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Preferential deletion in dynamic models of web-like networks

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Abstract

In this paper a discrete-time dynamic random graph process is studied that interleaves the birth of nodes and edges with the death of nodes. In this model, at each time step either a new node is added or an existing node is deleted. A node is added with probability p together with an edge incident on it. The node at the other end of this new edge is chosen based on a linear preferential attachment rule. A node (and all the edges incident on it) is deleted with probability $q = 1 - p$. The node to be deleted is chosen based on a probability distribution that favors small-degree nodes, in view of recent empirical findings. We analyze the degree distribution of this model and find that the expected fraction of nodes with degree k in the graph generated by this process decreases asymptotically as $k^{-1-(2p/2p-1)}$.

Keywords: Web-like networks; Interconnection networks; Dynamic random graph modeling; Preferential node deletion; Scale-free; Power-law degree distribution

1. Introduction

With the dramatic growth of the World Wide Web (Web) and the Internet, the study of large, random networks has acquired new prominence. Recent empirical studies have shown statistical similarities between these two and other complex, real-life networks such as the network of phone calls, power-distribution networks, citation network, science-collaboration network, movie-actor collaboration network, the network of sexual contacts, neural networks, and various infrastructure networks. In this paper, the term "*web-like*" has been used to refer to these real-life networks and others that are statistically similar. The ubiquity and the increasing importance of these networks have spawned a truly cross-disciplinary research aimed at understanding their fundamental properties and functions.

Many of these networks are very large, dynamic and arise without any centralized control [9,15]. Such networks need to be modeled as random graphs. But the classical Erdös–Rényi (ER) random graph $G_{n,m}$ [11] is not the correct model, as shown by some recent results [3,5,16]. Most notably, in many of these web-like networks the fraction of nodes with degree k follows a scale-free power-law, whereas in $G_{n,m}$ it follows approximately a Poisson distribution. In addition, web-like networks exhibit a significantly greater degree of clustering than $G_{n,m}$. Other major differences between web-like networks and $G_{n,m}$ have been observed (e.g., [15]).

The empirical discovery of such distinguishing properties of web-like networks has revived the study of

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random graphs in an effort to design models that exhibit the newly-discovered properties. Particular attention has been paid to the *dynamic* random graph models [1,4, 5,7,8,10,13,14]. A dynamic random graph model is a discrete-time process which starts out with a small fixed graph and in each subsequent time step a new node is added to the graph or an existing node is deleted from the graph. The changes applied during each step are controlled by certain stochastic rules.

By far, the most studied dynamic models are the birth-only ones (where only the addition of nodes and edges takes place). The underlying stochastic rule employed in virtually all birth-only models has been either *preferential attachment* [3] or *copying* [14] both of which have been shown to generate graphs with power-law degree distribution.

In contrast, models that combine birth and death (addition and deletion of nodes and edges) have been studied to a much lesser extent. Dorogovtsev and Mendes [10] studied a model which interleaves the addition of nodes and edges with a *uniform* deletion of edges. Chung and Lu [6] and Cooper et al. [8], independently, studied a dynamic model that combines the addition of nodes and edges with a *uniform* deletion of both nodes and edges. These birth-death models have also been found to generate graphs with power-law degree distribution.

In this paper, we investigate a dynamic random graph model which interleaves addition of nodes and edges with a *preferential* deletion of nodes that favors deletion of small-degree nodes. There is evidence that the small-degree nodes of some web-like networks, such as the Web and the Internet, die much more frequently than the high-degree ones [2,15]. As in many of the earlier papers, the focus in this paper is also on the asymptotic degree-distribution of the graph.

2. Notation

The notation used in this paper has been summarized in Table 1.

Table 1
List of symbols

Symbol	Meaning
$G(V,E)$	A graph with set of nodes V and set of edges E
$n(G)$ or n	Number of nodes in G
$m(G)$ or m	Number of edges in G
$d(u)$	Degree of node u
$\bar{d}(G)$	Average degree of graph G
G_t	Graph obtained after the tth step of a random graph generating procedure
n_t	Number of nodes in G_t
m_t	Number of edges in G_t
$d_t(u)$	Degree of node u in G_t
$\bar{d}_t$	Average degree of G_t
$N_{k,t}$	Number of nodes of degree k in G_t
$N_{k,t}^{(1)}$	Number of neighbors of degree k of a node chosen for deletion from G_t
$N_{k,t}^{(1)}(u)$	Number of neighbors of degree k of node u in G_t
$\mathbb{P}[k]$	Fraction of nodes of degree k
$\mathbb{P}_t[u]$	Probability of vertex u being selected as an end-node of a new edge or for deletion during the tth step of the random graph process
$\mathbb{P}[A]$	Probability of event A
$\mathbb{E}[X]$	Expectation of random variable X

3. Definition of the model

We begin by formally defining a dynamic random graph model with preferential deletion. Let the graph G_1 consists of a single node with a self-loop, and in each discrete time-step $t+1, t>0$, either one of the following two processes can take place:

(a) *Birth process*: with probability p, a new node is added, together with a new edge incident on it. The other end-node u of the new edge is chosen preferentially from among all the existing nodes based on a *linear preferential attachment* [3] rule:

$$\mathbb{P}_{t+1}[u] = \frac{d_t(u)}{\sum_{w\in V_t} d_t(w)} = \frac{d_t(u)}{2m_t}. \tag{1}$$

(b) *Death process*: with probability $q = 1-p$, a node u is chosen for deletion along with all the edges incident on it in G_t. To make small-degree nodes more likely candidates for deletion than the higher-degree ones, node u is chosen according to the following probability distribution:

$$\mathbb{P}_{t+1}[u] = \frac{n_t - d_t(u)}{n_t^2 - 2m_t}. \tag{2}$$

Note that the numerator of the ratio in the right-hand side of Eq. (2) subtracts the degree of node u from the number of nodes in the graph G_t. Therefore, the value assigned by Eq. (2) will be larger for small-degree nodes than for higher-degree ones. The division by $n_t^2 - 2m_t$ ensures that the values lie between 0 and 1. Naturally, there exist other alternative probability distributions that may achieve the same effect, such as:

$$\mathbb{P}_{t+1}[u] = \frac{2m_t - d_t(u)}{2m_t(n_t-1)}.$$

The distribution in Eq. (2) was chosen primarily because it was more convenient to work with.

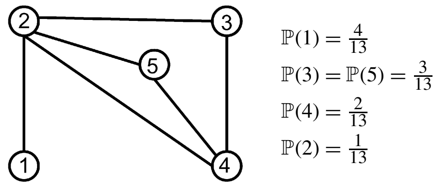

Fig. 1. A small graph illustrating the probability distribution used in the preferential deletion model.

As an illustration, a graph with deletion probabilities assigned to its nodes according to Eq. (2) is given in Fig. 1. In this paper, it is assumed that p is greater than q so that the graph continues to grow. There is a caveat to the two rules (a) and (b): If during some step $t > 0$ the number of nodes in G_t becomes zero, then the process rewinds, i.e., the graph G_{t+1} consists of a single node with a self-loop. However, as shown in the next section, this case occurs extremely rarely, and thus it may be ignored. We refer to the death process specified above as *preferential deletion* in analogy to preferential attachment.

4. Number of nodes

First, let us look at $\mathbb{P}[n_t = 0]$—the probability that the number of nodes becomes zero after some step $t > 0$. This event could occur only if t is even and exactly $t/2$ death processes have taken place during steps $1, \dots, t$ (note that the starting graph G_1 may be seen as the result of a birth process). Thus:

$$\mathbb{P}[n_t = 0] \leqslant \binom{t}{t/2} q^{t/2} p^{t/2} = \mathrm{O}\big[\big(2\sqrt{pq}\big)^t\big] = \mathrm{O}\left[\frac{1}{(1+\varepsilon)^t}\right],$$

for some $\varepsilon > 0$. Since the probability of reaching an empty graph decreases exponentially with the number of steps, it is assumed that $n_t > 0$ for all $t > 0$. Hence, for all $t > 0$, $n_{t+1} = n_t + X$, where X is a discrete random variable equal to 1 with probability p and equal to -1 with probability q. As a result, the conditional expectation of n_{t+1} with G_t fixed is

$$\mathbb{E}[n_{t+1} | G_t] = n_t + \mathbb{E}[X]. \tag{3}$$

By taking the expectations of both sides in Eq. (3), we obtain

$$\mathbb{E}[n_{t+1}] = \mathbb{E}[n_t] + (p - q), \quad \text{for } t > 1.$$

Solving this first-order linear difference equation with initial condition $\mathbb{E}[n_1] = 1$, yields:

$$\mathbb{E}[n_t] = (p - q)t + 2q, \tag{4}$$

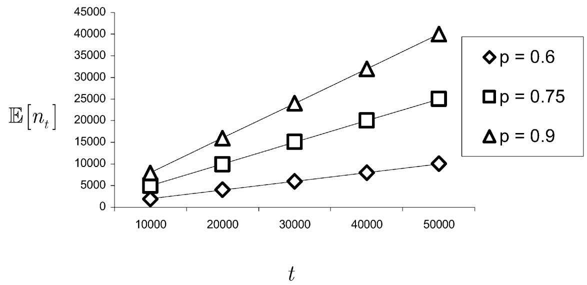

Fig. 2. Growth in the number of nodes of graph G_t with the number of steps t, for three different values of the birth probability p.

which implies that $\mathbb{E}[n_t] = \Theta[(p - q)t]$. Fig. 2 shows a comparison of the values of n_t predicted by Eq. (4) with those obtained by simulating the preferential deletion model. In this figure, the solid lines correspond to the analytical prediction of Eq. (4) while the data points correspond to the simulation result. To obtain these data points, for each value of p and t shown in Fig. 2, the number of nodes was computed by averaging over 30 realizations of the model. The analytical prediction and simulation results agree very well, as seen in Fig. 2.

5. Number of edges

The approach followed in this section is similar to that of Section 4. With G_t fixed, the number of edges after the $(t + 1)$th step may be expressed as $m_{t+1} = m_t + Y_{t+1}$, where Y_{t+1} is a random variable specified by

$$Y_{t+1} = \begin{cases} 1 & \text{with probability } p, \\ -k & \text{with probability } \frac{q(n_t - k)N_{k,t}}{n_t^2 - 2m_t}, \ k \geqslant 0. \end{cases}$$

Thus,

$$\mathbb{E}[Y_{t+1} | G_t] = p - q \sum_{k \geqslant 0} k N_{k,t} \frac{n_t - k}{n_t^2 - 2m_t} = p - q \sum_{k \geqslant 0} \frac{k N_{k,t}}{n_t - \bar{d}_t} + q \sum_{k \geqslant 0} \frac{k^2 N_{k,t}}{n_t^2 - 2m_t},$$

which implies that

$$\mathbb{E}[m_{t+1}] = \mathbb{E}[m_t] + p - q\mathbb{E}\left[\sum_{k \geqslant 0} \frac{k N_{k,t}}{n_t - \bar{d}_t}\right] + q\mathbb{E}\left[\sum_{k \geqslant 0} \frac{k^2 N_{k,t}}{n_t^2 - 2m_t}\right]. \tag{5}$$

We continue by evaluating the two expectations multiplied by q in Eq. (5). First, we have

$$\mathbb{E}\left[\sum_{k\geqslant 0}\frac{kN_{k,t}}{n_t-\bar{d}_t}\right]=\mathbb{E}\left[\frac{n_t}{n_t-\bar{d}_t}\sum_{k\geqslant 0}\frac{kN_{k,t}}{n_t}\right]=\mathbb{E}\left[\frac{2m_t}{n_t-\bar{d}_t}\right].$$

Second,

$$\mathbb{E}\left[\sum_{k\geqslant 0}\frac{k^2N_{k,t}}{n_t(n_t-\bar{d}_t)}\right]=\mathbb{E}\left[\frac{1}{n_t-\bar{d}_t}\sum_{k\geqslant 0}\frac{k^2N_{k,t}}{n_t}\right].$$

Now, using the approximation

$$\sum_{k\geqslant 0}\frac{k^2N_{k,t}}{n_t}\approx 2(\bar{d}_t)^2\approx\frac{8m_t^2}{n_t^2},$$

and then substituting it into Eq. (5) we get

$$\mathbb{E}[m_{t+1}]=\mathbb{E}[m_t]+p-q\left(\mathbb{E}\left[\frac{2m_t}{n_t-\bar{d}_t}\right]-\mathbb{E}\left[\frac{8m_t^2}{n_t^2(n_t-\bar{d}_t)}\right]\right),$$

or, equivalently

$$\mathbb{E}[m_{t+1}]-\left(1-\frac{2q}{\mathbb{E}[n_t-\bar{d}_t]}\right)\mathbb{E}[m_t]-\frac{4q}{\mathbb{E}[n_t^2(n_t-\bar{d}_t)]}\mathbb{E}[m_t]^2=p, \tag{6}$$

which is a non-linear difference equation. Methods for solving such equations are known only for a few special cases (these methods are usually based on transformations that convert non-linear equations into linear ones). To the best of our knowledge, Eq. (6) does not fall into any of the special cases. Therefore, we search for a solution of the form $\mathbb{E}[m_t]=\varepsilon t$, where ε is a constant that does not depend on t. Substituting into Eq. (6) and taking the limits as $t\to\infty$, we get:

$$(1+a)\varepsilon=p, \tag{7}$$

where $a=2q/(p-q)$ and $b=4q/(p-q)^3$. Hence:

$$\varepsilon=p(p-q), \tag{8}$$

i.e., $\mathbb{E}[m_t]=\Theta[p(p-q)t]$. To verify the result of Eq. (8), we computed the number of edges by simulating the model. The simulation results are shown in Fig. 3. The solid lines in this figure correspond to the analytical prediction for the number of edges m_t while the data points correspond to simulation results. Again, for each value of p and t the number of edges of G_t was computed by averaging over 30 realizations of the model. The two data sets are in very good agreement.

A direct consequence of Eqs. (4) and (8) is that the average degree of G_t tends to $2p$ as $t\to\infty$.

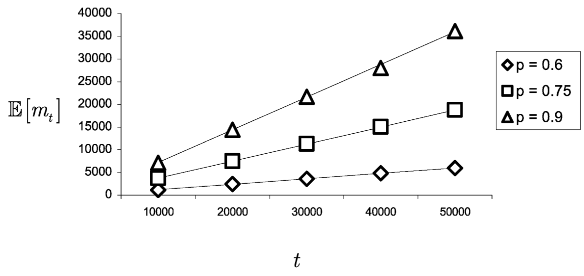

Fig. 3. Growth in the number of edges of graph G_t with the number of steps t, for three different values of the birth probability p.

6. Degree distribution in the first neighborhood of the deleted node

Before turning our attention to the degree distribution in G_t, we need to evaluate one more quantity, namely the expectation of $N_{k,t}^{(1)}$—the number of degree k nodes adjacent to the node chosen for deletion during step t.

This expectation is computed by conditioning on the node chosen for deletion. Indeed, with G_t fixed, one may write

$$\mathbb{E}\left[N_{k,t}^{(1)}|G_t\right]=\sum_{u\in V_t}N_{k,t}^{(1)}(u)\frac{n_t-d_t(u)}{n_t^2-2m_t}=\frac{1}{n_t-\bar{d}_t}\sum_{u\in V_t}N_{k,t}^{(1)}(u)-\frac{1}{n_t(n_t-\bar{d}_t)}\sum_{u\in V_t}N_{k,t}^{(1)}(u)d_t(u). \tag{9}$$

Next, note that

$$\sum_{u\in V_t}N_{k,t}^{(1)}(u)=kN_{k,t} \tag{10}$$

and

$$\sum_{u\in V_t}N_{k,t}^{(1)}(u)d_t(u)=\sum_{i=1}^{N_{k,t}}\sum_{j=1}^{k}d_{k,i,j,t}. \tag{11}$$

Here $d_{k,i,j,t}$ denotes the degree of the jth neighbor of the ith node of degree k after step t. It may be approximated by the average degree $\bar{d}_t^{(1)}$ of a random neighbor of a random node. This quantity is related to $\bar{d}_t$ by the identity $\bar{d}_t^{(1)}\approx 2\bar{d}_t$. Substituting into (9) we get

$$\mathbb{E}\left[N_{k,t}^{(1)}|G_t\right]\approx\frac{kN_{k,t}}{n_t-\bar{d}_t}-\frac{2kN_{k,t}\bar{d}_t}{n_t(n_t-\bar{d}_t)}=\frac{kN_{k,t}}{n_t-\bar{d}_t}\left[1-\frac{2\bar{d}_t}{n_t}\right],$$

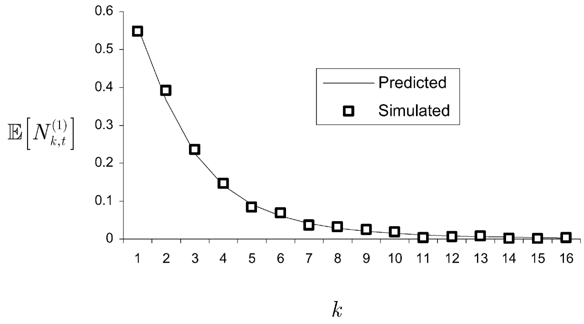

Fig. 4. The expected number of neighbors of degree k of a node chosen for deletion.

and finally, by taking the expectations of both sides in the last equation, we obtain

$$\mathbb{E}\left[N_{k,t}^{(1)}\right] \approx k\mathbb{E}[N_{k,t}]\mathbb{E}\left[\frac{1-2\bar{d}_t/n_t}{n_t-\bar{d}_t}\right]. \tag{12}$$

Eq. (12) was also verified numerically. The results are shown in Fig. 4 where the solid line corresponds to the prediction of Eq. (12) while the data points were computed by averaging over 1000 realizations of our model with $t = 40\,000$ and $p = 0.8$. The values of $\mathbb{E}[N_{k,t}^{(1)}]$ in Fig. 4 are shown in a normalized form after having been divided by the degree of the node chosen for deletion.

7. Degree distribution

Next, we turn to the degree distribution of the graph G_t. By analyzing the change in $N_{k,t}$ between the tth and $(t+1)$th steps, we get

$$\mathbb{E}[N_{k,t+1}] = \mathbb{E}[N_{k,t}] + pA + qB + p\delta_{k1}, \tag{13}$$

where

$$A = \frac{1}{2\mathbb{E}[m_t]}\big[(k-1)\mathbb{E}[N_{k-1,t}] - k\mathbb{E}[N_{k,t}]\big],$$

$$B = \frac{q}{\mathbb{E}[n_t(n_t-\bar{d}_t)]}\big\{(k+1)\mathbb{E}[N_{k+1,t}]\mathbb{E}[n_t-2\bar{d}_t] - \big[(k+1)n_t - k(1-2\bar{d}_t)\big]\mathbb{E}[N_{k,t}]\big\}.$$

Term A in Eq. (13) reflects the expected change in $N_{k,t}$ due to the birth process. The expression for term A was derived using standard techniques (e.g., [13]), and hence the details have been omitted.

Term B reflects the expected change in $N_{k,t}$ due to deletion. Its derivation takes into account the result of Section 6. Let us examine, for instance, the derivation of the term $[(k+1)n_t - k(1-2\bar{d}_t)]\mathbb{E}[N_{k,t}]/\mathbb{E}[n_t(n_t-\bar{d}_t)]$, which reflects the expected drop in $N_{k,t}$ due to a deletion. The deletion of a node can cause $N_{k,t}$ to decrease in two different ways: (i) if a node of degree k is deleted; or (ii) if the deleted node is adjacent to one or more nodes of degree k. The expected drop due to deletion of a node of degree k is $\mathbb{E}[N_{k,t}](n_t-k)/\mathbb{E}[n_t^2-2m_t]$. Furthermore, the result of Eq. (12) implies that the expected drop due to deletion of a node which has one or more neighbors of degree k is $k\mathbb{E}[N_{k,t}](n_t-2\bar{d}_t)/\mathbb{E}[(n_t^2-2m_t)]$. Adding these two drop terms yields the expected overall drop due to deletion in the number of nodes of degree k. In a similar fashion, one may also derive the expected *increase* in $N_{k,t}$ due to a deletion.

The last term in Eq. (13) comes from the fact that the degree of a newly-born node is always one.

We search for a solution to Eq. (13) of the form $\mathbb{E}[N_{k,t}] = a_k t$. Substituting for $\mathbb{E}[N_{k,t}]$ and taking the limits as $t \to \infty$, we get the following second-order, linear difference equation with non-constant coefficients:

$$-2q(k+2)a_{k+2} + \big[2p + (k+1)(2q+1)\big]a_{k+1} - ka_k = 2p(p-q)\delta_{k1}, \quad k \geqslant 0. \tag{14}$$

To solve Eq. (14) we have used the method of Laplace as described in [12]. Consider first the homogeneous equation which has the form $\sum_{i=0}^{2}(\alpha_i k + \beta_i)a_{k+i} = 0$, with

$$\alpha_0 = -1, \quad \beta_0 = 0, \quad \alpha_1 = 2q+1,$$
$$\beta_1 = 3, \quad \alpha_2 = -2q, \quad \beta_2 = -4q.$$

Following Laplace's method, it is assumed that the solution of the homogeneous equation is of the form:

$$a_k = \int_a^b t^{k-1} v(t)\,\mathrm{d}t, \tag{15}$$

where the function $v(t)$ and the limits of integration a, b are yet to be determined. As explained in [12], the relation

$$\frac{\mathrm{d}v}{v} = \frac{\sum t^i(\beta_i - i\alpha_i)}{\sum \alpha_i t^{i+1}} \tag{16}$$

must hold for any difference equation of the type under consideration. Furthermore, the limits of integration a, b are to be chosen among the roots of the function $t^k v(t)\sum \alpha_i t^i$. In the present case, Eq. (16) becomes

$$\frac{\mathrm{d}v}{v} = \frac{2p - 4pt}{-1 + (1+2q)t - 2qt^2}.$$

By integrating both sides of the preceding equation we get:

$$v(t) = (t-1)^{2p/2p-1}(2tq-1)^{2p^2/q(p-q)}.$$

The roots of $t^k v(t) \sum \alpha_i t^i$ are 0, 1, and $1/2q$. It follows that the two independent solutions of Eq. (13) are

$$a_k^{(1)} = \int_0^1 t^{k-1}(t-1)^{2p/2p-1}(2tq-1)^{2p^2/q(p-q)}\,\mathrm{d}t,$$

and

$$a_k^{(2)} = \int_0^{1/2q} t^{k-1}(t-1)^{2p/2p-1}(2tq-1)^{2p^2/q(p-q)}\,\mathrm{d}t.$$

By carrying out the first integration we get:

$$a_k^{(1)} = \Theta[k^{-1-(2p/2p-1)}]. \tag{17}$$

Note that as p increases from 0.5 to 1, the ratio $2p/2p-1$ decreases from ∞ to 2. Thus, asymptotically the degree distribution of G_t follows a power-law with exponent that varies between 3 (for $p=1$) and ∞ (for $p=0.5$). In the case when $p=1$ this result agrees with previous well-known results [3,5]. On the other hand, for values of p significantly smaller than 1 the exponent of the power-law becomes too big compared to the exponents observed for many web-like networks (which usually lie in the range between 2 and 3 [15]).

The second integral $a_k^{(2)}$ may be shown to diverge as $k \to \infty$, and is thus irrelevant.

The plot in Fig. 5 shows a comparison between the analytical prediction given by Eq. (17) and the data obtained by simulating our model, with $p=0.8$. The cumulative distribution $\mathbb{P}'(k) = \sum_{i \geqslant k} \mathbb{P}(i)$ has been plotted instead of the distribution $\mathbb{P}(k)$ itself in order to reduce the statistical noise in the tail of the distribution. As seen, there is a good agreement between the data obtained from the simulation results and the analytical prediction.

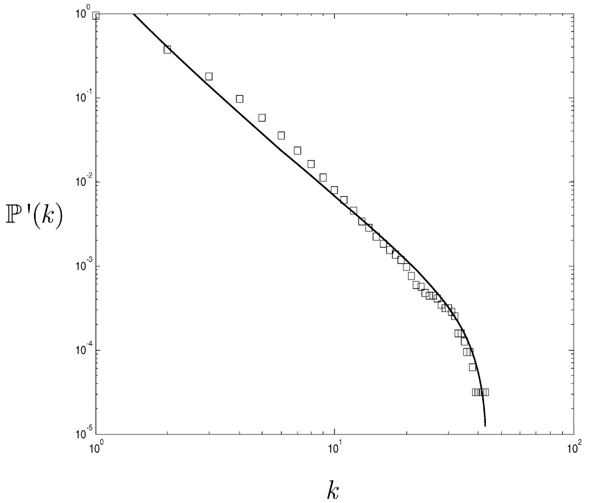

Fig. 5. Log–log plot of the cumulative degree distribution of the graph generated by the preferential deletion model.

8. Conclusion

In this paper, we studied the degree distribution of a random graph model that combines the addition of nodes and edges with a preferential deletion of nodes. In line with the results obtained earlier for several other dynamic random graph models, we found that the preferential deletion model generates graphs with asymptotically power-law degree distribution. This result reinforces our view that dynamic models of web-like networks are robust in the sense that a power-law degree distribution is obtained for a wide range of stochastic rules that control such models. A number of related investigations, such as analyzing alternative death processes and proving concentration results, will be undertaken in our future work.

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