CS 341: Algorithms
Module 7: Graph Algorithms

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Based on lecture notes by many previous CS 341 instructors

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Graph Algorithms

Graphs and their representation:

- A graph $G = (V, E)$ consists of a set of vertices $V = \{v_i \mid i = 1, 2, \cdots, n\}$ and a set of edges $E = \{e_j \mid j = 1, 2, \cdots, m\}$ that connect the vertices.
  - An edge e may be represented by the pair (u, v) where u and v are the vertices being connected by e.
    - Depending on the problem, the pair may be ordered or unordered.
- Many problems can be represented as graphs:
  - Traveling Salesman Problems
  - Airline flights
  - Friends who know each other
  - Moves in a game (each node= the state of the board or game).

Directed and Undirected Graphs

- An undirected graph, $G = (V, E)$ is a pair:
  - V = set of distinct vertices.
  - E = set of edges; each member is a set of 2 vertices
    - ⋆ For example:

$$V = \{t, u, v, w, x, y, z\}$$
$$E = \{\{u, v\}, \{u, w\}, \{v, w\}, \{v, y\}, \{x, z\}\}$$

- A directed graph, $G = (V, E)$:
  - Is the same as an undirected graph except E is a set of ordered pair:
    - ⋆ For example:

$$E = \{(a, b), (a, c), (c, b), (b, e), (e, b)\} \tag{1}$$

More Definitions

- Weighted graphs:
  - A weighted graph has value assigned to each of its edges.
  - More formally, there is a weight function $w : E \to R$.
    - $R =$ real numbers.
    - Depending on the syntax being used, we may see this as $w(e)$ with $e \in E$ or $w(u, v)$ with u and v representing the vertices for the edge.
- Degree:
  - The degree of vertex v, denoted by $\deg(v)$, is the number of edges that meet at v.
  - Let v be a vertex in a directed graph G, The number of vertices of G adjacent **from** v is called the outdegree of v. the number of vertices of G adjacent **to** v is called the indegree of v.

Representations of Graphs

- Adjacency matrix:
  - $M[i,j] = 1$ if i and j are neighbours, 0 otherwise.
  - Assign each vertex an integer index (in this example, $t = 7$, $u = 2$, etc.)
  - Assumes that any other info for a vertex and/or edge is in another data structure.
  - The matrix is symmetric for an undirected graph.
  - For a directed graph we will likely have an asymmetric matrix.

	u	v	w	x	y	z	t
u		1	1				
v	1		1		1		
w	1	1					
x						1	
y		1					
z				1			
t							

Representations of Graphs

- Adjacency matrix notes:
  - Blank row: no neighbours i.e. isolated vertex.
  - $M[i, i] =$ self-loop.
  - Undirected graphs are symmetrical
- Space
  - $|V|^2$ bits
  - $(|V|^2 + |V|)/2$ (if undirected, but harder to actually implement).
  - Additional information, such as cost of an edge, could be stored in the matrix. Another option is to store a pointer to this information.
- Cost of operations
  - Are vertices i and j adjacent? $O(1)$
  - Add or delete edge: $O(1)$
  - Add vertex: increases size of matrix.
  - Find neighbours of v: $O(|V|)$.

Representations of Graphs

- Adjacency list notes:
- Space:
  - $O(|V| + |E|)$
  - Usually much smaller than $O(|V|^2)$ for a sparse graph.
- Cost of operations:
  - Add an edge $O(1)$
  - Delete an edge: Search lists for each endpoint $O(|V|)$.
  - Add vertex: Depends on the implementation.
  - Find neighbours O(number of neighbours) (better than Adj. Matrix)
  - Are i, j adjacent? Search the list (worse than Adj. Matrix)

More Notation

- Path:
  - Given the graph $G(V, E)$, a path from vertex u to vertex v is a sequence of vertices $(v_0, v_1, \cdots, v_k)$ such that $v = v_0$, $u = v_k$, and (v_{i-1}, v_i) is in E for all $i = 1, 2, \cdots, k$.
- Simple path:
  - A path is simple if all the vertices in the path are distinct.
- Cycle:
  - A path $(v_0, v_1, \cdots, v_k)$ forms a cycle if $v_0 = v_k$, and the path contains at least one edge.
  - A graph with no cycles is acyclic.
- Connected graphs:
  - An undirected graph is connected if every pair of vertices is connected by a path.

BFS and DFS

- Breadth-first, depth-first search
- Each starts at an arbitrary node, explores the whole connected component
- Assume whole graph is connected
- General view: vertices start out coloured white (not visited)
- A visited vertex is coloured gray (visited, but may have white neighbours)
- When all neighbours of a vertex are visited, it is coloured black.

General view of searching

- Gray nodes form a “frontier”
- Can choose any neighbour of a gray node to be next visited
- In general, want to perform computation
  - Preprocess when colouring gray
  - Postprocess when colouring black
  - Analogous to tree traversal uses

Breadth-first search

- Use queue (first-in, first-out) to store gray nodes
- Start by taking any vertex, colouring it gray, add it to queue
- Repeat: find white node adjacent to head of queue, colour it gray and add it to queue
- When you can't find any such node, remove the head of the queue and colour it black

Pseudocode for BFS

```
colour_all_vertices_white()
while there is a white vertex s do
   BFS_tree(s)

BFS_tree(s: vertex)
  colour s gray (visited)
  enqueue(Q,s)
  while Q not empty do
     u <- dequeue(Q)
     for each v adjacent to u
        if v white then
           colour v gray
           enqueue(Q,v)
           \\(u,v) is a tree edge
        else
           \\(u,v) is non-tree edge
     colour u black
```

Analysis of BFS

- Assume adjacency list representation
- Each vertex enqueued once (colour changes from white to gray) and dequeued once (colour changes from gray to black)
- Body of inner while loop takes $\Theta(1)$ time
- Inner while loop implemented by scanning adjacency list of head of queue
- Therefore each edge looked at exactly twice
- Running time is $\Theta(|V| + |E|)$ or $\Theta(n + m)$

BFS trees

- BFS finds a spanning tree of the graph (edges representing first visits)
- Nontree edges are called cross edges
- A cross edge cannot connect a node to its ancestor in the tree
- A cross edge cannot connect a node to its descendant in the tree
- These are useful in proving properties of BFS searches

Single-source shortest path

- Can use BFS to compute distances $\delta(s, v)$ from source s to all other vertices v
- Compute quantity $d[v]$ in following way
  - $d[s] \leftarrow 0$
  - When adding v to queue because u is at head and there is an edge (u, v), set $d[v] \leftarrow d[u] + 1$

Pseudocode for shortest path

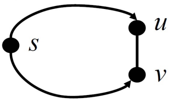
```
colour_all_vertices_white()
while there is a white vertex s do
   BFS_tree(s)

BFS_tree(s: vertex)
  colour s gray (visited)
  enqueue(Q,s);  d[s] <-- 0
  while Q not empty do
     u <- dequeue(Q)
     for each v adjacent to u
        if v white then
           colour v gray
           enqueue(Q,v); d[v] <-- d[u]+1
           \\(u,v) is a tree edge
        else
           \\(u,v) is non-tree edge
     colour u black
```

Why does this work?

Lemma 1

For any edge (u, v), we have $\delta(s, v) \leq \delta(s, u) + 1$



Lemma 2

For all v, $d[v] \geq \delta(s, v)$.

Proof: By induction on number of enqueues
At beginning, $d[s] = 0 = \delta(s, s)$ Suppose u is being visited and adjacent white v discovered.

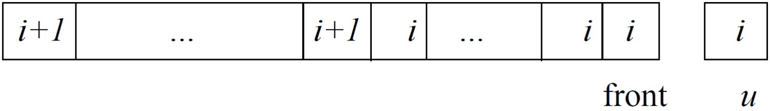
$$\delta(s, v) \leq \delta(s, u) + 1 \leq d[u] + 1 = d[v] \,.$$

Why does this work?

Lemma 2 proves $d[v] \geq \delta(s, v)$.

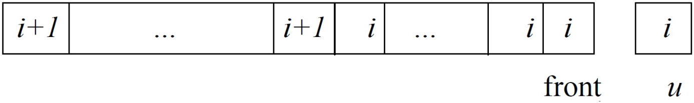
Lemma 3

At any point, the d-values of vertices in the queue are either i or $i + 1$ for some i, and all the i values are in front of all the $i + 1$ values.



Why does this work?

Proof: By induction on number of queue operations
True at beginning (only one item in queue)
True after $u \leftarrow \mathit{dequeue}(Q)$
True after $\mathit{enqueue}(Q, v)$, because $d[v] = d[u] + 1$.



Corollary 4

d values are assigned in increasing order.

Correctness continued

Theorem 5

$d[v] = \delta(s, v)$

Let v be closest vertex to s with wrong d;
$d[v] > \delta(s, v)$
Let u be vertex just before v on shortest $s - v$ path
$\delta(s, v) = \delta(s, u) + 1$, $d[u] = \delta(s, u)$, so $d[v] > d[u] + 1$.
What colour is v when u is dequeued?

- White? Then it should have been visited from u
- Black? Then u should have been visited from v
- Gray? Then it was visited from some w, $d[v] = d[w] + 1$, and $d[w] \leq d[u]$, which implies $d[v] \leq d[u] + 1$ contradicting above inequality.

Thus no such v exists.

BFS application

- Connected components via BFS ...
- Presence of cycles ...
- Bi-partite graphs ...
- ...