# CS 341: Algorithms Module 8: Intractability and Undecidability

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Spring 2019

#### Subset Sum

#### Subset Sum Problem

**Instance:** A list of sizes  $S = [s_1, \dots, s_n]$ ; and a **target sum**, T.

These are all positive integers.

**Question:** Does there exist a subset  $J\subseteq\{1,\cdots,n\}$  such that

 $\sum_{i\in J} s_i = T$ ?

# Vertex Cover $\leq_P$ Subset Sum

Suppose I = (G, k), where G = (V, E), |V| = n, |E| = m and  $1 \le k \le n$ . Suppose  $V = \{v_1, \dots, v_n\}$  and  $E = \{e_0, \dots, e_{m-1}\}$ . For 1 < i < n,  $0 < j \le m-1$ , let  $C = (c_{ii})$ , where

$$c_{ij} = egin{cases} 1 & ext{if } e_j ext{ is incident with } v_i \ 0 & ext{otherwise} \end{cases}$$

Define n + m sizes and a target sum W as follows:

$$a_i = 10^m + \sum_{j=0}^{m-1} c_{ij} 10^j \quad (1 \le i \le n)$$
 $b_j = 10^j \quad (0 \le j \le m-1)$ 
 $W = k \cdot 10^m + \sum_{j=0}^{m-1} 2 \cdot 10^j$ 

Then define  $f(I) = (a_1, \dots, a_n, b_0, \dots, b_{m-1}, W)$ .

#### Correctness of the Transformation

Suppose I is a yes-instance of Vertex Cover. There is a vertex cover  $V' \subseteq V$  such that |V'| = k. For i = 1, 2, let  $E^i$  denote the edges having exactly i vertices in V'. Then  $E = E^1 \cup E^2$  because V' is a vertex cover.

Let

$$A' = \{a_i : v_i \in V'\} \text{ and } B' = \{b_j : e_j \in E^1\}.$$

The sum of the sizes in A' is

$$k \cdot 10^m + \sum_{\{j: e_j \in E^1\}} 10^j + \sum_{\{j: e_j \in E^2\}} 2 \times 10^j$$
.

The sum of the sizes in B' is

$$\sum_{\{j:e_j\in E^1\}} 10^j.$$

Therefore the sum of all the chosen sizes is

$$k \cdot 10^m + \sum_{\{j: e_i \in E\}} 2 \times 10^j = k \cdot 10^m + \sum_{j=1}^m 2 \times 10^j = W.$$

# Correctness of the Transformation (cont.)

Conversely, suppose f(I) is a yes-instance of Subset Sum. We show that I is a yes-instance of Vertex Cover. Let  $A' \cup B'$  be the subset of chosen sizes. Define  $V' = \{v_i : a_i \in A'\}$ . We claim that V' is a vertex cover of size k. In order for the coefficient of  $10^m$  to be equal to k, we must have |V'| = k (there can't be any carries occurring). The coefficient of any other term  $10^j (0 \le j \le m-1)$  must be equal to 2. Suppose that  $e_j = v_i v_i'$ . There are two possible situations that can occur:

- $a_i$  and  $a'_i$  are both in A'. Then V' contains both vertices incident with  $e_j$ .
- ② exactly one of  $a_i$  or  $a_i'$  is in A' and  $b_j \in B'$ . In this case, V' contains exactly one vertex incident with  $e_i$ .

In both cases,  $e_i$  is incident with at least one vertex in V'.

# Subset Sum $\leq_P 0-1$ Knapsack

Let I be an instance of Subset Sum consisting of sizes  $[s_1, \dots, s_n]$  and target sum T. Define

$$p_i = s_i, \quad 1 \le i \le n$$
  
 $w_i = s_i, \quad 1 \le i \le n$   
 $M = T$ 

Then define f(I) to be the instance of 0-1 Knapsack consisting of profits  $[p_1, \dots, p_n]$ , weights  $[w_1, \dots, w_n]$ , capacity M and target profit T.

**Exercise:** Prove the correctness of this transformation.

# Hamiltonian Cycle $\leq_P$ TSP-Dec

Let I be an instance of Hamiltonian Cycle consisting of a graph G = (V, E).

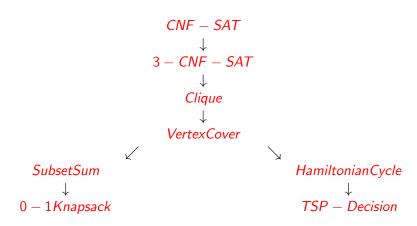
For the complete graph  $K_n$ , where n = |V|, define edge weights as follows:

$$w(uv) = \begin{cases} 1 & \text{if } uv \in E \\ 2 & \text{if } uv \notin E \end{cases}.$$

Then define f(I) to be the instance of TSP-Dec consisting of the graph  $K_n$ , edge weights w and target T = n.

**Exercise:** Prove the correctness of this transformation.

# Summary of Polynomial Transformations



In the above diagram, arrows denote polynomial transformations. The transformation Vertex Cover  $\leq_P$  Hamiltonian Cycle is complicated.

#### NP-hard Problems

A problem  $\prod$  is NP-hard if there exists a problem  $\prod' \in NPC$  such that  $\prod' \leq_P^T \prod$ .

Every NP-complete problem is automatically NP-hard, but there exist NP-hard problems that are not NP-complete.

Typical examples of NP-hard problems are optimization problems corresponding to NP-complete decision problems.

For example, TSP-Decision  $\leq_P^T$  TSP-Optimization and TSP-Decision  $\in$  NPC, so TSP-Optimization is NP-hard.

This is a "trivial" Turing reduction.