CS 341: Algorithms
Module 6: Dynamic Programming

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Based on lecture notes by many previous CS 341 instructors

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Integer Knapsack

Problem specification:
We are given n objects and a knapsack.
Each object i has a positive weight w_i and a positive value v_i.
The knapsack can carry a weight not exceeding W. Fill the knapsack so that the value of objects in the knapsack is maximized.
Brute force:
Try all possibilities. An object can be in or out and we sum weights to be sure we are not over W. This has complexity $\Theta(n2^n)$.
Greedy:
At each step add the object with the highest v_i/w_i ratio. Does not work. Counterexample?

Integer Knapsack - DP

Recall that objects are numbered from 1 to n.

Definition of a subproblem

Let $V[i, j]$ be the maximum value of the objects, selected from the first i objects, that can fit into a knapsack with upper weight limit j (the optimal value will be found in $V[n, W]$).

Key observation:

We either use object i in the optimal solution or we do not.

Suppose object i is not in the Knapsack. Then there is no difference between $V[i-1, j]$ and $V[i, j]$.

Suppose object i is in the Knapsack. Our claim, for this case, is that $V[i, j] = V[i-1, j-w_i] + v_i$.

Consider an optimal selection extracted from the first $i-1$ objects with a weight limitation of $j - w_i$.

Integer Knapsack: Derivation of the Recurrence

Looking at only these first $i-1$ objects, we can assume we have an optimal selection that is not more valuable than those chosen from the first $i-1$ objects as used in $V[i,j]$.

This is true because:

A more valuable selection from objects 1 to $i-1$ could be extended with object i and we would get a total value in excess of $V[i,j]$ in contradiction of the fact that $V[i,j]$ is optimal. So the value of $V[i,j]$ must be v_i plus the optimal solution for the first $i-1$ objects with a weight limitation of $j-w_i$.

Considering the above facts we are able to make up the following recurrence for $V[i,j]$:

$$V[i,j] = \max\{V[i-1,j], v_i + V[i-1,j-w_i]\}$$

Base case: $V[0,j] = 0$.

Order of computation:

Use row-order from top-left down to the bottom-right corner.

Knapsack Problem: Pseudo-code for DP

```
for j := 0 to W do
  V[0,j]:=0;
for i := 1 to n do
  for j := 1 to W do
    sol := V[i-1, j];
    if (w[i] <= j) then
      othersol := V[i-1, j-w[i]] + v[i];
      if (othersol > sol) then
        sol := othersol;
    V[i, j] := sol;
return V[n, W];
```

Complexity? $\Theta(nW)$. Is it good or bad???

Integer Knapsack: Notes on Pseudo-code

Note

We can make the program more memory efficient.
Note that to compute value $V[i,j]$, we need only the cells from the previous line and to the left of $V[i-1,j]$ (including $V[i-1,j]$).

```
for j := 0 to W do
  V[j] := 0;
for i := 1 to n do
  for j := W downto 1 do
    sol := V[j];
    if (w[i] <= j) then
      othersol := V[j-w[i]] + v[i];
      if (othersol > sol) then
        sol := othersol;
    V[j] := sol;
return V[W];
```

Integer Knapsack: Notes on Pseudo-code

More simplifications..

```
for j := 0 to W do
  V[j] := 0;
for i := 1 to n do
  for j := W downto 1 do
    if (w[i] <= j) then
      othersol := V[j-w[i]] + v[i];
      if (othersol > V[j]) then
        V[j] := othersol;
return V[W];
```

Integer Knapsack: Notes on Pseudo-code

Recovery of the solution added

```
for j := 0 to W do
   V[j] := 0; D[j] := 0;
for i := 1 to n do
  for j := W downto 1 do
    if (w[i] <= j) then
      othersol := V[j-w[i]] + v[i];
      if (othersol > V[j]) then
        V[j] := othersol; D[j]:= i;
print V[W];
\\ recover the items in knapsack
j:=W;
while (j>0) and (D[j]>0) do
  print(D[j]); j:=j-w[D[j]];
```

Minimum Length Triangulation

Problem 4.4

Minimum Length Triangulation v1
Instance: n points $q_1, \cdots, q_n$ in the Euclidean plane that form a convex $n - gon$ P.
Find: A triangulation of P such that the sum S_c of the lengths of the $n - 3$ chords is minimized.

Problem 4.5

Minimum Length Triangulation v2
Instance: n points $q_1, \cdots, q_n$ in the Euclidean plane that form a convex $n - gon$ P.
Find: A triangulation of P such that the sum S_p of the perimeters of the $n - 2$ triangles is minimized.

Let L denote the perimeter of P. Then we have that $S_p = L + 2S_c$. Hence the two versions have the **same optimal solutions**.

Problem Decomposition

We consider version 2 of the problem.
The edge $q_n q_1$ is in a triangle with a third vertex q_k, where $k \in 2, \cdots, n-1$.
For a given k, we have:

1. the triangle $q_1 q_k q_n$,
2. the polygon with vertices $q_1, \cdots, q_k$,
3. the polygon with vertices $q_k, \cdots, q_n$.

The optimal solution will consist of optimal solutions to the **two subproblems** in (2) and (3), along with the triangle in (1).

Recurrence Relation

For $1 \leq i < j \leq n$, let $S[i, j]$ denote the optimal solution to the subproblem consisting of the polygon having vertices $q_i, \cdots, q_j$.
Let $\Delta(q_i, q_k, q_j)$ denote the perimeter of the triangle having vertices q_i, q_k, q_j.
Then we have the recurrence relation

$$S[i, j] = \min \{\Delta(q_i, q_k, q_j) + S[i, k] + S[k, j] : i < k < j\}$$

the base cases are given by

$$S[i, i+1] = 0$$

for all i.
We compute all $S[i, j]$ with $j - i = c$, for $c = 2, 3, \cdots, n-1$.

Weighted Interval Scheduling

Problem 4.6

Problem: Weighted Interval Scheduling.
Instance: A set I of n intervals $[s_1, f_1], \cdots, [s_n, f_n]$ with weights $\omega_1, \cdots, \omega_n$.
Question: Find subset S of disjoint intervals that maximizes $\sum_{i \in S} \omega_i$.

Greedy approach does not work (example?)

Denote: $OPT(I)$ - optimum set S; $\omega_{OPT(I)}$ - corresponding weight.
The structure of optimal solution:
Consider interval i: it is either in $OPT(I)$ or not.
If $i \in OPT(I)$ then $OPT(I) = \{i\} \bigcup OPT(I')$, where I' denotes intervals disjoint from i.
If $i \notin OPT(I)$ then $OPT(I) = OPT(I - \{i\})$. Therefore

$$\omega_{OPT(I)} = \max\left\{\omega_{OPT(I-\{i\})}, \omega_i + \omega_{OPT(I')}\right\}$$

Using this directly one ends up with exponential running time (solving subproblems for 2^n subsets of I).

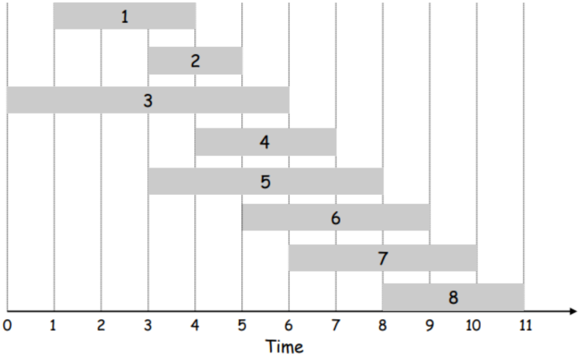
Rename the intervals, by sorting if necessary, so that $f_1 \leq f_2 \leq \cdots \leq f_n$.
Denote $p(j)$ the largest index $i < j$ such that interval i is disjoint from the interval j.

Let $opt(j)$ be the weight of optimal solution that considers intervals $1, 2, \cdots, j$.

Then $opt(0) = 0$ and

$$opt(j) = \max\{\omega_j + opt(p(j)), opt(j-1)\}$$

Ex: $p(8) = 5$, $p(7) = 3$, $p(2) = 0$.



j	p(j)
0	-
1	0
2	0
3	0
4	1
5	0
6	2
7	3
8	5

```
Sort intervals according to finish time
Compute p[j] for each j
opt[0]=0
for j from 1 to n
    opt[j]= max{opt[j-1], opt[p[j]]+w[j]}
Output opt[n]
```

Complexity?

Solution recovery ...

```
j = n
while (j>=0) do
if (opt[p[j]]+w[j] > opt[j-1])
print j
j = p[j]
else
j = j-1
```