Lecture 3: Linear Filters

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What we will learn today?

- Images as functions
- Linear systems (filters)
- Convolution and correlation
- Discrete Fourier Transform (DFT)
- Sampling and aliasing

Some background reading:
Forsyth and Ponce, Computer Vision, Chapter 7 & 8
Jae S. Lim, Two-dimensional signal and image processing, Chapter 1, 4, 5

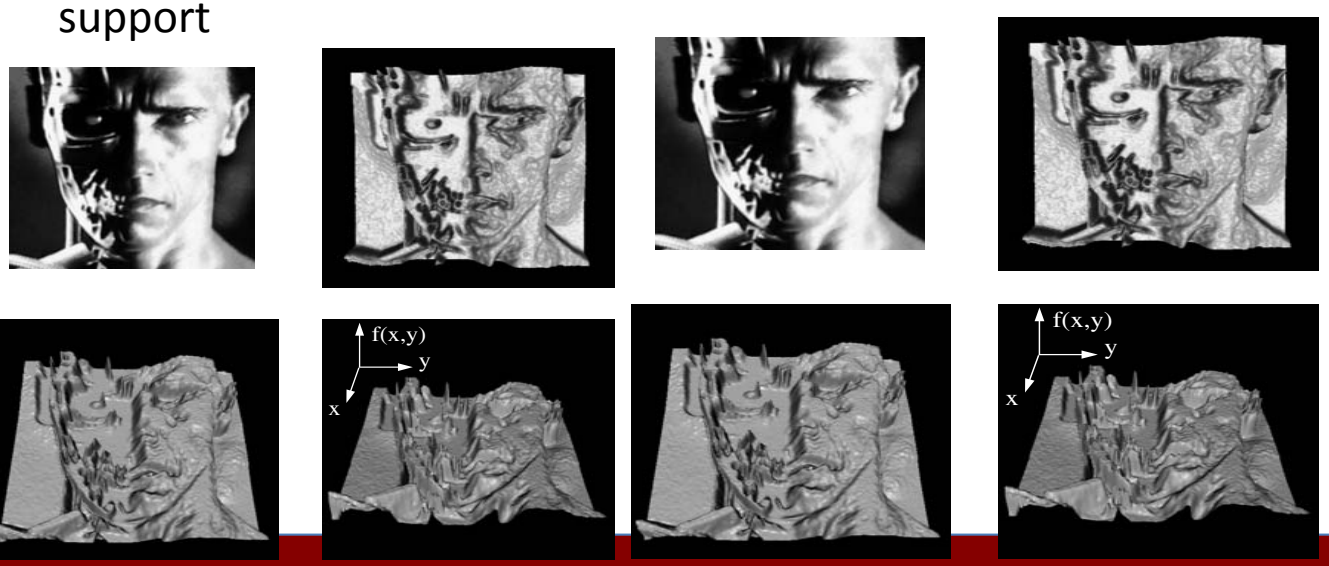
Images as functions

- **An Image** as a function f from R^2 to R^M:
  - $f(x, y)$ gives the **intensity** at position (x, y)
  - Defined over a rectangle, with a finite range:

$$f: [a,b] \times [c,d] \rightarrow [0,255]$$

Domain support

range



Images as functions

- **An Image** as a function f from R^2 to R^M:
  - $f(x, y)$ gives the **intensity** at position (x, y)
  - Defined over a rectangle, with a finite range:

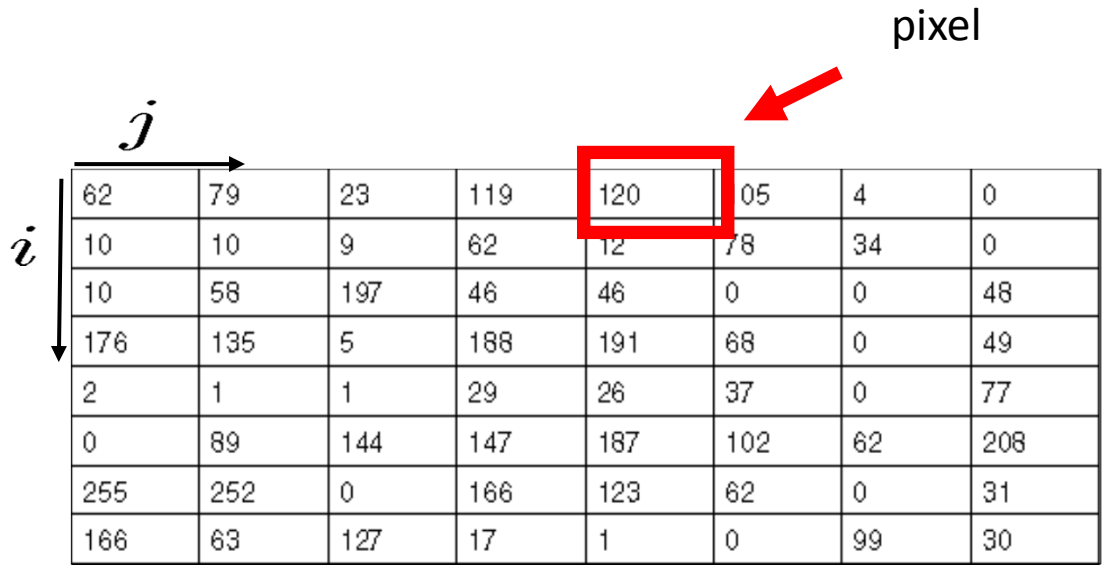
$$f: \underbrace{[a,b] \times [c,d]}_{\text{Domain support}} \rightarrow \underbrace{[0,255]}_{\text{range}}$$

- A color image: $$f(x,y) = \begin{bmatrix} r(x,y) \\ g(x,y) \\ b(x,y) \end{bmatrix}$$

Images as discrete functions

- Images are usually **digital** (**discrete**):
  - **Sample** the 2D space on a regular grid

- Represented as a matrix of integer values



Images as discrete functions

Cartesian coordinates

$$f[n,m] = \begin{bmatrix} \ddots & \vdots & \\ f[-1,1] & f[0,1] & f[1,1] \\ \dots \quad f[-1,0] & \underline{f[0,0]} & f[1,0] \quad \dots \\ f[-1,-1] & f[0,-1] & f[1,-1] \\ & \vdots & \ddots \end{bmatrix}$$

Notation for discrete functions

Images as discrete functions

Array coordinates

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & \dots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{N1} & \dots & a_{NM} \end{bmatrix}$$

Matlab notation

Systems and Filters

- **Filtering:**
  - Form a new image whose pixels are a combination original pixel values

Goals:

-Extract useful information from the images

- Features (edges, corners, blobs…)

- Modify or enhance image properties:

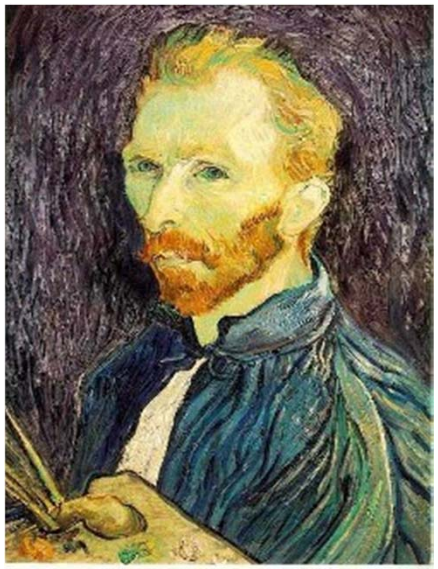
- super-resolution; in-painting; de-noising

De-noising

Original

Salt and pepper noise

Super-resolution



In-painting

Bertamio et al

2D discrete-space systems (filters)

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

$$g = \mathcal{S}[f], \quad g[n,m] = \mathcal{S}\{f[n,m]\}$$

$$f[n,m] \xrightarrow{\mathcal{S}} g[n,m]$$

Filters: Examples

- 2D DS moving average over a 3 × 3 window of neighborhood

$$g[n,m] = \frac{1}{9}\sum_{k=n-1}^{n+1}\sum_{l=m-1}^{m+1} f[k,l]$$

$$= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k,m-l]$$

h

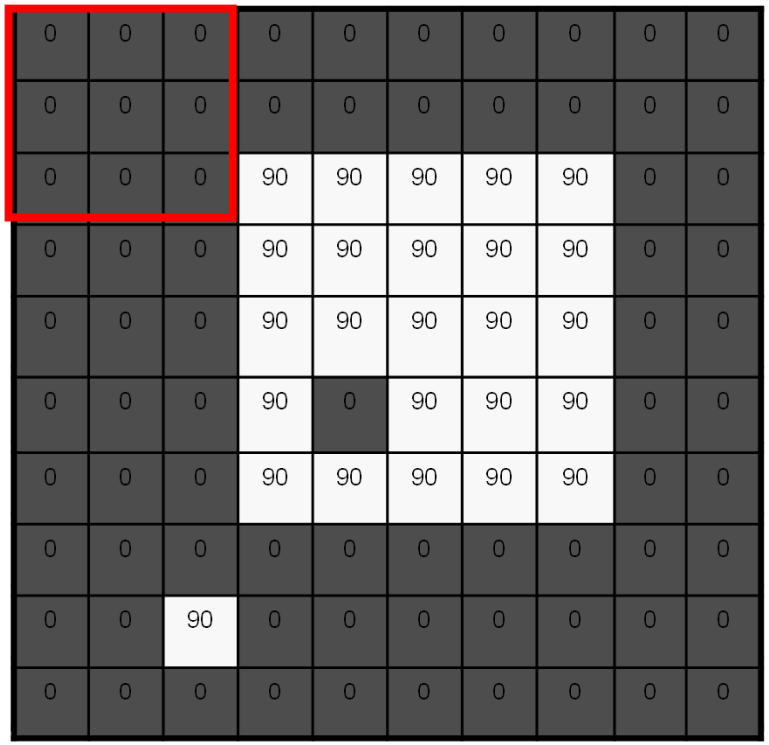
$\frac{1}{9}$

1	1	1
1	1	1
1	1	1

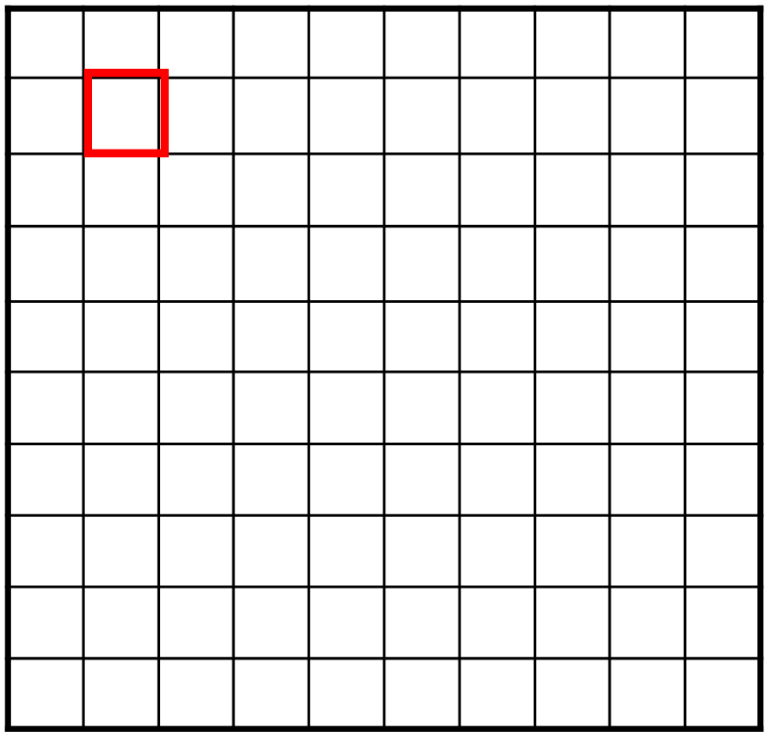
$$(f * h)[m,n] = \frac{1}{9}\sum_{k,l} f[k,l]\, h[m-k,n-l]$$

Moving average

$F[x, y]$



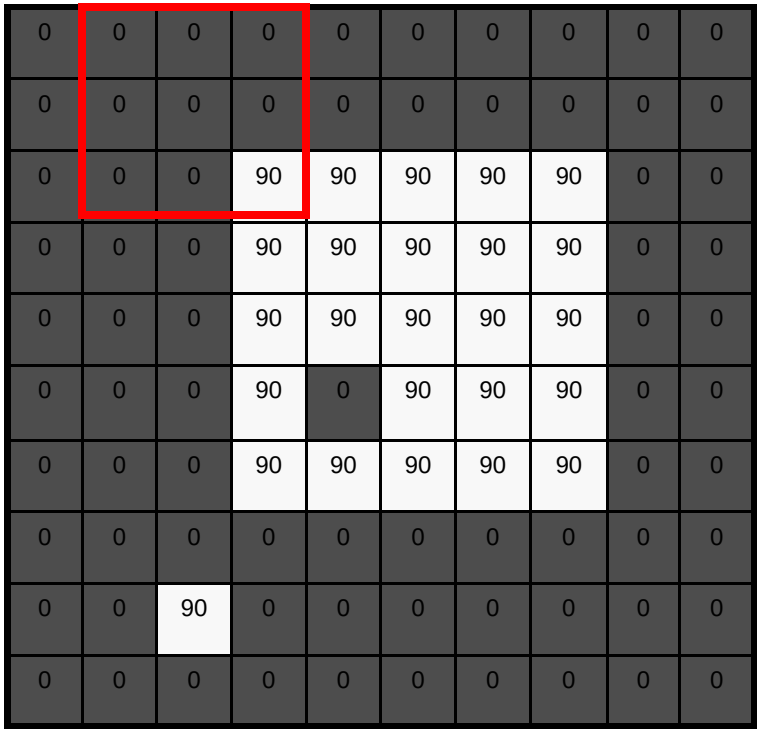
$G[x, y]$



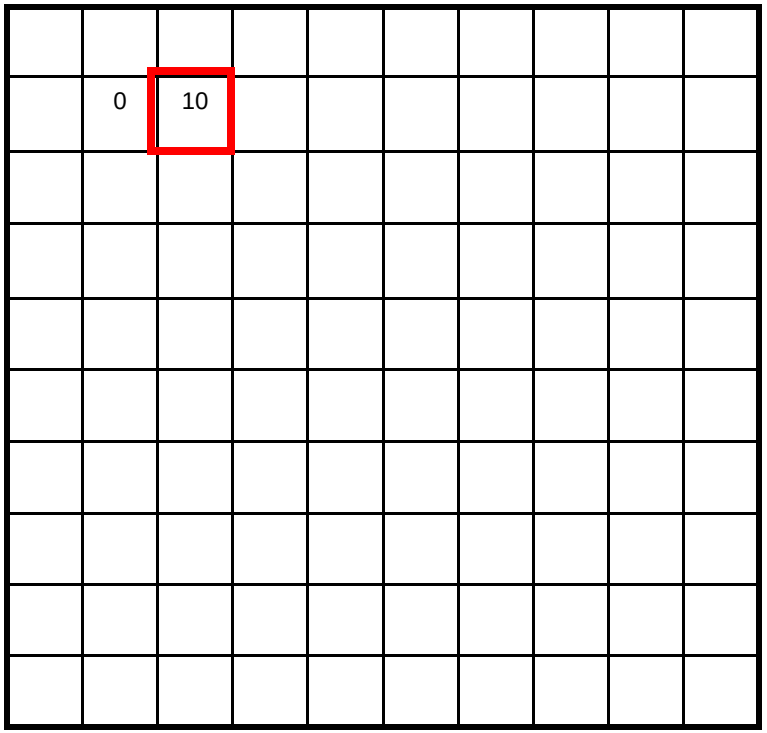
Courtesy of S. Seitz

Moving average

$F[x, y]$



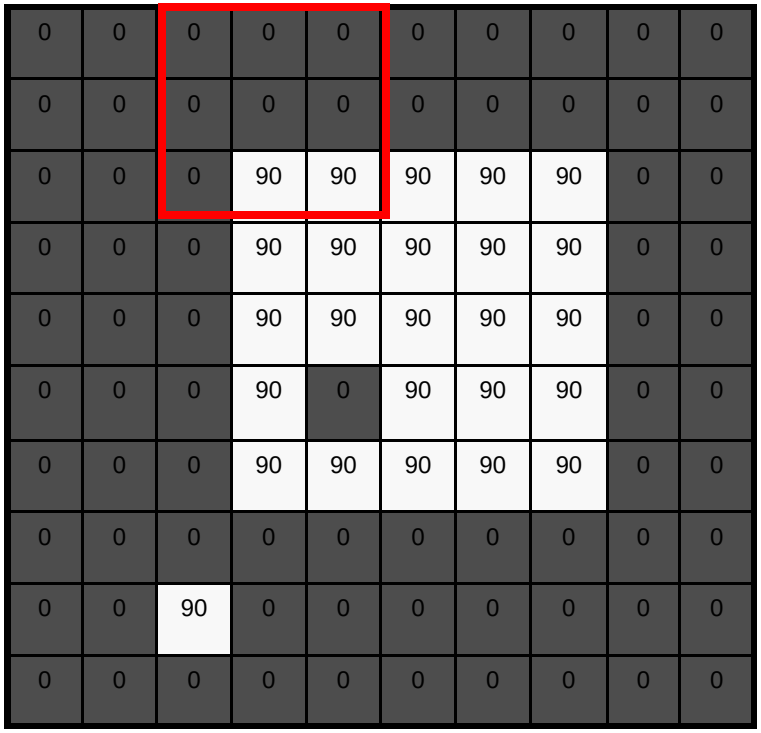
$G[x, y]$



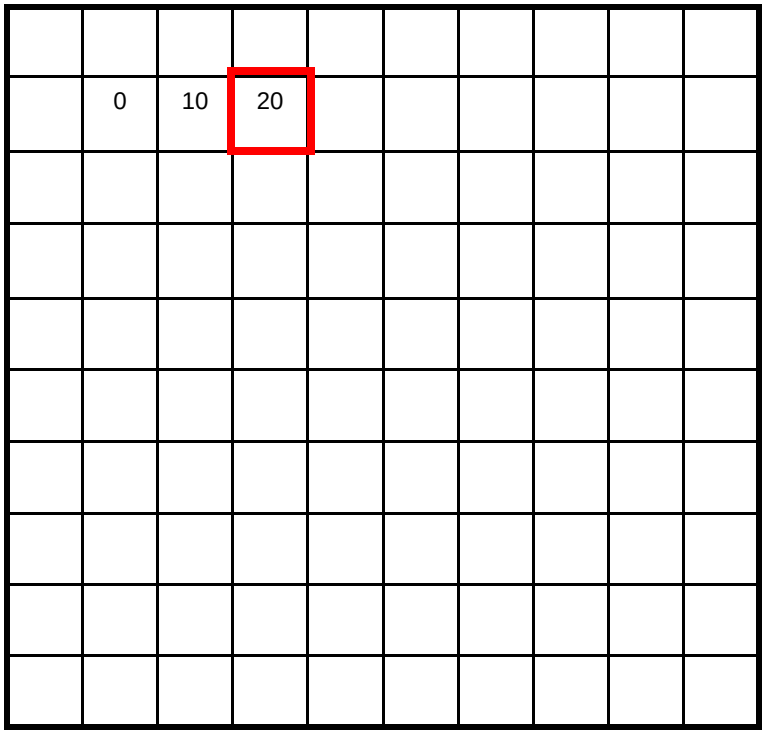
$$(f * g)[m, n] = \sum_{k,l} f[k, l]\, g[m - k, n - l]$$

Moving average

$F[x, y]$



$G[x, y]$



$$(f * g)[m, n] = \sum_{k,l} f[k, l]\, g[m - k, n - l]$$

Moving average

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

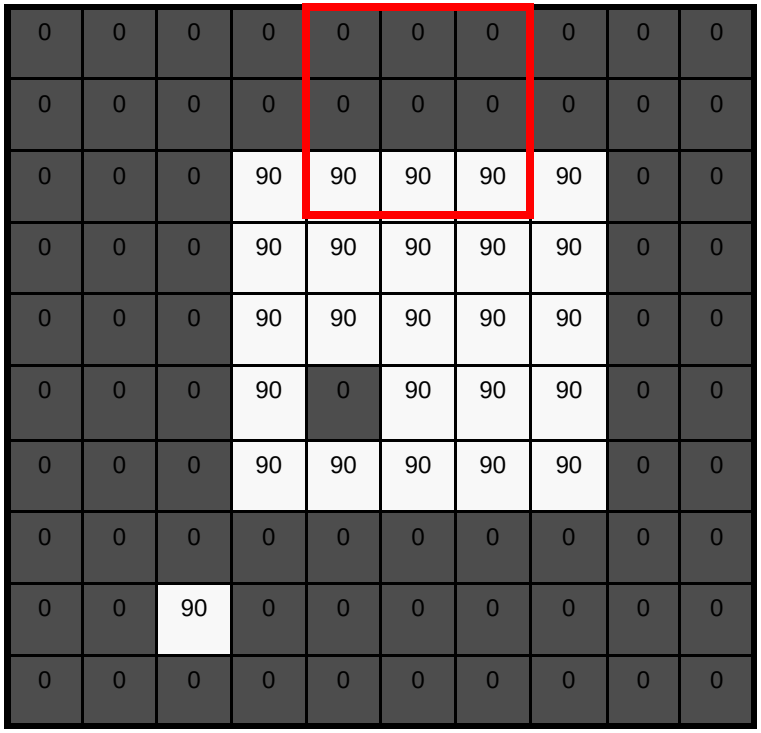
$G[x, y]$

	0	10	20	30					

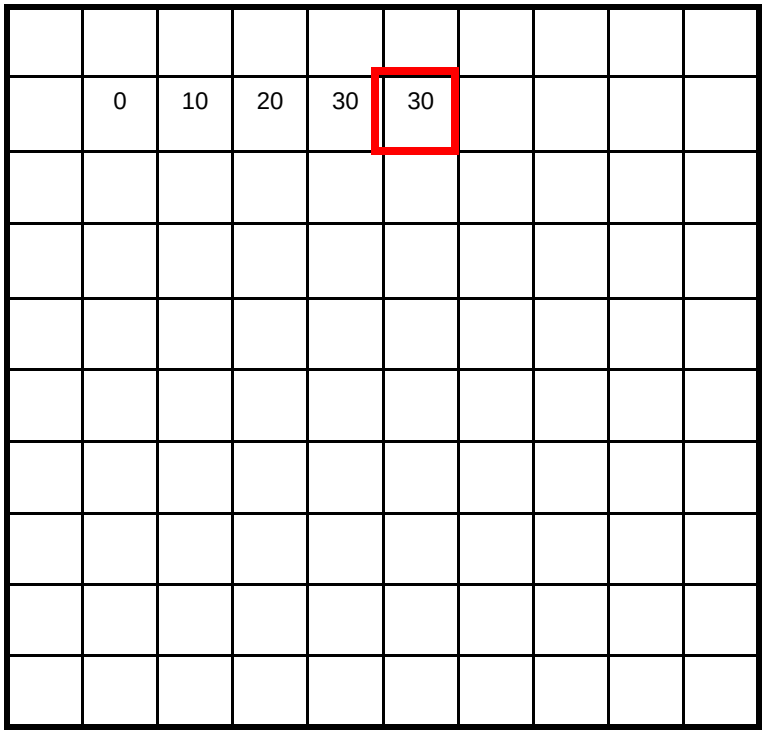
$$(f * g)[m, n] = \sum_{k,l} f[k, l]\, g[m - k, n - l]$$

Moving average

$F[x, y]$



$G[x, y]$



$$(f * g)[m, n] = \sum_{k,l} f[k, l]\, g[m - k, n - l]$$

Moving average

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

$$(f * g)[m, n] = \sum_{k,l} f[k, l]\, g[m - k, n - l]$$

Source: S. Seitz

Moving average

In summary:

- Replaces each pixel with an average of its neighborhood.

- Achieve smoothing effect (remove sharp features)

$g[\cdot,\cdot]$

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Moving average

Filters: Examples

- Image segmentation based on a simple threshold:

$$g[n,m] = \begin{cases} 1, & f[n,m] > 10 \\ 0, & \text{otherwise.} \end{cases}$$

Classification of systems

- Amplitude properties
  - Linearity
  - Stability
  - Invertibility
- Spatial properties
  - Causality
  - Separability
  - Memory
  - Shift invariance
  - Rotation invariance

Shift-invariance

- If $f[n,m] \xrightarrow{\mathcal{S}} g[n,m]$ then

$$f[n-n_0, m-m_0] \xrightarrow{\mathcal{S}} g[n-n_0, m-m_0]$$

for every input image f[n,m] and shifts n_0,m_0

- Is the moving average shift invariant a system ?

Is the moving average system is shift invariant?

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

Is the moving average system is shift invariant?

$$f[n,m] \quad \xrightarrow{\mathcal{S}} \quad g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

$$f[n-n_0, m-m_0]$$

$$\xrightarrow{\mathcal{S}} \quad g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

$$= \quad \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[(n-n_0)-k, (m-m_0)-l]$$

$$= \quad g[n-n_0, m-m_0]$$ Yes!

Linear Systems (filters)

$$f(x,y) \rightarrow \boxed{\mathcal{S}} \rightarrow g(x,y)$$

- Linear filtering:
  - Form a new image whose pixels are a weighted sum of original pixel values
  - Use the same set of weights at each point

- **S** is a linear system (function) iff it *S satisfies*

$$\mathcal{S}[\alpha f_1 + \beta f_2] = \alpha \mathcal{S}[f_1] + \beta \mathcal{S}[f_2]$$

superposition property

Linear Systems (filters)

$$f(x,y) \rightarrow \boxed{\mathcal{S}} \rightarrow g(x,y)$$

- Is the moving average a system linear?

- Is thresholding a system linear?
  - f1[n,m] + f2[n,m] > T
  - f1[n,m] < T
  - f2[n,m]<T

No!

LSI (linear *shift invariant*) systems

Impulse response

$$\delta_2[n, m] \rightarrow \boxed{\mathcal{S}} \rightarrow h[n, m]$$

$$\delta_2[n - k, m - l] \rightarrow \boxed{\mathcal{S}\,(\mathrm{SI})} \rightarrow h[n - k, m - l]$$

LSI (linear *shift invariant*) systems

Example: impulse response of the 3 by 3 moving average filter:

$$h[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1}\delta_2[n-k,m-l]$$

$$= \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

h

$\frac{1}{9}$

1	1	1
1	1	1
1	1	1

LSI (linear *shift invariant*) systems

An LSI system is completely specified by its impulse response.

sifting property of the delta function

$$f[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l]\, \delta_2[n-k, m-l]$$

superposition

$$\rightarrow \boxed{\mathcal{S}\ \text{LSI}} \rightarrow \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l]\, h[n-k, m-l]$$

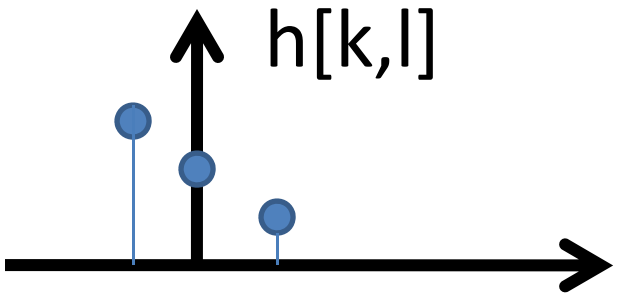
$$\delta_2[n,m] \rightarrow \boxed{\mathcal{S}} \rightarrow h[n,m]$$

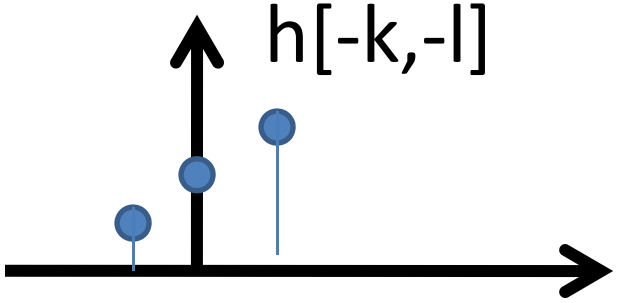
Discrete convolution

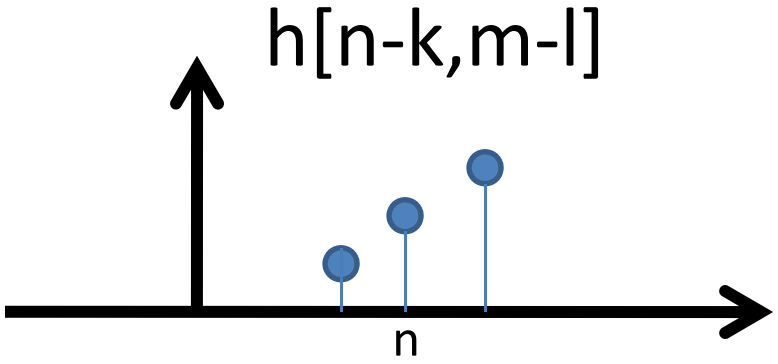
$$= f[n,m] ** h[n,m]$$

Discrete convolution

- Fold h[n,m] about origin to form h[−k,−l]
- Shift the folded results by n,m to form h[n − k,m − l]
- Multiply h[n − k,m − l] by f[k, l]
- Sum over all k,l
- Repeat for every n,m







Discrete convolution

- Fold h[n,m] about origin to form h[−k,−l]
- Shift the folded results by n,m to form h[n − k,m − l]
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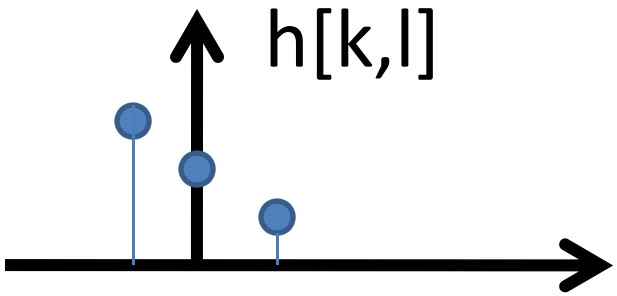

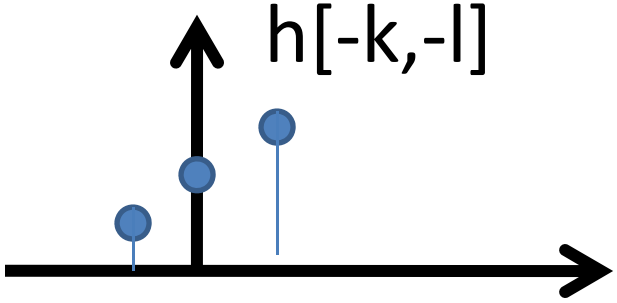

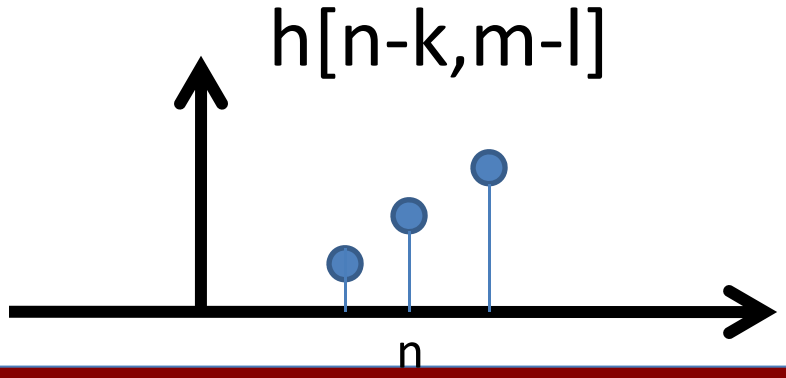

Discrete convolution

- Fold h[n,m] about origin to form h[−k,−l]
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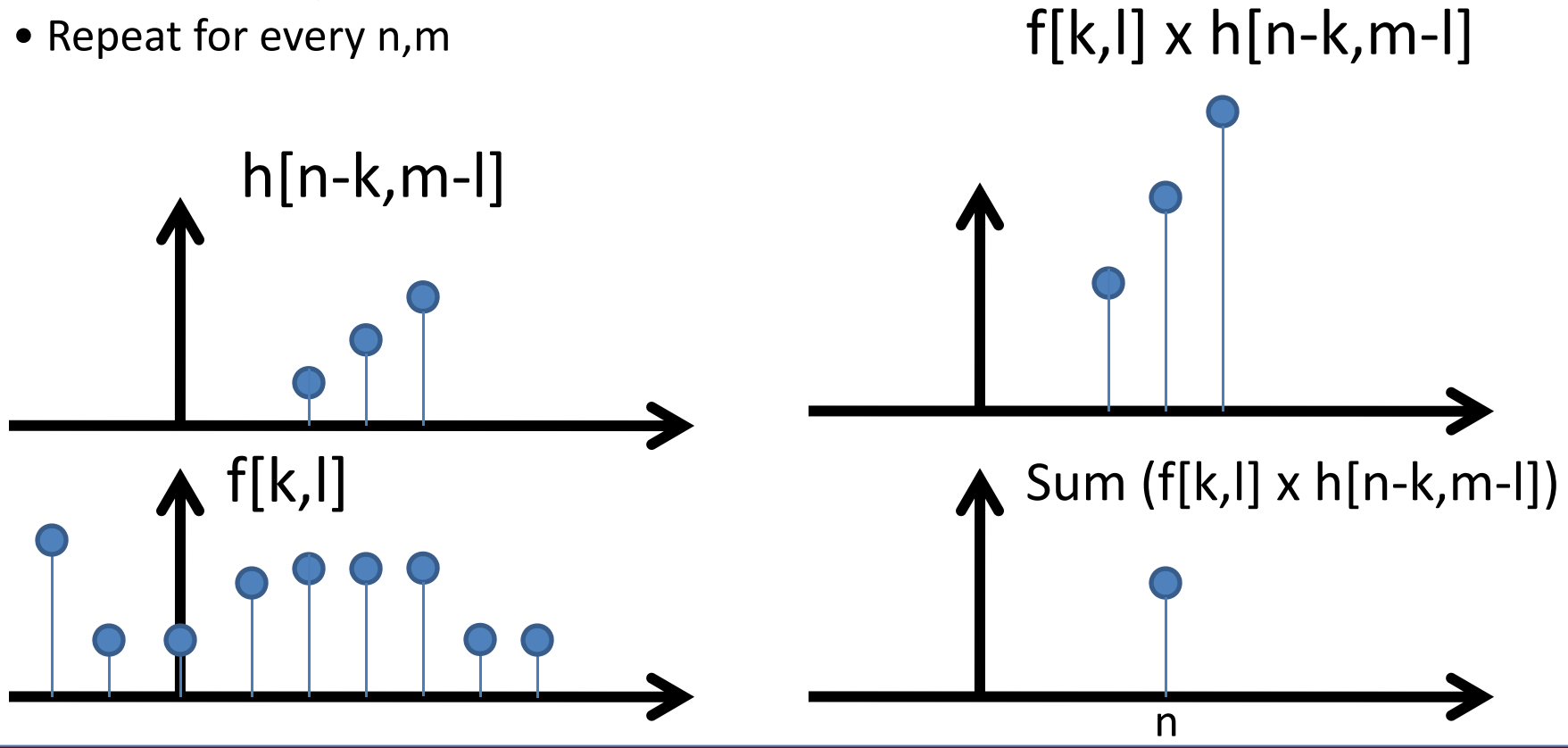
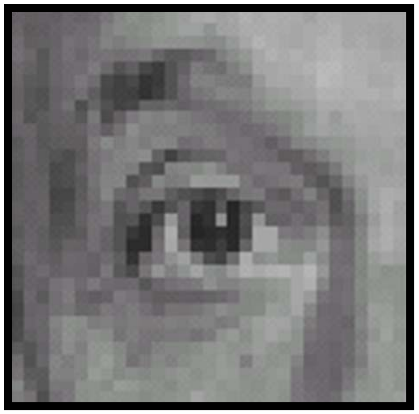


Convolution in 2D - examples



Original

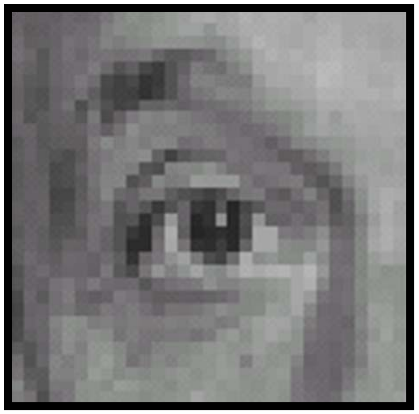
•0	•0	•0
•0	•1	•0
•0	•0	•0

=

?

Courtesy of D Lowe

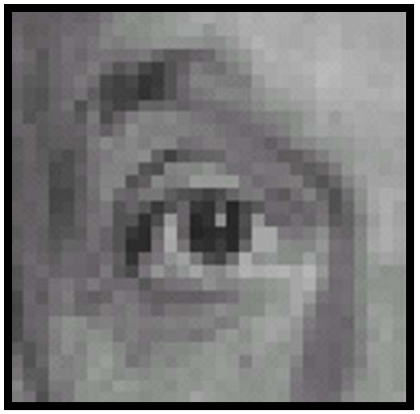
Convolution in 2D - examples



Original

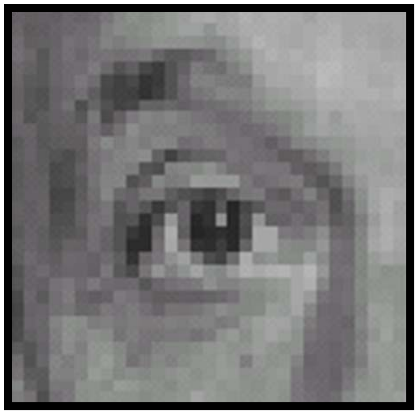
•0	•0	•0
•0	•1	•0
•0	•0	•0

=



Filtered
(no change)

Convolution in 2D - examples



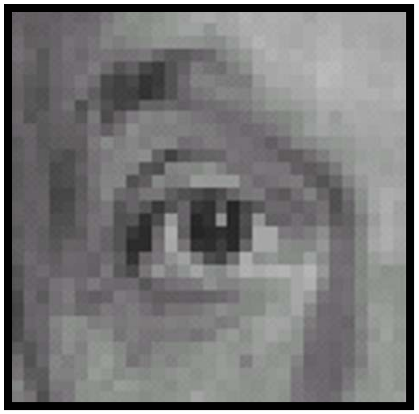
Original

•0	•0	•0
•0	•0	•1
•0	•0	•0

=

?

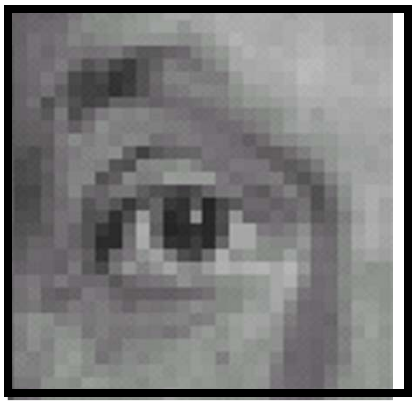
Convolution in 2D - examples



Original

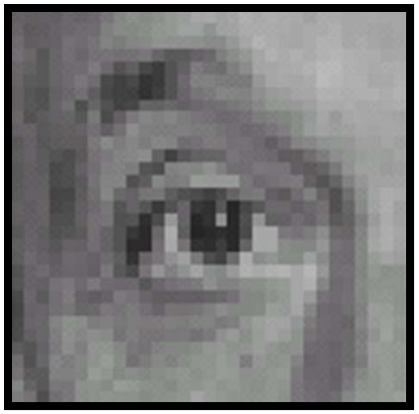
•0	•0	•0
•0	•0	•1
•0	•0	•0

=



Shifted left
By 1 pixel

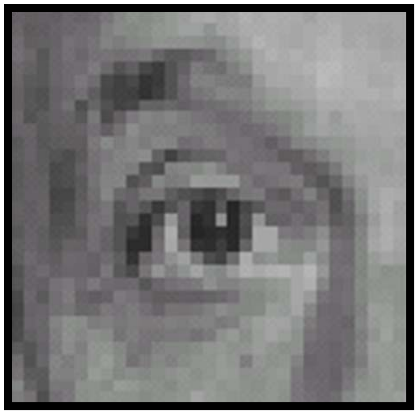
Convolution in 2D - examples



Original

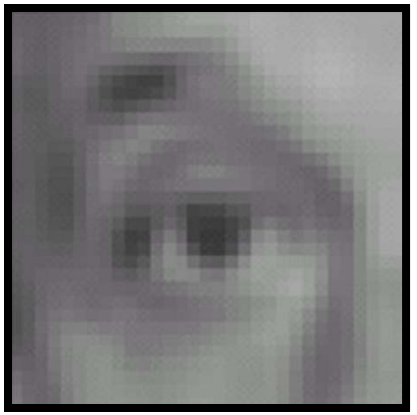
$$\frac{1}{9}\begin{bmatrix} \cdot 1 & \cdot 1 & \cdot 1 \\ \cdot 1 & \cdot 1 & \cdot 1 \\ \cdot 1 & \cdot 1 & \cdot 1 \end{bmatrix} = \; ?$$

Convolution in 2D - examples



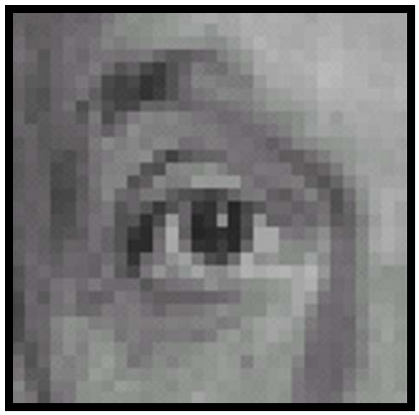
Original

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} =$$



Blur (with a box filter)

Convolution in 2D - examples



Original

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = ?$$

(Note that filter sums to 1)

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

details

- What does blurring take away?

–

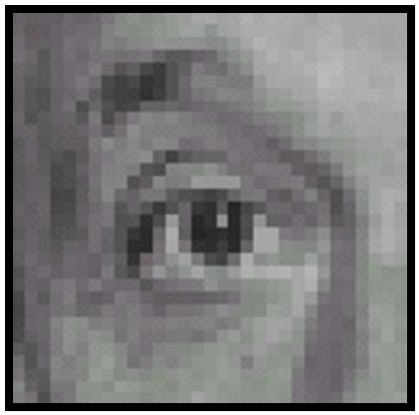
=

  - Let's add it back:

+ a

=

Convolution in 2D – Sharpening filter



Original

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} =$$

Sharpening filter: Accentuates differences with local average

Convolution properties

- **Commutative property:**

$$f * * h = h * * f$$

- **Associative property:**

$$(f * * h_1) * * h_2 = f * * (h_1 * * h_2)$$

- **Distributive property:**

$$f * * (h_1 + h_2) = (f * * h_1) + (f * * h_2)$$

The order doesn't matter! $h_1 * * h_2 = h_2 * * h_1$

Convolution properties

- **Shift property:**

$$f[n,m] ** \delta_2[n-n_0, m-m_0] = f[n-n_0, m-m_0]$$

- **Shift-invariance:**

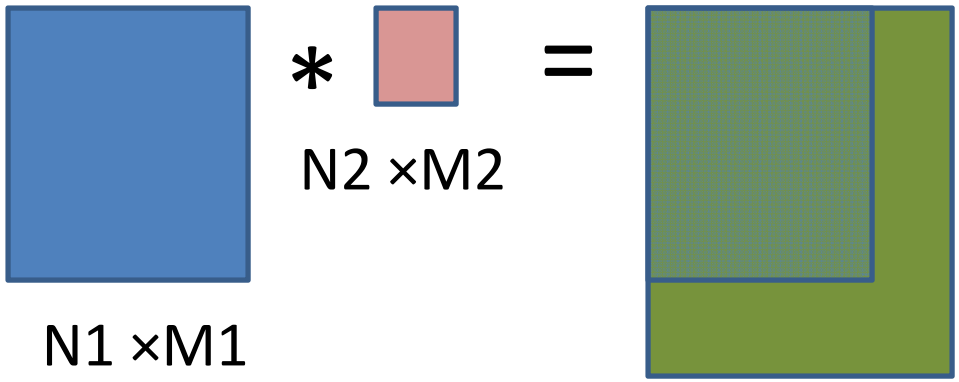
$$\begin{aligned} g[n,m] &= f[n,m] ** h[n,m] \\ &\Longrightarrow f[n-l_1, m-l_1] ** h[n-l_2, m-l_2] \\ &= g[n-l_1-l_2, m-l_1-l_2] \end{aligned}$$

Image support and edge effect

•A computer will only convolve **finite support signals.**

- That is: images that are zero for n,m outside some rectangular region

• MATLAB's conv2 performs 2D DS convolution of finite-support signals.

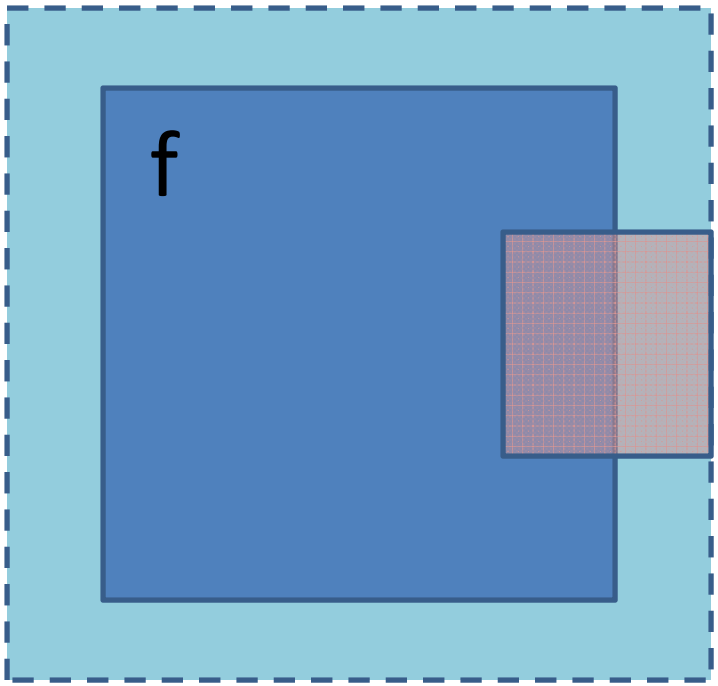


(N1 + N2 − 1) × (M1 +M2 − 1)

Image support and edge effect

•A computer will only convolve **finite support signals.**

• What happens at the edge?



h

- zero “padding”
- edge value replication
- mirror extension
- more (beyond the scope of this class)

-> Matlab conv2 uses zero-padding

Cross correlation

Cross correlation of two 2D signals f[n,m] and g[n,m]

$$r_{fg}[k,l] \triangleq \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f[n,m]\, g^*[n-k, m-l]$$

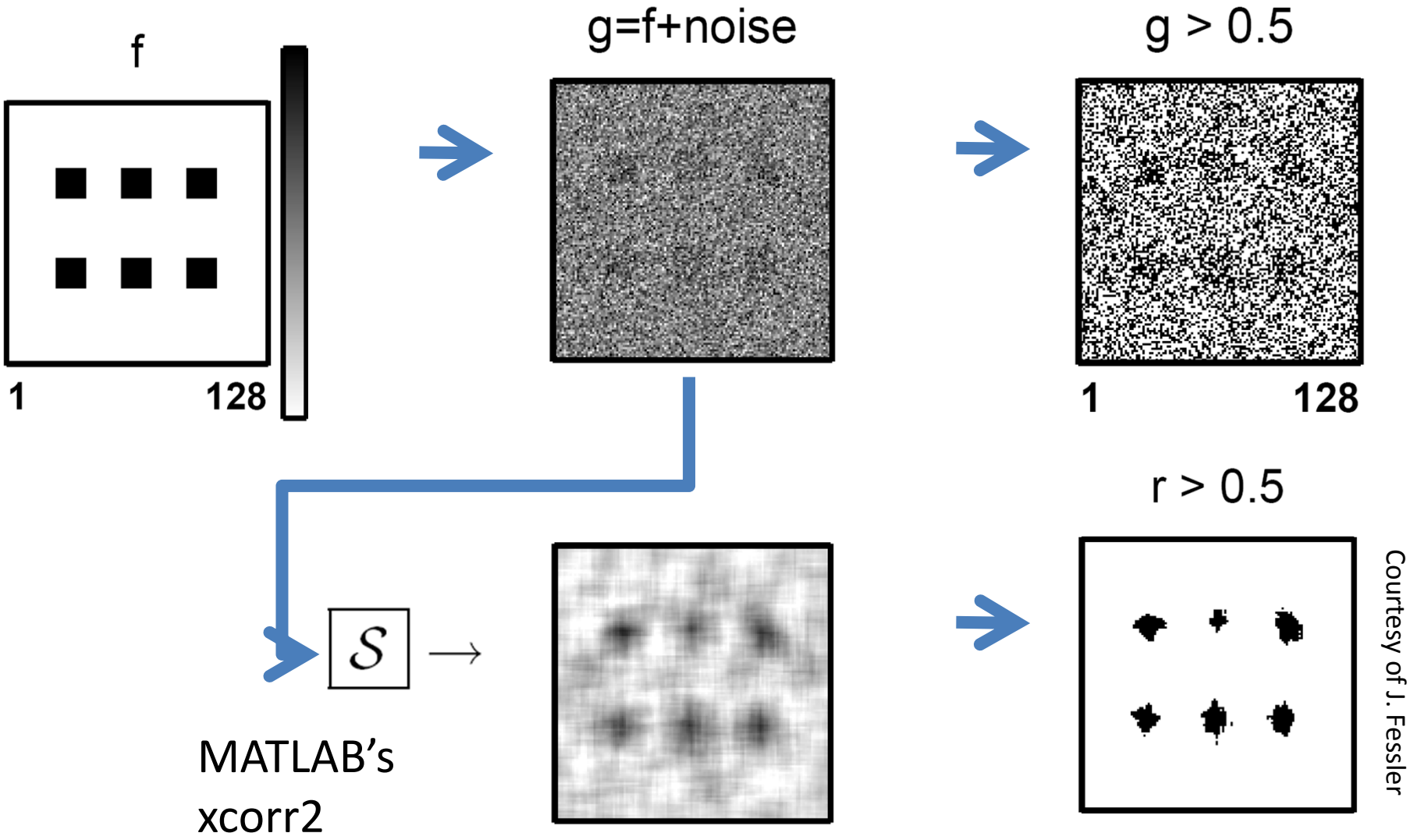
$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f[n+k, m+l]\, g^*[n,m], \quad k,l \in \mathbb{Z}$$

(k, l) is called the **lag**

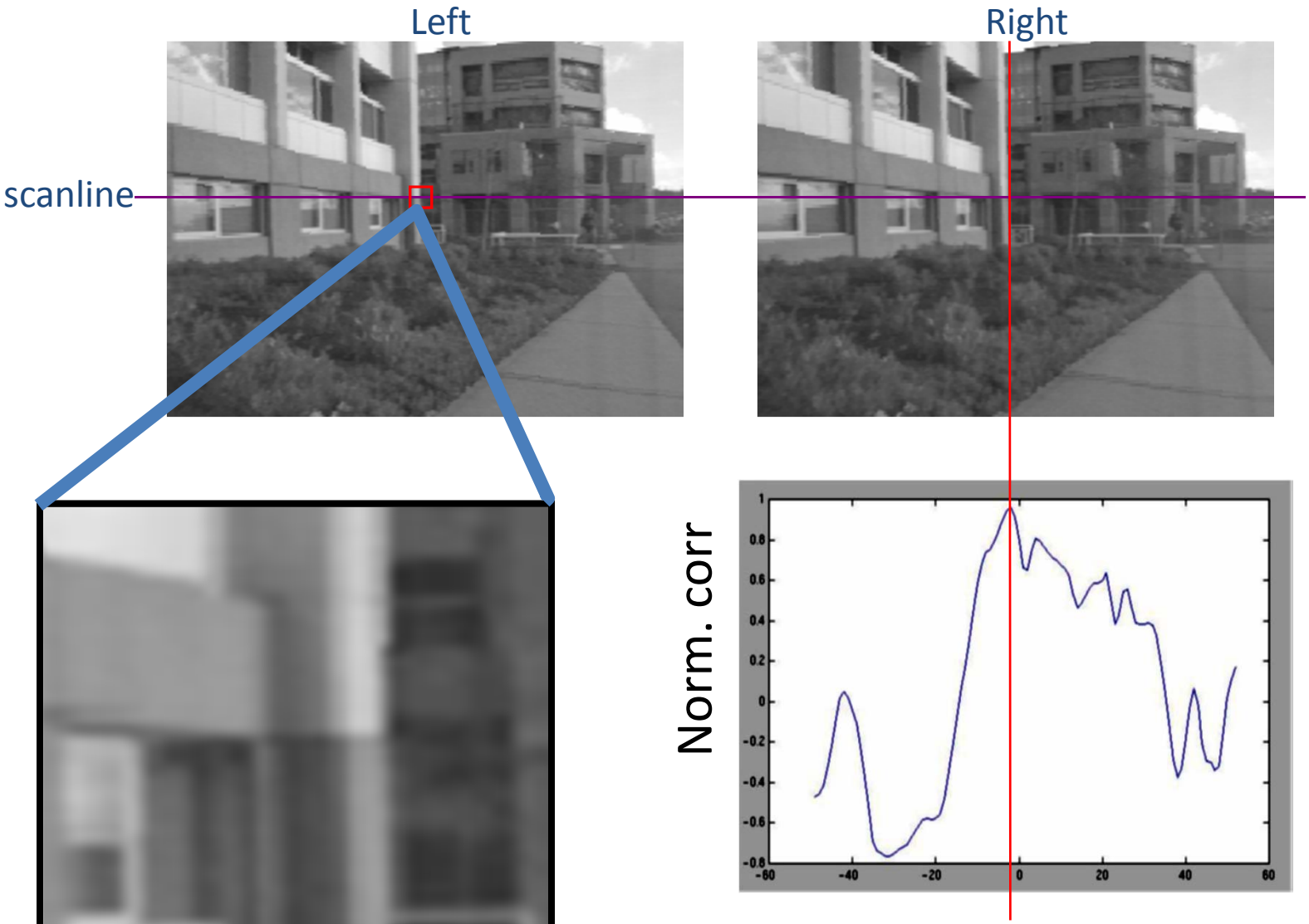
- Equivalent to a convolution without the flip

$$r_{fg}[n,m] = f[n,m] ** g^*[-n,-m]$$

Cross correlation – example



Cross correlation – example



Convolution vs. Correlation

- A **convolution** is an integral that expresses the amount of overlap of one function as it is shifted over another function.
  - convolution is a filtering operation

- **Correlation** compares the ***similarity*** *of* ***two*** *sets of* ***data.*** Correlation computes a measure of similarity of two input signals as they are shifted by one another. The correlation result reaches a maximum at the time when the two signals match best .
  - correlation is a measure of relatedness of two signals

2D Discrete-Signal Fourier Transform (DSFT)

2D discrete-signal Fourier transform (DSFT) of a 2D discrete-space signal g[n,m]:

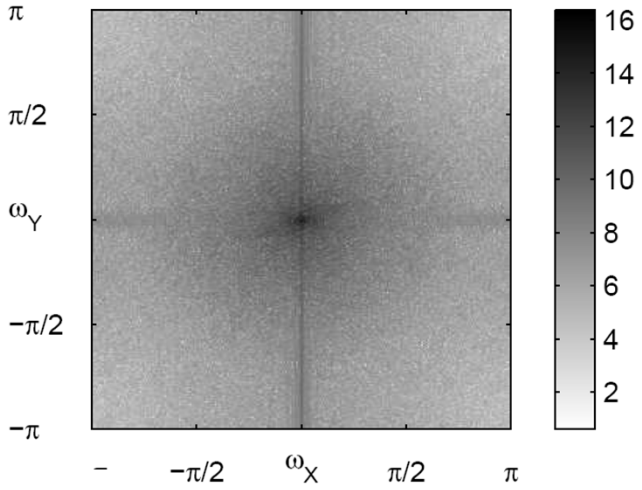
$$G(\omega_{\mathrm{X}}, \omega_{\mathrm{Y}}) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} g[n, m]\, \mathrm{e}^{-\imath(\omega_{\mathrm{X}} n + \omega_{\mathrm{Y}} m)}$$

Inverse 2D DSFT:

$$g[n, m] = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} G(\omega_{\mathrm{X}}, \omega_{\mathrm{Y}})\, \mathrm{e}^{\imath(\omega_{\mathrm{X}} n + \omega_{\mathrm{Y}} m)}\, \mathrm{d}\omega_{\mathrm{X}}\, \mathrm{d}\omega_{\mathrm{Y}}$$

Jean Baptiste Joseph Fourier

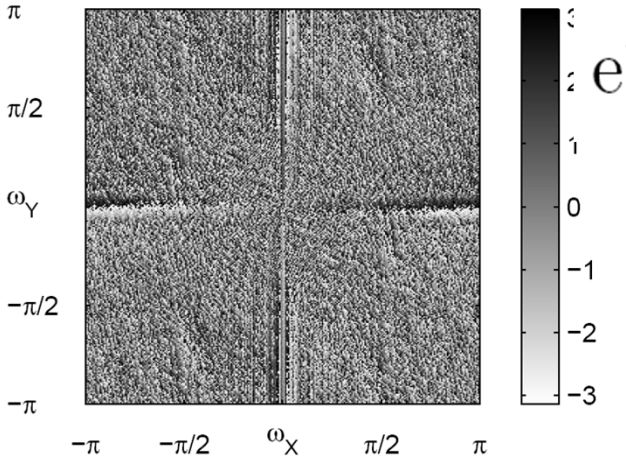
Fourier's Transform: log(Magnitude)



$|G(\omega_X, \omega_Y)|$ e

DSFT

Fourier's Transform: Phase



$e^{i\angle G(\omega_X,\omega_Y)}$

DSFT – properties

Shift:

$$g[n - n_0, m - m_0] \longleftrightarrow G(\omega_X, \omega_Y)\, e^{-\imath(\omega_X n_0 + \omega_Y m_0)}$$

Convolution:

$$f[n, m] ** h[n, m] \longleftrightarrow F(\omega_X, \omega_Y)\, H(\omega_X, \omega_Y)$$

Delta function:

$$\delta_2[n, m] \longleftrightarrow 1$$

$$\delta_2[n - n_0, m - m_0] \longleftrightarrow e^{-\imath(\omega_X n_0 + \omega_Y m_0)}$$

Example: DSFT of moving average filter

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

$$(f * h)[m,n] = \frac{1}{9}\sum_{k,l} f[k,l]\, h[m-k, n-l]$$

$$F(\omega_X, \omega_Y)\, H(\omega_X, \omega_Y)$$

$$= \frac{1}{9}\sum_{n=-1}^{1}\sum_{m=-1}^{1} e^{-\imath\omega_X n}\, e^{-\imath\omega_Y m}$$

$$= \frac{1}{9}[1 + 2\cos\omega_X][1 + 2\cos\omega_Y]$$

h

$\frac{1}{9}$

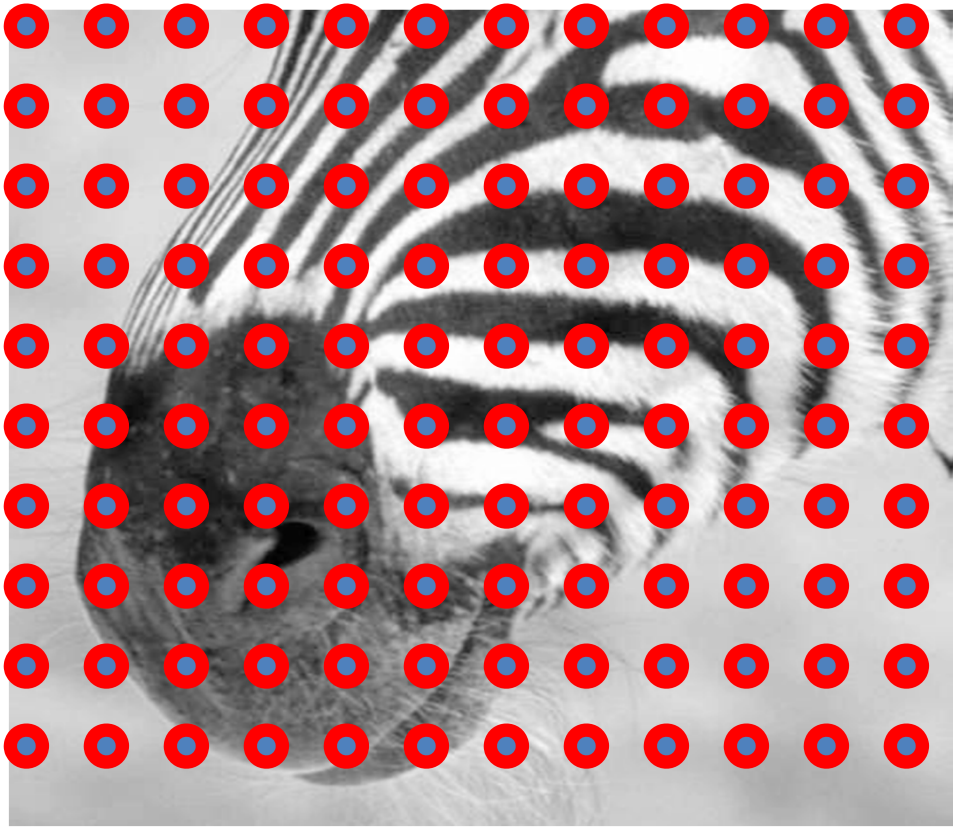
1	1	1
1	1	1
1	1	1

Function	Fourier transform
$g(x,y)$	$\int\int_{-\infty}^{\infty} g(x,y)e^{-i2\pi(ux+vy)}dxdy$
$\int\int_{-\infty}^{\infty} \mathcal{F}(g(x,y))(u,v)e^{i2\pi(ux+vy)}dudv$	$\mathcal{F}(g(x,y))(u,v)$
$\delta(x,y)$	1
$\frac{\partial f}{\partial x}(x,y)$	$u\mathcal{F}(f)(u,v)$
$0.5\delta(x+a,y)+0.5\delta(x-a,y)$	$\cos 2\pi au$
$e^{-\pi(x^2+y^2)}$	$e^{-\pi(u^2+v^2)}$
$box_1(x,y)$	$\frac{\sin u}{u}\frac{\sin v}{v}$
$f(ax,by)$	$\frac{\mathcal{F}(f)(u/a,v/b)}{ab}$
$\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\delta(x-i,y-j)$	$\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\delta(u-i,v-j)$
$(f**g)(x,y)$	$\mathcal{F}(f)\mathcal{F}(g)(u,v)$
$f(x-a,y-b)$	$e^{-i2\pi(au+bv)}\mathcal{F}(f)$
$f(x\cos\theta-y\sin\theta, x\sin\theta+y\cos\theta)$	$\mathcal{F}(f)(u\cos\theta-v\sin\theta, u\sin\theta+v\cos\theta)$

Why is DFT important?

- Perform efficient linear convolution as product of DFTs
- Each DFT can be implemented using the FFT (Fast Fourier Transform) [see appendix for details]

Sampling



Throw away every other row and column
to create a 1/2 size image

Sampling

- Down-sampling operation:

(trivial form of image compression)

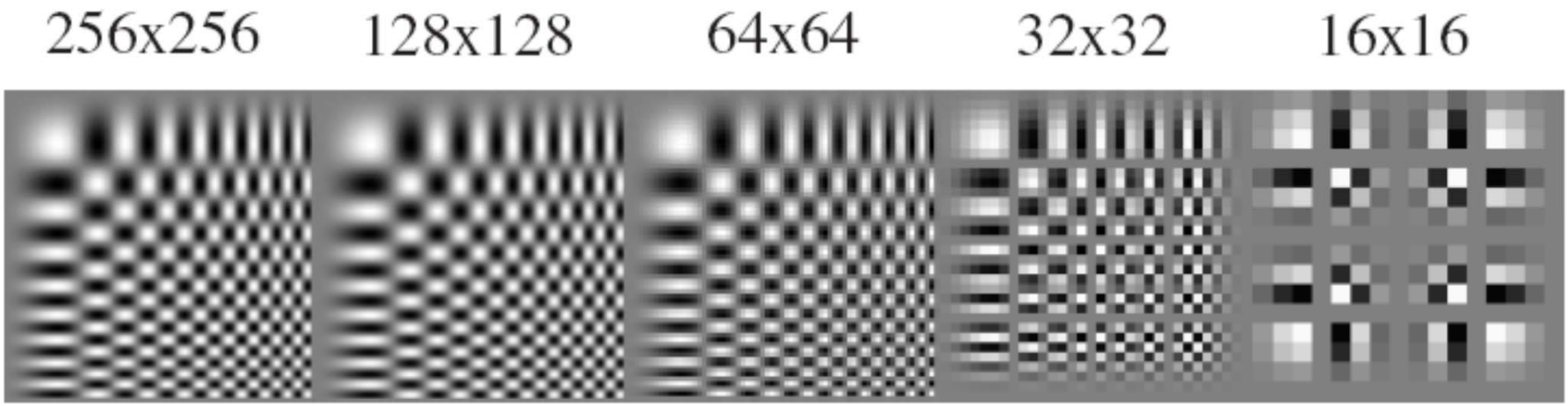
$$g[n,m] = f[2n,2m] = \begin{bmatrix} \ddots & & \vdots & & \\ & f[-2,2] & f[0,2] & f[2,2] & \\ \cdots & f[-2,0] & \underline{f[0,0]} & f[2,0] & \cdots \\ & f[-2,-2] & f[0,-2] & f[2,-2] & \\ & & \vdots & & \ddots \end{bmatrix}$$

Why is a multi-scale representation useful?

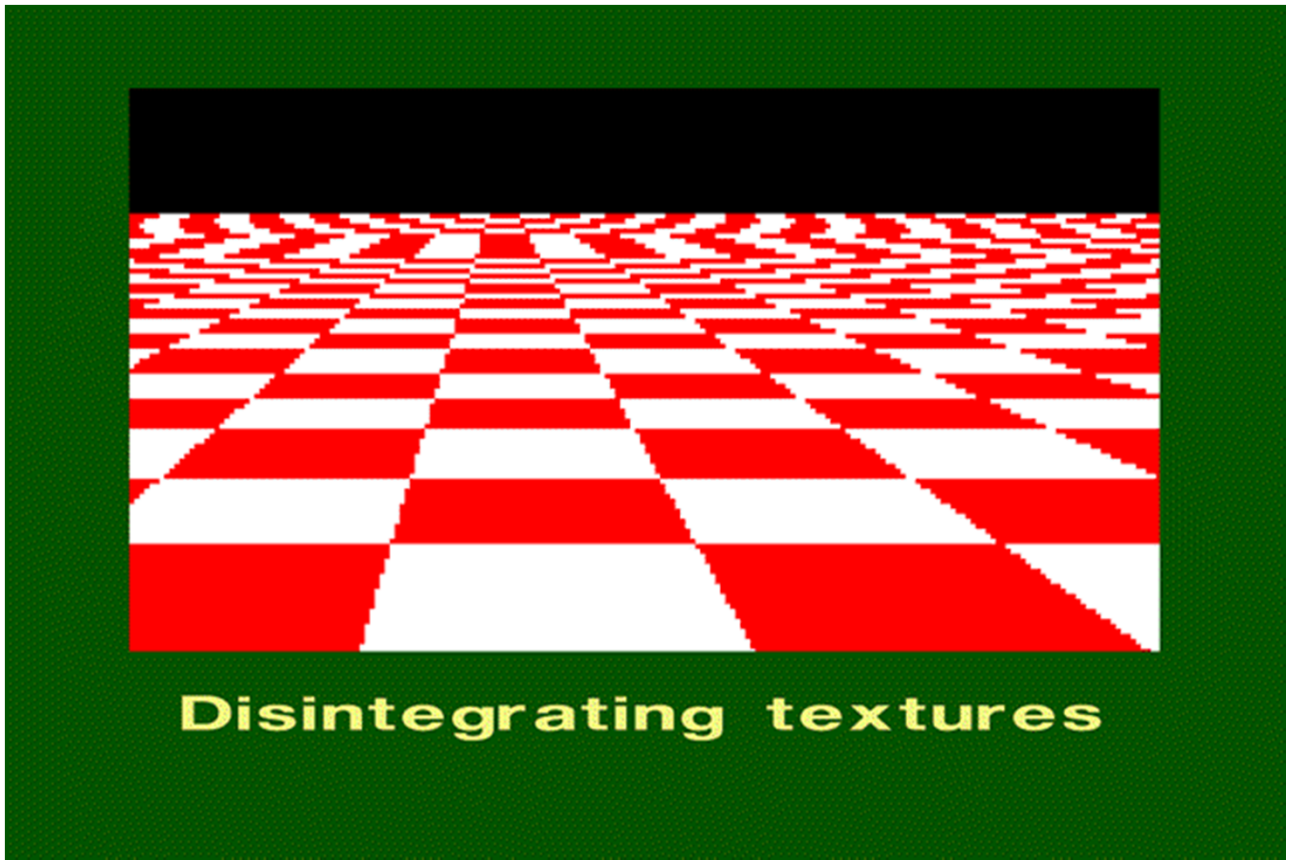
- Find template matches at all scales
  - e.g., when finding hands or faces, we don't know what size they will be in a particular image
  - Template size is constant, but image size changes
- Efficient search for correspondence
  - look at coarse scales, then refine with finer scales
- Examining all levels of detail
  - Find edges with different amounts of blur
  - Find textures with different spatial frequencies (levels of detail)

Slide credit: David Lowe (UBC)

Aliasing

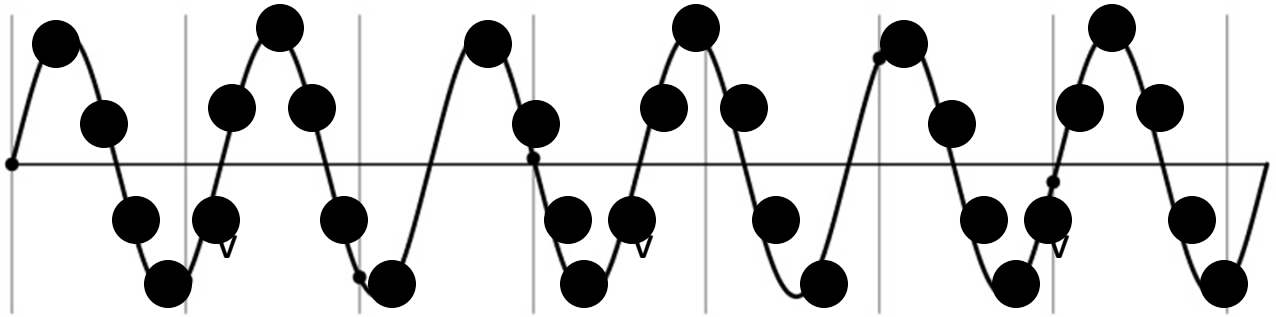


Aliasing

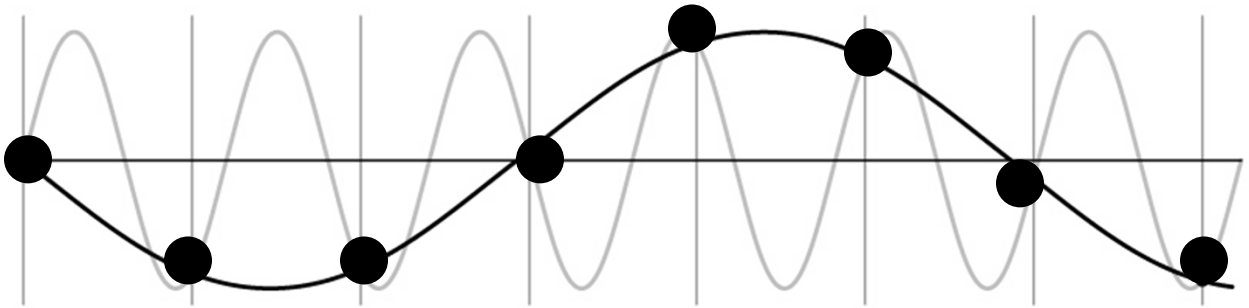


Sampling Theorem (Nyquist)

- When sampling a signal at discrete intervals, the sampling frequency must be $\geq 2 \times f_{max}$
- f_{max} = max frequency of the input signal.

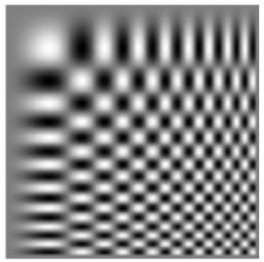


good



bad

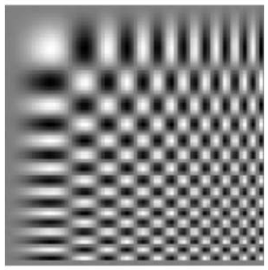
256x256



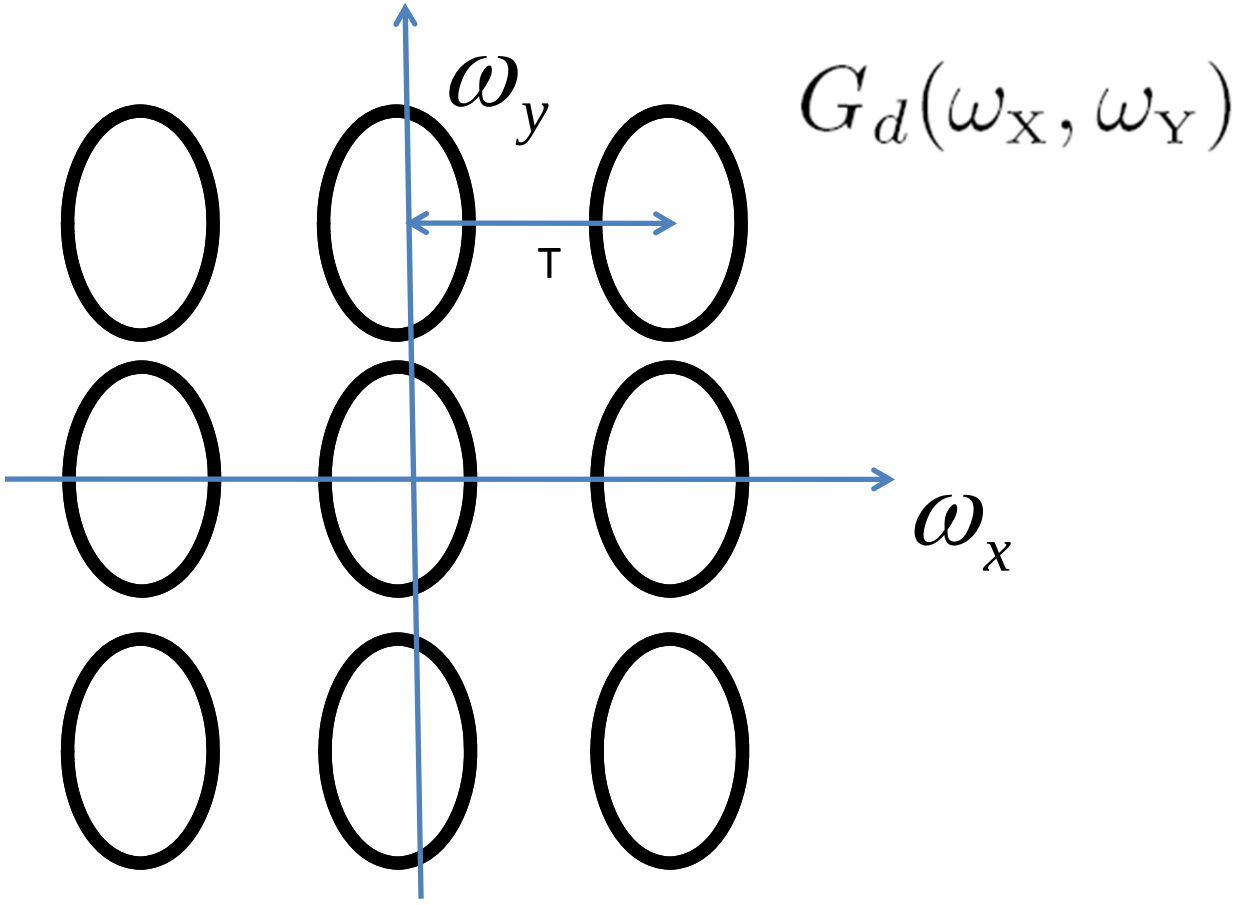
g [n,m]

Small sampling period N

64x64



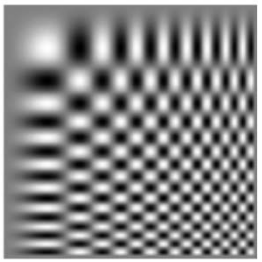
$g_{sampled}$ [n,m]



- N = sampling period
- T = periodicity of the replicas of the DFT (g)

$$T \sim 1/N$$

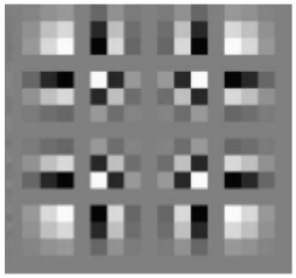
256x256



g [n,m]

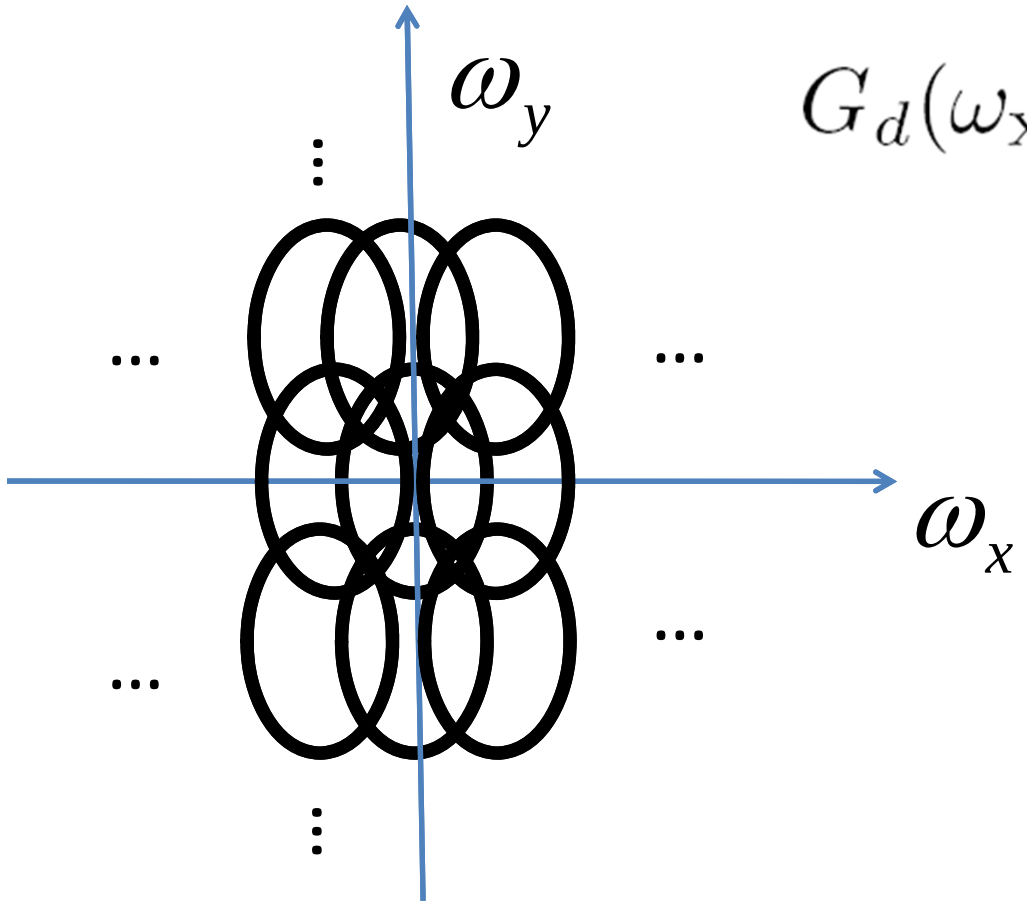
Large sampling period N

16x16



$g_{sampled}$ [n,m]

$G_d(\omega_X, \omega_Y)$



- N = sampling period
- T = periodicity of the replicas of the DFT (g)

$T \sim 1/N$

Anti-aliasing

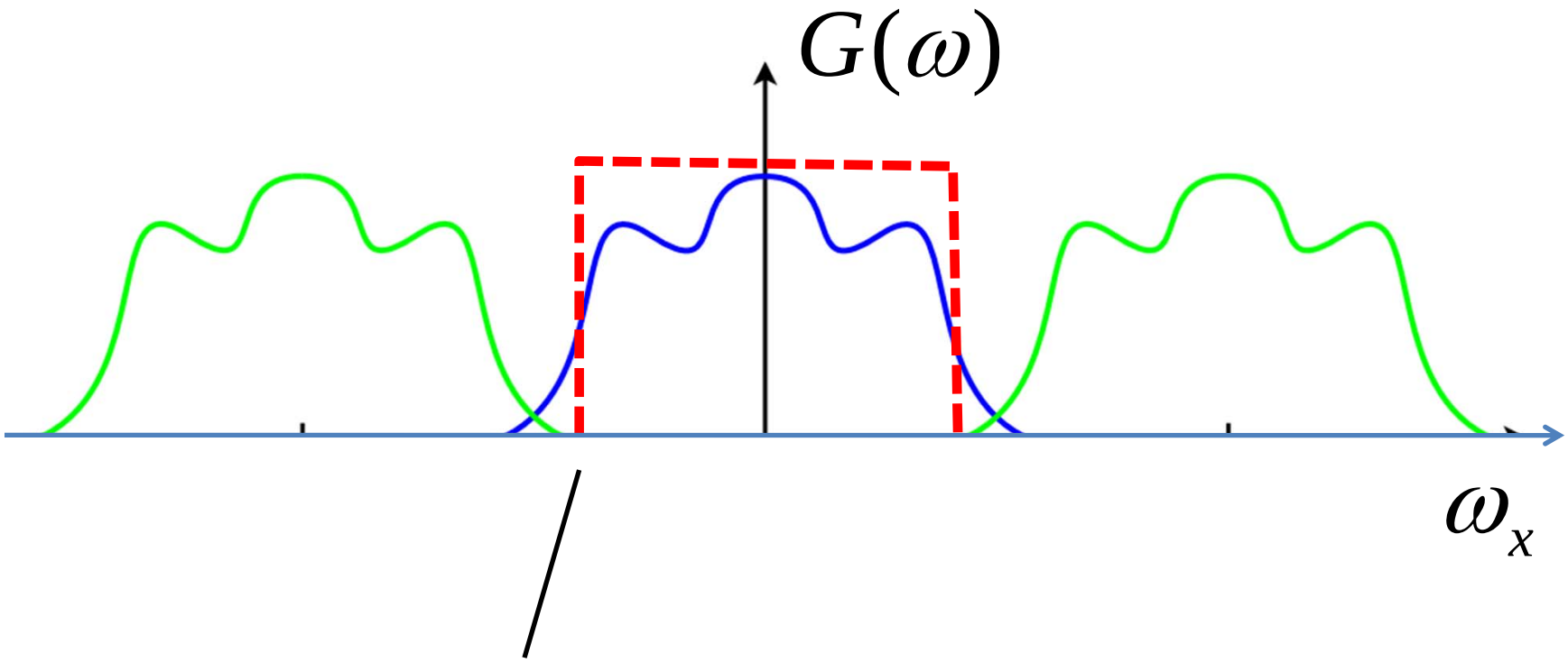
Solutions:

- Sample more often

- Get rid of all frequencies that are greater than half the new sampling frequency
  - Will lose information - but it's better than aliasing
  - Apply a smoothing filter to remove high frequencies

Anti-aliasing

Apply a smoothing filter to remove high frequencies:



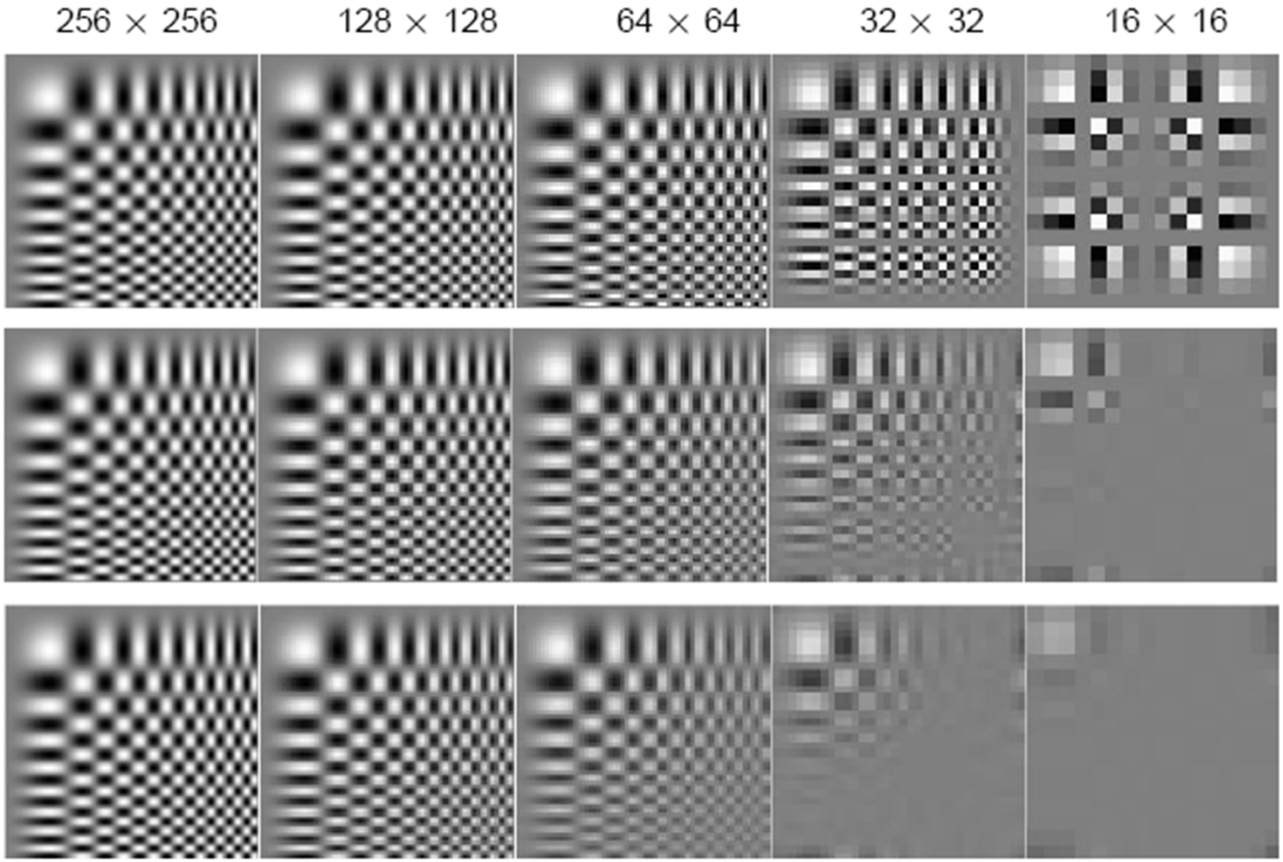
Low pass filter (smoothing)

Sampling algorithm

Algorithm 7.1: Sub-sampling an Image by a Factor of Two

Apply a low-pass filter to the original image
(a Gaussian with a σ of between one
and two pixels is usually an acceptable choice).
Create a new image whose dimensions on edge are half
those of the old image
Set the value of the i, j'th pixel of the new image to the value
of the $2i$, $2j$'th pixel of the filtered image

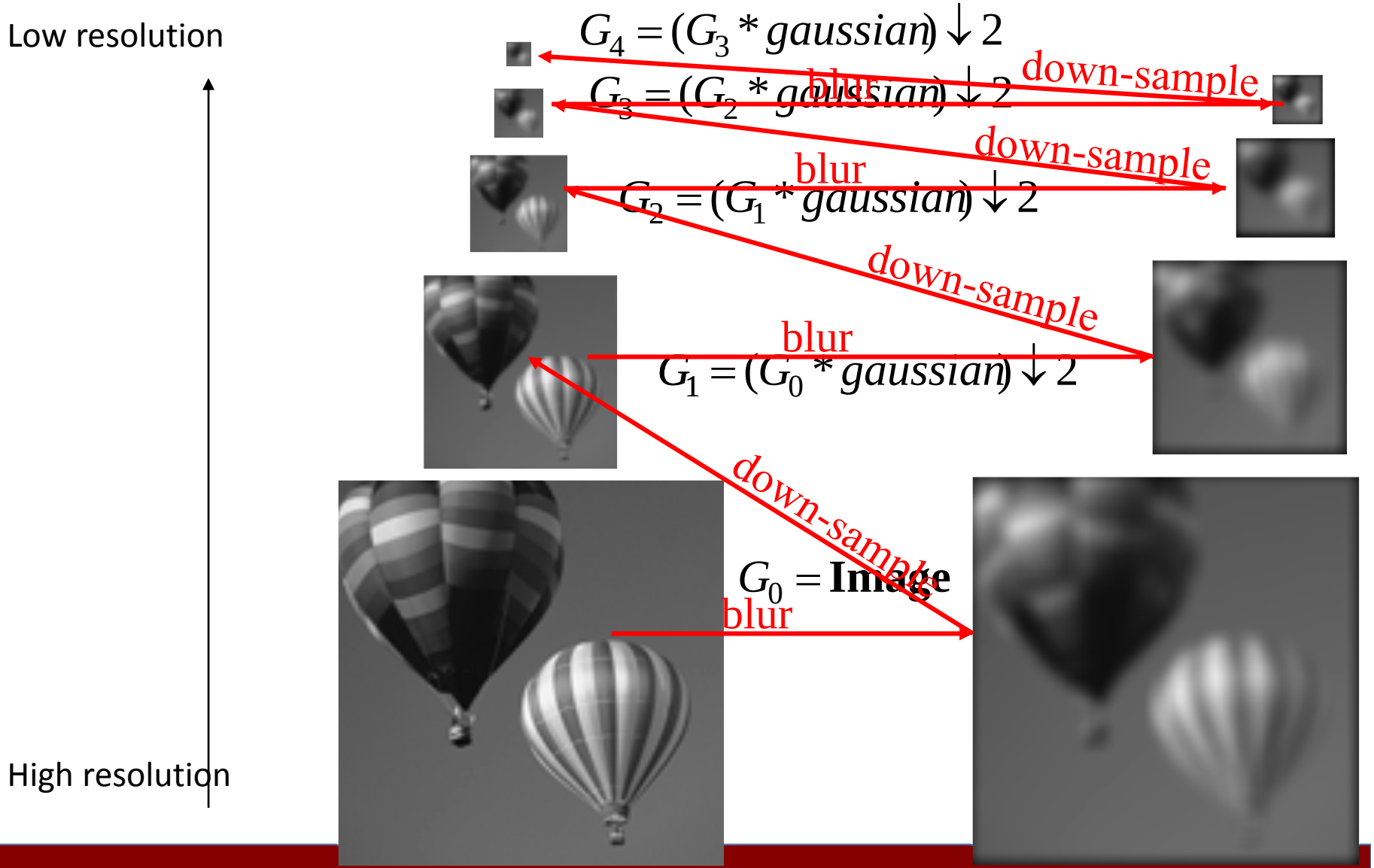
Resampling with Prior Smoothing



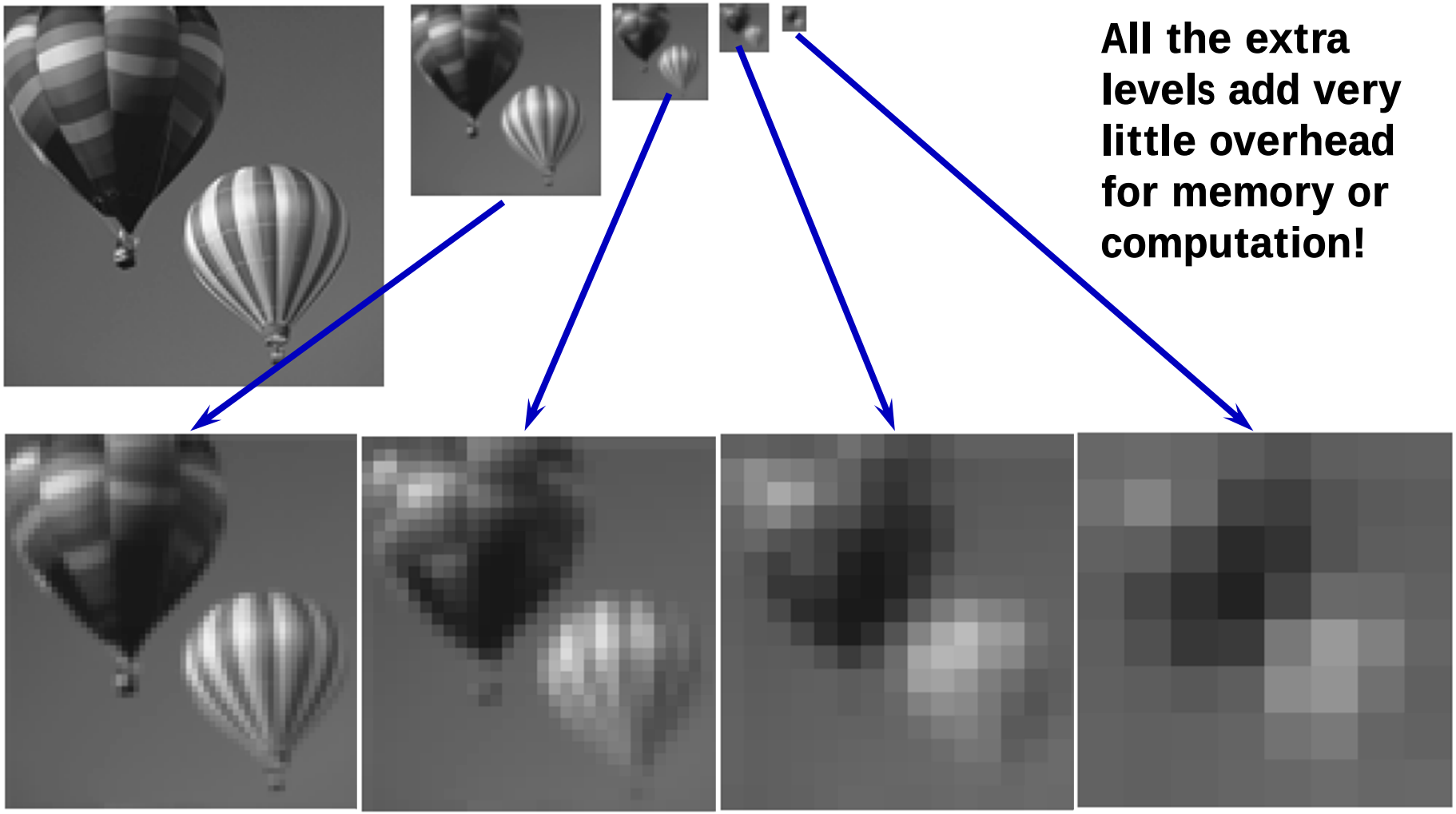
- Note: We cannot recover the high frequencies, but we can avoid artifacts by smoothing before resampling.

Image Source: Forsyth & Ponce

The Gaussian Pyramid



Gaussian Pyramid – Stored Information



All the extra levels add very little overhead for memory or computation!

Source: Irani & Basri

Summary: Gaussian Pyramid

- Construction: create each level from previous one
  - Smooth and sample
- Smooth with Gaussians, in part because
  - a Gaussian*Gaussian = another Gaussian
  - $G(\sigma_1) * G(\sigma_2) = G(\mathrm{sqrt}(\sigma_1^{\,2} + \sigma_2^{\,2}))$
- Gaussians are low-pass filters, so the representation is redundant once smoothing has been performed.

  $\Rightarrow$ There is no need to store smoothed images at the full original resolution.

Slide credit: David Lowe

Application: Vision system for TV remote control

- uses template matching

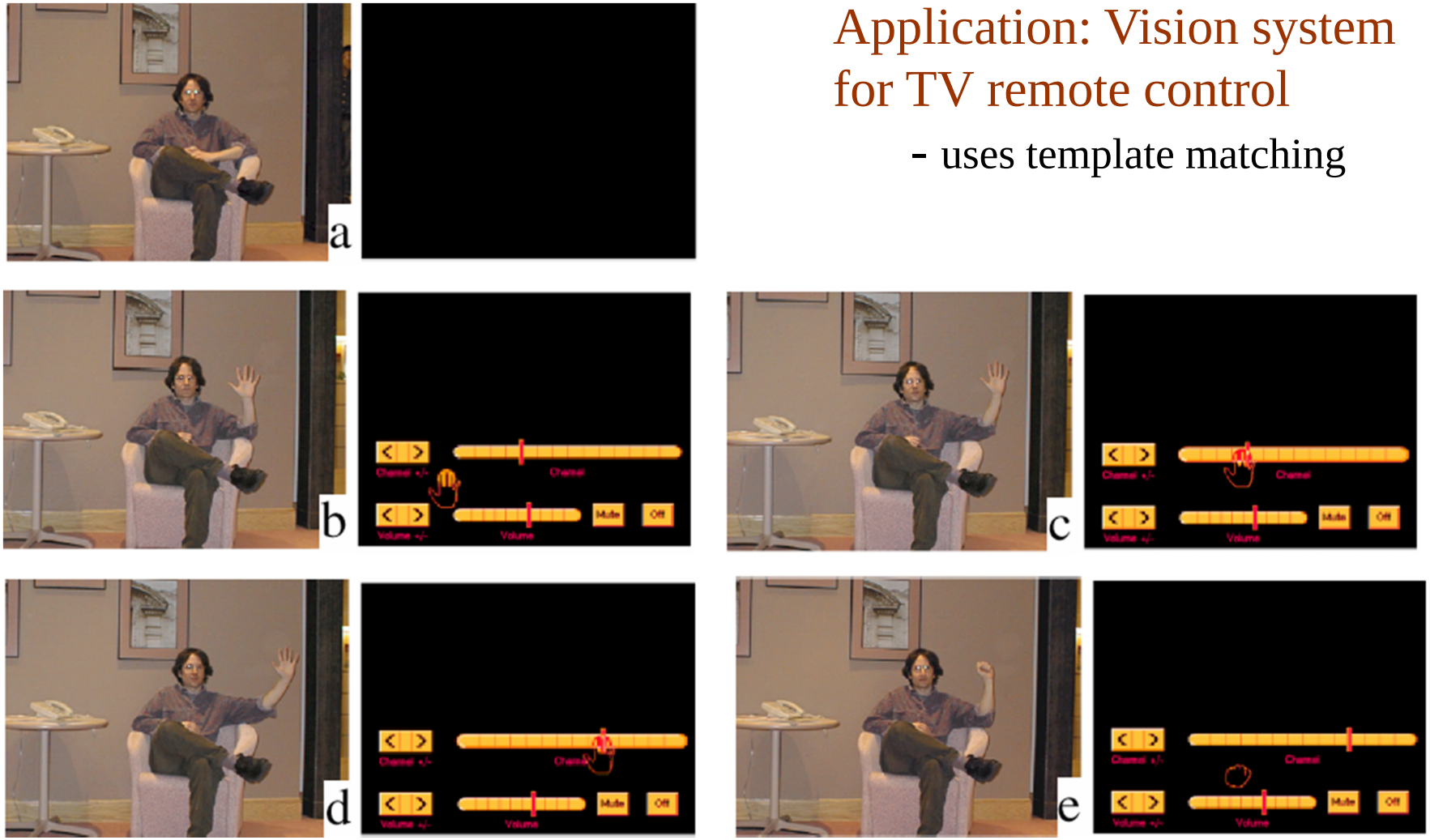


Figure from "Computer Vision for Interactive Computer Graphics," W.Freeman et al, IEEE Computer Graphics and Applications, 1998 copyright 1998, IEEE

What we have learned today?

- Images as functions
- Linear systems (filters)
- Convolution and correlation
- Discrete Fourier Transform (DFT)
- Sampling and aliasing

Appendix

Convergence

If g absolutely summable: $\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |g[n,m]| < \infty$

then $\lim_{N\to\infty} \sum_{n=-N}^{N} \sum_{m=-N}^{N} g[n,m]\, \mathrm{e}^{-\imath(\omega_{\mathrm{X}} n + \omega_{\mathrm{Y}} m)} = G(\omega_{\mathrm{X}}, \omega_{\mathrm{Y}})$

If g is square summable (energy signal): $E_g \triangleq \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |g[n,m]|^2 < \infty$

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| G_N(\omega_{\mathrm{X}}, \omega_{\mathrm{Y}}) - G(\omega_{\mathrm{X}}, \omega_{\mathrm{Y}}) \right|^2 \mathrm{d}\omega_{\mathrm{X}}\, \mathrm{d}\omega_{\mathrm{Y}} \to 0 \quad N \to \infty$$

$g_a(x,y)$
$G_a(\nu_X,\nu_Y)$
Sampling
Impulse Sampling
2D Sinc Convolution (bandlimited)
2D Sinc Interpolation (bandlimited)
Replicate and sum
Lowpass Filter (bandlimited)
$g_s(x,y)$
FT
$G_s(\nu_X,\nu_Y)$
Impulses
$\omega_X = 2\pi\nu_X\angle$
$\omega_Y = 2\pi\nu_Y\angle$
DSFT
$g_d[n,m]$
$G_d(\omega_X,\omega_Y)$
Replicate and sum
Truncate (space limited)
DFT
Sample
Dirichlet Interpolation (space limited)
DTFS
$\tilde{g}[n,m]$
$G[k,l]$

Fast Fourier transforms (FFT)

- Brute-force evaluation of the 2D DFT would require O((NM)2) flops

$$X[k,l] \triangleq$$

$$= \begin{cases} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} x[n,m]\, \mathrm{e}^{-\imath 2\pi(kn/N+lm/M)} \\ 0, \end{cases}$$

$$k = 0, \ldots, N-1, l = 0, \ldots, M-1$$
$$\text{otherwise.}$$

- DFT is a **separable operation**

  → we can reduce greatly the computation

Fast Fourier transforms (FFT)

- DFT is a **separable operation:**

$$X[k,l] = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} x[n,m]\, \mathrm{e}^{-\imath 2\pi(kn/N+lm/M)}$$

$$= \sum_{n=0}^{N-1} \mathrm{e}^{-\imath 2\pi kn/N} \left[\sum_{m=0}^{M-1} x[n,m]\, \mathrm{e}^{-\imath 2\pi lm/M}\right]$$

•Apply the 1D DFT to each column of the image, and then apply the 1D DFT to each row of the result.

- Use the **fast Fourier transform (1-D FFT)** for these 1D DFTs!

Fast Fourier transforms (FFT)

- FFT computational efficiency:

- **inner set of 1D FFTs** : N **O**(M logM)
- **outer set of 1D FFTs**: M **O**(N logN)

- **Total: O**(MN logMN) flops

- A critical property of FFT is that $N = 2^k$ with k =integer

If x = 512x512 image →
saving is a factor of 15000 relative to the brute-force 2D DFT!

Matlab: fft2 = fft(fft(x).').'

FFT & Efficiency

- **General goal:** perform efficient linear convolution
- Perform convolution as product of DFTs
- **Pros:** DFT can be implemented using the FFT (fast fourier transform)
  - FFT is very efficient (fast!)
- **Cons:** DFT perform circular convolution
  - Compensate the wrap-around effect
- **Cons:** Online-memory storage
  - Use the overlap-add method or overlap-save method

FFT & Efficiency

- Suppose we wish to convolve a **256 × 256** image with a **17 × 17** filter.
- The result will be **272×272**.
- The smallest prime factors of 272 is **2**.
- So one could pad to a **512 × 512 image**
- Note: only 28% of the final image would be the part we care about - the rest would be zero in exact arithmetic.

- Handling to a 512 × 512 image requires **much memory**

→ Use overlap-add method