CS 341: Algorithms
Module 8: Intractability and Undecidability

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Spring 2019

Subset Sum

Subset Sum Problem

Instance: A list of sizes $S = [s_1, \cdots, s_n]$; and a **target sum**, T. These are all positive integers.
Question: Does there exist a subset $\mathrm{J} \subseteq \{1, \cdots, n\}$ such that $\sum_{i \in J} s_i = T$?

Vertex Cover $\leq_P$ Subset Sum

Suppose $\mathrm{I} = (G, k)$, where $G = (V, E)$, $|V| = n$, $|E| = m$ and $1 \leq k \leq n$.

Suppose $V = \{v_1, \cdots, v_n\}$ and $E = \{e_0, \cdots, e_{m-1}\}$. For $1 \leq i \leq n$, $0 \leq j \leq m-1$, let $C = (c_{ij})$, where

$$c_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$

Define $n + m$ sizes and a target sum W as follows:

$$\begin{aligned} a_i &= 10^m + \sum_{j=0}^{m-1} c_{ij} 10^j \quad (1 \leq i \leq n) \\ b_j &= 10^j \quad (0 \leq j \leq m-1) \\ W &= k \cdot 10^m + \sum_{j=0}^{m-1} 2 \cdot 10^j \end{aligned}$$

Then define $f(\mathrm{I}) = (a_1, \cdots, a_n, b_0, \cdots, b_{m-1}, W)$.

Correctness of the Transformation

Suppose I is a yes-instance of Vertex Cover. There is a vertex cover $V' \subseteq V$ such that $|V'| = k$. For $i = 1, 2$, let E^i denote the edges having exactly i vertices in V'. Then $E = E^1 \cup E^2$ because V' is a vertex cover.

Let

$$A' = \{a_i : v_i \in V'\} \quad \text{and} \quad B' = \{b_j : e_j \in E^1\}.$$

The sum of the sizes in A' is

$$k \cdot 10^m + \sum_{\{j : e_j \in E^1\}} 10^j + \sum_{\{j : e_j \in E^2\}} 2 \times 10^j.$$

The sum of the sizes in B' is

$$\sum_{\{j : e_j \in E^1\}} 10^j.$$

Therefore the sum of all the chosen sizes is

$$k \cdot 10^m + \sum_{\{j : e_j \in E\}} 2 \times 10^j = k \cdot 10^m + \sum_{j=1}^{m} 2 \times 10^j = W.$$

Correctness of the Transformation (cont.)

Conversely, suppose $f(\mathrm{I})$ is a yes-instance of Subset Sum. We show that I is a yes-instance of Vertex Cover. Let $A' \cup B'$ be the subset of chosen sizes. Define $V' = \{v_i : a_i \in A'\}$. We claim that V' is a vertex cover of size k. In order for the coefficient of 10^m to be equal to k, we must have $|V'| = k$ (there can't be any carries occurring). The coefficient of any other term $10^j (0 \leq j \leq m-1)$ must be equal to 2. Suppose that $e_j = v_i v_i'$. There are two possible situations that can occur:

1. a_i and a_i' are both in A'. Then V' contains both vertices incident with e_j.
2. exactly one of a_i or a_i' is in A' and $b_j \in B'$. In this case, V' contains exactly one vertex incident with e_j.

In both cases, e_j is incident with at least one vertex in V'.

Subset Sum $\leq_P$ 0-1 Knapsack

Let I be an instance of Subset Sum consisting of sizes $[s_1, \cdots, s_n]$ and target sum T.
Define

$$\begin{aligned} p_i &= s_i, \;\; 1 \leq i \leq n \\ w_i &= s_i, \;\; 1 \leq i \leq n \\ M &= T \end{aligned}$$

Then define $f(\mathrm{I})$ to be the instance of 0-1 Knapsack consisting of profits $[p_1, \cdots, p_n]$, weights $[w_1, \cdots, w_n]$, capacity M and target profit T.
Exercise: Prove the correctness of this transformation.

Hamiltonian Cycle $\leq_P$ TSP-Dec

Let I be an instance of Hamiltonian Cycle consisting of a graph $G = (V, E)$.
For the complete graph K_n, where $n = |V|$, define edge weights as follows:

$$\mathrm{w}(\mathrm{uv}) = \begin{cases} 1 & \text{if } \mathrm{uv} \in E \\ 2 & \text{if } \mathrm{uv} \notin E \,. \end{cases}$$

Then define $f(\mathrm{I})$ to be the instance of TSP-Dec consisting of the graph K_n, edge weights w and target $T = n$.
Exercise: Prove the correctness of this transformation.

Summary of Polynomial Transformations

$$\begin{tikzcd}
 & CNF - SAT \arrow[d] & \\
 & 3 - CNF - SAT \arrow[d] & \\
 & Clique \arrow[d] & \\
 & VertexCover \arrow[dl] \arrow[dr] & \\
SubsetSum \arrow[d] & & HamiltonianCycle \arrow[d] \\
0 - 1 Knapsack & & TSP - Decision
\end{tikzcd}$$

In the above diagram, arrows denote polynomial transformations. The transformation Vertex Cover $\leq_P$ Hamiltonian Cycle is complicated.

NP-hard Problems

A problem $\prod$ is NP-hard if there exists a problem $\prod' \in NPC$ such that $\prod' \leq_P^T \prod$.

Every NP-complete problem is automatically NP-hard, but there exist NP-hard problems that are not NP-complete.

Typical examples of NP-hard problems are optimization problems corresponding to NP-complete decision problems.

For example, TSP-Decision $\leq_P^T$ TSP-Optimization and TSP-Decision $\in NPC$, so TSP-Optimization is NP-hard.

This is a "trivial" Turing reduction.