TTK4190 Guidance and Control of Vehicles

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Maren Kristine Eidal – 750664
Helene Hogstad Fossum – 750189

Problem 1 - Attitude Control of Satellite

Background Information

In this problem, a satellite with inertia matrix $\mathbf{I}_{CG} = mr^2\mathbf{I}_{3x3}, m = 100$ kg, and $r = 2.0$ m is considered. The equations of motions are as follows.

$$\dot{\mathbf{q}} = \mathbf{T}_q(\mathbf{q})\boldsymbol{\omega} \tag{1a}$$

$$\mathbf{I}_{CG}\dot{\boldsymbol{\omega}} - \mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega})\boldsymbol{\omega} = \boldsymbol{\tau} \tag{1b}$$

Where the angular velocity transformation given by (1a) is completed using the unit quaternions. With this in mind, the vectors and matrices for the equations of motions given above are shown by (2a - 2b).

$$\underbrace{\begin{bmatrix} \dot{\eta} \\ \dot{\epsilon_1} \\ \dot{\epsilon_2} \\ \dot{\epsilon_3} \end{bmatrix}}_{\dot{\mathbf{q}}} = \underbrace{\frac{1}{2}\begin{bmatrix} -\epsilon_1 & -\epsilon_2 & \epsilon_3 \\ \eta & -\epsilon_3 & \epsilon_2 \\ \epsilon_3 & \eta & -\epsilon_1 \\ -\epsilon_2 & \epsilon_1 & \eta \end{bmatrix}}_{\mathbf{T}_q(\mathbf{q})} \underbrace{\begin{bmatrix} p \\ q \\ \mathrm{r} \end{bmatrix}}_{\boldsymbol{\omega}} \tag{2a}$$

$$\underbrace{\begin{bmatrix} mr^2 & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & mr^2 \end{bmatrix}}_{\mathbf{I}_{CG}} \underbrace{\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{\mathrm{r}} \end{bmatrix}}_{\dot{\boldsymbol{\omega}}} - \underbrace{\begin{bmatrix} 0 & -\mathrm{r}mr^2 & qmr^2 \\ \mathrm{r}mr^2 & 0 & -pmr^2 \\ -qmr^2 & pmr^2 & 0 \end{bmatrix}}_{\mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega})} \underbrace{\begin{bmatrix} p \\ q \\ \mathrm{r} \end{bmatrix}}_{\boldsymbol{\omega}} = \underbrace{\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}}_{\boldsymbol{\tau}} \tag{2b}$$

After looking at (2b) more closely, it becomes clear that due to orthogonality (this could also have been calculated directly):

$$\mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega})\boldsymbol{\omega} = 0 \tag{3}$$

This fact is used to simplify the motions of equations in the proceeding problems.

Problem 1.1

Equilibrium point

In order to determine the equilibrium point, $\mathbf{x_0}$, of the closed loop system $\mathbf{x} = [\boldsymbol{\epsilon}^T \ \boldsymbol{\omega}^T]^T$, the equations of motions in (1) are rewritten in the following manner.

$$\underbrace{\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix}}_{\dot{\mathbf{x}}} = \begin{bmatrix} \mathbf{T}_q(\mathbf{q}) \\ I_{CG}^{-1}\ \mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega}) \end{bmatrix} \boldsymbol{\omega} + \begin{bmatrix} \mathbf{0}_{4x3} \\ I_{CG}^{-1} \end{bmatrix} \boldsymbol{\tau} \tag{4}$$

As given by the assignment, $\boldsymbol{\tau} = \mathbf{0}$ and $\mathbf{q} = [\eta\ \epsilon_1\ \epsilon_2\ \epsilon_3]^T = [1\ 0\ 0\ 0]^T$ is the equilibrium point. Further, it is not necessary to include η in the unit quaternion angles ($\mathbf{q}$), as it is a function of $\boldsymbol{\epsilon}$. With this in mind, the closed loop system in (4) simplifies to:

$$\underbrace{\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix}}_{\dot{\mathbf{x}}} = \begin{bmatrix} \mathbf{T}_q(\mathbf{q}) \\ I_{CG}^{-1}\ \mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega}) \end{bmatrix} \boldsymbol{\omega} \tag{5}$$

where

$$\mathbf{T}_q(\mathbf{q}) = \frac{1}{2}\begin{bmatrix} 1 & -\epsilon_3 & \epsilon_2 \\ \epsilon_3 & 1 & -\epsilon_1 \\ -\epsilon_2 & \epsilon_1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} \quad \text{and} \tag{6}$$

$$\mathbf{I}_{CG}^{-1}\mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega}) = \frac{1}{mr^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -\mathrm{r}mr^2 & qmr^2 \\ \mathrm{r}mr^2 & 0 & -pmr^2 \\ -qmr^2 & pmr^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \text{-r} & q \\ \mathrm{r} & 0 & -p \\ -q & p & 0 \end{bmatrix} \tag{7}$$

The equilibrium point is where $\dot{\mathbf{x}} = 0$, so (5) is set to zero and solved for $\mathbf{q}$ and $\boldsymbol{\omega}$ to determine $\mathbf{x}_0 = [\mathbf{q}_0\ \boldsymbol{\omega}_0]^T$.

$$\underbrace{\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix}}_{\dot{\mathbf{x}}} = \mathbf{0} \ \Rightarrow\ \frac{1}{2}\begin{bmatrix} 1 & -\epsilon_3 & \epsilon_2 \\ \epsilon_3 & 1 & -\epsilon_1 \\ -\epsilon_2 & \epsilon_1 & 1 \\ 0 & -2\mathrm{r} & q \\ 2\mathrm{r} & 0 & -2p \\ -2q & 2p & 0 \end{bmatrix}\underbrace{\begin{bmatrix} p \\ q \\ \mathrm{r} \end{bmatrix}}_{\boldsymbol{\omega}} = \mathbf{0} \tag{8}$$

With the given information that $\mathbf{q} = [\eta\ \epsilon_1\ \epsilon_2\ \epsilon_3]^T = [1\ 0\ 0\ 0]^T$, we get that $\mathbf{q}_0 = [0\ 0\ 0]^T$ by solving the respective equations from (8). Further, taking that $\mathbf{q}_0 = [\epsilon_{1,0}\ \epsilon_{2,0}\ \epsilon_{3,0}] = [0\ 0\ 0]$ into account and multiplying the top three rows in (8) with $\boldsymbol{\omega}$, gives that $\boldsymbol{\omega}_0 = [p_0\ q_0\ \mathrm{r}_0] = [0\ 0\ 0]$. Therefore, the equilibrium point becomes:

$$\mathbf{x}_0 = \begin{bmatrix} \mathbf{q}_0 \\ \boldsymbol{\omega}_0 \end{bmatrix} = \begin{bmatrix} [0 & 0 & 0]^T \\ [0 & 0 & 0]^T \end{bmatrix} \tag{9}$$

Linearizing the Model

Keeping the fact that η is disregarded in $\mathbf{q}$, such that $\mathbf{q} = [\epsilon_1\ \epsilon_2\ \epsilon_3]^T$, the nonlinear model of $\mathbf{x} = [\epsilon, \omega]^T$ in (4) is reduced to the following.

$$\underbrace{\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix}}_{\dot{\mathbf{x}}} = \begin{bmatrix} \mathbf{T}_q(\mathbf{q}) \\ I_{CG}^{-1}\ \mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega}) \end{bmatrix} \boldsymbol{\omega} + \begin{bmatrix} \mathbf{0}_{3x3} \\ I_{CG}^{-1} \end{bmatrix} \boldsymbol{\tau} \tag{10}$$

Our aim in the linearization is to get this system on the form:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\boldsymbol{\tau} \tag{11}$$

Where the linearization occurs around a specific point, which in this case is the equilibrium point in (9). The A and B matrices are determined by using the Jacobian matrix, where the six equations resulting from matrix multiplication in equation (10) represent $\mathbf{f} = [f_1\ f_2\ f_3\ f_4\ f_5\ f_6]^T$. The details of this calculation are given below.

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial \epsilon_1} & \frac{\partial f_1}{\partial \epsilon_2} & \frac{\partial f_1}{\partial \epsilon_3} & \frac{\partial f_1}{\partial p} & \frac{\partial f_1}{\partial q} & \frac{\partial f_1}{\partial \mathrm{r}} \\ \vdots & \ddots & & & & \vdots \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ \frac{\partial f_6}{\partial \epsilon_1} & \cdots & \cdots & \cdots & \cdots & \frac{\partial f_6}{\partial \mathrm{r}} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & \mathrm{r} & -q & 1 & -\epsilon_3 & \epsilon_2 \\ -\mathrm{r} & 0 & p & \epsilon_3 & 1 & -\epsilon_1 \\ q & -p & 0 & -\epsilon_2 & \epsilon_1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{12a}$$

$$\Rightarrow A = \frac{1}{2} \begin{bmatrix} \mathbf{0}_{3x3} & \mathbf{I}_{3x3} \\ \mathbf{0}_{3x3} & \mathbf{0}_{3x3} \end{bmatrix} \tag{12b}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial \tau_1} & \frac{\partial f_1}{\partial \tau_2} & \frac{\partial f_1}{\partial \tau_3} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \frac{\partial f_6}{\partial \tau_1} & \cdots & \frac{\partial f_6}{\partial \tau_3} \end{bmatrix} = \frac{1}{mr^2} \begin{bmatrix} \mathbf{0}_{3x3} \\ \mathbf{I}_{3x3} \end{bmatrix} \tag{12c}$$

Where the shift from (12a) to (12b) is when the A-matrix is evaluated in $\mathbf{x}_0$. The linearization is now complete and the matrices A and B are used in (11).

Problem 1.2

With the control law in (13) added to the linearized system in Problem 1.1, the system can now be rewritten as (14).

$$\boldsymbol{\tau} = -\mathbf{K}_d \boldsymbol{\omega} - k_p \boldsymbol{\epsilon} \Rightarrow \boldsymbol{\tau} = -\underbrace{\begin{bmatrix} k_p \mathbf{I}_{3x3} & k_d \mathbf{I}_{3x3} \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ p \\ q \\ \mathrm{r} \end{bmatrix}}_{\mathbf{x}} \tag{13}$$

$$\dot{\mathbf{x}} = A\mathbf{x} + B\boldsymbol{\tau} \Rightarrow \dot{\mathbf{x}} = \underbrace{(A - B\mathbf{K})}_{\bar{A}} \mathbf{x} \tag{14}$$

The gain matrix in (13) was set to contain $k_p = 1$ and $k_d = 20$. In order to determine the stability of the system around the equilibrium point ($\mathbf{x}_0$), the eigenvalues of $\bar{A}$ were investigated.

$$\bar{A} = \begin{bmatrix} \mathbf{0}_{3x3} & \frac{1}{2}\mathbf{I}_{3x3} \\ -\frac{1}{mr^2}\mathbf{I}_{3x3} & -\frac{20}{mr^2}\mathbf{I}_{3x3} \end{bmatrix}$$

Using the MATLAB function $eig(\mathbf{X})$, which returns a vector of the eigenvalues of a matrix $\mathbf{X}$, the eigenvalues of $\bar{A}$ were determined to the following values.

$$\boldsymbol{\lambda} = \begin{bmatrix} -0.025 + 0.025i \\ -0.025 - 0.025i \\ -0.025 + 0.025i \\ -0.025 - 0.025i \\ -0.025 + 0.025i \\ -0.025 - 0.025i \end{bmatrix} \tag{15}$$

As all the eigenvalues in $\bar{A}$, and thereby the poles of the system in (14), are located on the negative real axis of the imaginary plane, the system is stable. Thus, the imaginary part of the unit quaternion ($\mathbf{q} = [\epsilon_1\ \epsilon_2\ \epsilon_3]^T$) and the angular velocities ($\boldsymbol{\omega}$) will have bounded responses and not go towards infinity.

Complex conjugated poles causes the system to overshoot the desired values for the states. Yet, for controlling a satellite's attitude, overshooting the desired position will not have disastrous consequences. So, the need for extreme accuracy is not vital. Therefore having complex conjugated poles will be preferred as the response will approach the desired values faster. Yet, the complex conjugated poles should not be located too far away from the real axis, as this would cause oscillations about the desired position.

The optimal placement of the poles is along a semi-circle in the imaginary plane, as poles close together will create a sluggish response and require large input values ($\boldsymbol{\tau}$). This means that the satellite would have to use a lot of energy to reach the desired positions, which is not desired.

Problem 1.3

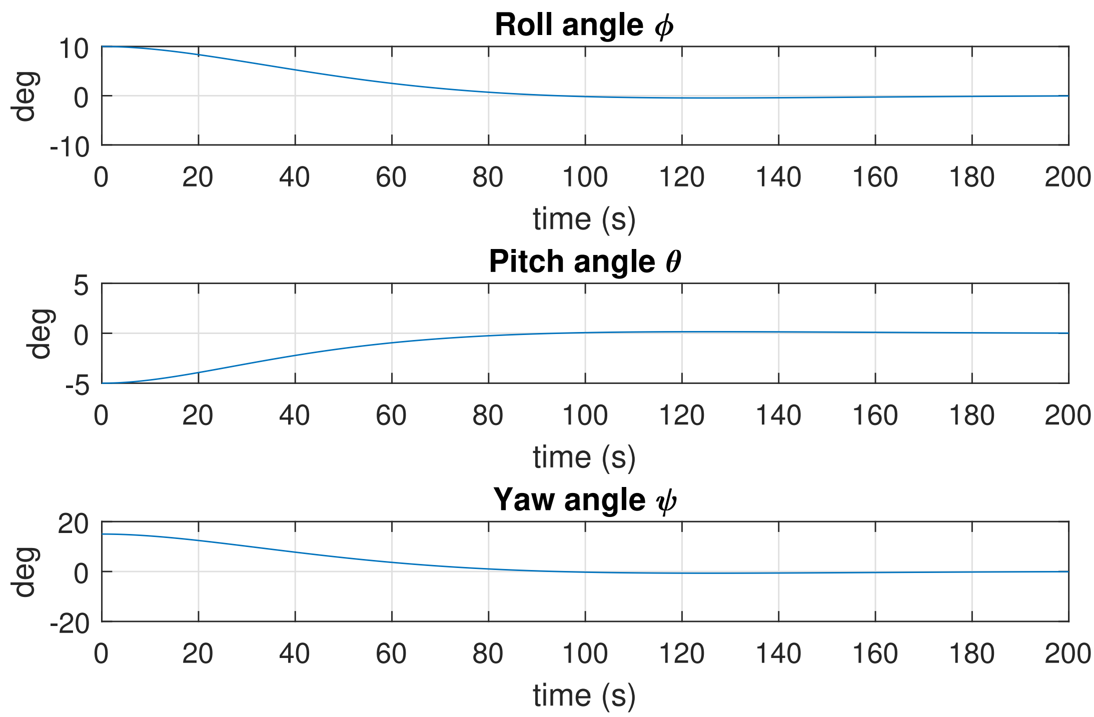
Using the attitude control law

$$\boldsymbol{\tau} = -\mathbf{K_p}\boldsymbol{\omega} - k_p\boldsymbol{\epsilon} \tag{16}$$

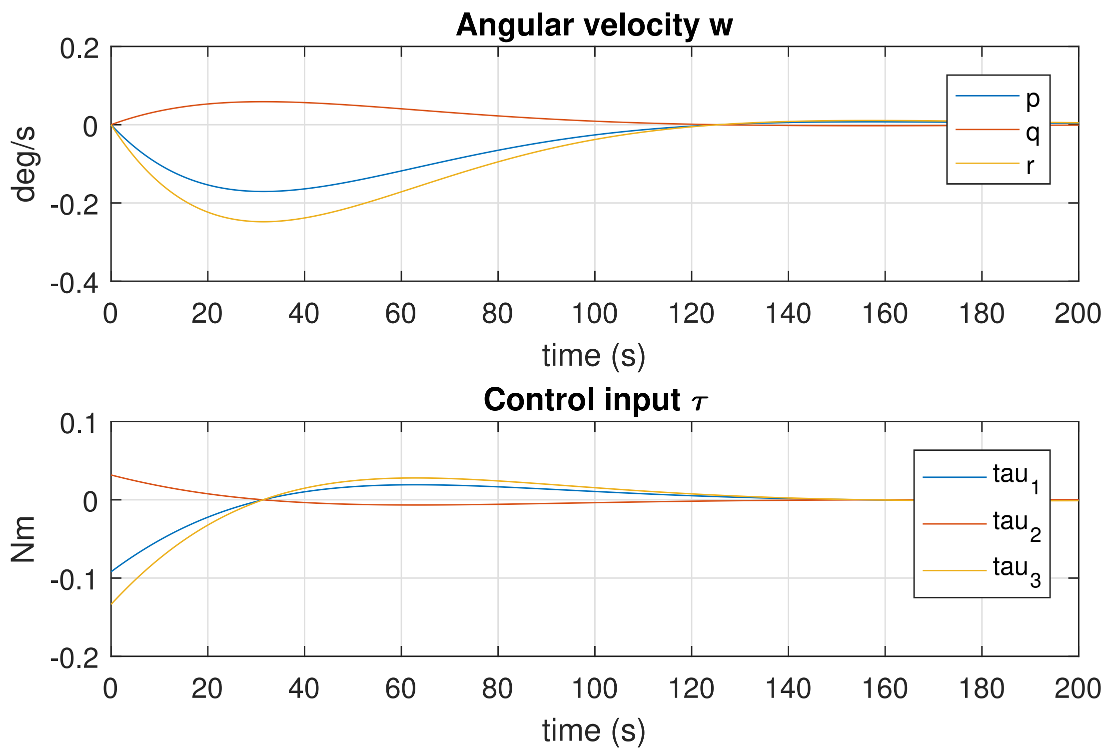
With the control law and the gains (k_d and k_p) the same as in Problem 1.2, the initial conditions for the angular position of the satellite were set to: $\phi(0) = 10°$, $\theta(0) = -5°$ and $\psi(0) = 15°$, whereas the initial angular velocities of the satellite were zero. The response of the satellite with the control input is shown in figure 1.

The response (in figure 1a) are as expected, as the complex conjugated poles will cause the system to overshoot the desired position before settling to their correct values. This slight overshoot is apparent in the response of the angular velocity (in figure 1b), as the satellite speeds up to reach its desired position and then slows back down again after it has overshot it. This behaviour is also evident in the control input, where the input is at a maximum at the start of the simulation, and slowly shifts sign to overcome the overshoot of the satellite's desired position.

The convergence of $\mathbf{q}$(t) shown in figure 2. As a unit quaternion must satisfy $\mathbf{q}^T\mathbf{q} = 1$, meaning that $\eta^2 + \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 = 1$. One way to satisfy this, is by having $\mathbf{q} = [1\ 0\ 0\ 0]$, which is what the unit quaternion in figure 2 converges to.



(a) Attitude in Euler Angles



(b) Angular velocity ($\boldsymbol{\omega}$) and control input ($\boldsymbol{\tau}$)

Figure 1: Plots showing stability of system with the given initial conditions initial conditions.

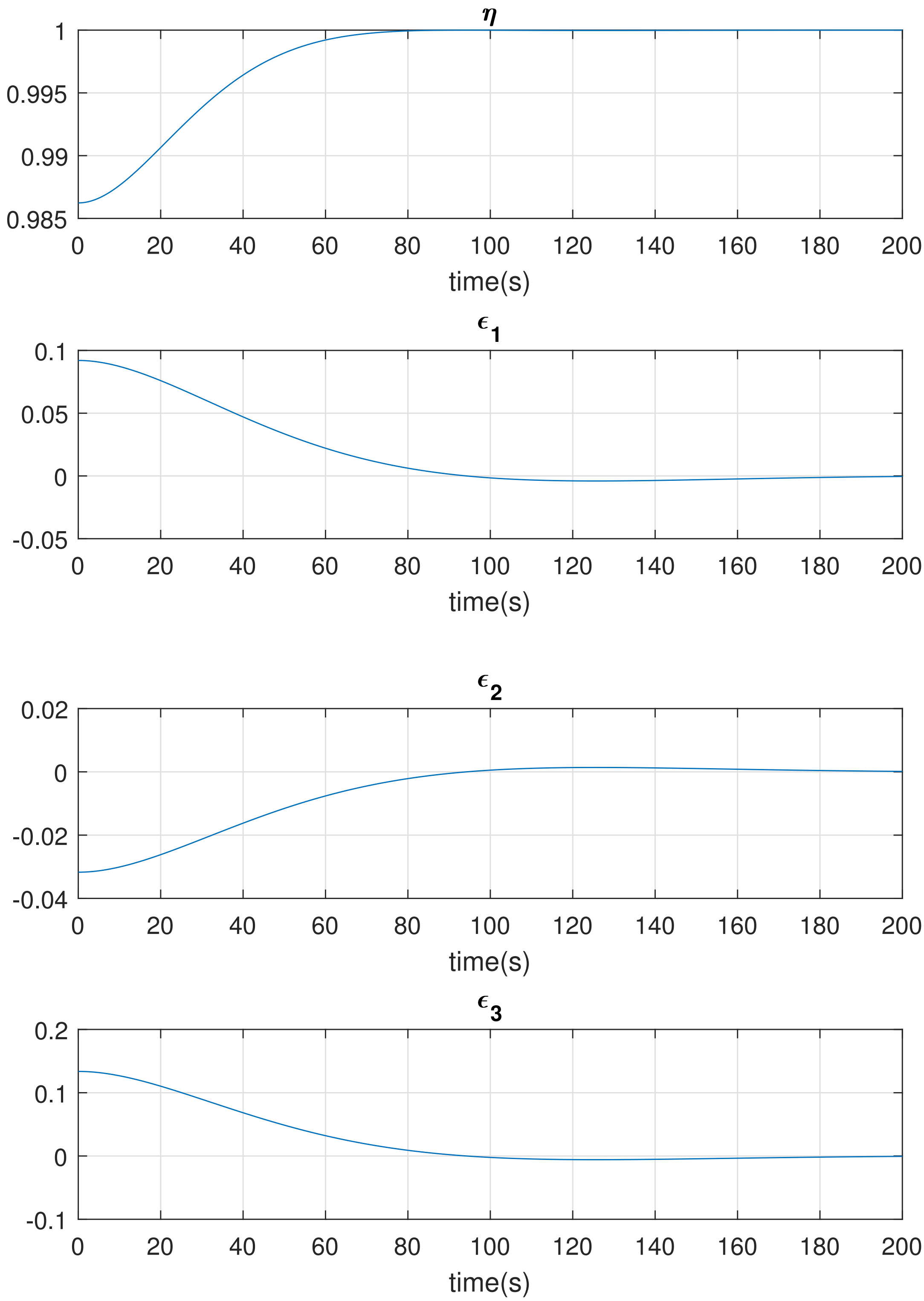


Figure 2: Plots showing the convergence of $\mathbf{q}(t)$ toward the desired $\mathbf{q_0}$

Problem 1.4

The simulation in problem 1.5 uses a new control law where the error in the imaginary part of the quaternion ($\tilde{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon} - \boldsymbol{\epsilon}_d$) is used as input:

$$\boldsymbol{\tau} = -\mathbf{K_d}\boldsymbol{\omega} - k_p\tilde{\boldsymbol{\epsilon}} \tag{17}$$

In order to use this the expression for the error $\tilde{\boldsymbol{\epsilon}}$ has to be defined. The quaternion error is defined in (18), where $\mathbf{q_d}$ represents the desired angles expressed using a quaternion. The expression involves the use of both the conjugate of $\mathbf{q}$ and the quaterion product. The calculation of $\bar{\mathbf{q}}$ is expressed in equation (19) and finally equation (20) expresses the quaternian error on component form calculated using equation (18).

$$\tilde{\mathbf{q}} = \bar{\mathbf{q}}_d \otimes \mathbf{q} \tag{18}$$

$$\mathbf{q_d} = \begin{bmatrix} \eta_d \\ \boldsymbol{\epsilon_d} \end{bmatrix} \Rightarrow \bar{\mathbf{q}}_d = \begin{bmatrix} \eta \\ -\boldsymbol{\epsilon_d} \end{bmatrix} \tag{19}$$

$$\begin{aligned} &\Rightarrow \tilde{\mathbf{q}} = \begin{bmatrix} \eta_d\eta - (-\boldsymbol{\epsilon_d})^T\boldsymbol{\epsilon} \\ \eta_d\boldsymbol{\epsilon} + \eta(-\boldsymbol{\epsilon_d}) + \mathbf{S}(-\boldsymbol{\epsilon_d})\boldsymbol{\epsilon} \end{bmatrix} \\ &\Rightarrow \tilde{\mathbf{q}} = \begin{bmatrix} \eta_d\eta + \boldsymbol{\epsilon_d^T}\epsilon \\ \eta_d\epsilon - \eta\boldsymbol{\epsilon_d} - \mathbf{S}(\boldsymbol{\epsilon_d})\epsilon \end{bmatrix} \end{aligned} \tag{20}$$

If $\mathbf{q} = \mathbf{q_d}$, we have that

$$\tilde{\boldsymbol{q}} = \begin{bmatrix} \eta^2 + \boldsymbol{\epsilon}^T\boldsymbol{\epsilon} \\ \eta\boldsymbol{\epsilon} - \eta\boldsymbol{\epsilon} - \mathbf{S}(\boldsymbol{\epsilon})\boldsymbol{\epsilon} \end{bmatrix} \Rightarrow \tilde{\boldsymbol{q}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{21}$$

So when $\mathbf{q} = \mathbf{q_d}$ the error in the imaginary part of the quaternion is naturally equal to zero, where η is equal to one. This also corresponds to the equilibrium point of our closed-loop system.

Problem 1.5

From here on out the control law that will be used takes the error in the imaginary part of the quaternion, $\tilde{\boldsymbol{\epsilon}}$, as input instead of the imaginary part $\boldsymbol{\epsilon}$ directly (17).

In addition to the new control law we now want the system to follow a time varying reference signal instead of just trying to reach the equilibrium point. In Euler angles, the reference signal is $\phi(t)$ = 10sin(0.1t), $\theta(t)$ = 0 and $\psi(t)$ = 15cos(0.05t). Since we operate in quaternions instead of Euler angles the desired angles had to be expressed as a (time varying) quaternion $\mathbf{q_d}$. This conversion was done directly using the *euler2q* function in MATLAB. The controller gains and the initial conditions are the same as in the previous simulations.

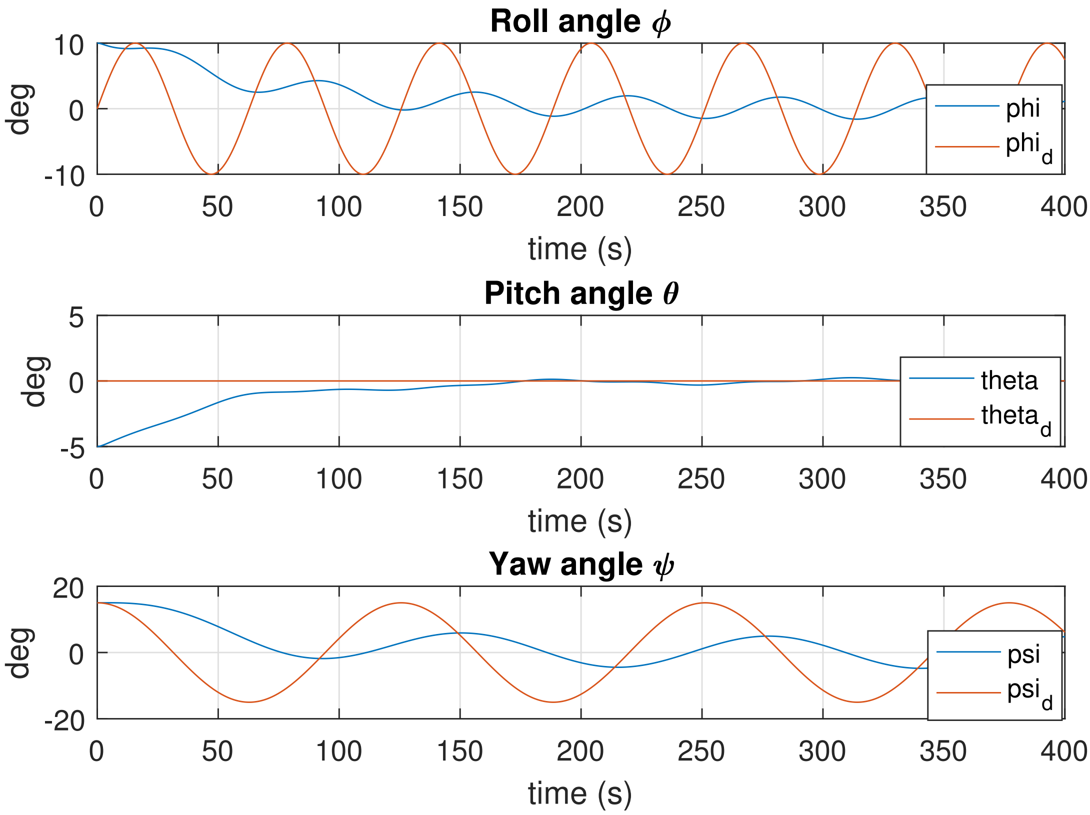


Figure 3: Compares the actual and the desired angles, here expressed using Euler angles

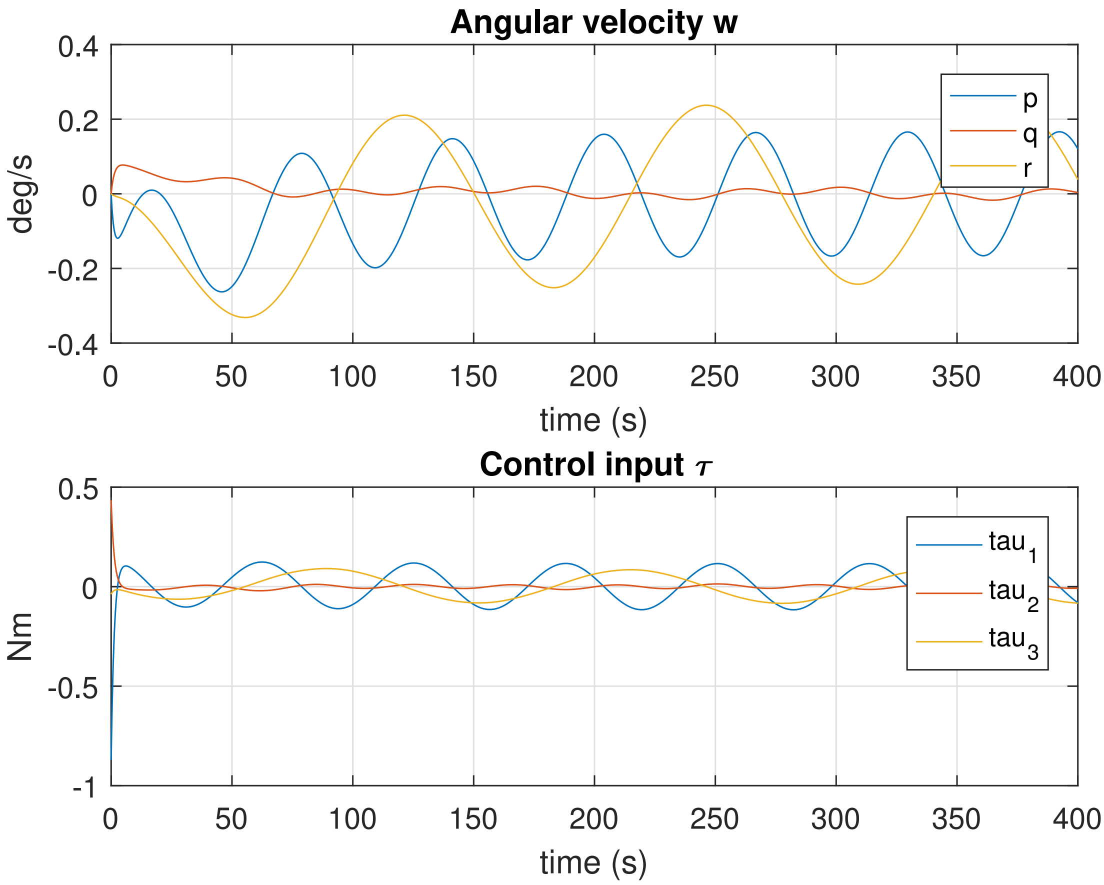


Figure 4: The angular velocities and the control output

After running the simulation, figure 3 shows how the controller is able to control the pitch angle to the desired point, however, it is not able to track the desired position for the roll or the yaw angle properly. In both cases the response oscillates with an amplitude that is too small.

This is as expected as the control law uses the angular velocity $\boldsymbol{\omega}$ directly, instead of using the error as is later done in Problem 1.6. Using the current control law, the controller will try to regulate $\boldsymbol{\omega}$ to zero in $\boldsymbol{\tau}_1 = -\mathbf{K_d}\boldsymbol{\omega}$, whereas $\boldsymbol{\tau}_2 = -k_p\tilde{\boldsymbol{\epsilon}}$ will regulate the imaginary part of the quaternions towards their time-varying reference signals. The resulting controller input, $\boldsymbol{\tau} = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2$, will then give an input that is inconsistent with the actual desired position for the satellite. Therefore, in order to follow a time varying signal, the angular velocity in the control law must incorporate the changing reference point. If this is not taken into account, then the controller will produce inaccurate responses from the system, as seen in the simulation results presented by figures 3 and 4.

The pitch angle θ is desired to be zero, and so is the angular velocity. As $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ will both regulate θ to zero, we see that the controller is able to bring θ to its set point. This would not have been the case if θ had a time-varying reference signal.

Problem 1.6

In order to improve the controller a new control law was now tested. Now the error between the actual and the desired angular velocity was used in the control loop instead of using the angular velocity directly as was done in problem 1.5. The new control law was given as:

$$\boldsymbol{\tau} = -\mathbf{K_d}\tilde{\boldsymbol{\omega}} - k_p\tilde{\boldsymbol{\epsilon}} \tag{22}$$

The expression for the error in the angular velocity ($\tilde{\boldsymbol{\omega}}$) was based on the desired angular velocity ($\boldsymbol{\omega}_d$), which was determined using:

$$\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega} - \boldsymbol{\omega}_d \Rightarrow \boldsymbol{\omega_d} = \boldsymbol{T}_{\boldsymbol{\Theta_d}}^{-1}(\Theta_d)\dot{\Theta}_d \tag{23}$$

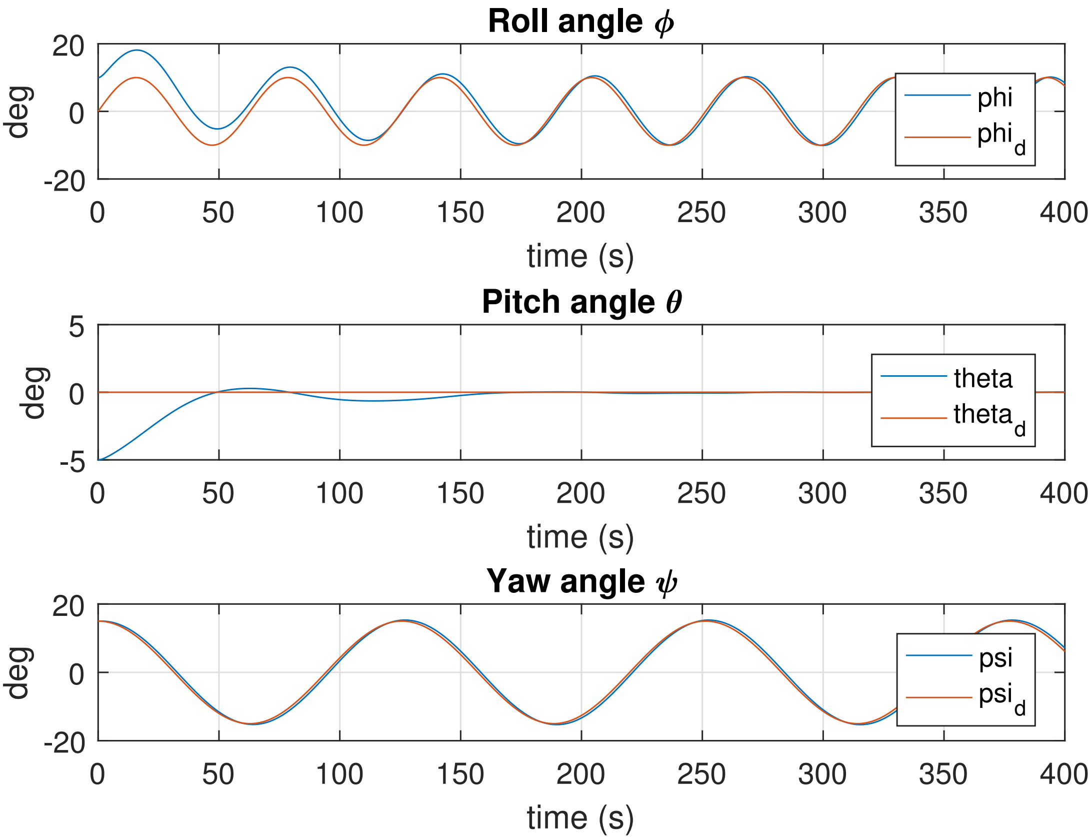


Figure 5: Compares the actual and the desired angles, here expressed using Euler angles

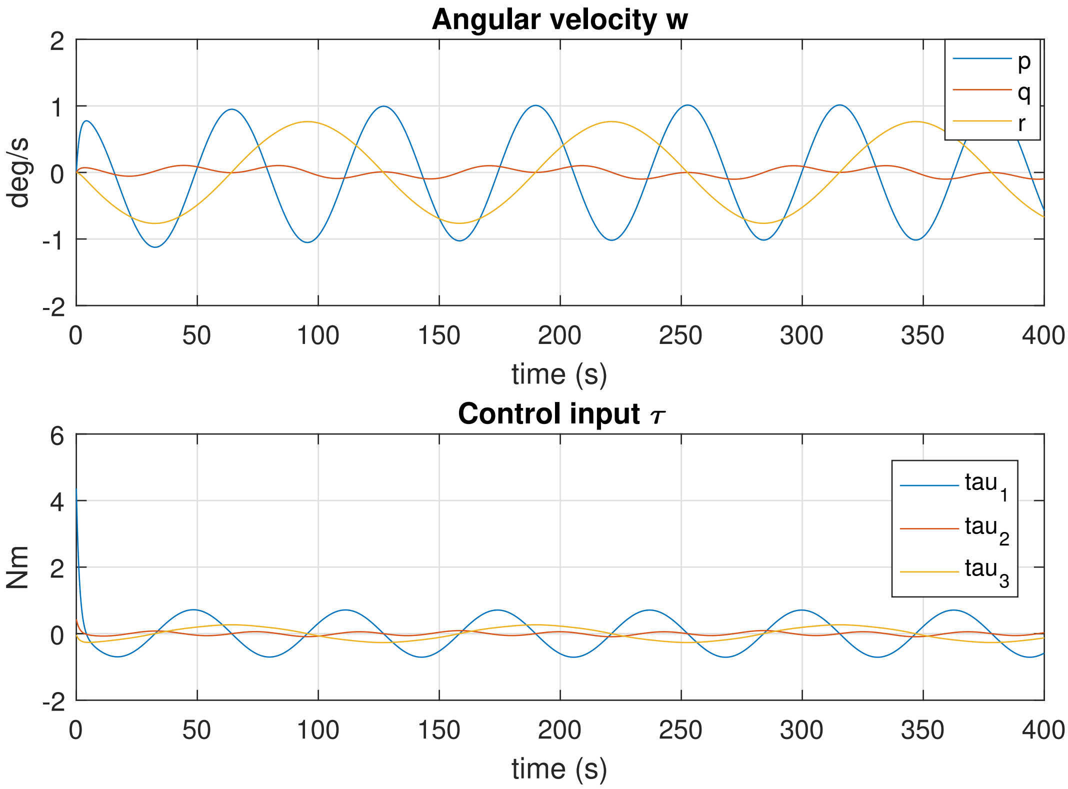


Figure 6: The angular velocities and the control output

The results using this control law was significantly improved compared to the one used in problem 1.5. Here the controller is able to follow the reference signal very well and provides overall satisfactory results. It takes longer time for the roll angle to converge to the desired angle compared to in pitch and yaw, but this must be expected when the initial position is further from the reference position.

As explained in problem 1.5, the reason for this improvement is expected and can be traced back to now employing the time-varying desired angular velocity ($\boldsymbol{\omega}_d$) as part of the input in the control law. Within $\boldsymbol{\tau}$, both $\boldsymbol{\tau}_1 = -\mathbf{K_d}\tilde{\boldsymbol{\omega}}$ and $\boldsymbol{\tau}_2 = -k_p\tilde{\boldsymbol{\epsilon}}$, will now give proportional outputs with regard to each other. As both $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ will aim to follow the time-varying reference signal. By using the error between the actual angular velocity ($\boldsymbol{\omega}$) and the desired angular velocity ($\boldsymbol{\omega}_d$), the controller will now output a value that tries to minimize the error in the angular velocity instead of minimizing the angular velocity altogether. As a result, the system response eventually matches that of its reference signal.

Problem 1.7

In this problem we used the control law in (17) on our system in (1), and assumed set point regulation where $\boldsymbol{\omega}_d = \mathbf{0}$, $\boldsymbol{\epsilon}_d$ = constant and η_d = constant. With this in mind, the following Lyapunov function candidate was examined.

$$V = \underbrace{\frac{1}{2}\tilde{\boldsymbol{\omega}}^T \mathbf{I}_{CG}\tilde{\boldsymbol{\omega}}}_{Z} + \underbrace{2k_p(1-\tilde{\eta})}_{Q} \tag{24}$$

Ensuring that V is Positive and Radially Unbounded

V in (24) represents the energy in our system, it is always positive as both its elements Z and Q will never be negative.

Z is always positive as the matrix multiplication represented by $\tilde{\boldsymbol{\omega}}^T \mathbf{I}_{CG} \tilde{\boldsymbol{\omega}}$ is like squaring the values within the $\tilde{\boldsymbol{\omega}}$-matrix. Based on this, even if the angular velocity error represented by $\tilde{\boldsymbol{\omega}}$ is negative, the matrix multiplication in Z will ensure that this first part of V is positive. The inertia matrix $\mathbf{I_{CG}}$ is know, constant and always positive.

Next, in order for Q to always remain positive $\tilde{\eta}$ cannot exceed 1. As $\tilde{\eta} = cos(\frac{\beta}{2})$ [1], it is clear that $\tilde{\eta}$ will vary between -1 and 1, ensuring that Q remains positive. Therefore, as Z and Q are positive, then V will be positive.

A function, $V(x)$, is radially unbounded if:

$$V(\mathbf{x}) \to \infty \;\text{ as }\; ||\mathbf{x}|| \to \infty \tag{25}$$

$||\mathbf{x}||$ represents the norm of the vector $\mathbf{x}$, where a norm is related to the length of the vector. Once the length of the vector goes to infinity, then at least one of the vector's elements must go to infinity.

In order to compare this to the Lyapunov function in (24), $\mathbf{x}$ in (25) has to be replaced with $\tilde{\mathbf{x}}$:

$$\tilde{\mathbf{x}} = \begin{bmatrix} \tilde{\mathbf{q}} \\ \tilde{\boldsymbol{\omega}} \end{bmatrix} = \begin{bmatrix} \tilde{\eta} \\ \tilde{\boldsymbol{\epsilon}} \\ \tilde{\boldsymbol{\omega}} \end{bmatrix}$$

Expanding the matrices within (24) gives the following:

$$V(\tilde{\mathbf{x}}) = \frac{1}{2} \begin{bmatrix} \tilde{p} & \tilde{q} & \tilde{\mathrm{r}} \end{bmatrix} \begin{bmatrix} mr^2 & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & mr^2 \end{bmatrix} \begin{bmatrix} \tilde{p} \\ \tilde{q} \\ \tilde{\mathrm{r}} \end{bmatrix} + 2k_p(1 - \tilde{\eta}) \tag{26a}$$

$$\Rightarrow V(\tilde{\mathbf{x}}) = \frac{1}{2} mr^2(\tilde{p}^2 + \tilde{q}^2 + \tilde{r}^2) + 2k_p(1 - \tilde{\eta}) \tag{26b}$$

Based on (26b), we see that once $||\tilde{\mathbf{x}}|| = [\tilde{\eta}\ \tilde{\boldsymbol{\epsilon}}\ \tilde{p}\ \tilde{q}\ \tilde{r}] \to \infty$ the value for $V(\tilde{\mathbf{x}})$ will follow.

$$\lim_{||\tilde{\mathbf{x}}|| \to \infty} V(\tilde{\mathbf{x}}) \to \infty \tag{27}$$

Using (27), we conclude that $V(\tilde{\mathbf{x}})$ is radially unbounded.

Determining $\dot{V}$

V in (24) represents the energy of the system, in order to determine the power of the system, we derivate (24). In order to derivate the matrix expression, the following tools are used:

$$\mathbf{I}_{CG} \text{ is constant} \Rightarrow \dot{\mathbf{I}}_{CG} = 0 \tag{28}$$

$$\mathbf{I}_{CG} \text{ is symmetric} \Rightarrow \mathbf{I}_{CG}^T = \mathbf{I}_{CG} \tag{29}$$

$$\text{For a scalar, } a = a^T \Rightarrow \tilde{\boldsymbol{\omega}}^T \mathbf{I}_{CG} \tilde{\boldsymbol{\omega}} = (\tilde{\boldsymbol{\omega}}^T \mathbf{I}_{CG} \tilde{\boldsymbol{\omega}})^T \tag{30}$$

$$(ABC)^T = C^T B^T A^T \Rightarrow \dot{\tilde{\boldsymbol{\omega}}}^T \mathbf{I}_{CG} \tilde{\boldsymbol{\omega}} = (\dot{\tilde{\boldsymbol{\omega}}}^T \mathbf{I}_{CG} \tilde{\boldsymbol{\omega}})^T = \tilde{\boldsymbol{\omega}}^T \mathbf{I}_{CG} \dot{\tilde{\boldsymbol{\omega}}} \tag{31}$$

Now, to the actual derivation of V to obtain the $\dot{V}$.

$$\begin{aligned}
\frac{d}{dt}V &= \frac{1}{2}\frac{d}{dt}(\tilde{\boldsymbol{\omega}}^T\mathbf{I}_{CG}\tilde{\boldsymbol{\omega}}) + 2k_p\frac{d}{dt}(1-\tilde{\eta}) \\
&= \frac{1}{2}(\dot{\tilde{\boldsymbol{\omega}}}^T\mathbf{I}_{CG}\tilde{\boldsymbol{\omega}} + \underbrace{\tilde{\boldsymbol{\omega}}^T\dot{\mathbf{I}}_{CG}\tilde{\boldsymbol{\omega}}}_{=0} + \tilde{\boldsymbol{\omega}}^T\mathbf{I}_{CG}\dot{\tilde{\boldsymbol{\omega}}}) - 2k_p\dot{\tilde{\eta}} \\
&= \frac{1}{2}((\dot{\tilde{\boldsymbol{\omega}}}^T\mathbf{I}_{CG}\tilde{\boldsymbol{\omega}})^T + \tilde{\boldsymbol{\omega}}^T\mathbf{I}_{CG}\dot{\tilde{\boldsymbol{\omega}}}) - 2k_p\dot{\tilde{\eta}} \\
&= \frac{1}{2}(2\tilde{\boldsymbol{\omega}}^T\mathbf{I}_{CG}\dot{\tilde{\boldsymbol{\omega}}}) - 2k_p\dot{\tilde{\eta}} \\
&= \tilde{\boldsymbol{\omega}}^T\underbrace{\mathbf{I}_{CG}\dot{\tilde{\boldsymbol{\omega}}}}_{\boldsymbol{\tau}} - 2k_p\underbrace{(-\frac{1}{2}\tilde{\boldsymbol{\epsilon}}^T\tilde{\boldsymbol{\omega}})}_{\dot{\tilde{\eta}}}
\end{aligned}$$

$\mathbf{I}_{CG}\dot{\tilde{\boldsymbol{\omega}}}$ can be solved from (1b), and is therefore replaced by $\mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega})\boldsymbol{\omega} + \boldsymbol{\tau}$. Yet, from (3), $\mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega}) = 0$, therefore $\mathbf{I}_{CG}\dot{\tilde{\boldsymbol{\omega}}}$ reduces to $\boldsymbol{\tau}$.

$$\begin{aligned}
\Rightarrow &= \tilde{\boldsymbol{\omega}}^T(-\mathbf{K}_d\boldsymbol{\omega} - k_p\tilde{\boldsymbol{\epsilon}}) + k_p\tilde{\boldsymbol{\epsilon}}^T\tilde{\boldsymbol{\omega}} \\
&= -k_d\tilde{\boldsymbol{\omega}}^T\mathbf{I}\boldsymbol{\omega} - k_p\tilde{\boldsymbol{\omega}}^T\tilde{\boldsymbol{\epsilon}} + k_p(\tilde{\boldsymbol{\epsilon}}^T\tilde{\boldsymbol{\omega}})^T \\
&= -k_d\tilde{\boldsymbol{\omega}}^T\mathbf{I}\boldsymbol{\omega} - k_p\tilde{\boldsymbol{\omega}}^T\tilde{\boldsymbol{\epsilon}} + k_p\tilde{\boldsymbol{\omega}}^T\tilde{\boldsymbol{\epsilon}} \\
&= -k_d\tilde{\boldsymbol{\omega}}^T\boldsymbol{\omega}
\end{aligned}$$

Using that $\boldsymbol{\omega}_d = 0$, we get that $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega} - \boldsymbol{\omega}_d = \boldsymbol{\omega}$. As a result, the equation for the power of the system is:

$$\dot{V} = -k_d\boldsymbol{\omega}^T\boldsymbol{\omega} \tag{32}$$

Convergence and Stability

If there exists a uniformly continuous function $V : \mathbf{R}^n \times \mathbf{R}_+ \to \mathbf{R}$ satisfying:

1. $V(\mathbf{x}, t) \geq 0$
2. $\dot{V}(\mathbf{x}, t) \leq 0$
3. $\dot{V}(\mathbf{x}, t)$ is uniformly continuous

Then, in accordance to Barbalat's Lemma, one can deduce that $\lim_{t\to\infty}\dot{V}(\mathbf{x}, t) = 0$ [1].

Using the Lyapunov function candidate in (24) and its derivative in (32), it is clear that the first two conditions for using Barbalat's Lemma are satisfied. V was proven to be greater than or equal to zero earlier in this section, and based on (32), $\dot{V}$ will always be less than or equal to zero. For the third condition, $\dot{V}$ is uniformly continuous if its derivative, $\ddot{V}$ is bounded.

$$\begin{aligned}
\frac{d}{dt}\dot{V}(\mathbf{x}) &= -k_d\frac{d}{dt}(\boldsymbol{\omega}^T\boldsymbol{\omega}) \\
&= -k_d(\dot{\boldsymbol{\omega}}^T\boldsymbol{\omega} + \boldsymbol{\omega}^T\dot{\boldsymbol{\omega}}) \\
&= -2k_d\boldsymbol{\omega}^T\dot{\boldsymbol{\omega}}
\end{aligned}$$

Based on the derivation above, $\ddot{V}$ will be bounded so long as $\boldsymbol{\omega}$ remains bounded. As the previously written V seemingly only been dependent on the states $(\mathbf{x})$, note that V has also been dependent upon time as its states, such as $\boldsymbol{\omega}$, vary with time.

With the conditions for using Barbalat's Lemma satisfied, it is possible to deduce that $\lim_{t\to\infty} \dot{V}(\mathbf{x},t) = 0$. This, alongside (32), is then used to prove that $\boldsymbol{\omega}$ will converge to zero as time goes to infinity:

$$\lim_{t\to\infty} \dot{V}(\mathbf{x},t) = 0 \Rightarrow \lim_{t\to\infty} -k_d\boldsymbol{\omega}^T\boldsymbol{\omega} \Rightarrow \lim_{t\to\infty} \boldsymbol{\omega} = 0 \tag{33}$$

The equilibrium point of the closed-loop in (9) shows that as $\boldsymbol{\omega} \to 0$, then the closed-loop is reaching its equilibrium point. Therefore, with the proof that $\boldsymbol{\omega} \to 0$ over time, where Barbalat's Lemma was used, the equilibrium point of the closed-loop is globally convergent. So with this in mind, we know that $\boldsymbol{\omega}$ will eventually go towards its equilibrium point, but we do not know exactly when this will occur.

In order to determine asymptotic stability, the Krazovskii-LaSalle theorem is used. The Lyapunov's direct method for determining stability is not used as $\dot{V} \leq 0$, meaning it is a negative semi-definite. It is viable to use the Krazovskii-LaSalle theorem [1] for this V, as the following conditions are satisfied.

1. $V(\mathbf{x}) = \infty$ as $||\mathbf{x}|| \to \infty$
2. $\dot{V}(\mathbf{x}) \leq 0, \forall \mathbf{x}$

The first condition was proven in (27) and the second condition was verified whilst using Barbalat's Lemma.

Now, in order to prove global asymptotic stability (GAS), we must make sure that the system does not "get stuck" as $\dot{V} = 0$, but that it will continuously move towards the equilibrium point. A test for this is completed below.

$$\dot{V} = -k_d\boldsymbol{\omega}^T\boldsymbol{\omega} = 0 \Rightarrow \dot{V} \neq 0 \text{ when } \boldsymbol{\omega} \neq 0 \tag{34}$$

Based on (34), as long as $\boldsymbol{\omega} \neq 0$, meaning that as long as the system has not reached its equilibrium point, then $\dot{V} \neq 0$, and the system will continue to move. Therefore, the system is globally asymptotically stable around its equilibrium point.

Problem 1.8

The problem could have been parametrized using Euler angles, however, in the Euler angle parametrization there exist a singularity when the pitch angle θ is $\pm 90°$. Euler angles is often preferred in naval and aircraft applications because roll, pitch and yaw have a physical meanings and it is easy to see what the represent. In these applications, pitch angle equal to $pm90°$ occur very seldom, that is, then the bow of the ship/aircraft is either pointing directly upwards or downwards. In most cases this is not supposed to occur, and if it does one might frankly argue that you have bigger problems.

However, for a satellite, especially a completely round one like this one, pitch angle equal to $\pm 90°$ should not result in a physical problem and might happen at a regular basis. Therefor an Euler angle paramterization might be problematic to use.

The other option is to use quaternions as was done in this assignment. This representation contains the same information, but without the problem of the possible singularity. However,the price for this is having to use four variables and therefore having to add an extra constraint, that is $\mathbf{q^T q}$ must be equal to 1. In order to assure this in simulations the unit quaternion has to be normalized for each iteration. This involves integration and will therefor introduce additional numerical errors. Introducing measures to counteract this is recommended in order to minimize the errors.

Related to convergence and stability one must keep in mind that both the Euler angle and the quaternion parametrizations represents a configuration in 3D-space. Using Euler angles there is only one way to do this, but since the quaternion has four variables it is not certain that a specific representation is globally unique.

References

[1] T. Fossen, *Handbook of Marine Craft Hydrodynamics and Motion Control*. John Wiley & Sons, 2011.