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EE 5320

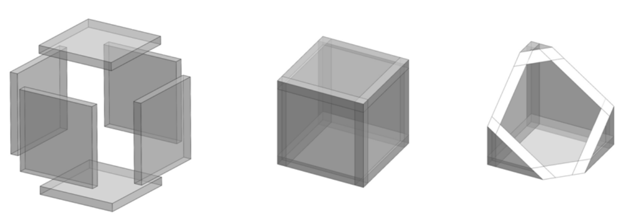
Computational Electromagnetics

Lecture #9

Perfectly Matched Layer

Outline

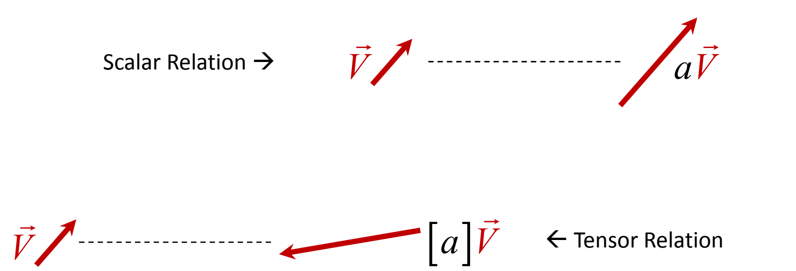
- Background Information
- The Uniaxial Perfectly Matched Layer (UPML)
- Incorporating a UPML into Maxwell's Equations
- Implementing the UPML
- Stretched Coordinate PML (SC-PML)
- PML Performance
- UPML vs SC-PML



Background Information

Tensors

Tensors are a generalization of a scaling factor where the direction of a vector can be altered in addition to its magnitude.



$$[a]\vec{V} = \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$

Reflectance from a Surface with Loss

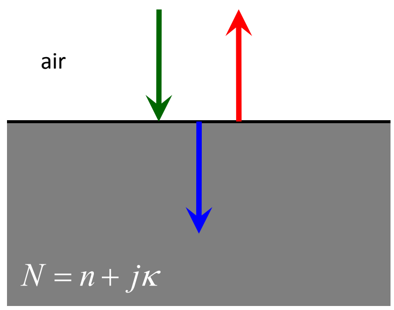
Complex Refractive Index

$$\tilde{n} = n + j\kappa$$

$n \equiv$ ordinary refractive index (oscillation)

$\kappa \equiv$ extinction coefficient (decay)

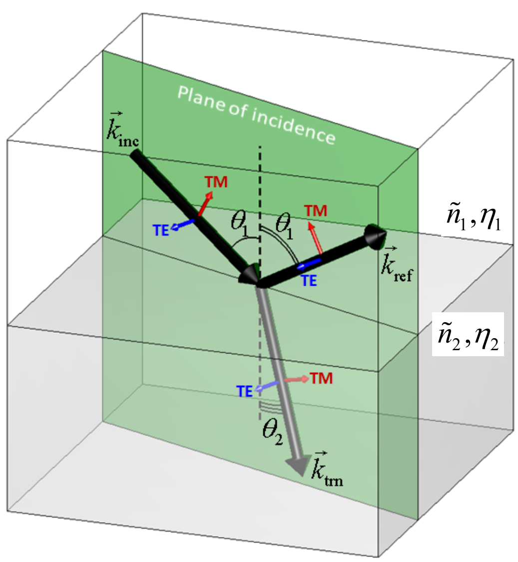
Reflectance from a Lossy Surface



$$R = \frac{(1-n)^2 + \kappa^2}{(1+n)^2 + \kappa^2}$$

** Loss contributes to reflections

Reflection, Transmission and Refraction at an Interface



$n_i \equiv$ refractive index in region i

$\eta_i \equiv$ impedance in region i

Angles

$$\theta_{\text{inc}} = \theta_{\text{ref}} = \theta_1$$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \quad \text{Snell's Law}$$

TE Polarization

$$r_{\text{TE}} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

$$t_{\text{TE}} = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

TM Polarization

$$r_{\text{TM}} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2}$$

$$t_{\text{TM}} = \frac{2\eta_2 \cos\theta_1}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2}$$

Maxwell's Equations in Anisotropic Media

Maxwell's curl equations in anisotropic media are:

$$\nabla \times \vec{H} = j\omega\varepsilon_0 \left[\varepsilon_r\right]\vec{E} \qquad \nabla \times \vec{E} = -j\omega\mu_0 \left[\mu_r\right]\vec{H}$$

These can also be written in a matrix form that makes the tensor aspect of μ and ε more obvious.

$$\begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = j\omega\varepsilon_0 \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = -j\omega\mu_0 \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$$

Types of Anisotropic Media

There are three basic types of anisotropic media

$$\begin{bmatrix} \varepsilon_{\text{iso}} & 0 & 0 \\ 0 & \varepsilon_{\text{iso}} & 0 \\ 0 & 0 & \varepsilon_{\text{iso}} \end{bmatrix} \quad \text{isotropic}$$

$$\begin{bmatrix} \varepsilon_{\text{o}} & 0 & 0 \\ 0 & \varepsilon_{\text{o}} & 0 \\ 0 & 0 & \varepsilon_{\text{e}} \end{bmatrix} \quad \text{uniaxial}$$

$$\begin{bmatrix} \varepsilon_{\text{a}} & 0 & 0 \\ 0 & \varepsilon_{\text{b}} & 0 \\ 0 & 0 & \varepsilon_{\text{c}} \end{bmatrix} \quad \text{biaxial}$$

Note: terms only arise in the off-diagonal positions when the tensor is rotated relative to the coordinate system.

Maxwell's Equations in Doubly-Diagonally Anisotropic Media

Maxwell's equations for diagonally anisotropic media can be written as

$$\begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = j\omega\varepsilon_0 \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \qquad \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = -j\omega\mu_0 \begin{bmatrix} \mu_{xx} & 0 & 0 \\ 0 & \mu_{yy} & 0 \\ 0 & 0 & \mu_{zz} \end{bmatrix}\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$$

We can generalize further by incorporating loss.

$$\begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = j\omega\varepsilon_0 \begin{bmatrix} \varepsilon_x + \sigma_x^E / j\omega & 0 & 0 \\ 0 & \varepsilon_y + \sigma_y^E / j\omega & 0 \\ 0 & 0 & \varepsilon_z + \sigma_z^E / j\omega \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = -j\omega\mu_0 \begin{bmatrix} \mu_x + \sigma_x^H / j\omega & 0 & 0 \\ 0 & \mu_y + \sigma_y^H / j\omega & 0 \\ 0 & 0 & \mu_z + \sigma_z^H / j\omega \end{bmatrix}\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$$

Lecture 9 Slide 9

Scattering at a Doubly-Anisotropic Interface

<u>Refraction</u> into a diagonally anisotropic materials is described by

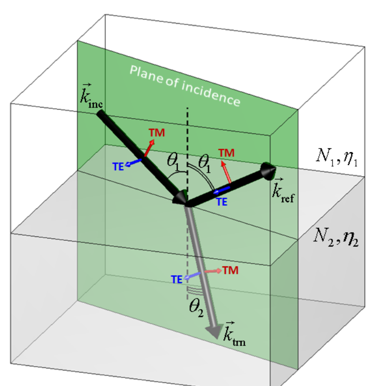
$$\sin\theta_1 = \sqrt{bc}\sin\theta_2$$

$$[\mu_r] = [\varepsilon_r] = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

<u>Reflection</u> from a diagonally anisotropic material is

$$r_{\text{TE}} = \frac{\sqrt{a}\cos\theta_1 - \sqrt{b}\cos\theta_2}{\sqrt{a}\cos\theta_1 + \sqrt{b}\cos\theta_2}$$

$$r_{\text{TM}} = \frac{-\sqrt{a}\cos\theta_1 + \sqrt{b}\cos\theta_2}{\sqrt{a}\cos\theta_1 + \sqrt{b}\cos\theta_2}$$



Sacks, Zachary S., et al. "A perfectly matched anisotropic absorber for use as an absorbing boundary condition." IEEE Trans. Antennas and Propagation, Vol. 43, No. 12, pp. 1460-1463, 1995.

Lecture 9 Slide 10

Notes on a Single Interface

- It is a change in impedance that causes reflections
- Snell's Law quantifies the angle of transmission
- Angle of transmission and reflection does not depend on polarization.
- The Fresnel equations quantify the amount of reflection and transmission
- Amount of reflection and transmission depends on the polarization

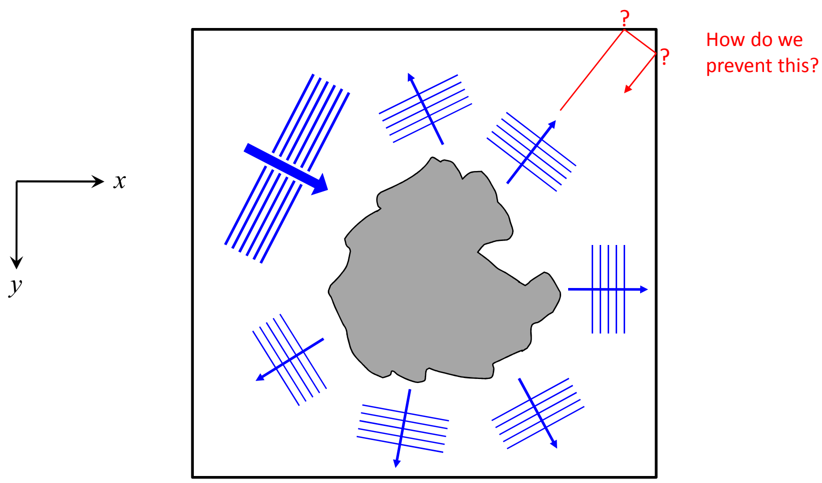
Uniaxial Perfectly Matched Layer (UPML)

S. Zachary, D. Kingsland, R. Lee, J. Lee, "A Perfectly Matched Anisotropic Absorber for Use as an Absorbing Boundary Condition," IEEE Trans. on Ant. and Prop., Vol. 43, No. 12, pp 1460–1463, 1995.

Boundary Condition Problem

CEM

If we model a wave hitting some device or object, it will scatter the applied wave into potentially many directions. We do NOT want these scattered waves to reflect from the boundaries of the grid. We also don't want them to reenter from the other side of the grid (periodic boundaries).

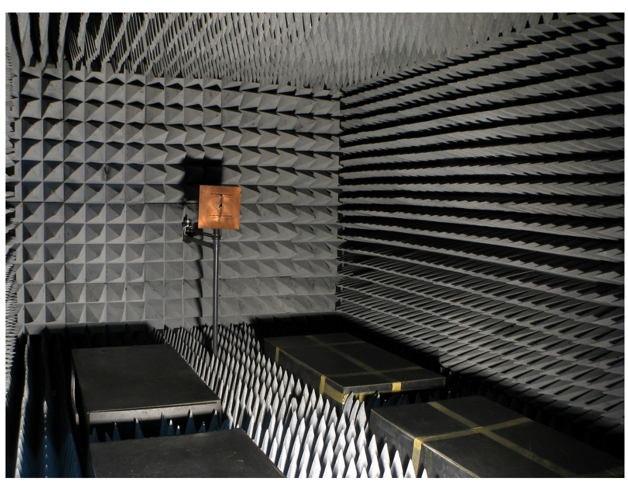


Lecture 9 Slide 13

How We Prevent Reflections in Lab

CEM

In the lab, we use anechoic foam to absorb outgoing waves.

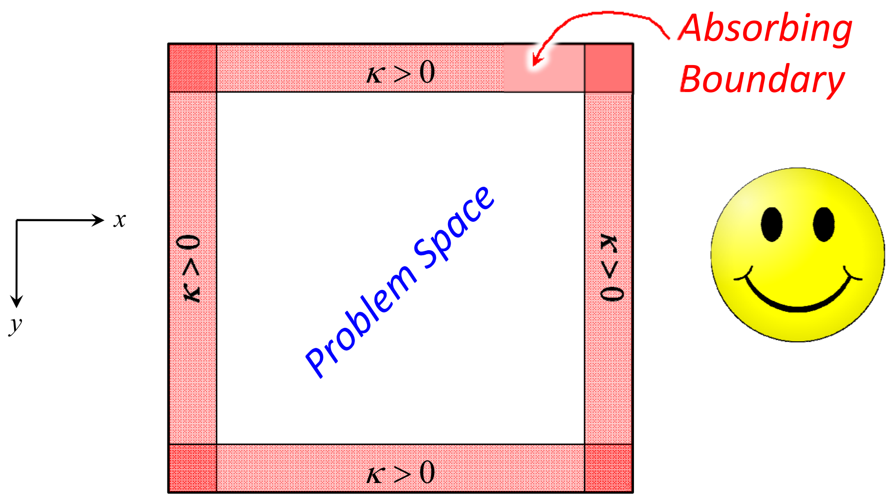


Lecture 9 Slide 14

Absorbing Boundary Conditions

CEM

We can introduce loss at the boundaries of the grid!

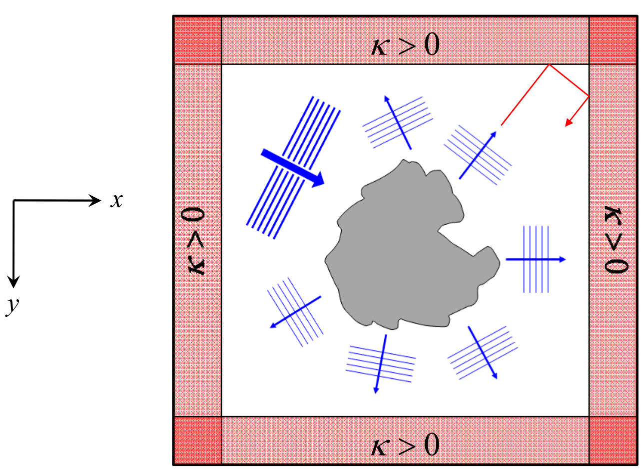


Lecture 9 Slide 15

Oops!!

CEM

But if we introduce loss, we also introduce reflections from the lossy regions!!



$$R = \frac{(1-n)^2 + \kappa^2}{(1+n)^2 + \kappa^2}$$

Lecture 9 Slide 16

Match the Impedance

We need to introduce loss to absorb outgoing waves, but we also need to match the impedance to the problem space to prevent reflections.

introduce loss here

$$\tilde{\varepsilon}_r = \varepsilon_r' + j\varepsilon_r''$$

adjust this to control impedance

Lecture 9 Slide 17

More Trouble?

By examining the Fresnel equations, we see that we can only prevent reflections from the interface at one frequency, one angle of incident, and one polarization.

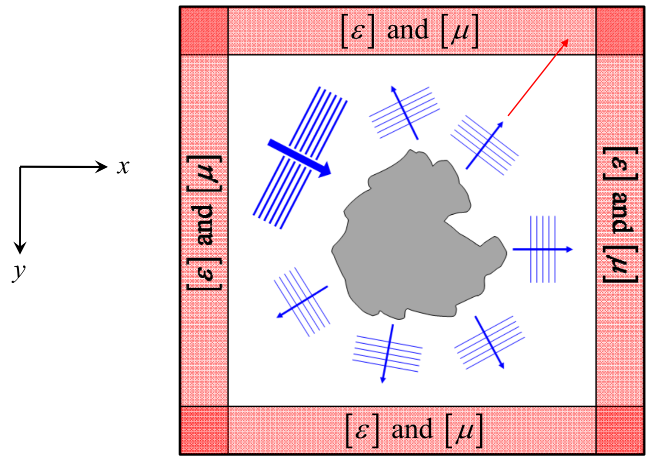
$$r_{\mathrm{TE}} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2} = 0 \quad \rightarrow \quad \eta_2 = \eta_1 \frac{\cos\theta_2}{\cos\theta_1}$$

$$r_{\mathrm{TM}} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2} = 0 \quad \rightarrow \quad \eta_2 = \eta_1 \frac{\cos\theta_1}{\cos\theta_2}$$

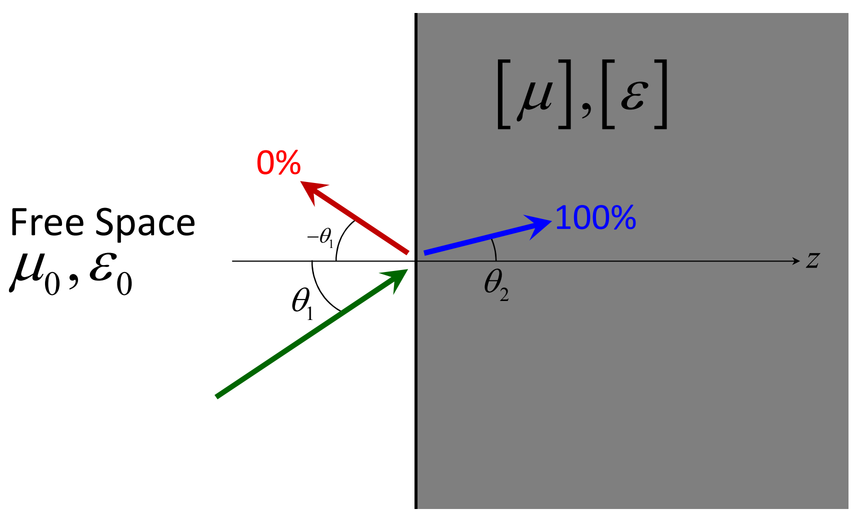
Lecture 9 Slide 18

Anisotropy to the Rescue!!

It turns out we can prevent reflections at all angles and for all polarizations if we allow our absorbing material to be doubly-diagonally anisotropic.



Problem Statement for the PML



Designing Anisotropy for Zero Reflection (1 of 3)

We need to perfectly match the impedance of the grid to the impedance of the absorbing region.

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \qquad \text{everywhere}$$

One easy way to ensure impedance is perfectly matched is:

$$[\mu_r] = [\varepsilon_r] = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

Lecture 9 Slide 21

Designing Anisotropy for Zero Reflection (2 of 3)

If we choose $\sqrt{bc} = 1$, then the refraction equation reduces to

$$\sin\theta_1 = \sqrt{bc}\sin\theta_2 = \sin\theta_2 \quad \rightarrow \quad \theta_1 = \theta_2$$

No refraction*!*

The reflection coefficients now reduce to

$$r_{\text{TE}} = \frac{\sqrt{a}\cos\theta_1 - \sqrt{b}\cos\theta_2}{\sqrt{a}\cos\theta_1 + \sqrt{b}\cos\theta_2} = \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}}$$

$$r_{\text{TM}} = \frac{-\sqrt{a}\cos\theta_1 + \sqrt{b}\cos\theta_2}{\sqrt{a}\cos\theta_1 + \sqrt{b}\cos\theta_2} = \frac{-\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}}$$

These are no longer a function of angle*!!* ☺

Lecture 9 Slide 22

Designing Anisotropy for Zero Reflection (3 of 3)

If we further choose $a = b$, the reflection equations reduce to

$$r_{\text{TE}} = \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}} = 0$$

$$r_{\text{TM}} = \frac{-\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}} = 0$$

Reflection will always be zero regardless of frequency, angle of incidence, or polarization!! ☺

Recall the necessary conditions: $\boxed{\sqrt{bc} = 1 \quad \text{and} \quad a = b}$

The PML Parameters (1 of 3)

So far, we have

$$[\mu_r] = [\varepsilon_r] = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \qquad a = b = \frac{1}{c}$$

Thus, we can write our PML in terms of just one parameter s_z.

$$[S_z] = \begin{bmatrix} s_z & 0 & 0 \\ 0 & s_z & 0 \\ 0 & 0 & s_z^{-1} \end{bmatrix} \qquad s_z = \alpha - j\beta$$

This form of tensor is why we call this a uniaxial PML.

This is for a wave travelling in the $+z$ direction incident on a z-axis boundary.

The PML Parameters (2 of 3)

We potentially want a PML along all the borders.

$$[S_x]=\begin{bmatrix} s_x^{-1} & 0 & 0 \\ 0 & s_x & 0 \\ 0 & 0 & s_x \end{bmatrix} \qquad [S_y]=\begin{bmatrix} s_y & 0 & 0 \\ 0 & s_y^{-1} & 0 \\ 0 & 0 & s_y \end{bmatrix} \qquad [S_z]=\begin{bmatrix} s_z & 0 & 0 \\ 0 & s_z & 0 \\ 0 & 0 & s_z^{-1} \end{bmatrix}$$

These can be combined into a single tensor quantity.

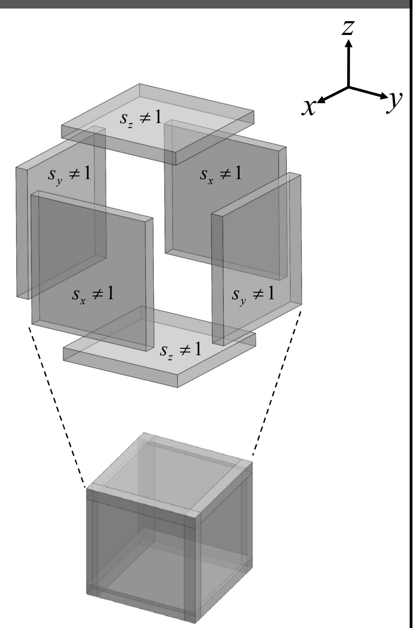
$$[S]=[S_x]\cdot[S_y]\cdot[S_z]=\begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix}$$

Lecture 9 Slide 25

The PML Parameters (3 of 3)

The 3D PML can be visualized this way…

$$[S]=\begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix}$$



Lecture 9 Slide 26

UPML in Cylindrical and Spherical Coordinates

Cylindrical Coordinates

$$[S]=\begin{bmatrix} \frac{\tilde{\rho}}{\rho}\frac{s_z}{s_\rho} & 0 & 0 \\ 0 & \frac{\rho}{\tilde{\rho}}s_z s_\rho & 0 \\ 0 & 0 & \frac{\tilde{\rho}}{\rho}\frac{s_\rho}{s_z} \end{bmatrix}$$

Spherical Coordinates

$$[S]=\begin{bmatrix} \left(\frac{\tilde{r}}{r}\right)^2 \frac{1}{s_r} & 0 & 0 \\ 0 & s_r & 0 \\ 0 & 0 & s_r \end{bmatrix}$$

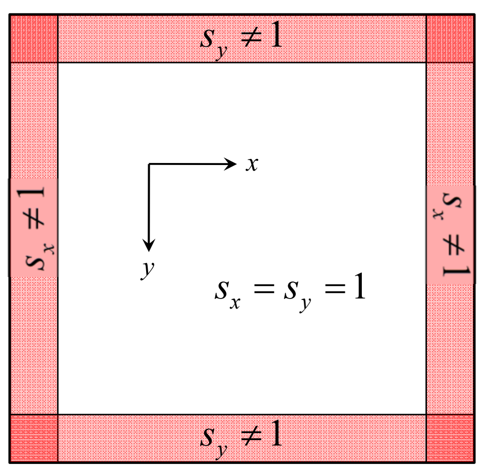
F. L. Teixeira, W. C. Chew, "Systematic Derivation of Anisotropic PML Absorbing Media in Cylindrical and Spherical Coordiantes," IEEE Microwave and Guided Wave Letters, Vol. 7, No. 11, pp. 371-373, 1997.

Lecture 9 Slide 27

Two-Dimensional UPML

For 2D simulations in the x-y plane, sz = 1 and the UPML tensor reduces to

$$[S]=\begin{bmatrix} \frac{s_y}{s_x} & 0 & 0 \\ 0 & \frac{s_x}{s_y} & 0 \\ 0 & 0 & s_x s_y \end{bmatrix}$$



Lecture 9 Slide 28

Incorporating a UPML into Maxwell's Equations

Lecture 9 Slide 29

Incorporating the UPML Into Maxwell's Eqs.

Maxwell's Equations

This set of equations does includes devices, but no UPML at the boundary to absorb outgoing waves.

$$\nabla \times \vec{E} = k_0 [\mu_r] \vec{\tilde{H}}$$
$$\nabla \times \vec{\tilde{H}} = k_0 [\varepsilon_r] \vec{E}$$

UPML

This set of equations includes the UPML to absorb outgoing waves, but does not include devices or real materials.

$$\nabla \times \vec{E} = k_0 [S] \vec{\tilde{H}}$$
$$\nabla \times \vec{\tilde{H}} = k_0 [S] \vec{E}$$

Maxwell's Equations with UPML

$$\nabla \times \vec{E} = k_0 [\mu_r][S] \vec{\tilde{H}}$$
$$\nabla \times \vec{\tilde{H}} = k_0 [\varepsilon_r][S] \vec{E}$$

This approach incorporates the PML in a way that is independent of the materials. It keeps the PML impedance matched to the background materials automatically.

Lecture 9 Slide 30

Maxwell's Equations with a UPML

Maxwell's equations with a UPML

$$\nabla\times\vec{E}=k_0[\mu_r][S]\vec{\tilde{H}}\qquad \nabla\times\vec{\tilde{H}}=k_0[\varepsilon_r][S]\vec{E}$$

$$[\varepsilon_r]=\begin{bmatrix}\varepsilon_{xx}&\varepsilon_{xy}&\varepsilon_{xz}\\\varepsilon_{yx}&\varepsilon_{yy}&\varepsilon_{yz}\\\varepsilon_{zx}&\varepsilon_{zy}&\varepsilon_{zz}\end{bmatrix}\qquad [\mu_r]=\begin{bmatrix}\mu_{xx}&\mu_{xy}&\mu_{xz}\\\mu_{yx}&\mu_{yy}&\mu_{yz}\\\mu_{zx}&\mu_{zy}&\mu_{zz}\end{bmatrix}$$

The UPML can be incorporated into the material tensors directly.

$$\nabla\times\vec{E}=k_0[\mu_r']\vec{\tilde{H}}\qquad \nabla\times\vec{\tilde{H}}=k_0[\varepsilon_r']\vec{E}$$

$$[\mu_r']=[\mu_r][S]\qquad [\varepsilon_r']=[\varepsilon_r][S]$$

$$[S]=\begin{bmatrix}\frac{s_ys_z}{s_x}&0&0\\0&\frac{s_xs_z}{s_y}&0\\0&0&\frac{s_xs_y}{s_z}\end{bmatrix}$$

This let's us formulate and implement a numerical algorithm without having to explicitly consider the PML. It is simply incorporated into the material tensors.

Vector Expansion

Assuming only diagonal tensors

$$[\varepsilon_r]=\begin{bmatrix}\varepsilon_{xx}&0&0\\0&\varepsilon_{yy}&0\\0&0&\varepsilon_{zz}\end{bmatrix}\qquad [\mu_r]=\begin{bmatrix}\mu_{xx}&0&0\\0&\mu_{yy}&0\\0&0&\mu_{zz}\end{bmatrix}$$

Maxwell's equations expand to

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=k_0\mu_{xx}\frac{s_ys_z}{s_x}\tilde{H}_x\qquad \frac{\partial \tilde{H}_z}{\partial y}-\frac{\partial \tilde{H}_y}{\partial z}=k_0\varepsilon_{xx}\frac{s_ys_z}{s_x}E_x$$

$$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=k_0\mu_{yy}\frac{s_xs_z}{s_y}\tilde{H}_y\qquad \frac{\partial \tilde{H}_x}{\partial z}-\frac{\partial \tilde{H}_z}{\partial x}=k_0\varepsilon_{yy}\frac{s_xs_z}{s_y}E_y$$

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=k_0\mu_{zz}\frac{s_xs_y}{s_z}\tilde{H}_z\qquad \frac{\partial \tilde{H}_y}{\partial x}-\frac{\partial \tilde{H}_x}{\partial y}=k_0\varepsilon_{zz}\frac{s_xs_y}{s_z}E_z$$

Absorb UPML into μ and ε (3D Grid)

We can absorb the UPML parameters into the material functions.

$$\mu'_{xx} = \mu_{xx}\frac{s_y s_z}{s_x} \qquad \varepsilon'_{xx} = \varepsilon_{xx}\frac{s_y s_z}{s_x}$$

$$\mu'_{yy} = \mu_{yy}\frac{s_x s_z}{s_y} \qquad \varepsilon'_{yy} = \varepsilon_{yy}\frac{s_x s_z}{s_y}$$

$$\mu'_{zz} = \mu_{zz}\frac{s_x s_y}{s_z} \qquad \varepsilon'_{zz} = \varepsilon_{zz}\frac{s_x s_y}{s_z}$$

We can now write Maxwell's equations as

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = k_0 \mu'_{xx} \tilde{H}_x \qquad \frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = k_0 \varepsilon'_{xx} E_x$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = k_0 \mu'_{yy} \tilde{H}_y \qquad \frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} = k_0 \varepsilon'_{yy} E_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = k_0 \mu'_{zz} \tilde{H}_z \qquad \frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = k_0 \varepsilon'_{zz} E_z$$

This means we can formulate a code as if there was no PML. All we have to do is modify the materials being modeled near the boundaries.

Lecture 9 Slide 33

Absorb UPML into μ and ε (2D Grid)

Let z be the uniform direction, then $d/dz = 0$ and $s_z = 1$.

We can still absorb the UPML parameters into the material functions.

$$\mu'_{xx} = \mu_{xx}\frac{s_y}{s_x} \qquad \varepsilon'_{xx} = \varepsilon_{xx}\frac{s_y}{s_x}$$

$$\mu'_{yy} = \mu_{yy}\frac{s_x}{s_y} \qquad \varepsilon'_{yy} = \varepsilon_{yy}\frac{s_x}{s_y}$$

$$\mu'_{zz} = \mu_{zz} s_x s_y \qquad \varepsilon'_{zz} = \varepsilon_{zz} s_x s_y$$

We can now write Maxwell's equations as

E Mode

$$\frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = k_0 \varepsilon'_{zz} E_z$$

$$\frac{\partial E_z}{\partial y} = k_0 \mu'_{xx} \tilde{H}_x$$

$$-\frac{\partial E_z}{\partial x} = k_0 \mu'_{yy} \tilde{H}_y$$

H Mode

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = k_0 \mu'_{zz} \tilde{H}_z$$

$$\frac{\partial \tilde{H}_z}{\partial y} = k_0 \varepsilon'_{xx} E_x$$

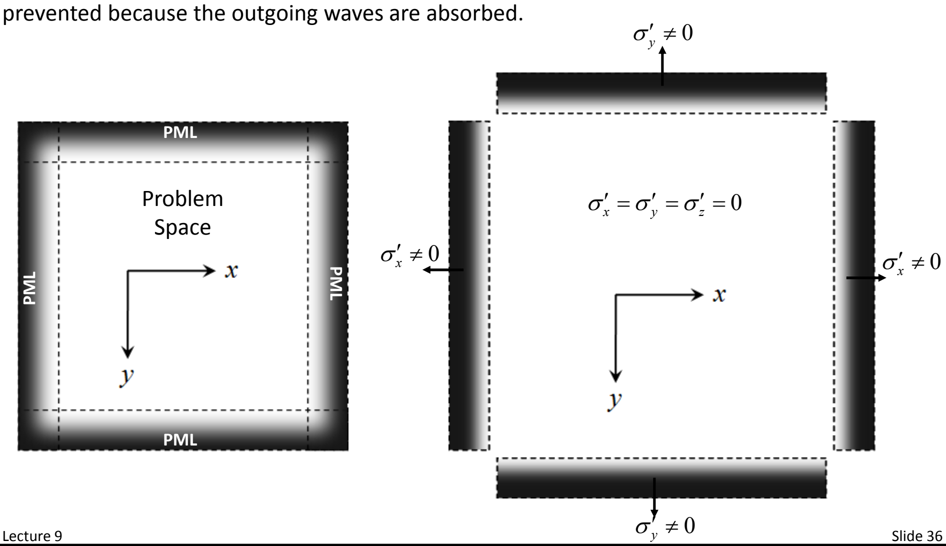
$$-\frac{\partial \tilde{H}_z}{\partial x} = k_0 \varepsilon'_{yy} E_y$$

Lecture 9 Slide 34

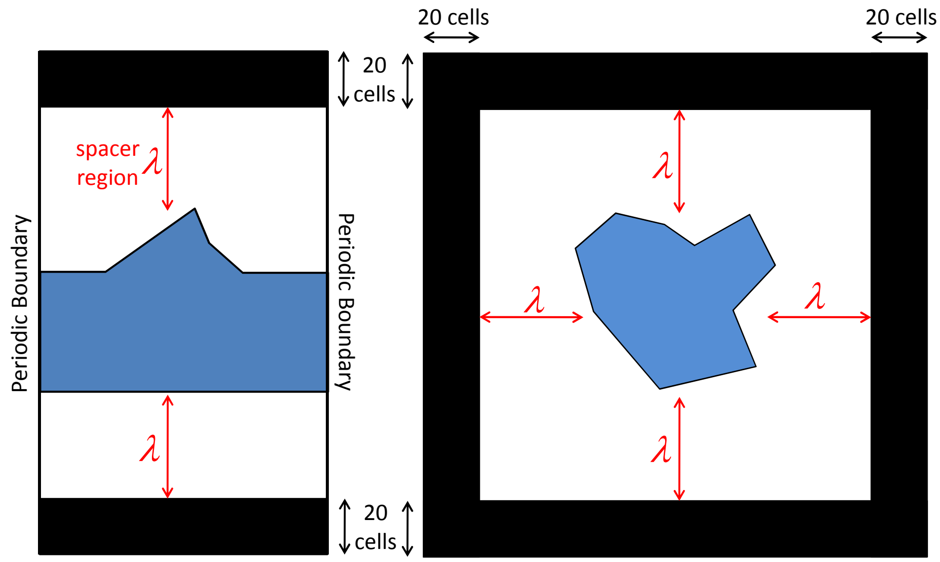
Implementing the PML

The Perfectly Matched Layer (PML)

The perfectly matched layer (PML) is an absorbing boundary condition (ABC) where the impedance is perfectly matched to the problem space. Reflections entering the lossy regions are prevented because impedance is matched. Reflections from the grid boundary are prevented because the outgoing waves are absorbed.



Typical Grid Schemes



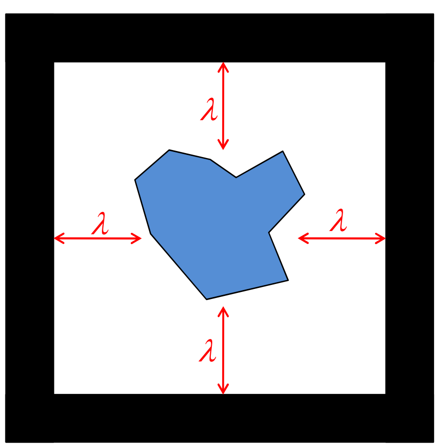
Periodic Devices

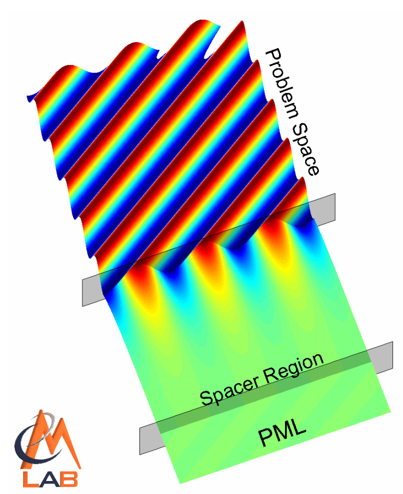
Finite Devices

Lecture 9 Slide 37

Justification for the Spacer Regions

The refractive index is high inside the PML so evanescent waves can become propagating waves, giving an escape path for power.





Lecture 9 Slide 38

How to Calculate the PML Parameters

Maxwell's Eqs. with PML

$$\nabla \times \vec{E} = k_0 [\mu_r][s]\vec{\tilde{H}}$$

$$\nabla \times \vec{\tilde{H}} = k_0 [\varepsilon_r][s]\vec{E}$$

$$[s] = \begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix}$$

Computing PML Parameters

$$s_x(x) = a_x(x)\left[1 + j\eta_0 \sigma'_x(x)\right]$$

$$s_y(y) = a_y(y)\left[1 + j\eta_0 \sigma'_y(y)\right]$$

$$s_z(z) = a_z(z)\left[1 + j\eta_0 \sigma'_z(z)\right]$$

$\eta_0 = 376.73... \equiv$ free space impedance

$$a_x(x) = 1 + a_{\max} \cdot (x/L_x)^p$$

$$a_y(y) = 1 + a_{\max} \cdot (y/L_y)^p$$

$$a_z(z) = 1 + a_{\max} \cdot (z/L_z)^p$$

$$\sigma'_x(x) = \sigma'_{\max} \sin^2\left(\frac{\pi x}{2L_x}\right)$$

$$\sigma'_y(y) = \sigma'_{\max} \sin^2\left(\frac{\pi y}{2L_y}\right)$$

$$\sigma'_z(z) = \sigma'_{\max} \sin^2\left(\frac{\pi z}{2L_z}\right)$$

$$0 \le a_{\max} \le 5$$

$$3 \le p \le 5$$

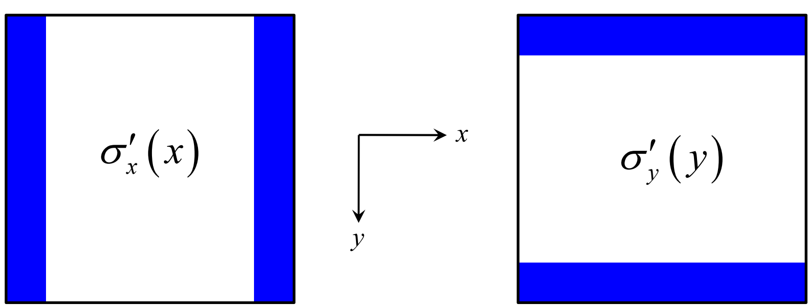
$$\sigma'_{\max} \approx 1$$

```
NGRID = [Nx Ny];
NPML  = [0 0 20 20];
[sx,sy] = calcpml2d(NGRID,NPML);
```

Writing this function will be in homework ☺

Visualizing the PML Loss Terms – 2D

For best performance, the loss terms should increase gradually into the PMLs.

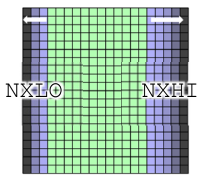


Procedure for Calculating s_x and s_y on a 2D Grid

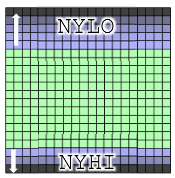
1. Initialize s_x and s_y to all ones.

$$s_x(x,y) = s_y(x,y) = 1$$

2. Fill in x-axis PML regions using two `for` loops.



3. Fill in y-axis PML regions using two `for` loops.



Lecture 9 Slide 41

Note About x/L_x, y/L_y, and z/L_z

The following ratios provide a single quantity that goes from 0 to 1 as you move through a PML region.

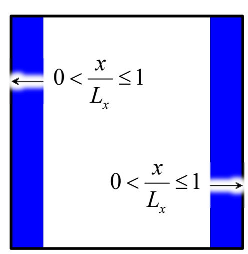
$$\frac{x}{L_x} \text{ and } \frac{y}{L_y} \text{ and } \frac{z}{L_z} \qquad x,y,z \equiv \text{position within PML} \quad L_x, L_y, L_z \equiv \text{size of PML}$$

We can calculate the same ratio using integer indices from our grid.

$$\frac{x}{L_x} \approx \frac{\text{nx}}{\text{NXLO}} \text{ or } \frac{\text{nx}}{\text{NXHI}} \qquad \begin{matrix} \text{nx} = 1, 2, \ldots, \text{NXLO} \\ \text{nx} = 1, 2, \ldots, \text{NXHI} \end{matrix}$$

$$\frac{y}{L_y} \approx \frac{\text{ny}}{\text{NYLO}} \text{ or } \frac{\text{ny}}{\text{NYHI}} \qquad \begin{matrix} \text{ny} = 1, 2, \ldots, \text{NYLO} \\ \text{ny} = 1, 2, \ldots, \text{NYHI} \end{matrix}$$

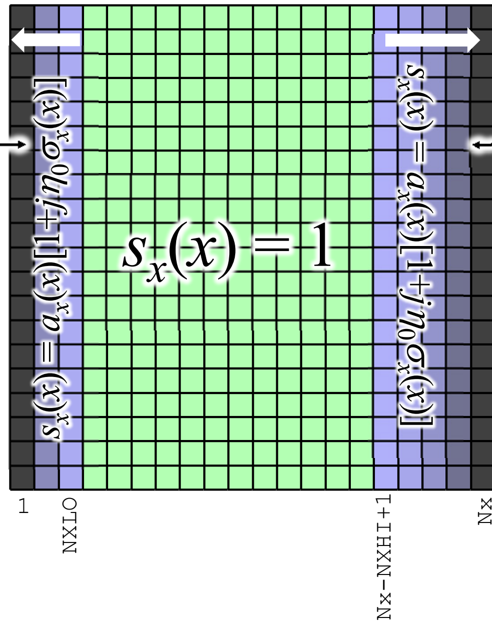
$$\frac{z}{L_z} \approx \frac{\text{nz}}{\text{NZLO}} \text{ or } \frac{\text{nz}}{\text{NZHI}} \qquad \begin{matrix} \text{nz} = 1, 2, \ldots, \text{NZLO} \\ \text{nz} = 1, 2, \ldots, \text{NZHI} \end{matrix}$$



Lecture 9 Slide 42

Visualizing s_x in 2D

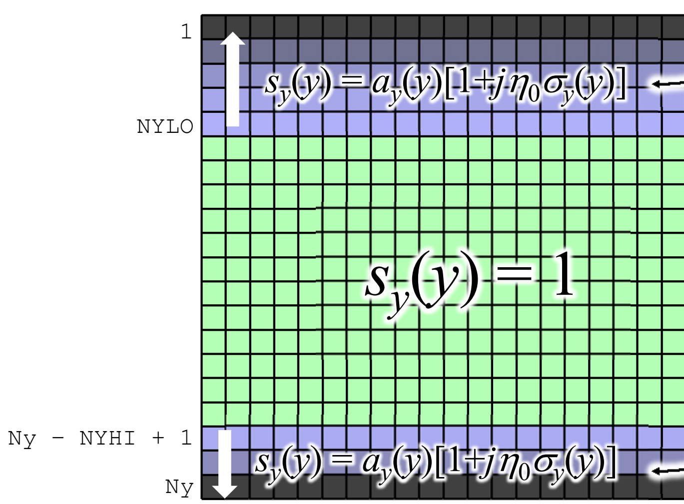
```
% ADD XLO PML
for nx = 1 : NXLO
    …
    sx(NXLO-nx+1,:) = ...
end
```




```
% ADD XHI PML
for nx = 1 : NXHI
    …
    sx(Nx-NXHI+nx,:) = ...
end
```

Lecture 9 Slide 43

Visualizing s_y in 2D




```
% ADD YLO PML
for ny = 1 : NYLO
    …
    sy(:,NYLO-ny+1) = ...
end
```

```
% ADD YHI PML
for ny = 1 : NYHI
    …
    sy(:,Ny-NYHI+ny) = …
end
```

Lecture 9 Slide 44

Example Data for 2D

```
NGRID   = [7 4];
NPML    = [2 3 1 2];
[sx,sy] = calcpml2d(NGRID,NPML);
```

$a_{max} = 3$
$p = 3$
$\sigma'_{max} = 1$

```
sx =  1.0e+03 *

   0.0040 + 1.5069i   0.0040 + 1.5069i   0.0040 + 1.5069i   0.0040 + 1.5069i
   0.0014 + 0.2590i   0.0014 + 0.2590i   0.0014 + 0.2590i   0.0014 + 0.2590i
   0.0010             0.0010             0.0010             0.0010
   0.0010             0.0010             0.0010             0.0010
   0.0011 + 0.1046i   0.0011 + 0.1046i   0.0011 + 0.1046i   0.0011 + 0.1046i
   0.0019 + 0.5337i   0.0019 + 0.5337i   0.0019 + 0.5337i   0.0019 + 0.5337i
   0.0040 + 1.5069i   0.0040 + 1.5069i   0.0040 + 1.5069i   0.0040 + 1.5069i


sy =  1.0e+03 *

   0.0040 + 1.5069i   0.0010             0.0014 + 0.2590i   0.0040 + 1.5069i
   0.0040 + 1.5069i   0.0010             0.0014 + 0.2590i   0.0040 + 1.5069i
   0.0040 + 1.5069i   0.0010             0.0014 + 0.2590i   0.0040 + 1.5069i
   0.0040 + 1.5069i   0.0010             0.0014 + 0.2590i   0.0040 + 1.5069i
   0.0040 + 1.5069i   0.0010             0.0014 + 0.2590i   0.0040 + 1.5069i
   0.0040 + 1.5069i   0.0010             0.0014 + 0.2590i   0.0040 + 1.5069i
   0.0040 + 1.5069i   0.0010             0.0014 + 0.2590i   0.0040 + 1.5069i
```


PML is Not a Boundary Condition

A numerical boundary condition is the rule you follow when an equation references a field from outside the grid.

The PML does not address this issue.

It is simply a way of incorporating loss while preventing reflections so as to absorb outgoing waves.

Sometimes it is called an absorbing boundary condition, but this is still misleading as **the PML is not a true boundary condition**.

Stretched Coordinate Perfectly Matched Layer (SC-PML)

The Uniaxial PML

Maxwell's equations with uniaxial PML are:

$$\nabla \times \vec{E} = k_0 [\mu_r][S]\vec{\tilde{H}} \qquad \nabla \times \vec{\tilde{H}} = k_0 [\varepsilon_r][S]\vec{E}$$

$$[S] = \begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix}$$

Rearrange the Terms

We can bring the PML tensor to the left side of the equations and associate it with the curl operator.

$$[S]^{-1}\nabla\times\vec{E}=k_0[\mu_r]\vec{\tilde{H}} \qquad [S]^{-1}\nabla\times\vec{\tilde{H}}=k_0[\varepsilon_r]\vec{E}$$

The curl operator is now

$$[S]^{-1}\nabla\times=\begin{bmatrix} s_z^{-1}s_y^{-1}s_x & 0 & 0 \\ 0 & s_z^{-1}s_y s_x^{-1} & 0 \\ 0 & 0 & s_z s_y^{-1}s_x^{-1}\end{bmatrix}\begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0\end{bmatrix}$$

$$=\begin{bmatrix} 0 & -\frac{s_x}{s_y}\left(\frac{1}{s_z}\frac{\partial}{\partial z}\right) & \frac{s_x}{s_z}\left(\frac{1}{s_y}\frac{\partial}{\partial y}\right) \\ \frac{s_y}{s_x}\left(\frac{1}{s_z}\frac{\partial}{\partial z}\right) & 0 & -\frac{s_y}{s_z}\left(\frac{1}{s_x}\frac{\partial}{\partial x}\right) \\ -\frac{s_z}{s_x}\left(\frac{1}{s_y}\frac{\partial}{\partial y}\right) & \frac{s_z}{s_y}\left(\frac{1}{s_x}\frac{\partial}{\partial x}\right) & 0\end{bmatrix}$$

"Stretched" Coordinates

Our new curl operator is

$$[S]^{-1}\nabla\times=\begin{bmatrix} 0 & -\frac{s_x}{s_y}\left(\frac{1}{s_z}\frac{\partial}{\partial z}\right) & \frac{s_x}{s_z}\left(\frac{1}{s_y}\frac{\partial}{\partial y}\right) \\ \frac{s_y}{s_x}\left(\frac{1}{s_z}\frac{\partial}{\partial z}\right) & 0 & -\frac{s_y}{s_z}\left(\frac{1}{s_x}\frac{\partial}{\partial x}\right) \\ -\frac{s_z}{s_x}\left(\frac{1}{s_y}\frac{\partial}{\partial y}\right) & \frac{s_z}{s_y}\left(\frac{1}{s_x}\frac{\partial}{\partial x}\right) & 0\end{bmatrix}$$

The factors s_x, s_y, and s_z are effectively "stretching" the coordinates, but they are "stretching" into a complex space.

Drop the Other Terms

We drop the non-stretching terms.

$$\nabla_s \times = \begin{bmatrix} 0 & -\cancel{\frac{s_x}{s_y}}\left(\frac{1}{s_z}\frac{\partial}{\partial z}\right) & \cancel{\frac{s_x}{s_z}}\left(\frac{1}{s_y}\frac{\partial}{\partial y}\right) \\ \cancel{\frac{s_y}{s_x}}\left(\frac{1}{s_z}\frac{\partial}{\partial z}\right) & 0 & -\cancel{\frac{s_y}{s_z}}\left(\frac{1}{s_x}\frac{\partial}{\partial x}\right) \\ -\cancel{\frac{s_z}{s_x}}\left(\frac{1}{s_y}\frac{\partial}{\partial y}\right) & \cancel{\frac{s_z}{s_y}}\left(\frac{1}{s_x}\frac{\partial}{\partial x}\right) & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{s_z}\frac{\partial}{\partial z} & \frac{1}{s_y}\frac{\partial}{\partial y} \\ \frac{1}{s_z}\frac{\partial}{\partial z} & 0 & -\frac{1}{s_x}\frac{\partial}{\partial x} \\ -\frac{1}{s_y}\frac{\partial}{\partial y} & \frac{1}{s_x}\frac{\partial}{\partial x} & 0 \end{bmatrix}$$

<u>Justification</u>

$$\frac{s_y}{s_x}\left(\frac{1}{s_z}\frac{\partial}{\partial z}\right) = \frac{1}{s_z}\frac{\partial}{\partial z}$$

Inside the z-PML, $s_x = s_y = 1$. This is valid everywhere except at the extreme corners of the grid where the PMLs overlap.

This also implies that the UPML and SC-PML have nearly identical performance in terms of reflections, sensitivity to angle of incidence, polarization, etc.

Lecture 9 Slide 51

Maxwell's Equations with a SC-PML

Maxwell's equations before the PML is added are

$$\nabla \times \vec{E} = k_0 [\mu_r] \vec{\tilde{H}} \qquad \nabla \times \vec{\tilde{H}} = k_0 [\varepsilon_r] \vec{E}$$

$$[\varepsilon_r] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \qquad [\mu_r] = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix}$$

The SC-PML is incorporated as follows.

$$\nabla_s \times \vec{E} = -j\omega [\mu] \vec{H}$$
$$\nabla_s \times \vec{H} = j\omega [\varepsilon] \vec{E}$$

$$\nabla_s \times = \begin{bmatrix} 0 & -\frac{1}{s_z}\frac{\partial}{\partial z} & \frac{1}{s_y}\frac{\partial}{\partial y} \\ \frac{1}{s_z}\frac{\partial}{\partial z} & 0 & -\frac{1}{s_x}\frac{\partial}{\partial x} \\ -\frac{1}{s_y}\frac{\partial}{\partial y} & \frac{1}{s_x}\frac{\partial}{\partial x} & 0 \end{bmatrix}$$

Lecture 9 Slide 52

Vector Expansion

Maxwell's equations with a SC-PML expand to

Fully Anisotropic

$$\frac{1}{s_y}\frac{\partial \tilde{H}_z}{\partial y}-\frac{1}{s_z}\frac{\partial \tilde{H}_y}{\partial z}=k_0\left(\varepsilon_{xx}E_x+\varepsilon_{xy}E_y+\varepsilon_{xz}E_z\right)$$

$$\frac{1}{s_z}\frac{\partial \tilde{H}_x}{\partial z}-\frac{1}{s_x}\frac{\partial \tilde{H}_z}{\partial x}=k_0\left(\varepsilon_{yx}E_x+\varepsilon_{yy}E_y+\varepsilon_{yz}E_z\right)$$

$$\frac{1}{s_x}\frac{\partial \tilde{H}_y}{\partial x}-\frac{1}{s_y}\frac{\partial \tilde{H}_x}{\partial y}=k_0\left(\varepsilon_{zx}E_x+\varepsilon_{zy}E_y+\varepsilon_{zz}E_z\right)$$

$$\frac{1}{s_y}\frac{\partial E_z}{\partial y}-\frac{1}{s_z}\frac{\partial E_y}{\partial z}=k_0\left(\mu_{xx}\tilde{H}_x+\mu_{xy}\tilde{H}_y+\mu_{xz}\tilde{H}_z\right)$$

$$\frac{1}{s_z}\frac{\partial E_x}{\partial z}-\frac{1}{s_x}\frac{\partial E_z}{\partial x}=k_0\left(\mu_{yx}\tilde{H}_x+\mu_{yy}\tilde{H}_y+\mu_{yz}\tilde{H}_z\right)$$

$$\frac{1}{s_x}\frac{\partial E_y}{\partial x}-\frac{1}{s_y}\frac{\partial E_x}{\partial y}=k_0\left(\mu_{zx}\tilde{H}_x+\mu_{zy}\tilde{H}_y+\mu_{zz}\tilde{H}_z\right)$$

Diagonally Anisotropic

$$\frac{1}{s_y}\frac{\partial \tilde{H}_z}{\partial y}-\frac{1}{s_z}\frac{\partial \tilde{H}_y}{\partial z}=k_0\varepsilon_{xx}E_x$$

$$\frac{1}{s_z}\frac{\partial \tilde{H}_x}{\partial z}-\frac{1}{s_x}\frac{\partial \tilde{H}_z}{\partial x}=k_0\varepsilon_{yy}E_y$$

$$\frac{1}{s_x}\frac{\partial \tilde{H}_y}{\partial x}-\frac{1}{s_y}\frac{\partial \tilde{H}_x}{\partial y}=k_0\varepsilon_{zz}E_z$$

$$\frac{1}{s_y}\frac{\partial E_z}{\partial y}-\frac{1}{s_z}\frac{\partial E_y}{\partial z}=k_0\mu_{xx}\tilde{H}_x$$

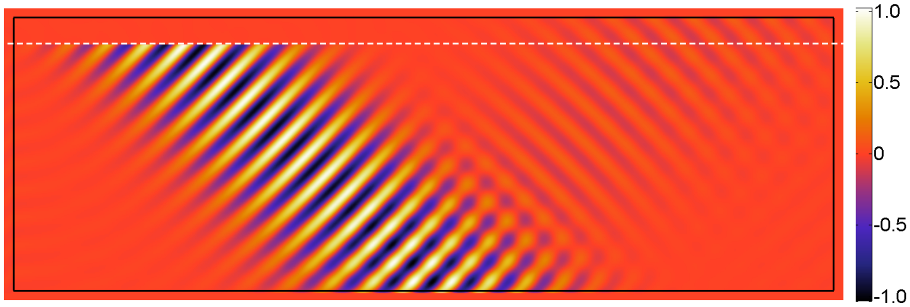
$$\frac{1}{s_z}\frac{\partial E_x}{\partial z}-\frac{1}{s_x}\frac{\partial E_z}{\partial x}=k_0\mu_{yy}\tilde{H}_y$$

$$\frac{1}{s_x}\frac{\partial E_y}{\partial x}-\frac{1}{s_y}\frac{\partial E_x}{\partial y}=k_0\mu_{zz}\tilde{H}_z$$

PML Performance

PMLs Are Not Perfect

PML absorbing boundary conditions are not perfect absorbers. They still reflecte waves!



Theoretical Performance

Given the following choice of PML parameters

$$\nabla_s = \frac{1}{s_x}\frac{\partial}{\partial x}\hat{a}_x + \frac{1}{s_y}\frac{\partial}{\partial y}\hat{a}_y + \frac{1}{s_z}\frac{\partial}{\partial z}\hat{a}_z$$

$$s_x(x) = 1 + j\frac{\sigma_x(x)}{\omega\varepsilon_0} \qquad \sigma_x(x) = \sigma_{x,\max}\cdot\left(\frac{x}{L_x}\right)^m$$

$$s_y(y) = 1 + j\frac{\sigma_y(y)}{\omega\varepsilon_0} \qquad \sigma_y(y) = \sigma_{y,\max}\cdot\left(\frac{y}{L_y}\right)^m$$

$$s_z(z) = 1 + j\frac{\sigma_z(z)}{\omega\varepsilon_0} \qquad \sigma_z(z) = \sigma_{z,\max}\cdot\left(\frac{z}{L_z}\right)^m$$

We choose $\sigma_{i,\max}$ to achieve a target maximum reflectance R at normal incidence according to

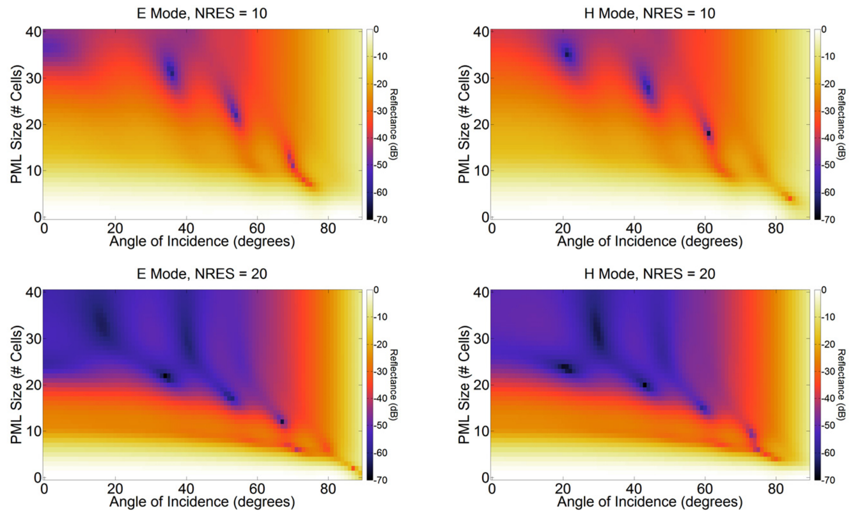
$$\sigma_{i,\max} = -\frac{(m+1)\ln R}{2\eta_0 L_i}$$

We typically choose

$$3 \le m \le 4$$

$$\sigma_{i,\max} \approx \frac{4}{\eta_0 \Delta_i}$$

UPML Performance in FDFD



UPML performance if affected by] and its size.
H mode UPML exhibits slightly poorer performance.

$$a_{\max} = 3$$
$$p = 3$$
$$\sigma'_{\max} = 1$$

UPML Vs. SC-PML

UPML Vs. SC-PML

Uniaxial PML

Benefits

- Has a physical interpretation
- Models can be formulated and implemented without considering the PML in the frequency-domain

Drawbacks

- Can be more computationally intensive to implement in time-domain
- Resulting matrices are less well conditioned in the frequeny-domain

Stretched-Coordinate PML

Benefits

- Less computationally intensive in time-domain
- More efficient implementation in the time-domain
- Matrices are better conditioned.

Drawbacks

- Must be accounted for in the formulation and implementation of the numerical method.
- Not intuitive to understand