CS 341: Algorithms
Module 7: Graph Algorithms

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Based on lecture notes by many previous CS 341 instructors

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Depth-first search

- Idea: instead of a queue to store gray nodes, use a stack.
- Algorithm visits new (white) vertices before dealing with older gray ones.
- Hence it tends to explore deeply first.
- More valuable than BFS, especially for directed graphs.
- We add a timestamp of colour changes to indicate when node turned gray ($d[u]$) and black ($f[u]$).

Pseudocode for DFS

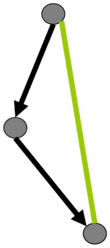
```
DFS (G)
  colour_all_vertices_white(); time<-0
  while there is a white vertex s do
     DFS_visit(s)
  done
DFS_visit(v)
  colour v gray; time++; d[v]<-time
  for each w adjacent to v do
     if w white then
        \\(v,w) tree edge
        DFS_visit(w)
     else
        \\(v,w) is non-tree edge
        colour v black; time++; f[v]<-time
```

Analysis of DFS

- Note stack is implicit here (stores parameters for recursive calls)
- "v on stack" means call to DFS-Visit(v) has not terminated
- DFS-Visit called once on every white node
- Each adjacency list run through once
- As with BFS, running time is $\Theta(|V| + |E|)$ or $\Theta(n + m)$.

DFS on undirected graphs

- Let (v, w) be an edge, $d[v] < d[w]$
- If w found first on v's adjacency list
  - w must have been white
  - (v, w) is a tree edge
- If v found first on w's adjacency list
  - v is gray
  - (v, w) is a back edge



Tree Edges and Back Edges

- DFS on an undirected graph:
  - For undirected graphs we have tree edges and all other edges not in the spanning tree are called back edges.

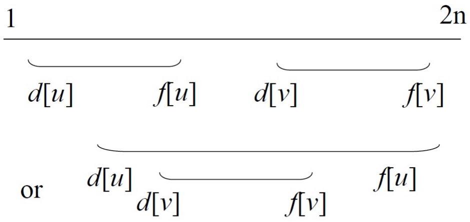
Absence of Cross Links

- Again, consider DFS on an undirected graph:
  - Let u and v be two vertices such that neither is a descendent of the other. Then there is no back edge between any descendent of u and any descendent of v.

The parenthesis theorem

Theorem 1

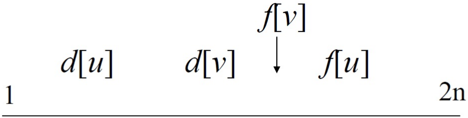
The intervals $[d[u], f[u]]$ and $[d[v], f[v]]$ are either nested (in which case the inner one is a descendant of the outer) or disjoint.



The parenthesis theorem

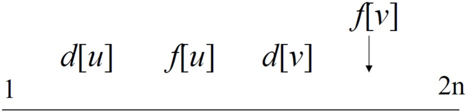
Proof: WLOG assume $d[u] < d[v]$,

- If $d[v] < f[u]$, v was discovered while u was gray (on the stack), so v is a descendant of u and $f[v] < f[u]$ (nested)



The parenthesis theorem

If $f[u] < d[v]$, then intervals are disjoint



Corollary 3

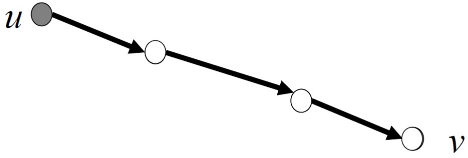
v is descendant of u if and only if

$$d[u] < d[v] < f[v] < f[u]$$

The white-path theorem

Theorem 2

v is a descendant of u if and only if at time $d[u]$, v is reachable by a white path from u



The white-path theorem

Proof: If v a descendant of u, by Corollary 3, every vertex on tree path from u to v has higher dvalue, so is white at time $d[u]$.
If v reachable by white path at time $d[u]$ but does not become descendant, assume every other vertex on path does (otherwise repeat argument for closest one to u that doesnt)

The white-path theorem

Predecessor w of v in path is descendant of u, so $f[w] \leq f[u]$ (w could be u)

$d[u] < d[v]$ (v white when u discovered)
$< f[w]$(v must be discovered before w finished)
$\leq f[u]$ (by above)

The white-path theorem

$d[u] < d[v] < f[w] \leq f[u]$ (from last slide)

Since $[d[v], f[v]]$ nested inside $[d[u], f[u]]$, the parenthesis theorem says that v is a descendant of u (contradiction to the assumption that it was not).

Articulations

- Definition:
  - A node v of a connected graph G is an **articulation** point (also called a cut vertex) if the removal of v and all its incident edges causes G to become disconnected.
- Motivation for articulations:
  - Articulations are important in communication networks.
  - In traffic flows they identify places which will stop traffic between two areas of a city if they become blocked.

Finding Articulations

- Problem:
  - Given any graph $G = (V, E)$, find all the articulation points.
- Possible strategy:
  - For all vertices v in V:
    - Remove v and its incident edges
    - Test connectivity using a DFS.
  - Execution time: $\Theta(n(n+m))$.
    - Can we do better?

Finding Articulation Points

- A DFS tree can be used to discover articulation points in $\Theta(n+m)$ time.
  - We start with a program that computes a DFS tree labeling the vertices with their discovery times.
  - We also compute a function called $low(v)$ that can be used to characterize each vertex as an articulation or non-articulation point.
    - The root of the DFS tree (the root has a $d[\,]$ value of 1) will be treated as a special case:

Finding Articulation Points

- The root of the DFS tree is an articulation point if and only if it has two children.
  - Suppose the root has two or more children.
    - Recall that the back edges never link the vertices in two different subtrees.
    - So, the subtrees are only linked through the root vertex and if it is removed we will get two or more connected components (i.e. the root is an articulation point).
  - Suppose the root is an articulation point.
    - This means that its removal would produce two or more connected components each previously connected to this root vertex.
    - So, the root has two or more children.

Computation of $low(v)$

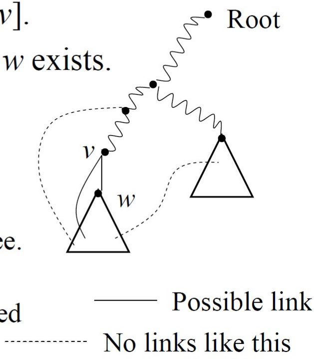
- We need another function defined on vertices: This quantity will be used in our articulation finding algorithm:

$low(v) = min\{d[v], d[w] \ : \ (u, w)$ is a back edge for some descendent u of v$\}$

- So, $low(v)$ is the discovery time of the vertex closest to the root and reachable from v by following zero or more edges downward, and then at most one back edge.

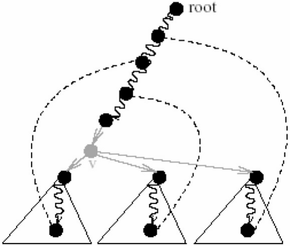
Finding Articulation Points

- For non-root vertices we have a different test.
  - Suppose v is a non-root vertex of the DFS tree T. Then v is an articulation point of G if and only if there is a child w of v in T with $low(w) \geq d[v]$.
  - Sufficiency: Assume such a child w exists.
    - There is no descendent vertex of v that has a back edge going "above" vertex v.
    - Also, there is no cross link from a descendent of v to any other subtree.
    - So, when v is removed the subtree with w as its root will be disconnected from the rest of the graph.



Finding Articulation Points

– Necessity: Assume no such child w exists.
  - In this case all children of v have a descendent with a back edge going to an ancestor of v.
  - When v is removed each of the children of v will still be connected to some vertex on the path going from the root to the vertex
  - The graph stays connectec and so v would not be an articulation point in this ca



Finding Articulation Points Pseudocode

```
function dfs-visit(v)
   status[v] := gray; time := time+1; d[v] := time;
   low[v] := d[v];
   for each w in out(v)
       if status[w] = white
          //--- (v,w) is a TREE edge
          dfs-visit(w); // low[w] is now computed!
          if low[w] >= d[v] then
             record that vertex v is an articulation
          if low[w] < low[v] then low[v] := low[w];
       else if w is not the parent of v then
          //--- (v,w) is a BACK edge
          if d[w] < low[v] then low[v] := d[w];
   status[v] := black;
```