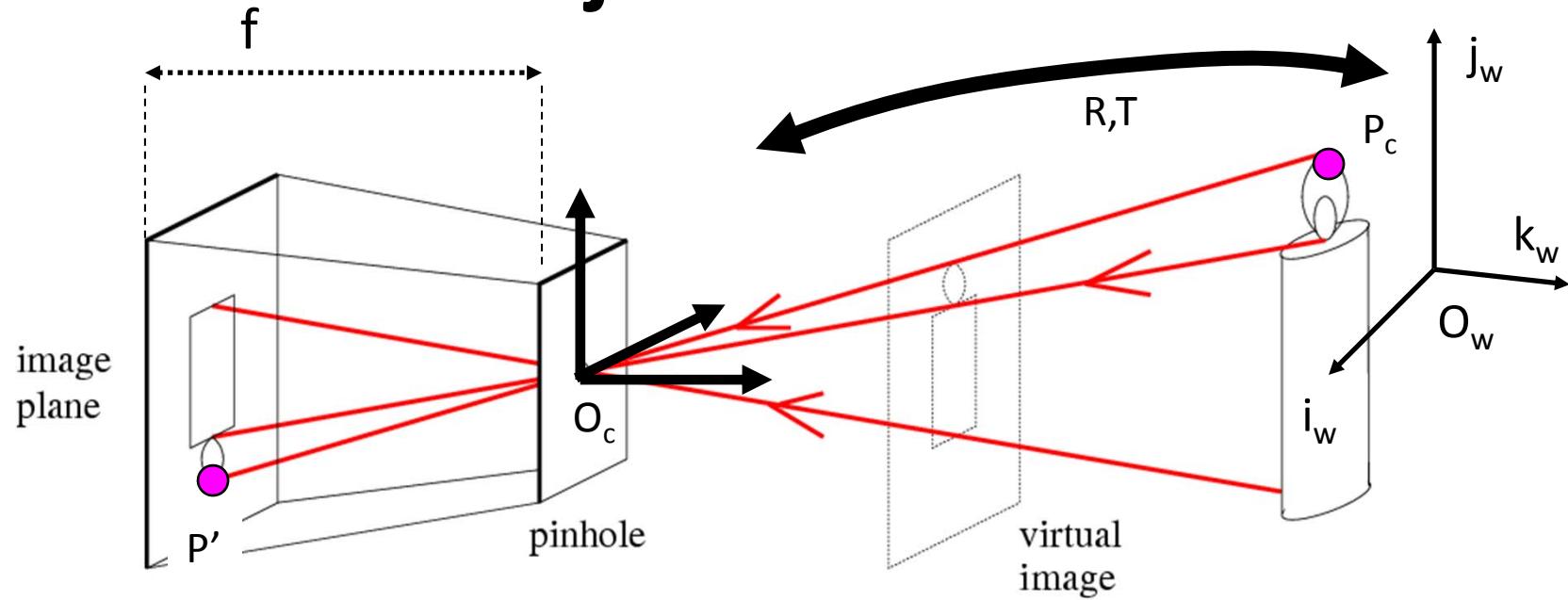
$u_o, v_o$  = offset

$\alpha, \beta \rightarrow$  non-square pixels

$\theta$  = skew angle

$K$  has 5 degrees of freedom!

# Projective camera

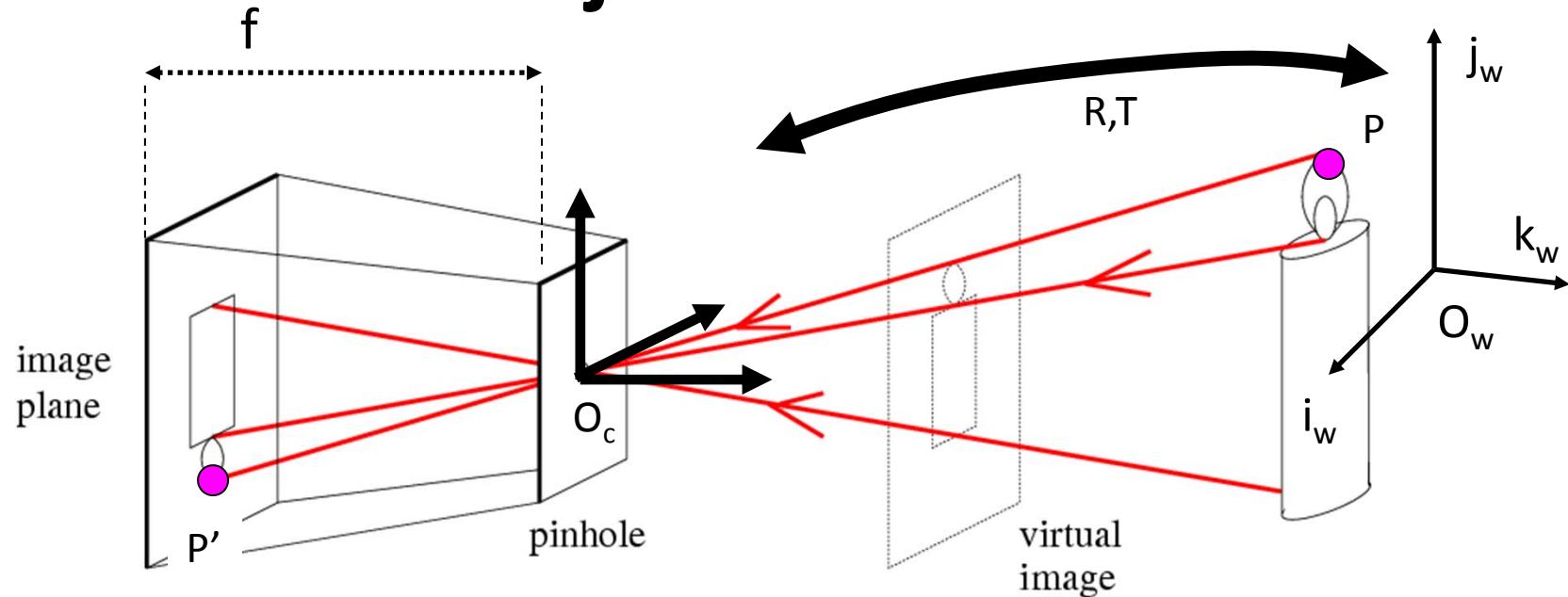


$$P = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}_{4 \times 4} P_w$$

$$T = -R \tilde{O}_c$$

$f$  = focal length  
 $u_o, v_o$  = offset  
 $\alpha, \beta$  → non-square pixels  
 $\theta$  = skew angle  
 $R, T$  = rotation, translation

# Projective camera



$$P' = M P_w$$

$$= \boxed{K} \begin{bmatrix} R & T \end{bmatrix} P_w$$

Internal (intrinsic) parameters

External (extrinsic) parameters

$f$  = focal length

$u_o, v_o$  = offset

$\alpha, \beta$  → non-square pixels

$\theta$  = skew angle

$R, T$  = rotation, translation

# Projective camera

$$P' = M P_w = \begin{bmatrix} K & [R \quad T] \end{bmatrix} P_w$$

Internal (intrinsic) parameters                      External (extrinsic) parameters

# Projective camera

$$P' = M P_w = K [R \quad T] P_w$$

$$M = \begin{pmatrix} \alpha \mathbf{r}_1^T - \alpha \cot \theta \mathbf{r}_2^T + u_0 \mathbf{r}_3^T & \alpha t_x - \alpha \cot \theta t_y + u_0 t_z \\ \frac{\beta}{\sin \theta} \mathbf{r}_2^T + v_0 \mathbf{r}_3^T & \frac{\beta}{\sin \theta} t_y + v_0 t_z \\ \mathbf{r}_3^T & t_z \end{pmatrix}_{3 \times 4}$$

$$K = \begin{bmatrix} \alpha & -\alpha \cot \theta & u_o \\ 0 & \frac{\beta}{\sin \theta} & v_o \\ 0 & 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} \mathbf{r}_1^T \\ \mathbf{r}_2^T \\ \mathbf{r}_3^T \end{bmatrix} \quad T = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

# Goal of calibration

$$P' = M P_w = K[R \ T] P_w$$

$$M = \begin{pmatrix} \alpha \mathbf{r}_1^T - \alpha \cot \theta \mathbf{r}_2^T + u_0 \mathbf{r}_3^T & \alpha t_x - \alpha \cot \theta t_y + u_0 t_z \\ \frac{\beta}{\sin \theta} \mathbf{r}_2^T + v_0 \mathbf{r}_3^T & \frac{\beta}{\sin \theta} t_y + v_0 t_z \\ \mathbf{r}_3^T & t_z \end{pmatrix}_{3 \times 4}$$

$$K = \begin{bmatrix} \alpha & -\alpha \cot \theta & u_o \\ 0 & \frac{\beta}{\sin \theta} & v_o \\ 0 & 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} \mathbf{r}_1^T \\ \mathbf{r}_2^T \\ \mathbf{r}_3^T \end{bmatrix} \quad T = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Estimate intrinsic and extrinsic parameters  
from 1 or multiple images

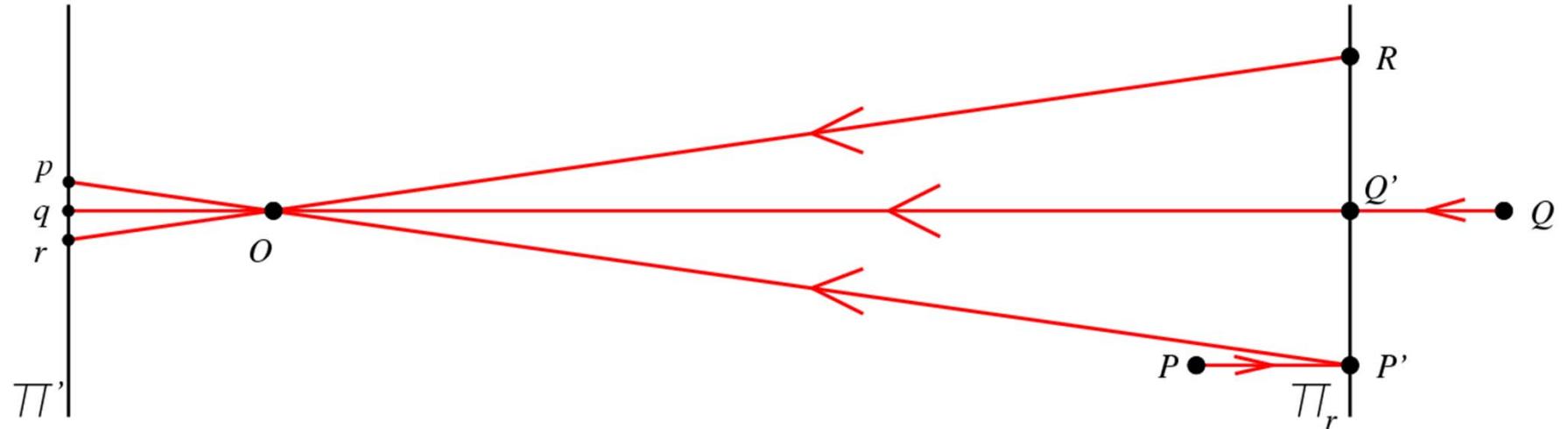
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- Review camera parameters
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Reading:

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# Weak perspective projection

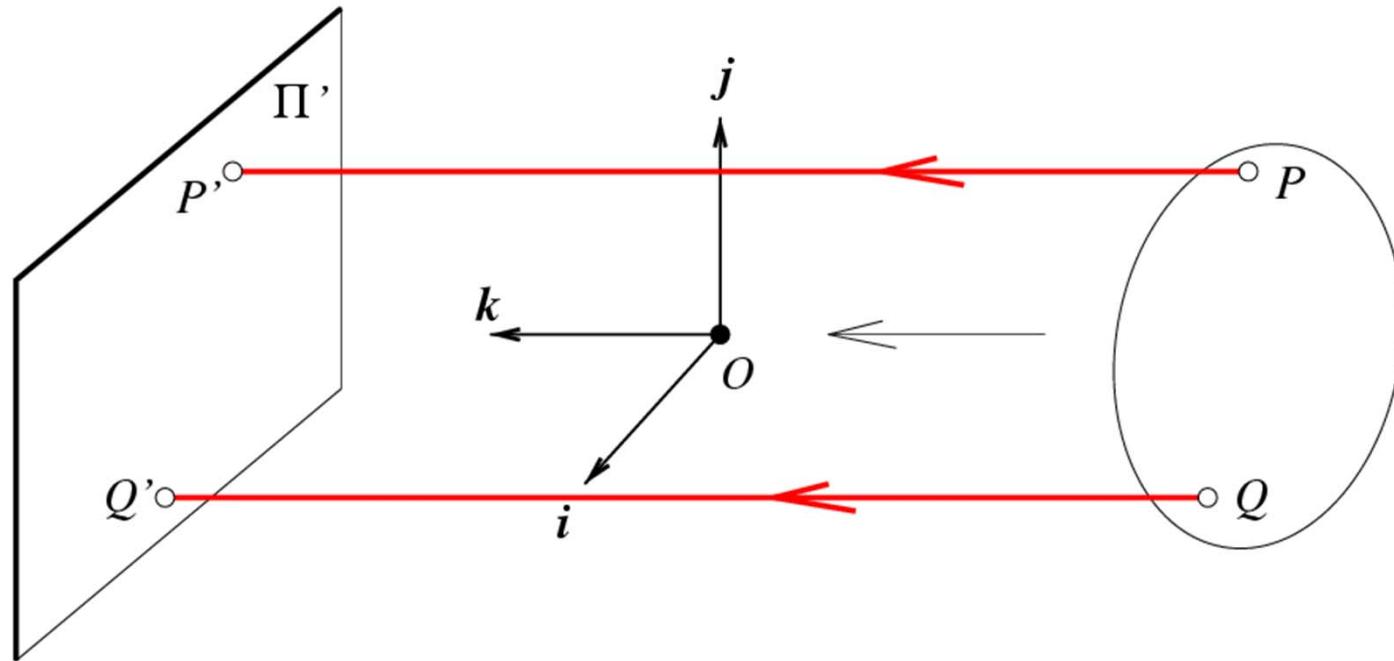


$$\begin{cases} x' = -mx \\ y' = -my \end{cases}$$

where  $m = -\frac{f'}{z_0}$  = magnification

Relative scene depth is small compared to its distance from the camera

# Orthographic (affine) projection



$$\begin{cases} x' = x \\ y' = y \end{cases}$$

Distance from center of projection to image plane is infinite

# Affine cameras

$$P' = K \begin{bmatrix} R & T \end{bmatrix} P$$

Affine case

$$K = \begin{bmatrix} \alpha_x & s & 0 \\ 0 & \alpha_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M = K \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$$

Parallel projection matrix

Compared to

Projective case

$$K = \begin{bmatrix} \alpha_x & s & x_o \\ 0 & \alpha_y & y_o \\ 0 & 0 & 1 \end{bmatrix}$$

$$M = K \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$$

# Remember....

Affinities:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_a \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Projectivities:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & t \\ v & b \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_p \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

# Affine cameras

We can obtain a more compact formulation than:  $P' = K \begin{bmatrix} R & T \end{bmatrix} P$

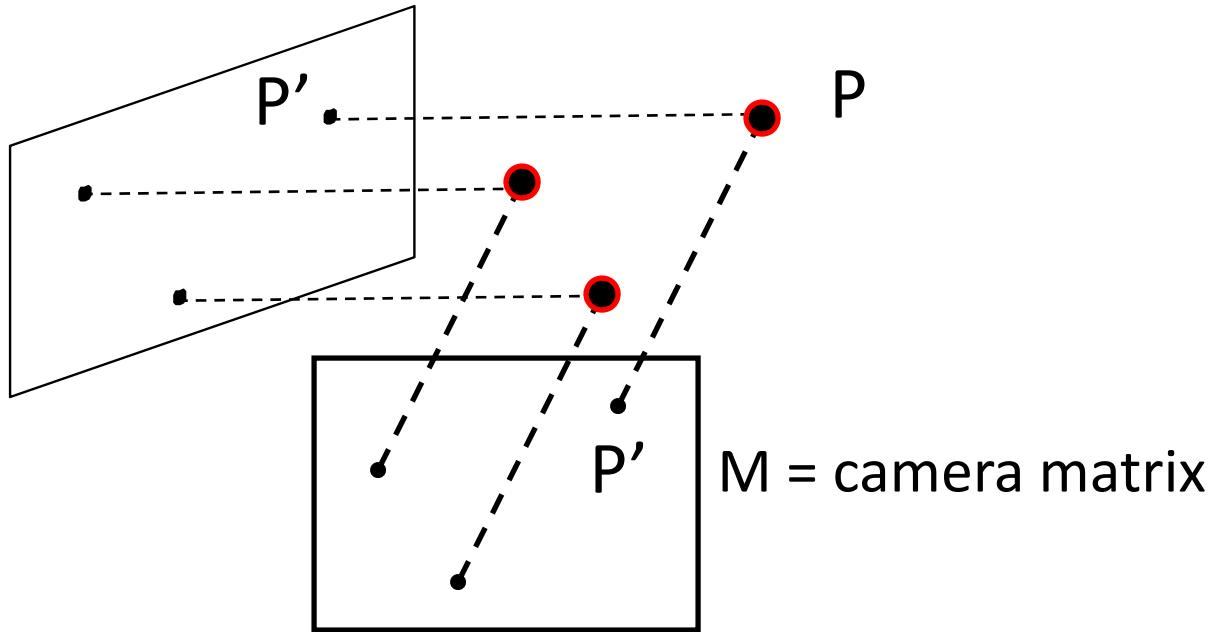
$$K = \begin{bmatrix} \alpha_x & 0 & 0 \\ 0 & \alpha_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M = K \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$$

$$M = [3 \times 3 \text{ affine}] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} [4 \times 4 \text{ affine}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$P' = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \mathbf{A}P + \mathbf{b} = M_{Euc} \begin{bmatrix} P \\ 1 \end{bmatrix}$$
$$M_{Euc} = M = [\mathbf{A} \quad \mathbf{b}]$$

# Affine cameras

To recap:



$M$  = camera matrix

$$P' = \begin{pmatrix} u \\ v \end{pmatrix} = AP + b = M \begin{bmatrix} P \\ 1 \end{bmatrix}; \quad M = [A \quad b]$$

[non-homogeneous image coordinates]

This notation is useful when we'll discuss affine structure from motion

# Affine cameras

- Weak perspective much simpler math.
  - Accurate when object is small and distant.
  - Most useful for recognition.
- Pinhole perspective much more accurate for scenes.
  - Used in structure from motion.

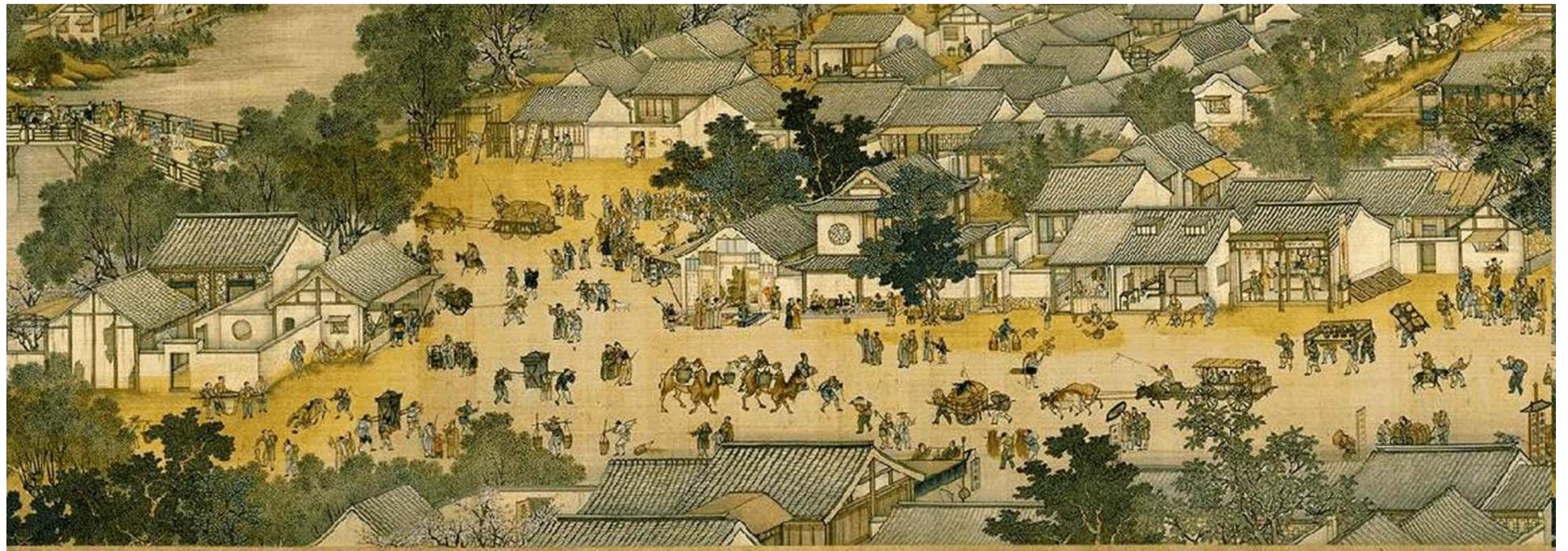
# Weak perspective projection - examples



*The Kangxi Emperor's Southern Inspection Tour (1691-1698)* By Wang Hui

You tube video – click here

# Weak perspective projection - examples



*Qingming Festival by the Riverside*

Zhang Zeduan ~900 AD

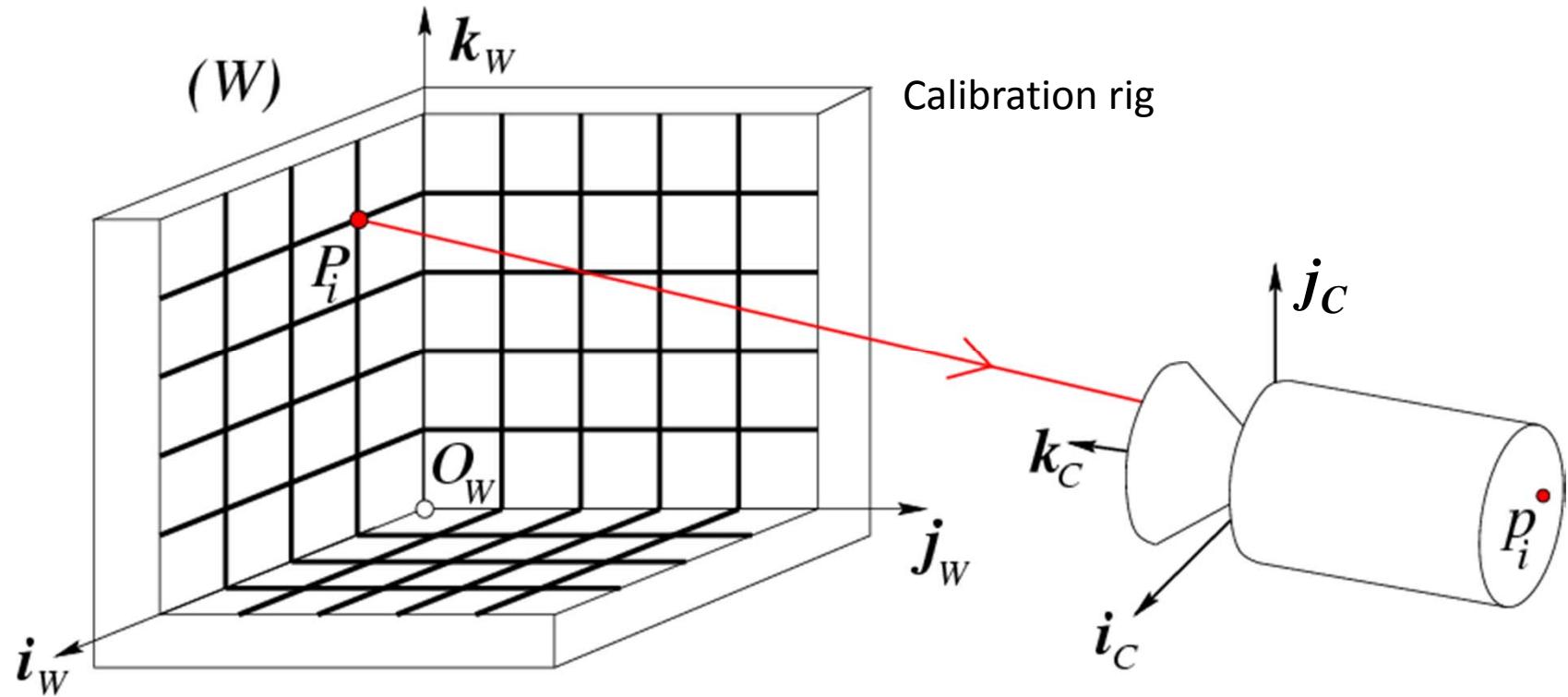
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# Calibration Problem

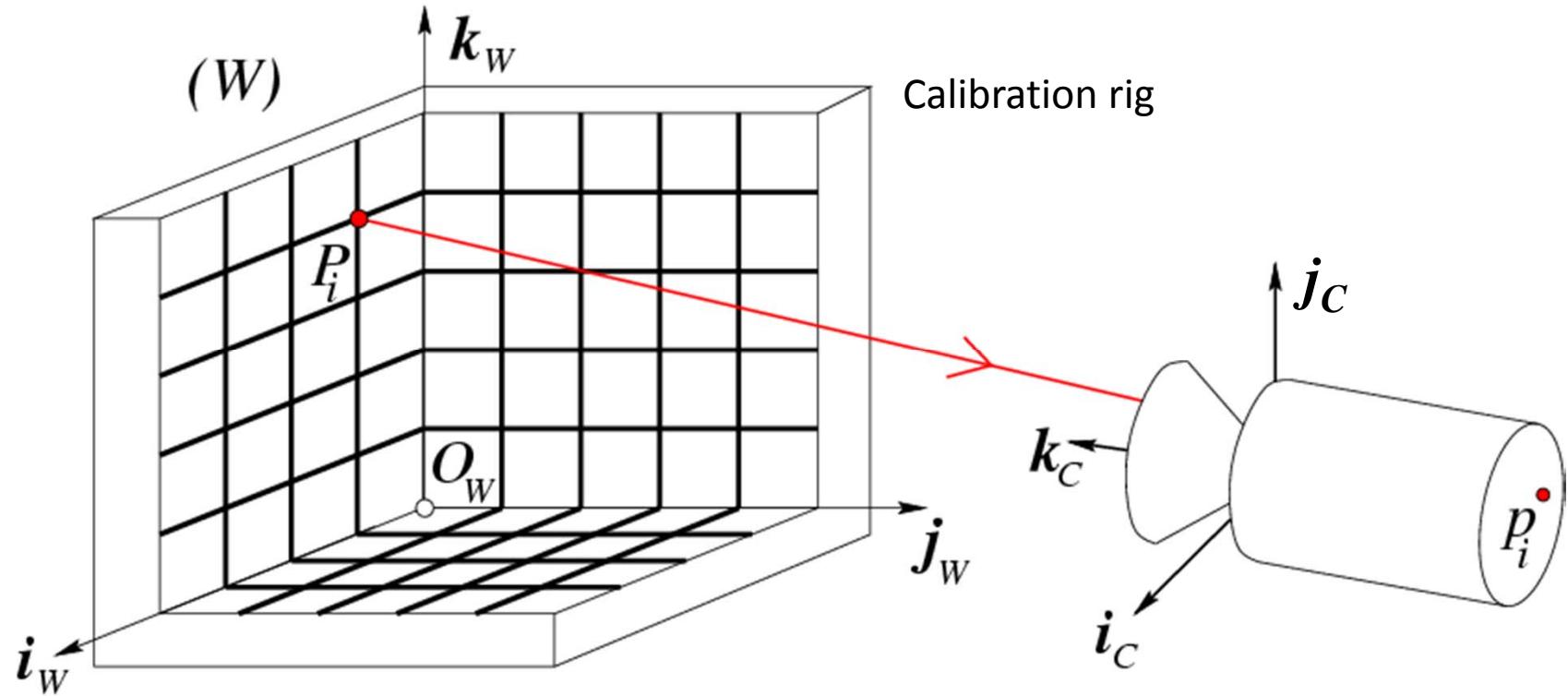


- $P_1 \dots P_n$  with **known** positions in  $[O_w, i_w, j_w, k_w]$
  - $p_1, \dots p_n$  **known** positions in the image
- Goal:** compute intrinsic and extrinsic parameters

Remember the “digital Michelangelo project”?



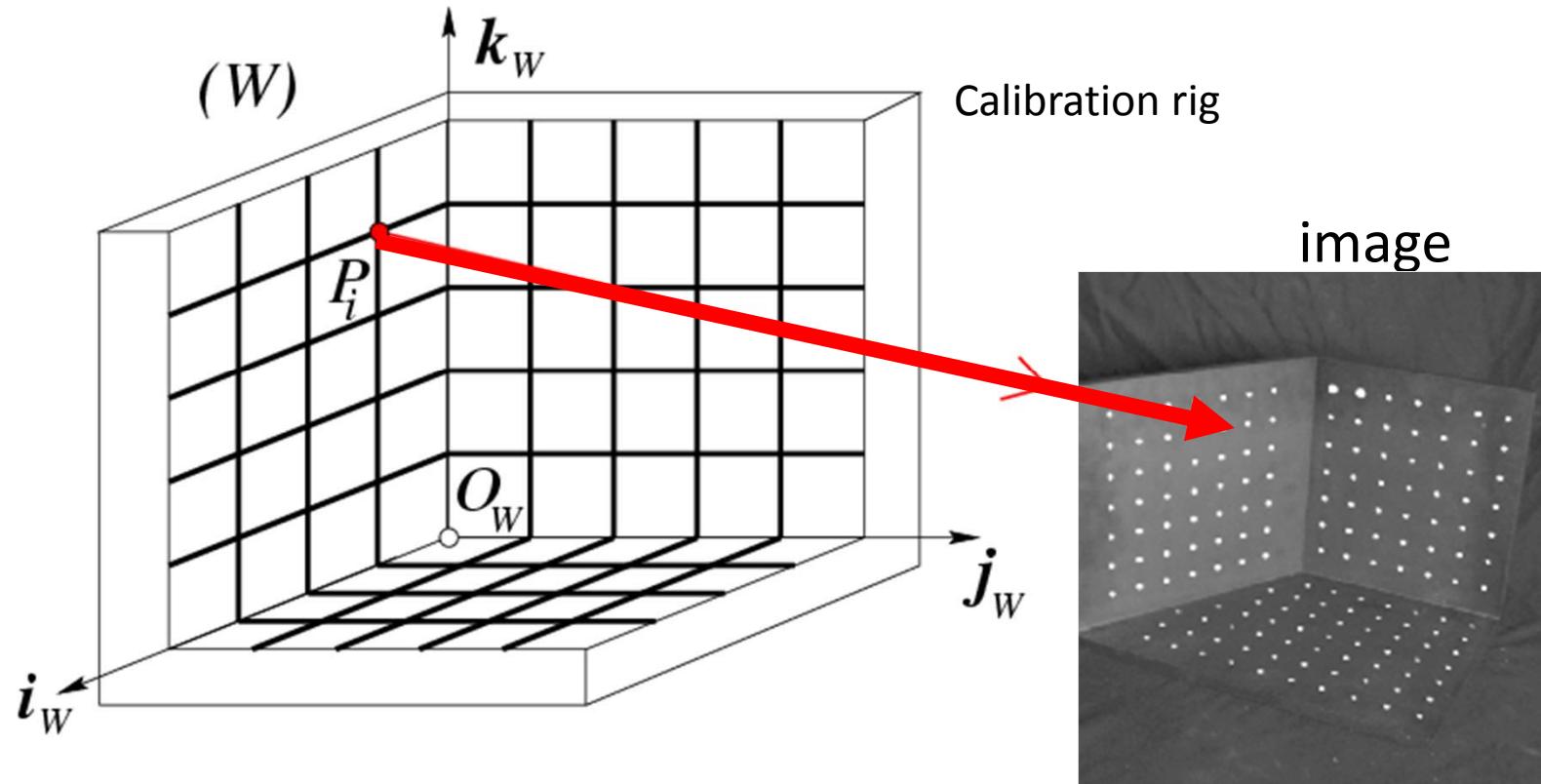
# Calibration Problem



How many correspondences do we need?

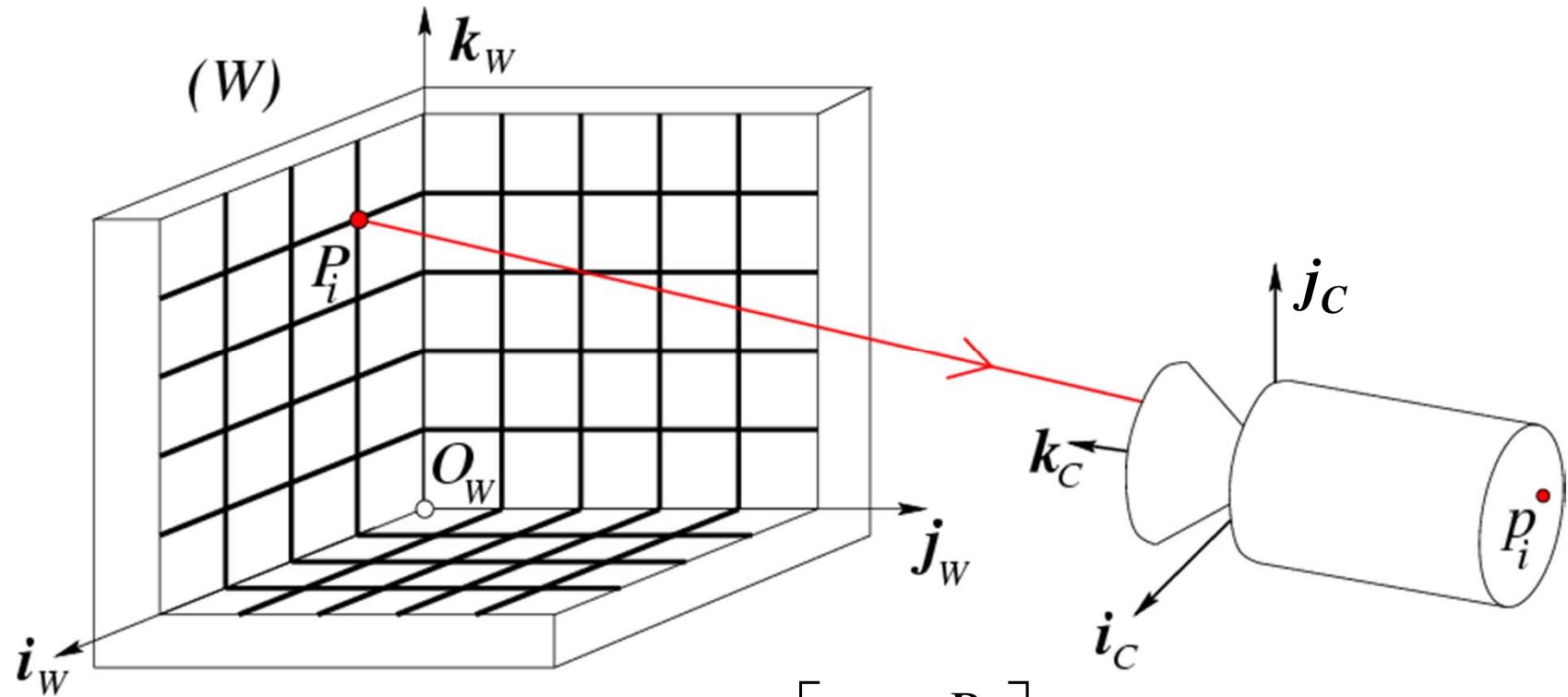
- M has 11 unknowns
- We need 11 equations
- 6 correspondences would do it

# Calibration Problem



In practice: user may need to look at the image and select the  $n \geq 6$  correspondences

# Calibration Problem



$$p_i \rightarrow M \ P_i \rightarrow p_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} \mathbf{m}_1 \cdot P_i \\ \mathbf{m}_3 \cdot P_i \\ \mathbf{m}_2 \cdot P_i \\ \mathbf{m}_3 \cdot P_i \end{bmatrix}$$

in pixels

$$M = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix}$$

# Calibration Problem

$$\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{m}_1 P_i}{\mathbf{m}_3 P_i} \\ \frac{\mathbf{m}_2 P_i}{\mathbf{m}_3 P_i} \end{bmatrix}$$

$$u_i = \frac{\mathbf{m}_1 P_i}{\mathbf{m}_3 P_i} \rightarrow u_i(\mathbf{m}_3 P_i) = \mathbf{m}_1 P_i \rightarrow u_i(\mathbf{m}_3 P_i) - \mathbf{m}_1 P_i = 0$$

$$v_i = \frac{\mathbf{m}_2 P_i}{\mathbf{m}_3 P_i} \rightarrow v_i(\mathbf{m}_3 P_i) = \mathbf{m}_2 P_i \rightarrow v_i(\mathbf{m}_3 P_i) - \mathbf{m}_2 P_i = 0$$

# Calibration Problem

$$\left\{ \begin{array}{l} u_1(\mathbf{m}_3 P_1) - \mathbf{m}_1 P_1 = 0 \\ v_1(\mathbf{m}_3 P_1) - \mathbf{m}_2 P_1 = 0 \\ \vdots \\ u_i(\mathbf{m}_3 P_i) - \mathbf{m}_1 P_i = 0 \\ v_i(\mathbf{m}_3 P_i) - \mathbf{m}_2 P_i = 0 \\ \vdots \\ u_n(\mathbf{m}_3 P_n) - \mathbf{m}_1 P_n = 0 \\ v_n(\mathbf{m}_3 P_n) - \mathbf{m}_2 P_n = 0 \end{array} \right.$$

# Block Matrix Multiplication

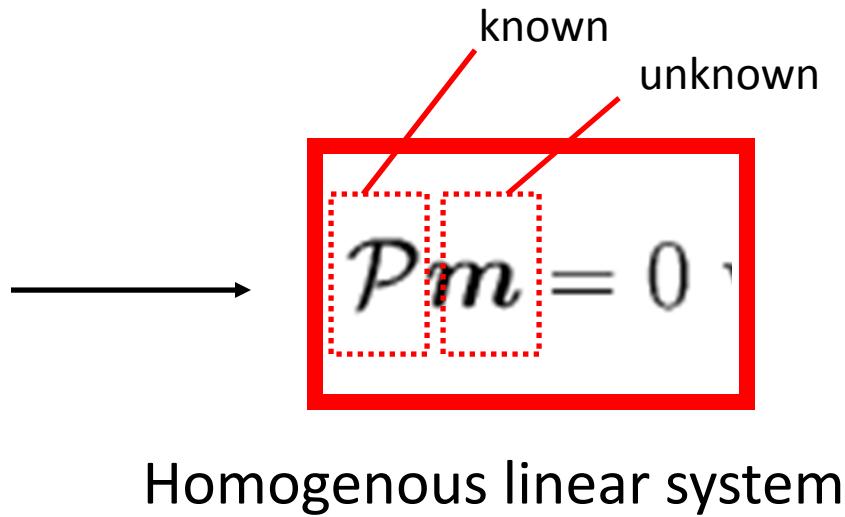
$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

What is  $AB$  ?

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

# Calibration Problem

$$\left\{ \begin{array}{l} -u_1(\mathbf{m}_3 P_1) + \mathbf{m}_1 P_1 = 0 \\ -v_1(\mathbf{m}_3 P_1) + \mathbf{m}_2 P_1 = 0 \\ \vdots \\ -u_n(\mathbf{m}_3 P_n) + \mathbf{m}_1 P_n = 0 \\ -v_n(\mathbf{m}_3 P_n) + \mathbf{m}_2 P_n = 0 \end{array} \right.$$



$$\mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{P}_1^T & \mathbf{0}^T & -u_1 \mathbf{P}_1^T \\ \mathbf{0}^T & \mathbf{P}_1^T & -v_1 \mathbf{P}_1^T \\ \dots & \dots & \dots \\ \mathbf{P}_n^T & \mathbf{0}^T & -u_n \mathbf{P}_n^T \\ \mathbf{0}^T & \mathbf{P}_n^T & -v_n \mathbf{P}_n^T \end{pmatrix}_{2n \times 12}^{1 \times 4}$$

$$\mathbf{m} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{m}_1^T \\ \mathbf{m}_2^T \\ \mathbf{m}_3^T \end{pmatrix}_{12 \times 1}^{4 \times 1}$$

# Homogeneous $M \times N$ Linear Systems

$M$ =number of equations

$N$ =number of unknown

$$\begin{matrix} A \\ \times \\ = \\ 0 \end{matrix}$$

Rectangular system ( $M > N$ )

- 0 is always a solution
- To find non-zero solution

Minimize  $|Ax|^2$

under the constraint  $|x|^2 = 1$

# Calibration Problem

$$\mathcal{P}\mathbf{m} = 0$$

How do we solve this homogenous linear system?

Singular Value Decomposition (SVD)

# Calibration Problem

$$\boxed{\mathcal{P}\mathbf{m} = 0} \quad \text{Compute SVD decomposition of P}$$

$$\boxed{\mathbf{U}_{2n \times 12} \ \mathbf{D}_{12 \times 12} \ \mathbf{V}^T_{12 \times 12}}$$

Last column of V gives

$$\mathbf{m}$$



$$M$$

$$M \ P_i \rightarrow p_i$$

Why? See page 593 of  
Hartley & Zisserman

# Extracting camera parameters

$$\frac{\mathcal{M}}{\rho} = \begin{pmatrix} \alpha \mathbf{r}_1^T - \alpha \cot \theta \mathbf{r}_2^T + u_0 \mathbf{r}_3^T & \alpha t_x - \alpha \cot \theta t_y + u_0 t_z \\ \frac{\beta}{\sin \theta} \mathbf{r}_2^T + v_0 \mathbf{r}_3^T & \frac{\beta}{\sin \theta} t_y + v_0 t_z \\ \mathbf{r}_3^T & t_z \end{pmatrix} = \mathbf{K} [\mathbf{R} \quad \mathbf{T}]$$

$$A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \mathbf{a}_3^T \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Estimated values

$$\text{Intrinsic}$$

$$\rho = \frac{\pm 1}{|\mathbf{a}_3|} \quad u_o = \rho^2 (\mathbf{a}_1 \cdot \mathbf{a}_2)$$

$$v_o = \rho^2 (\mathbf{a}_2 \cdot \mathbf{a}_3)$$

$$\cos \theta = \frac{(\mathbf{a}_1 \times \mathbf{a}_3) \cdot (\mathbf{a}_2 \times \mathbf{a}_3)}{|\mathbf{a}_1 \times \mathbf{a}_3| \cdot |\mathbf{a}_2 \times \mathbf{a}_3|}$$

# Theorem (Faugeras, 1993)

$$M = K \begin{bmatrix} R & T \end{bmatrix} = \begin{bmatrix} K R & K T \end{bmatrix} = [A \quad b]$$

$$A = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix}$$

Let  $\mathcal{M} = (\mathcal{A} \quad \mathbf{b})$  be a  $3 \times 4$  matrix and let  $\mathbf{a}_i^T$  ( $i = 1, 2, 3$ ) denote the rows of the matrix  $\mathcal{A}$  formed by the three leftmost columns of  $\mathcal{M}$ .

$$K = \begin{bmatrix} \alpha & s & c_x \\ 0 & \beta & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

- A necessary and sufficient condition for  $\mathcal{M}$  to be a perspective projection matrix is that  $\text{Det}(\mathcal{A}) \neq 0$

- A necessary and sufficient condition for  $\mathcal{M}$  to be a zero-skew perspective projection matrix is that  $\text{Det}(\mathcal{A}) \neq 0$  and

$$(\mathbf{a}_1 \times \mathbf{a}_3) \cdot (\mathbf{a}_2 \times \mathbf{a}_3) = 0.$$

$$\boldsymbol{\alpha} = f k;$$

$$\boldsymbol{\beta} = f l$$

- A necessary and sufficient condition for  $\mathcal{M}$  to be a perspective projection matrix with zero skew and unit aspect-ratio is that  $\text{Det}(\mathcal{A}) \neq 0$  and

$$\begin{cases} (\mathbf{a}_1 \times \mathbf{a}_3) \cdot (\mathbf{a}_2 \times \mathbf{a}_3) = 0, \\ (\mathbf{a}_1 \times \mathbf{a}_3) \cdot (\mathbf{a}_1 \times \mathbf{a}_3) = (\mathbf{a}_2 \times \mathbf{a}_3) \cdot (\mathbf{a}_2 \times \mathbf{a}_3). \end{cases}$$

# Extracting camera parameters

$$\frac{\mathcal{M}}{\rho} = \begin{pmatrix} \alpha \mathbf{r}_1^T - \alpha \cot \theta \mathbf{r}_2^T + u_0 \mathbf{r}_3^T \\ \frac{\beta}{\sin \theta} \mathbf{r}_2^T + v_0 \mathbf{r}_3^T \\ \mathbf{r}_3^T \end{pmatrix} = \mathbf{K} [\mathbf{R} \quad \mathbf{T}]$$

$\mathbf{A}$                                      $\mathbf{b}$

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \mathbf{a}_3^T \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}$$

Estimated values

Intrinsic

$$\alpha = \rho^2 |\mathbf{a}_1 \times \mathbf{a}_3| \sin \theta \rightarrow f$$
$$\beta = \rho^2 |\mathbf{a}_2 \times \mathbf{a}_3| \sin \theta$$

# Extracting camera parameters

$$\frac{\mathcal{M}}{\rho} = \begin{pmatrix} \alpha \mathbf{r}_1^T - \alpha \cot \theta \mathbf{r}_2^T + u_0 \mathbf{r}_3^T \\ \frac{\beta}{\sin \theta} \mathbf{r}_2^T + v_0 \mathbf{r}_3^T \\ \mathbf{r}_3^T \end{pmatrix} = \mathbf{K} [\mathbf{R} \quad \mathbf{T}]$$

$\mathbf{A}$                                      $\mathbf{b}$

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \mathbf{a}_3^T \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}$$

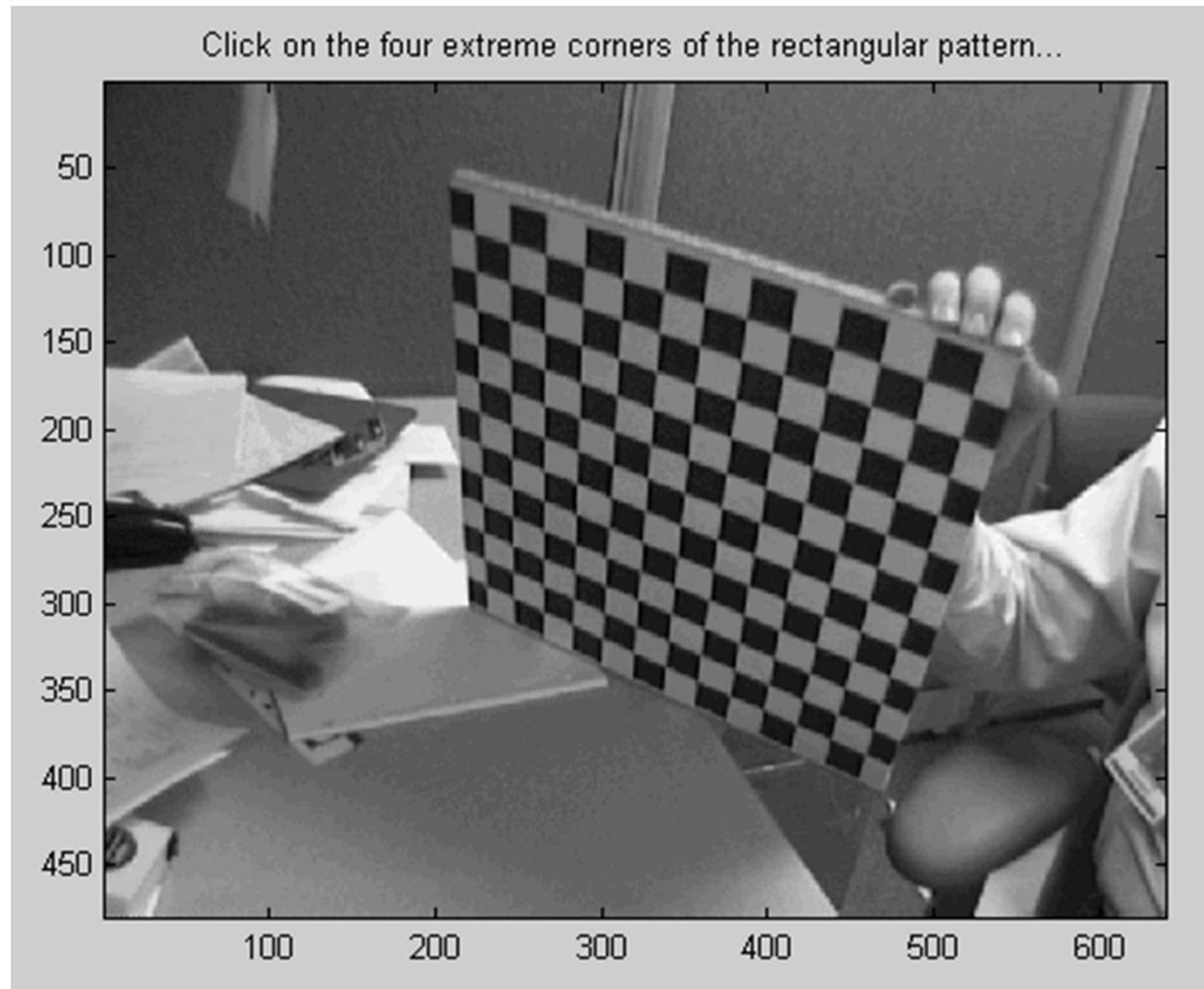
Estimated values

**Extrinsic**

$$\mathbf{r}_1 = \frac{(\mathbf{a}_2 \times \mathbf{a}_3)}{|\mathbf{a}_2 \times \mathbf{a}_3|} \quad \mathbf{r}_3 = \frac{\pm 1}{|\mathbf{a}_3|}$$
$$\mathbf{r}_2 = \mathbf{r}_3 \times \mathbf{r}_1 \quad \mathbf{T} = \rho \mathbf{K}^{-1} \mathbf{b}$$

# Calibration Demo

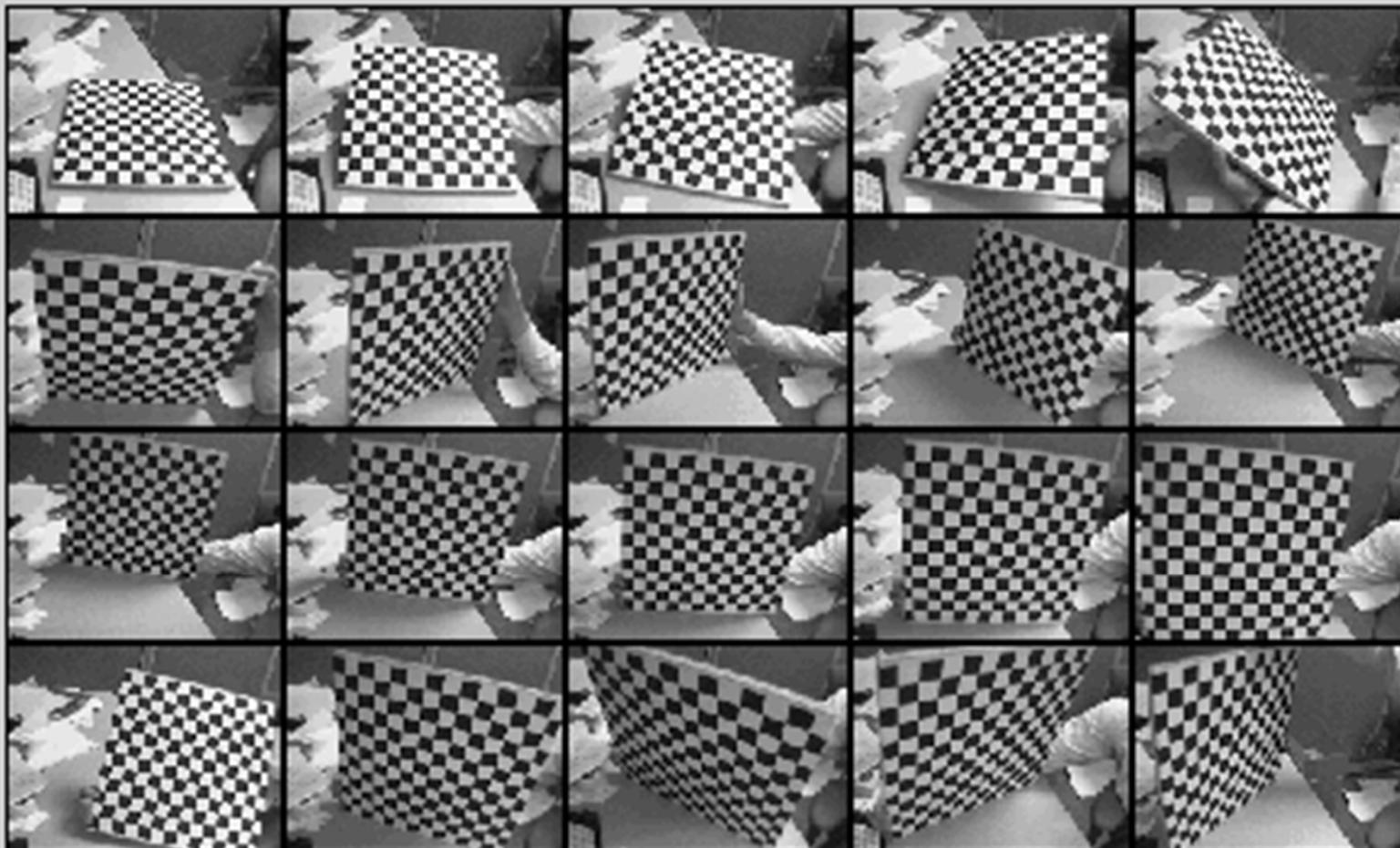
*Camera Calibration Toolbox for Matlab J. Bouguet – [1998-2000]*



[http://www.vision.caltech.edu/bouguetj/calib\\_doc/index.html#examples](http://www.vision.caltech.edu/bouguetj/calib_doc/index.html#examples)

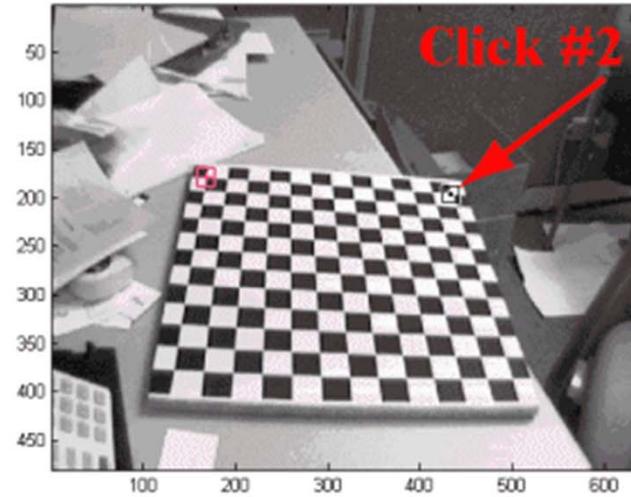
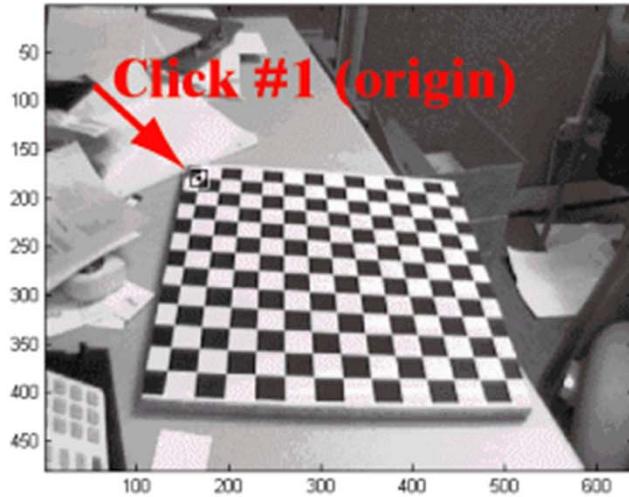
# Calibration Demo

Calibration images

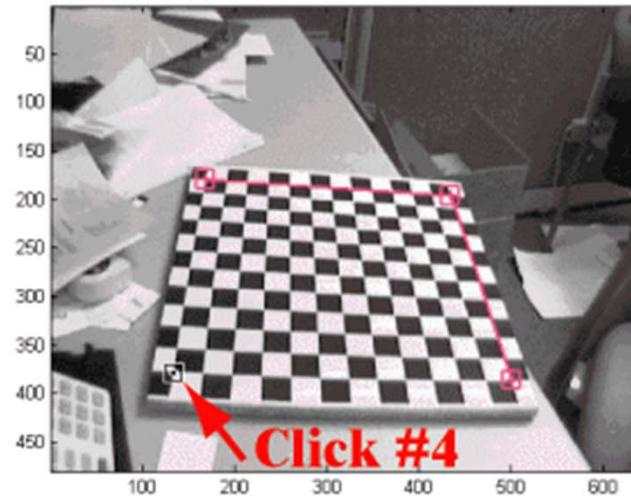
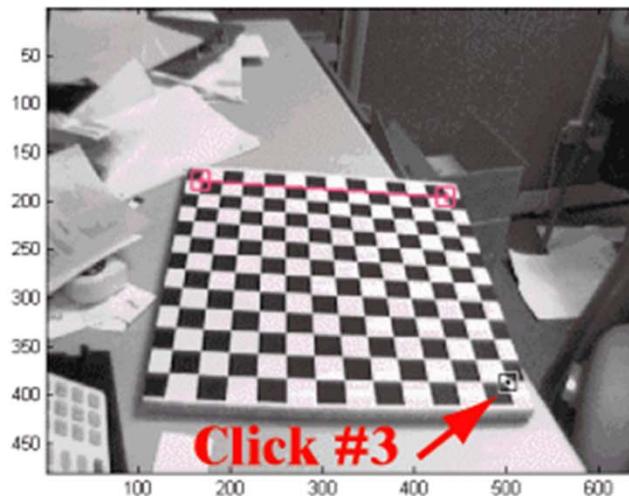


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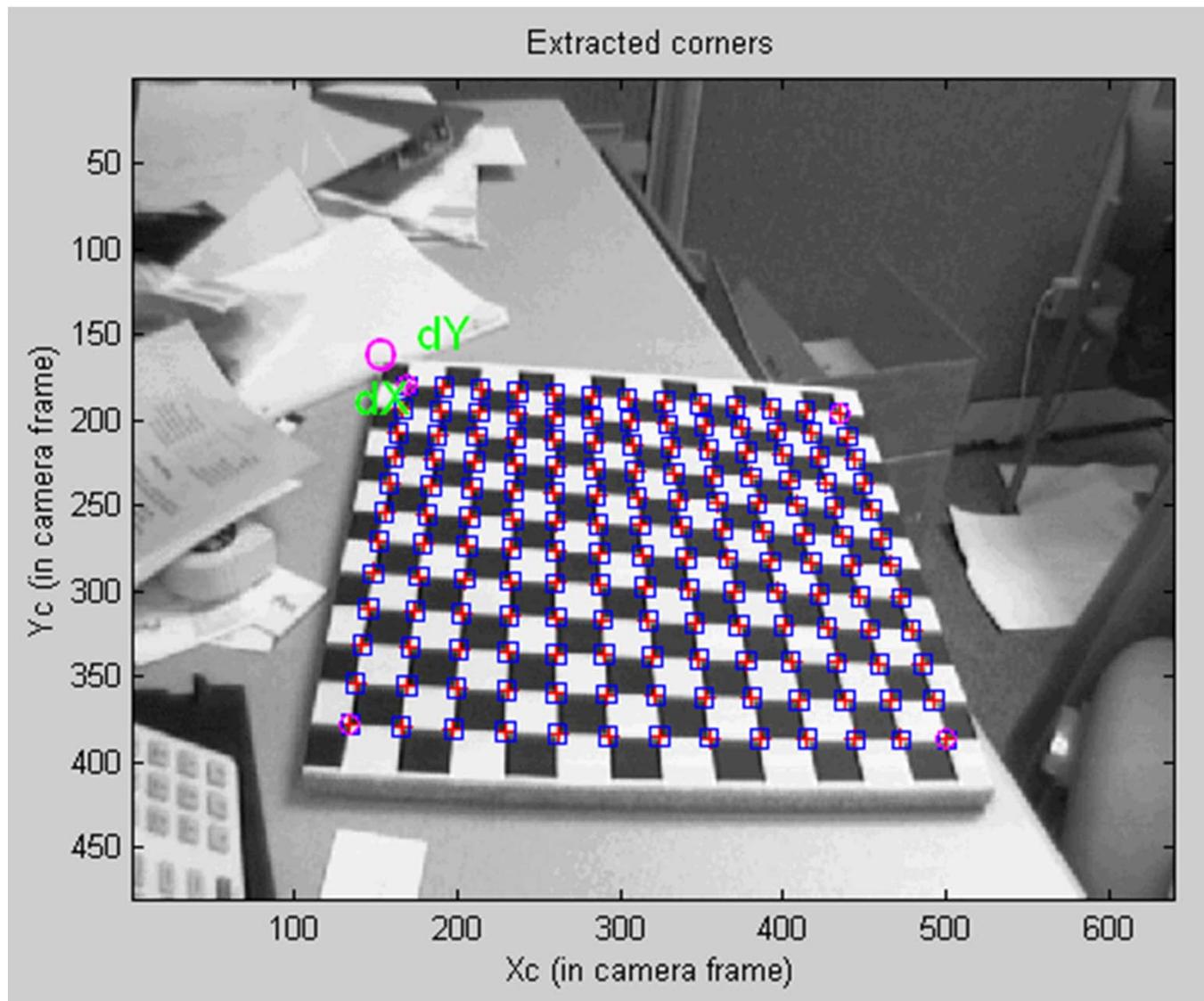
Click on the four extreme corners of the rectangular pattern (first corner = origin)... Image 1 Click on the four extreme corners of the rectangular pattern (first corner = origin)... Image 1



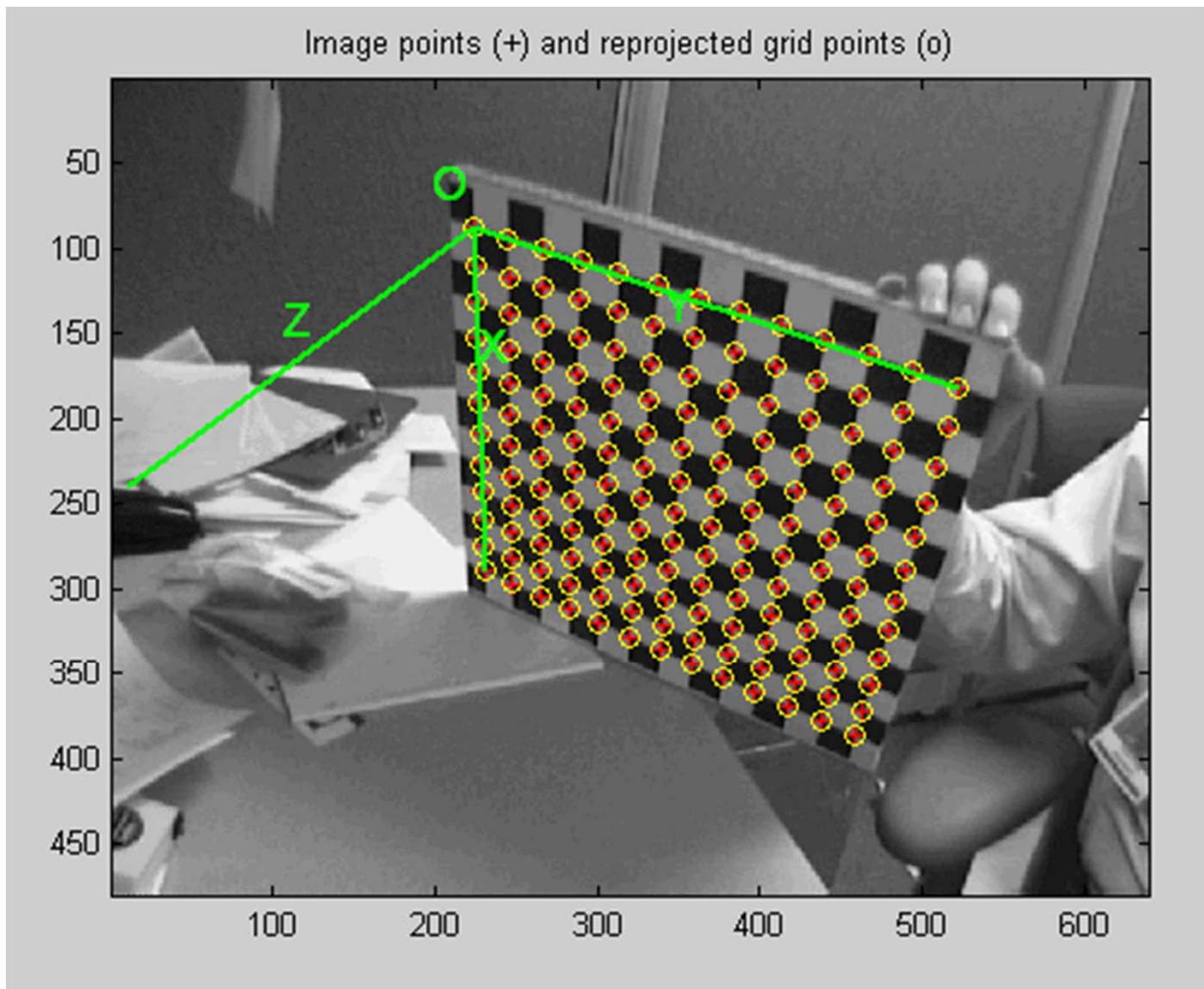
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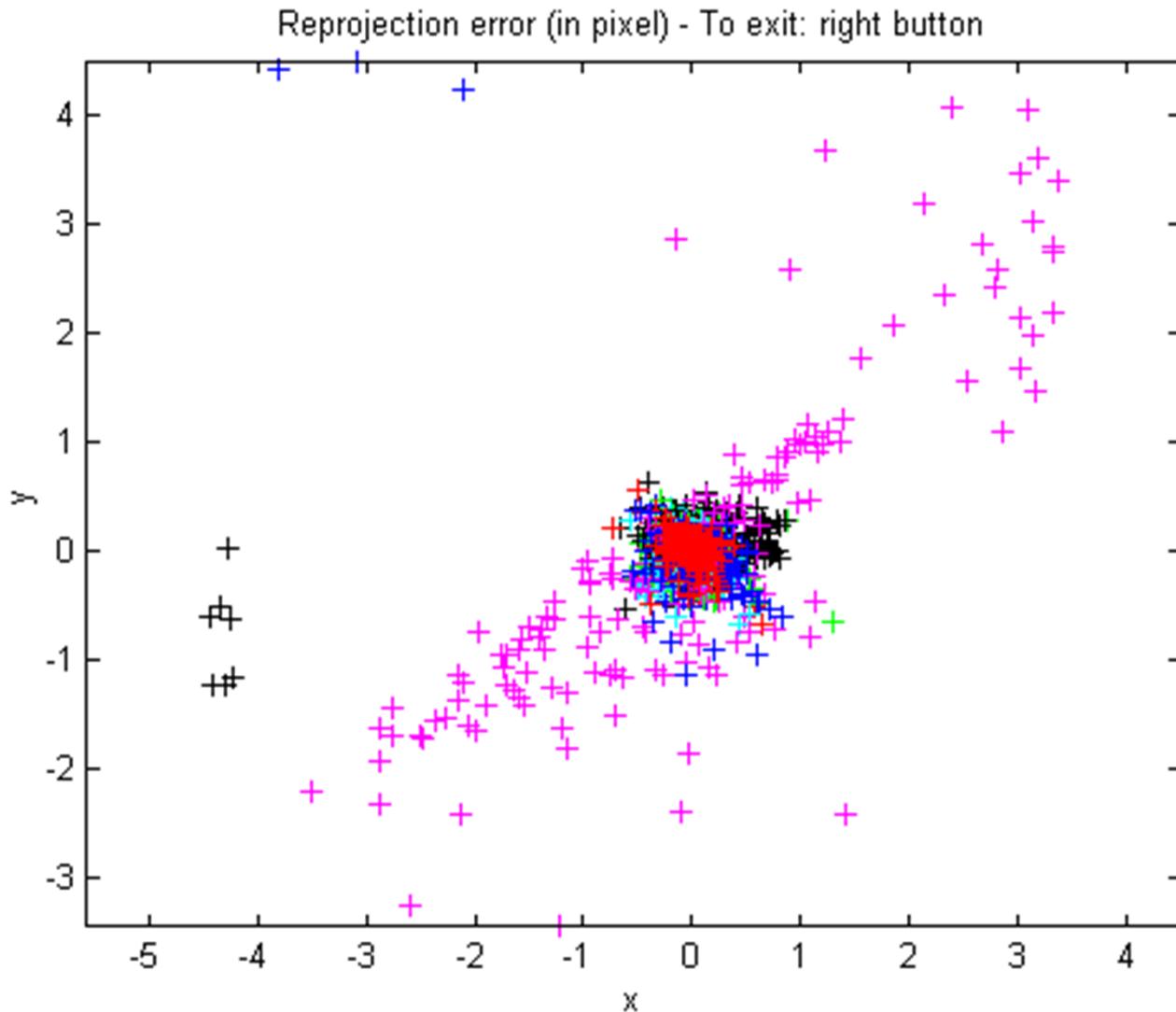
# Calibration Demo



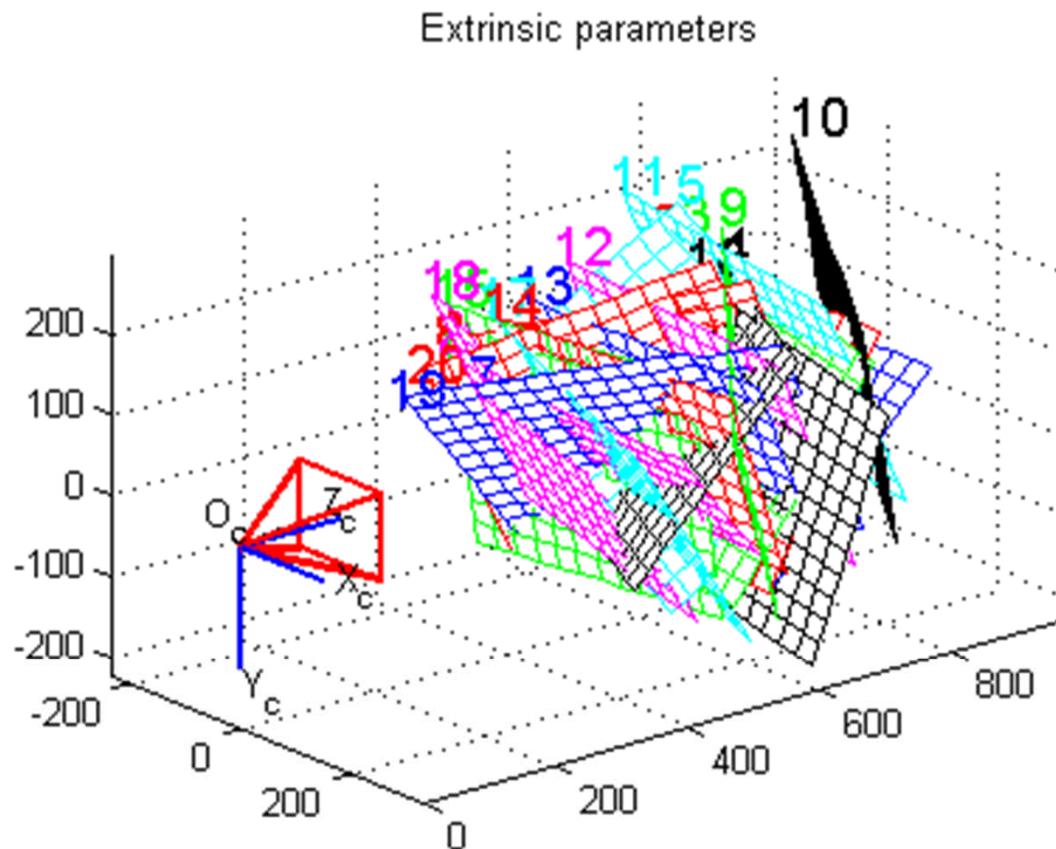
# Calibration Demo



# Calibration Demo

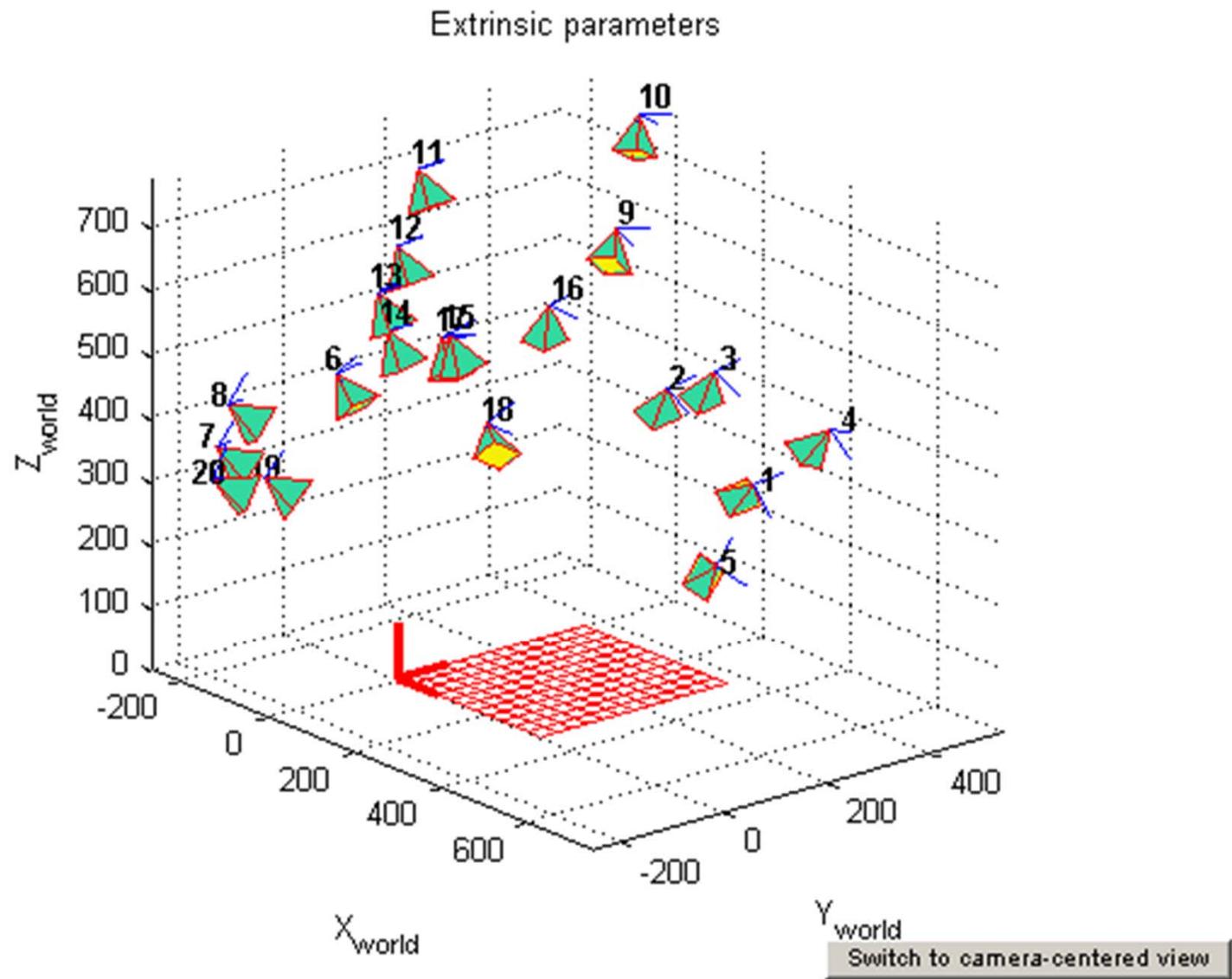


# Calibration Demo



[Switch to world-centered view](#)

# Calibration Demo



# What we will learn today?

- Review camera parameters
- Affine camera model
- Camera calibration
- Vanishing points and lines (**Problem Set 2 (Q1)**)

Reading:

- [FP] Chapter 3
- [HZ] Chapter 7, 8.6

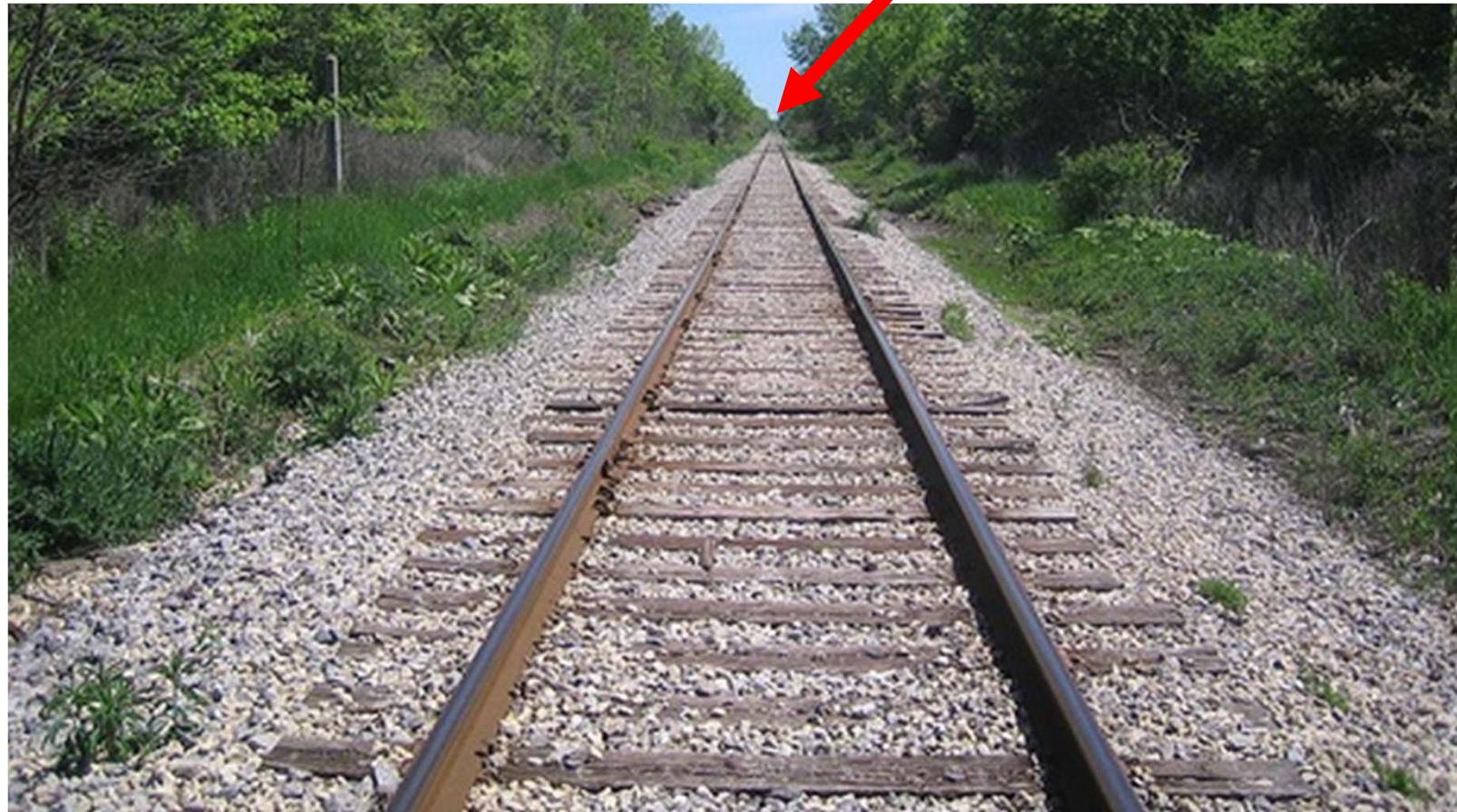
# Properties of Projection

- Points project to points
- Lines project to lines



# Properties of Projection

- Angles are not preserved
- Parallel lines meet



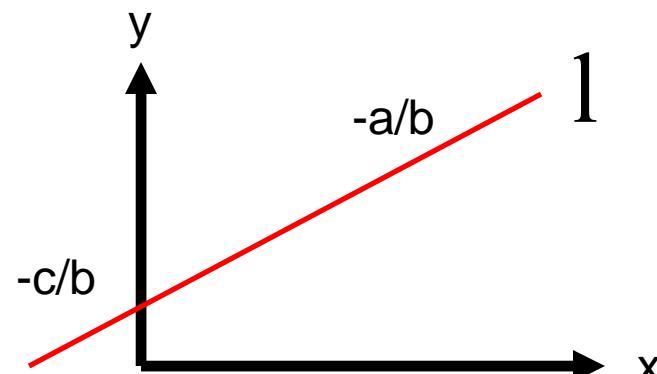
Vanishing point

# Lines in a 2D plane

$$ax + by + c = 0$$

$$l = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

If  $x = [x_1, x_2]^T \in l$



$$\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}^T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

# Lines in a 2D plane

Intersecting lines

$$x = l \times l'$$

Proof

$$l \times l' \perp l \rightarrow (l \times l') \cdot l = 0 \rightarrow x \in l$$

$$l \times l' \perp l' \rightarrow \underbrace{(l \times l') \cdot l'}_x = 0 \rightarrow x \in l'$$

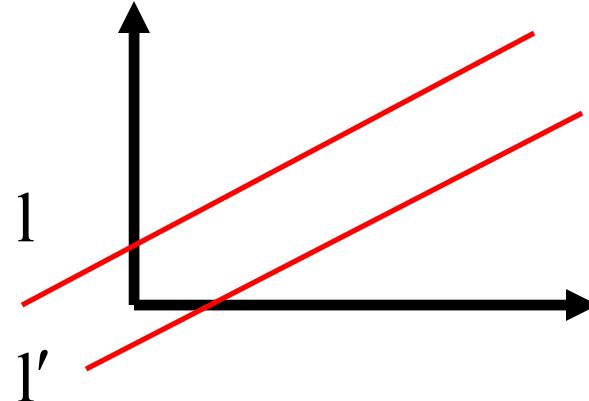
→  $x$  is the intersecting point

# Points at infinity (ideal points)

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_3 \neq 0$$

$$x_\infty = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

Let's intersect two parallel lines:  $\rightarrow l \times l' = (c - c') \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$



$$l = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$l' = \begin{bmatrix} a \\ b \\ c' \end{bmatrix}$$

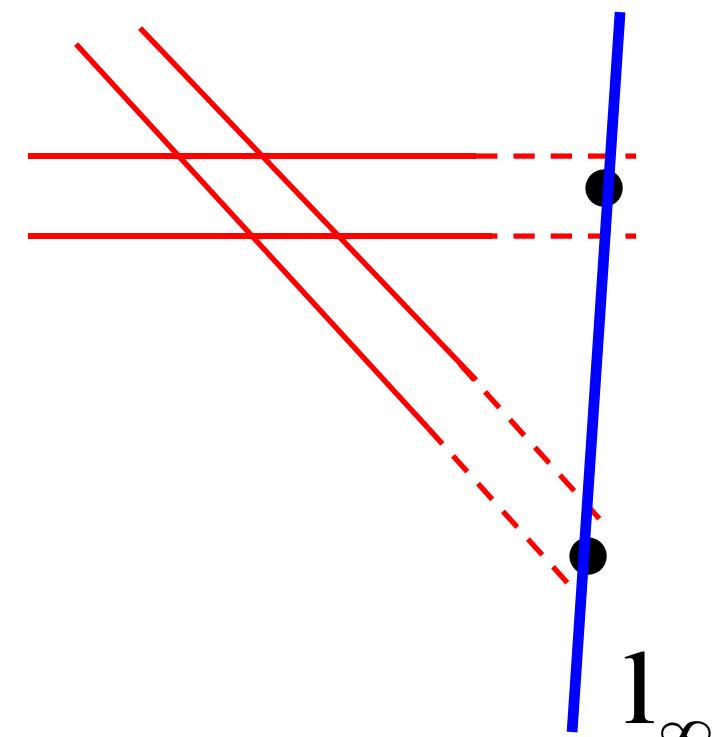
# Lines at infinity $l_\infty$

Set of ideal points lies on a line called the line at infinity  
How does it look like?

$$l_\infty = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

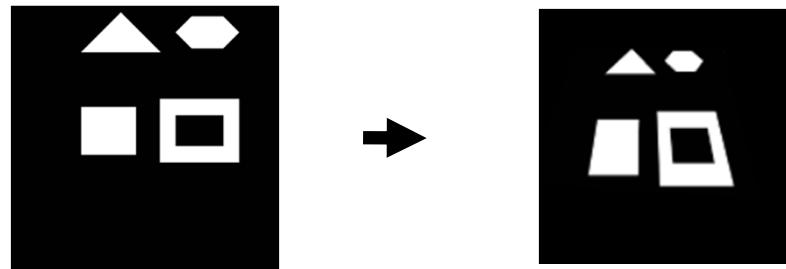
Indeed:

$$\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$



# Projective projections of lines at infinity (2D)

$$H = \begin{bmatrix} A & t \\ v & b \end{bmatrix}$$



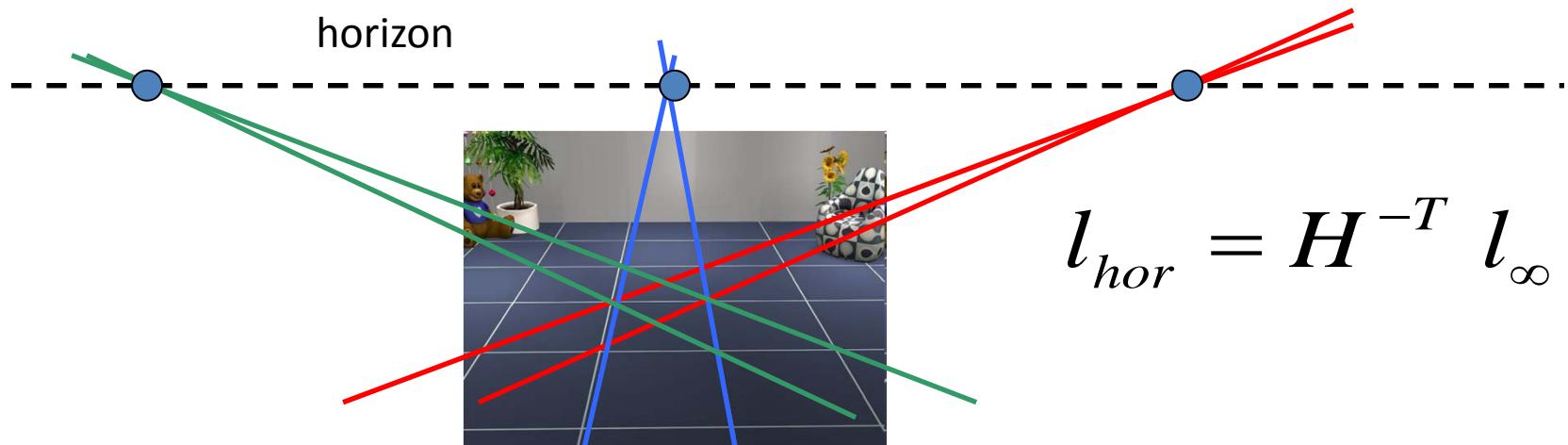
$$l' = H^{-T} l$$

is it a line at infinity?

$$H_A^{-T} l_\infty = ? = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix}^{-T} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} A^{-T} & 0 \\ -t^T A^{-T} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$H^{-T} l_\infty = ? = \begin{bmatrix} A & t \\ v & b \end{bmatrix}^{-T} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \\ b \end{bmatrix} \quad \dots \text{no!}$$

# Projective projections of lines at infinity (2D)

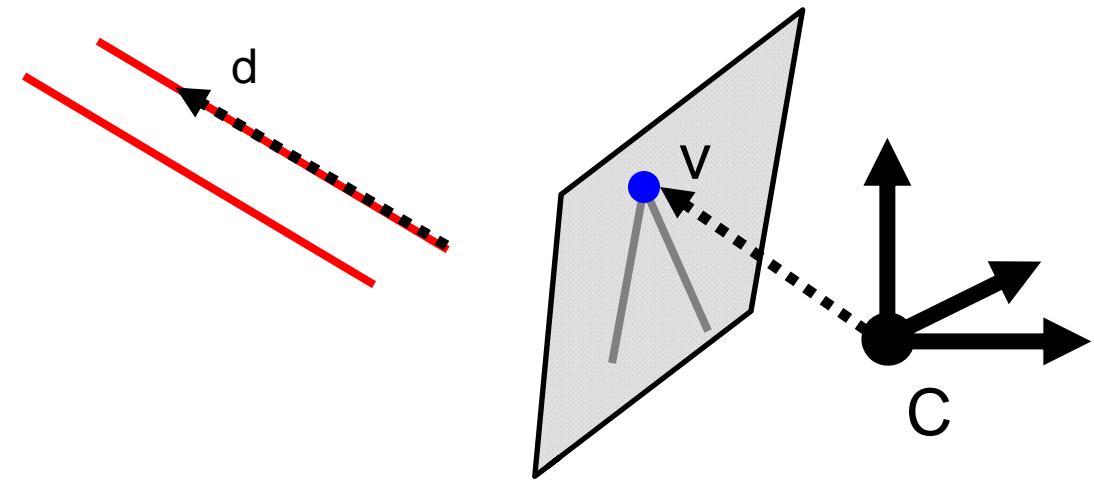


Are these two lines parallel or not?

- Recognize the horizon line
- Measure if the 2 lines meet at the horizon
- if yes, these 2 lines are //

# Vanishing points (= ideal points in 2D)

**Vanishing points**  
= points where  
parallel lines  
intersect in 3D



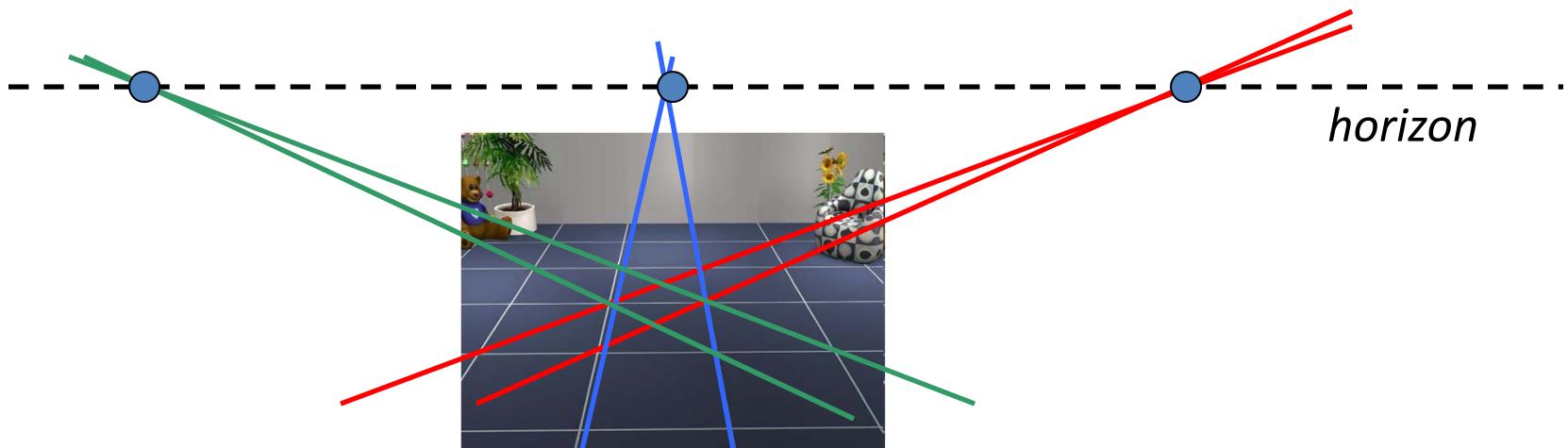
$d$ =direction of the line

$$M = K[R \ T]$$

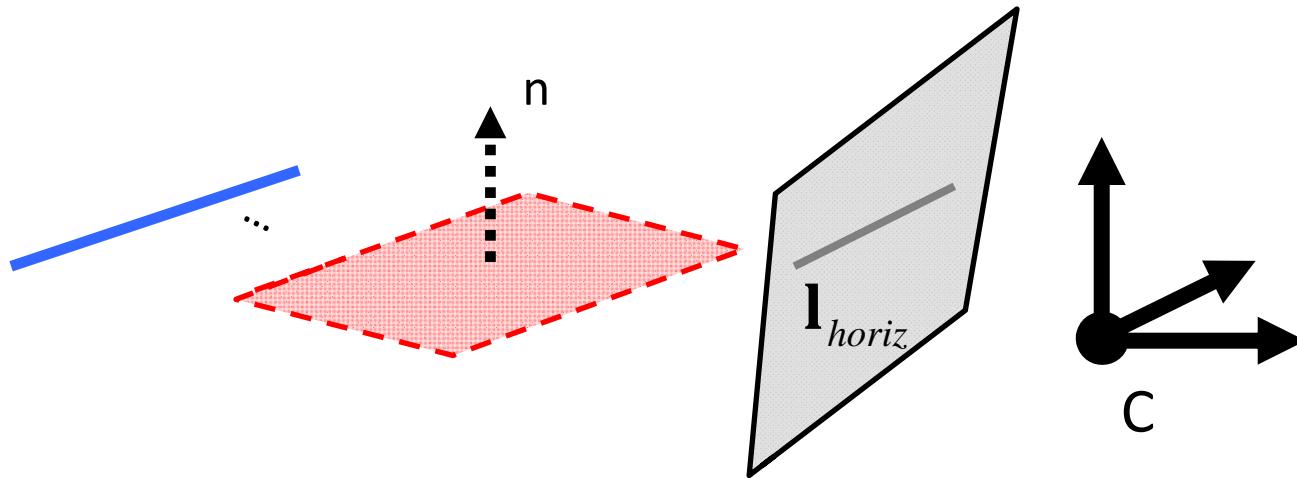
Image of a vanishing point =  $\mathbf{v} = \mathbf{K} \ \mathbf{d}$

# Horizon

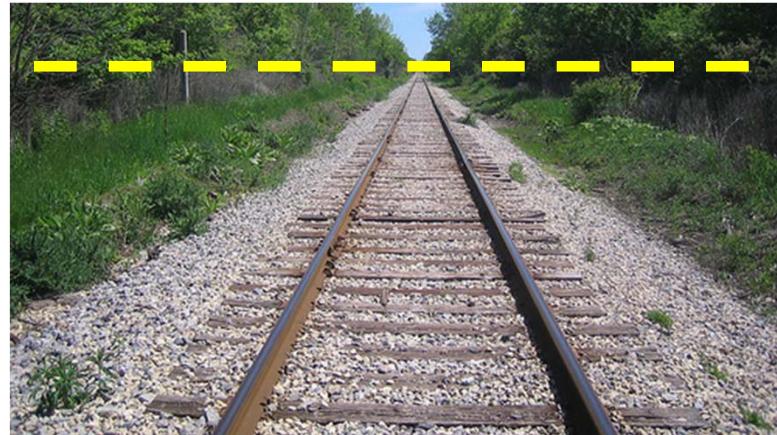
- Sets of parallel lines on the same plane lead to *collinear* vanishing points [The line is called the *horizon* for that plane]



# Horizon



$$\mathbf{n} = \mathbf{K}^T \mathbf{l}_{horiz}$$



# Application

These transformations are used in single view metrology

Criminisi & Zisserman, 99



# Application

these transformations are used in single view metrology

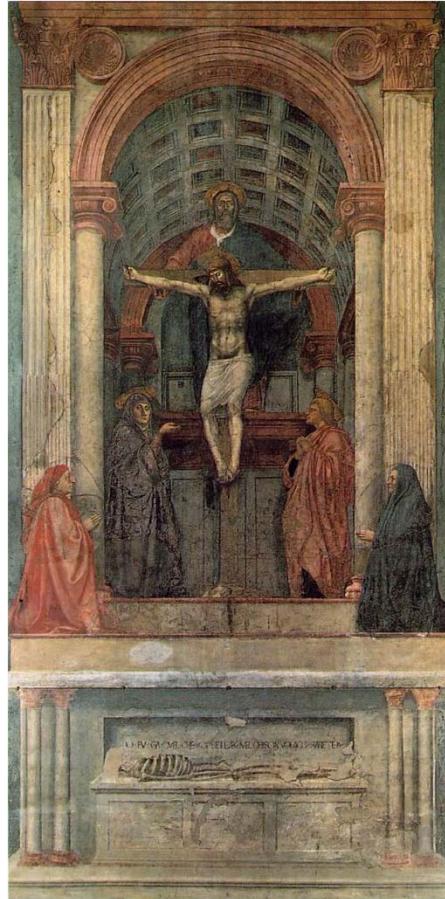
Criminisi & Zisserman, 99



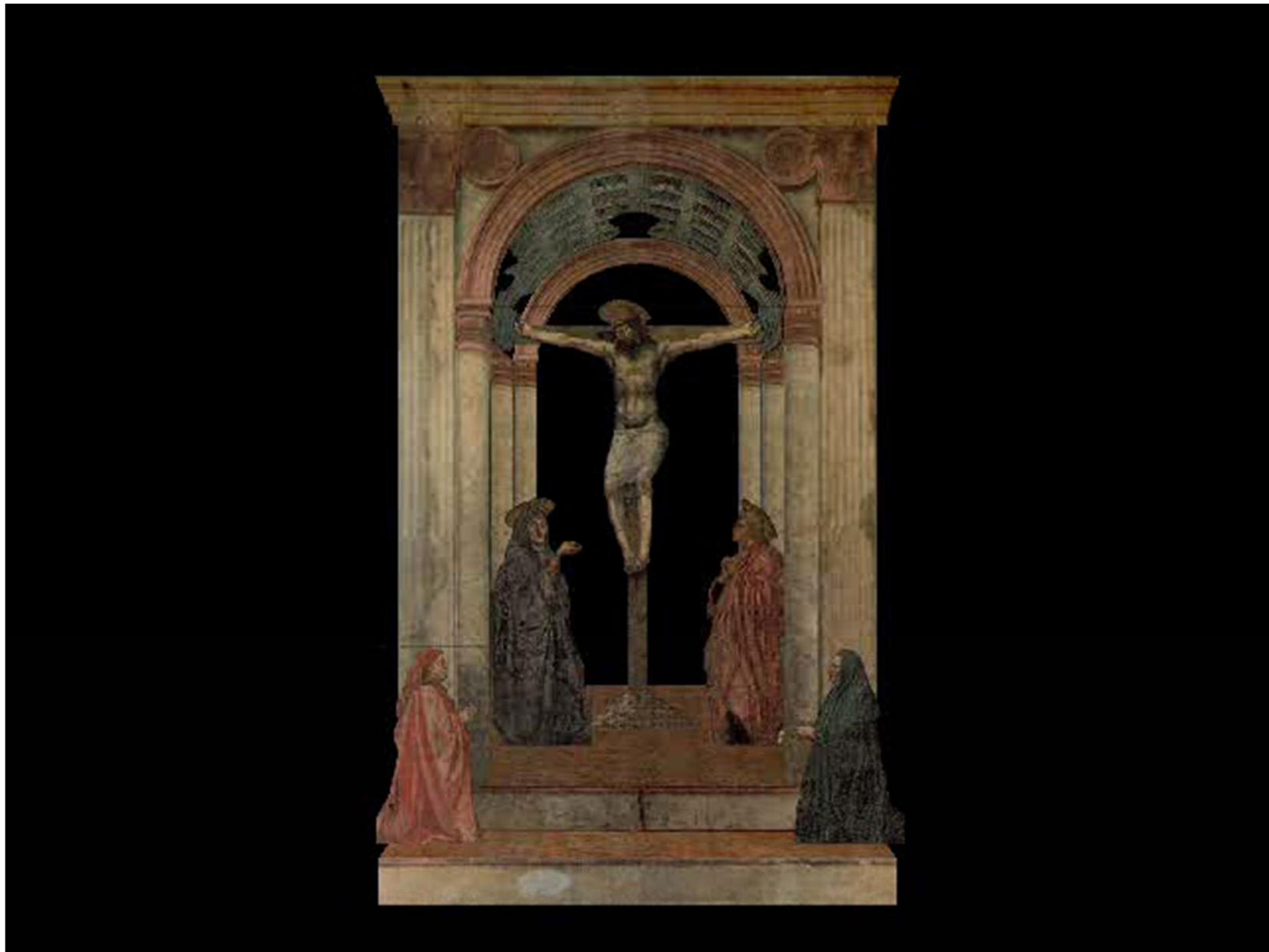
# Application

these transformations are used in single view metrology

Criminisi & Zisserman, 99



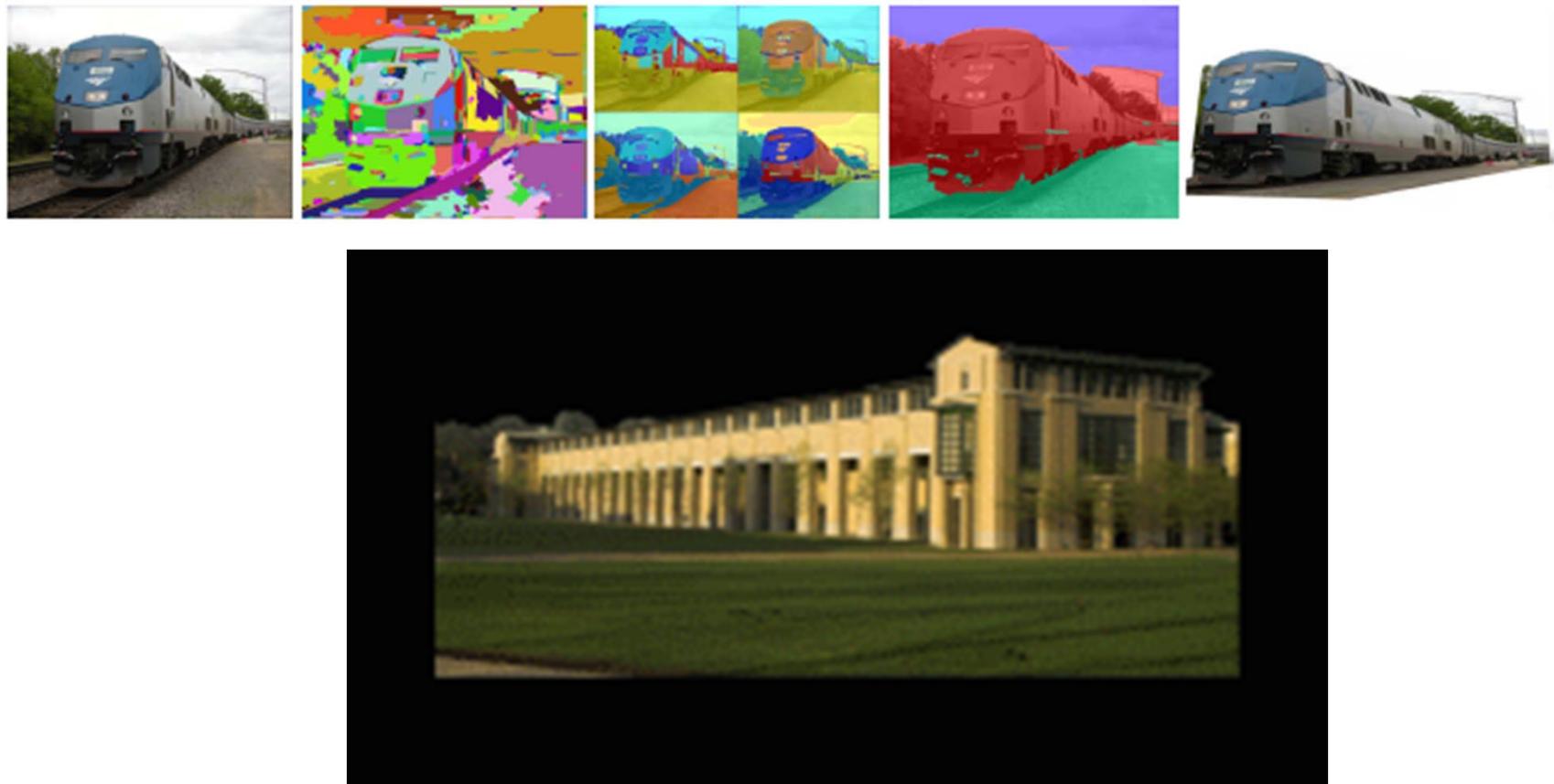
*La Trinità* (1426)  
Firenze, Santa Maria  
Novella; by Masaccio  
(1401-1428)



# Application

these transformations are used in single view metrology

Hoiem et al, 05...



<http://www.cs.uiuc.edu/homes/dhoiem/projects/software.html>

# Application

these transformations are used in single view metrology

Saxena, Sun, Ng, 05...



A software: **Make3D**

“Convert your image into 3d model” <http://make3d.stanford.edu/>

<http://make3d.stanford.edu/images/view3D/185>

<http://make3d.stanford.edu/images/view3D/931?noforward=true>

<http://make3d.stanford.edu/images/view3D/108>

# What we have learned today

- Review camera parameters
- Affine camera model (**Problem Set 2 (Q4)**)
- Camera calibration
- Vanishing points and lines (**Problem Set 2 (Q1)**)

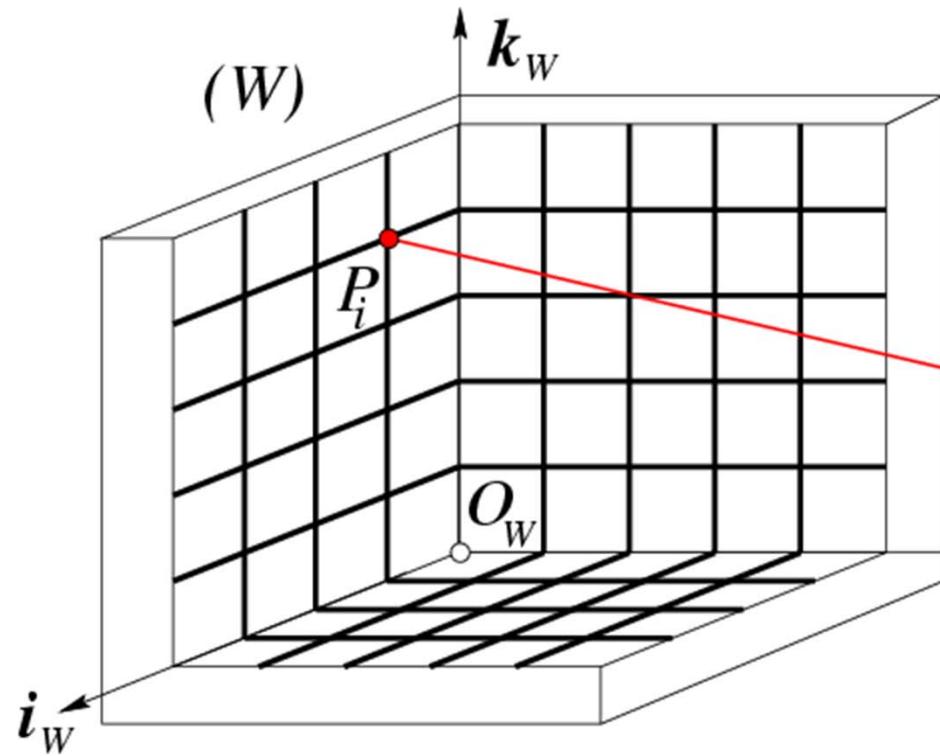
Reading:

- [FP] Chapter 3
- [HZ] Chapter 7, 8.6

# Supplementary Materials

# Degeneracy and distortion in real-world camera calibration

# Degenerate cases

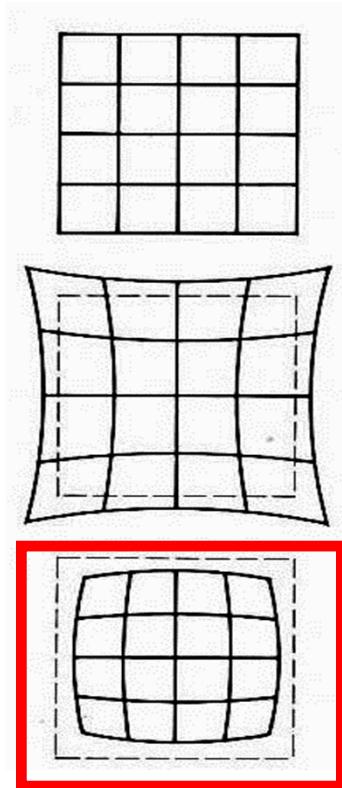


$$\mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{P}_1^T & \mathbf{0}^T & -u_1 \mathbf{P}_1^T \\ \mathbf{0}^T & \mathbf{P}_1^T & -v_1 \mathbf{P}_1^T \\ \dots & \dots & \dots \\ \mathbf{P}_n^T & \mathbf{0}^T & -u_n \mathbf{P}_n^T \\ \mathbf{0}^T & \mathbf{P}_n^T & -v_n \mathbf{P}_n^T \end{pmatrix}$$

- $\mathbf{P}_i$ 's cannot lie on the same plane!
- Points cannot lie on the intersection curve of two quadric surfaces

# Radial Distortion

- Caused by imperfect lenses
- Deviations are most noticeable for rays that pass through the edge of the lens



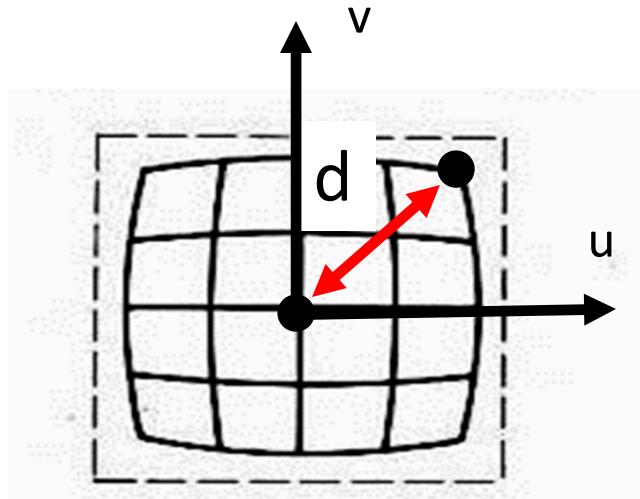
No distortion

Pin cushion

Barrel



# Radial Distortion



$$\begin{bmatrix} \frac{1}{\lambda} & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 1 \end{bmatrix} M \cdot P_i \rightarrow \begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix} = p_i$$

$$d^2 = a u^2 + b v^2 + c u v$$

To model radial behavior

Distortion coefficient

$$\lambda = 1 \pm \sum_{p=1}^3 \kappa_p d^{2p}$$

Polynomial function

# Radial Distortion

$$M = \begin{bmatrix} \frac{1}{\lambda} & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_i \rightarrow \begin{bmatrix} u_i \\ v_i \end{bmatrix} = p_i$$

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Q

$$p_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} \frac{q_1 P_i}{q_3 P_i} \\ \frac{q_2 P_i}{q_3 P_i} \end{bmatrix}$$

Non-linear system of equations

$$\begin{cases} u_i q_3 P_i = q_1 P_i \\ v_i q_3 P_i = q_2 P_i \end{cases}$$

# General Calibration Problem

$$X = f(P)$$

↑                      ↑  
measurement          parameter

f( ) is nonlinear

- Newton Method
- Levenberg-Marquardt Algorithm

- Iterative, starts from initial solution
  - May be slow if initial solution far from real solution
  - Estimated solution may be function of the initial solution
  - Newton requires the computation of  $J$ ,  $H$
  - Levenberg-Marquardt doesn't require the computation of  $H$

# General Calibration Problem

$$X = f(P)$$

↑                      ↑  
measurement            parameter

f( ) is nonlinear

# A possible algorithm

1. Solve linear part of the system to find approximated solution
  2. Use this solution as initial condition for the full system
  3. Solve full system (including distortion) using Newton or L.M.

# General Calibration Problem

$$X = f(P)$$

↑                      ↑  
measurement          parameter

f( ) is nonlinear

Typical assumptions for computing initial condition :

- zero-skew, square pixel
  - $u_o, v_o$  = known center of the image
  - no distortion

Just estimate f  
and R, T

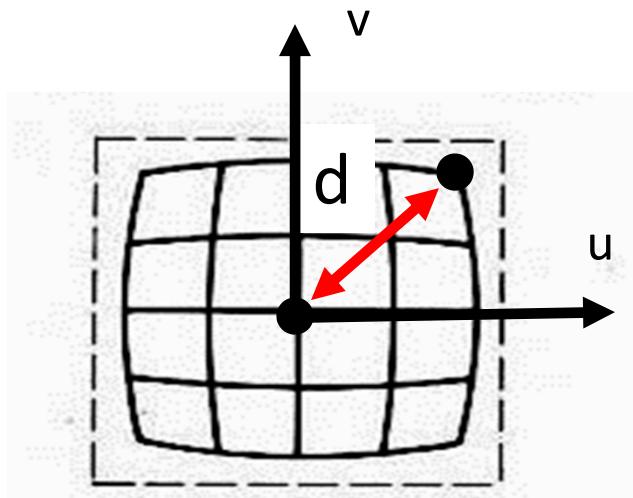
# Tsai's calibration technique

1. Estimate  $\mathbf{m}_1$  and  $\mathbf{m}_2$  first:

$$\mathbf{p}_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} \frac{\mathbf{m}_1 P_i}{\mathbf{m}_3 P_i} \\ \frac{\mathbf{m}_2 P_i}{\mathbf{m}_3 P_i} \end{bmatrix}$$

How to do that?

Hint:



$$\frac{u_i}{v_i} = \text{slope}$$

# Tsai's calibration technique

1. Estimate  $\mathbf{m}_1$  and  $\mathbf{m}_2$  first:

$$\mathbf{p}_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} \frac{\mathbf{m}_1 P_i}{\mathbf{m}_3 P_i} \\ \frac{\mathbf{m}_2 P_i}{\mathbf{m}_3 P_i} \end{bmatrix}$$
$$\frac{u_i}{v_i} = \frac{\frac{(\mathbf{m}_1 P_i)}{(\mathbf{m}_3 P_i)}}{\frac{(\mathbf{m}_2 P_i)}{(\mathbf{m}_3 P_i)}} = \frac{\mathbf{m}_1 P_i}{\mathbf{m}_2 P_i}$$

$$\left\{ \begin{array}{l} v_1(\mathbf{m}_1 P_1) - u_1(\mathbf{m}_2 P_1) = 0 \\ v_i(\mathbf{m}_1 P_i) - u_i(\mathbf{m}_2 P_i) = 0 \\ \vdots \\ v_n(\mathbf{m}_1 P_n) - u_n(\mathbf{m}_2 P_n) = 0 \end{array} \right. \quad Q \mathbf{n} = 0 \quad \mathbf{n} = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{bmatrix}$$

# Tsai's calibration technique

2. Once that  $\mathbf{m}_1$  and  $\mathbf{m}_2$  are estimated, estimate  $\mathbf{m}_3$ :

$$\mathbf{p}_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} \frac{\mathbf{m}_1 P_i}{\mathbf{m}_3 P_i} \\ \frac{\mathbf{m}_2 P_i}{\mathbf{m}_3 P_i} \end{bmatrix}$$

$\mathbf{m}_3$  is non linear function of  $\mathbf{m}_1$   $\mathbf{m}_2$   $\lambda$

There are some degenerate configurations for which  $\mathbf{m}_1$  and  $\mathbf{m}_2$  cannot be computed