Using LaTeX and Gnuplot: The Fresnel Integrals

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The Fresnel integrals are defined as the following definite integrals:

$$S(x) = \int_0^x \sin\left(t^2\right) dt, \tag{1}$$

and

$$C(x) = \int_0^x \cos\left(t^2\right) dt, \tag{2}$$

where $S(x)$ is called the Fresnel sine integral and $C(x)$ is called the Fresnel cosine integral. These functions are shown in the figure.

They are named after the French engineer and physicist, Augustin-Jean Fresnel (1788-1827), who originally used them in optics calculations.

The limit of the Fresnel functions as x approaches infinity are

$$\int_0^\infty \cos t^2 dt = \int_0^\infty \sin t^2 dt = \frac{\sqrt{2\pi}}{4} \approx 0.6267. \tag{3}$$

In some cases, the argument t^2 is replaced with $\pi t^2/2$ and the functions are known as normalized Fresnel integrals. These functions converge to 1/2.

The Fresnel integrals can be expanded in the following power series that converge for all x:

$$S(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)!(4n+3)}, \tag{4}$$

and

$$C(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n)!(4n+1)}. \tag{5}$$

The Fresnel integrals can be extended to the domain of complex numbers, $S(z)$ and $C(z)$, where they become analytic functions of a complex variable z and can be expressed using the error function.

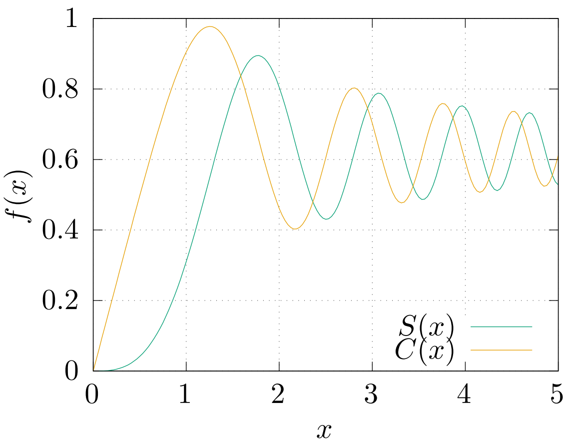

Figure 1: Plots of the Fresnel functions, $S(x)$ and $C(x)$.

References

[1] Wikipedia, Fresnel integral, `https://en.wikipedia.org/wiki/Fresnel_integral`, accessed: 2020-05-05.

[2] Wikipedia, Augustin-Jean Fresnel, `https://en.wikipedia.org/wiki/Augustin-Jean_Fresnel`, accessed: 2020-05-05.