Homework 4 - Stochastic Simulation

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1 Exercise 1

See separate pdf and R-markdown file. Programming done in R.

2 Exercise 2

We want to estimate $z(x) = \mathbf{P}(\tau(x) < \infty)$ for $x > 0$, where $\tau(x) = inf\{n : S_n > x\}$, $S_n = \sum_{i=1}^n X_i$ and the X_i are i.i.d copies of an r.v. X. X has a non-lattice, light-tailed distribution with a negative mean. X can take positive values.

We have that the expression for R that is necessary to obtain a 95% confidence interval with a width of 20% of the target value, z, estimated by repetitions of Z is

$$R = \frac{100 \cdot 1.96^2 \mathrm{Var}[Z(x)]}{z(x)^2}$$

So we need to find an expression for

$$\frac{\mathrm{Var}[Z(x)]}{z(x)^2}$$

in our case, as $x \to \infty$. From Siegmund's algorithm, we have that

$$z(x) = e^{-\gamma x}\mathbf{E}_\gamma[e^{-\gamma\xi(x)}] \Rightarrow z(x)^2 = e^{-2\gamma x}\mathbf{E}_\gamma[e^{-\gamma\xi(x)}]^2$$

and

$$\begin{aligned} &Z(x) = e^{-\gamma x} \cdot e^{-\gamma\xi(x)} \\ &\Rightarrow \mathrm{Var}[Z(x)] = \mathrm{Var}[e^{-\gamma x} \cdot e^{-\gamma\xi(x)}] = e^{-2\gamma x}\mathrm{Var}[e^{-\gamma\xi(x)}]. \end{aligned}$$

So, we get

$$\begin{aligned} \frac{\mathrm{Var}[Z(x)]}{z(x)^2} &= \frac{e^{-2\gamma x}\mathrm{Var}[e^{-\gamma\xi(x)}]}{e^{-2\gamma x}\mathbf{E}_\gamma[e^{-\gamma\xi(x)}]^2} = \frac{\mathrm{Var}[e^{-\gamma\xi(x)}]}{\mathbf{E}_\gamma[e^{-\gamma\xi(x)}]^2} \\ &= \frac{\mathbf{E}_\gamma[e^{-2\gamma\xi(x)}] - \mathbf{E}_\gamma[e^{-\gamma\xi(x)}]^2}{\mathbf{E}_\gamma[e^{-\gamma\xi(x)}]^2} \\ &= \frac{\mathbf{E}_\gamma[e^{-2\gamma\xi(x)}]}{\mathbf{E}_\gamma[e^{-\gamma\xi(x)}]^2} - 1. \end{aligned}$$

This gives us the expression for R as $x \to \infty$

$$\lim_{x\to\infty} R = 100 \cdot 1.96^2 \cdot \frac{\mathbf{E}_\gamma[e^{-2\gamma\xi(\infty)}]}{\mathbf{E}_\gamma[e^{-\gamma\xi(\infty)}]^2} - 1.$$

We now assume that X is decomposable into $X = X^+ - X^-$, where $X^+ \sim \exp(\frac{1}{\mu})$ and $X^- \sim \exp(\frac{1}{\lambda})$, with $\frac{1}{\mu} < \frac{1}{\lambda}$. We will use this information to evaluate the two expressions $\mathbf{E}_\gamma[e^{-2\gamma\xi(\infty)}]$ and $\mathbf{E}_\gamma[e^{-\gamma\xi(\infty)}]$.

We know that in Siegmund's algorithm, γ is the value that solves the equation $\hat{F}[\gamma]=1$, where $\hat{F}[\theta]$ is the moment generating function of $X = X^+ - X^-$. This expressions comes from the exponential tilting in the importance sampling that Siegmund's algorithm is based on. We get

$$1 = \hat{F}[\gamma] = \mathbf{E}[e^{\gamma X^+}]\mathbf{E}[e^{-\gamma X^-}] = \frac{\mu}{\mu-\gamma}\frac{\lambda}{\lambda+\gamma},$$

which gives us $\gamma = \mu - \lambda$. We recall that $\mu > \lambda$, and so $\gamma > 0$ as required. This gives us

$$\mathbf{E}_\gamma[e^{\gamma\xi(\infty)}] = \mathbf{E}_\gamma[e^{(-\mu)\xi(\infty)}].$$

We will now examine $\xi(\infty)$. This is the overshoot, the amount that we go over the threshold x the first time that happens. We know that this event will take place du to the realization $X_{\tau(\infty)} > 0$, where

$$\sum_{i=1}^{\tau(\infty)-1} X_i < x \quad \text{and} \quad \sum_{i=1}^{\tau(\infty)-1} X_i + X_{\tau(\infty)} > x.$$

This means that we can expect

$$\xi(\infty) = S_{\tau(x)} - x|_{x\to\infty} \leq X_{\tau(\infty)}.$$

We insert this into our expression and get

$$\mathbf{E}_\gamma[e^{(\lambda-\mu)\xi(\infty)}] = \mathbf{E}_\gamma[e^{(\lambda-\mu)X_{\tau(\infty)}}].$$

Furthermore, we have as described in lecture notes from week 11, the following property for $\mathbf{E}_\gamma[e^{aX}]$ for some constant a and our γ; since

$$\begin{aligned}
&\mathbf{E}[e^{aX}] = \mathbf{E}_\gamma[e^{aX}L_{1,\gamma}] = \mathbf{E}_\gamma[e^{aX}e^{-\gamma X}] = \mathbf{E}_\gamma[e^{(a-\gamma)X}] \\
&\Rightarrow \mathbf{E}_\gamma[e^{aX}] = \mathbf{E}[e^{(a+\gamma)X}] = \mathbf{E}[e^{(a+\gamma)X^+}]\mathbf{E}[e^{-(a+\gamma)X^-}] = \frac{\mu}{\mu-(a+\gamma)}\frac{\lambda}{\lambda+(a+\gamma)} = \frac{\mu}{a+\mu}\frac{\lambda}{\lambda-a}.
\end{aligned}$$

Applying this to our expression, we get

$$\mathbf{E}_\gamma[e^{(\lambda-\mu)X_{\tau(\infty)}}] = \frac{\mu}{\lambda-\mu+\mu}\frac{\lambda}{\lambda-\lambda+\mu} = \frac{\mu}{\lambda}\frac{\lambda}{\mu} = 1,$$

which provides the result

$$\mathbf{E}_\gamma[e^{-\gamma\xi(\infty)}] = 1.$$

Similarly, we get that

$$\mathbf{E}_\gamma[e^{-2\gamma\xi(\infty)}] = \mathbf{E}_\gamma[e^{2(\lambda-\mu)X_{\tau(\infty)}}] = \frac{\mu}{2\lambda-2\mu+\mu}\frac{\lambda}{\lambda-2\lambda+2\mu} = \frac{\mu}{2\lambda-\mu}\frac{\lambda}{2\mu-\lambda}$$