CS 341: Algorithms
Module 5: Greedy Algorithms

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Based on lecture notes by many previous CS 341 instructors

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Optimization Problems

Problem: Given a problem instance, find a feasible solution that maximizes (or minimizes) a certain objective function.

Problem Instance: Input for the specified problem.

Problem Constraints: Requirements that must be satisfied by any feasible solution.

Feasible Solution: For any problem instance I, $feasible(I)$ is the set of all outputs (i.e., solutions) for the instance I that satisfy the given constraints.

Objective Function: A function $f : feasible(I) \rightarrow \mathbb{R}^+ \bigcup 0$. We often think of f as being a profit or a cost function.

Optimal Solution: A feasible solution $X \in feasible(I)$ such that the profit $f(X)$ is maximized (or the cost $f(X)$ is minimized).

The Greedy Method

partial solution

Given a problem instance I, it should be possible to write a feasible solution X as a tuple $[x_1, x_2, \ldots x_n]$ for some integer n. A tuple $[x_1, x_2, \ldots x_i]$ where $i < n$ is a partial solution if no constraints are violated.

Note: it may be the case that a partial solution cannot be extended to a feasible solution.

choice set

For a partial solution $X = [x_1, x_2, \ldots x_i]$ where $i < n$, we define the choice set

$$choice(X) = \{y \in \mathcal{X} : [x_1, x_2, \ldots x_i, y] \text{ is a partial solution}\}.$$

The Greedy Method (cont.)

local evaluation criterion

A local evaluation criterion is a function g such that, for any partial solution $X = [x_1, ..., x_i]$ and any $y \in choice(X)$, $g(x_1, ..., x_i, y)$ measures the cost or profit of extending the partial solution X to include y.

extension

Given a partial solution $X = [x_1, x_2, \ldots x_i]$ where $i < n$, choose $y \in choice(X)$ so that $g(y)$ is as small (or large) as possible. Update X to be the $(i+1)$-tuple $[x_1, x_2, \ldots x_i, y]$.

greedy algorithm

Starting with the "empty" partial solution, repeatedly extend it until a feasible solution X is constructed. This feasible solution may or may not be optimal.

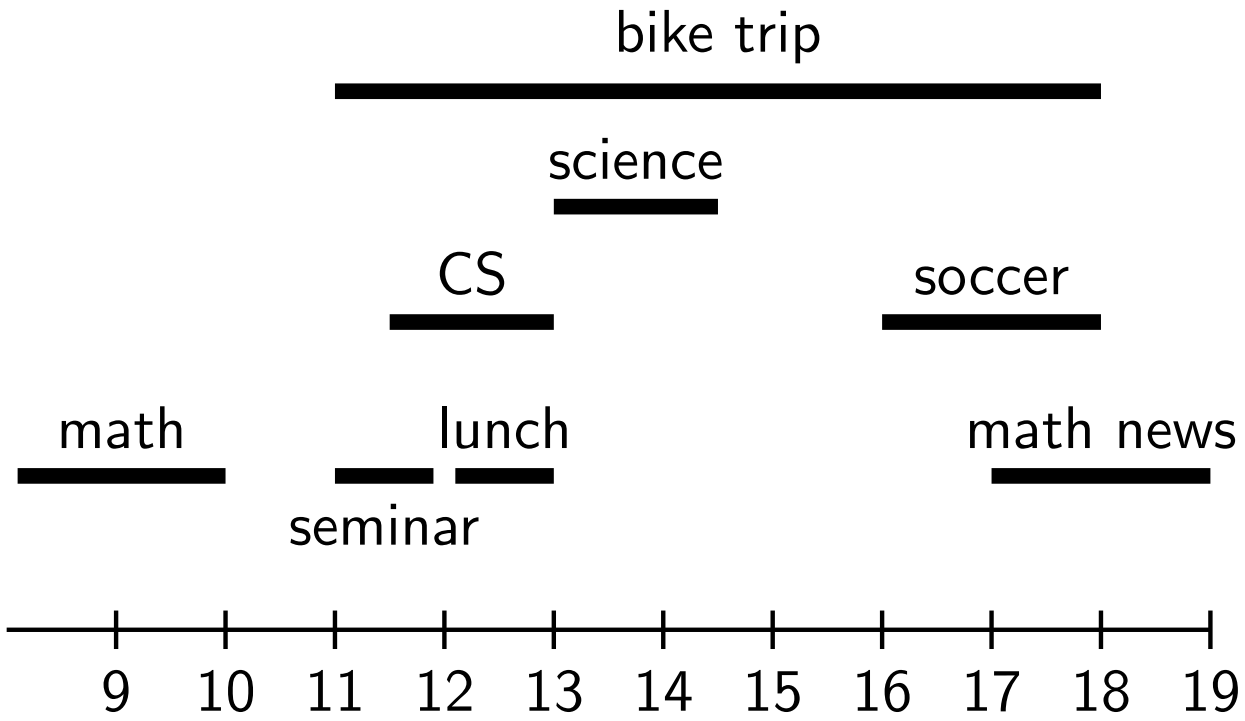
For a greedy algorithm to be efficient, we need a fast way to find the best extension.
Note that for a greedy algorithm to be "correct", it has to find the optimal solution for every problem instance.

Interval Scheduling or "Activity Selection"

Problem

Given a set of activities, each with a specified time interval, select a maximum set of disjoint (non-intersecting) intervals.

Example:



There are several possible greedy approaches.

- select activity that starts earliest (find counterexample).
- select the shortest interval (find counterexample).
- select the interval with fewest conflict (find counterexample).
- select the interval that ends earliest.

In our example we get math, seminar, lunch, science, soccer.

Algorithm 1: ActivitySelection

```
1 Sort activities 1..n by end time
2 A ← ∅
3 for i = 1..n
4     if {activity i does not overlap with any previous
5        (check the last one) activity in A } then
6         A ← A ∪ {i}
7 return(A)
```

Correctness

Lemma

ActivitySelection algorithm returns a max size set A of disjoint intervals.

Proof

Let $A = \{a_1, \ldots, a_k\}$ sorted by end time.
compare to an optimum solution $B = \{b_1, \ldots, b_\ell\}$ ordered by end time. Thus $\ell \geq k$ and we want to prove $\ell = k$.
Idea: at every step we can do better with a's.
Claim: $\forall i$, $a_1, \ldots, a_i, b_{i+1}, \ldots, b_\ell$ is an opt. solution.

Correctness (cont.)

Proof (cont.)

Proof by induction:
basis: $i = 1$. a_1 had earliest end time of all intervals so,

$$end(a_1) \leq end(b_1).$$

Hence replacing b_1 by a_1 gives disjoint intervals.
Induction step: suppose $a_1, \ldots, a_{i-1}, b_i, \ldots, b_\ell$ is an opt. solution. b_i does not intersect a_{i-1} so the greedy algorithm could have chosen it. Instead, it chose a_i so

$$end(a_i) \leq end(b_i)$$

and replacing b_i by a_i leaves disjoint intervals.
To finish proving the lemma note that if $k < \ell$ then $a_1, \ldots, a_k, b_{k+1}, \ldots, b_\ell$ is an opt. solution. But then the greedy algorithm had more choices after a_k.

Coin Changing

Problem

Instance: A list of *coin denominations*, $d_1, d_2, \ldots, d_n$, and a positive integer T, which is called the *target sum*.
Question: Find n-tuple of non-negative integers, say $A = [a_1, \ldots, a_n]$, such that $T = \sum_{i=1}^{n} a_i d_i$, and such that $N = \sum_{i=1}^{n} a_i$ is minimized.

In the Coin Changing problem, a_i denotes the number of coins of denomination d_i that are used, for $i = 1, \ldots, n$. The total value of all the chosen coins must be exactly equal to T. We want to **minimize** the number of coins used, which is denoted by N.

Divide and Conquer? How?

Algorithm 2: GreedyCoins1 (What's wrong?)

```
1  rename the coins, by sorting if necessary, so that d_1 > ... > d_n and set
     all a_i to 0;
2  i := 1; N := 0;
3  while T > 0 & i ≤ n do
4      if T > d_i then
5          T := T − d_i; a_i++; N++;
6      else
7          i++;
8  if T > 0 then
9      return (FAIL)
10 else
11     return ([a_1, ..., a_n], N)
```

GreedyCoins2

Algorithm 3: GreedyCoins2

1 rename the coins, by sorting if necessary, so that $d_1 > \ldots > d_n$;
2 $N := 0$;
3 **for** $i = 1$ **to** n **do**
4 $\quad a_i := \left\lfloor \frac{T}{d_i} \right\rfloor$;
5 $\quad T := T - a_i d_i$;
6 $\quad N := N + a_i$;
7 **if** $T > 0$ **then**
8 $\quad$ **return** (FAIL)
9 **else**
10 $\quad$ **return** $([a_1, \ldots, a_n], N)$

We will generally assume that there is a coin with value 1, so we can make change for any T. For some sets of coins, the greedy algorithm always gives the correct solution. Some examples:

1. If $d_j | d_{j-1}$ for all $j, 2 \leq j \leq n$, then greedy = optimal.
2. If $D = [200, 100, 25, 10, 5, 1]$ (Canadian coin system, including pennies) then greedy = optimal.
3. If $D = [30, 24, 12, 6, 3, 1]$ (old English system), then (sometimes) greedy $\neq$ optimal. For example, suppose $T = 48$. The greedy algorithm yields $30, 12, 6$ whereas the optimal solution is $24, 24$.

Features of the Greedy Method

Greedy algorithms do no looking ahead and no backtracking.
Greedy algorithms can usually be implemented efficiently. Often they consist of a preprocessing step based on the function g, followed by a single pass through the data.
In a greedy algorithm, only one feasible solution is constructed.
The execution of a greedy algorithm is based on local criteria (i.e., the values of the function g).
Correctness: For certain greedy algorithms, it is possible to prove that they always yield optimal solutions. However, these proofs can be tricky and complicated!