CS 341: Algorithms
Module 8: Intractability and Undecidability

Armin Jamshidpey, Eugene Zima

Based on lecture notes by many previous CS 341 instructors

David R. Cheriton School of Computer Science, University of Waterloo

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Certificates

Certificate: Informally, a certificate for a yes-instance I is some "extra information" C which makes it easy to verify that I is a yes-instance.

Certificate Verification Algorithm: Suppose that Ver is an algorithm that verifies certificates for yes-instances. Then Ver(I, C) outputs "yes" if I is a yes-Instance and C is a valid certificate for I. If Ver(I, C) outputs "no", then either I is a no-instance, or I is a yes-instance and C is an invalid certificate.

Polynomial-time Certificate Verification Algorithm: A certificate verification algorithm Ver is a polynomial-time certificate verification algorithm if the complexity of Ver is $O(n^k)$, where k is a positive integer and $n = \mathit{Size}(\mathrm{I})$.

The Complexity Class NP

Certificate Verification Algorithm: A certificate verification algorithm Ver is said to solve a decision problem Π provided that

- for every yes-instance I, there exists a certificate C such that Ver(I, C) outputs "yes".
- for every no-instance I and for every certificate C, Ver(I, C) outputs "no".

The Complexity Class NP denotes the set of all decision problems that have polynomial-time certificate verification algorithms solving them. We write $\Pi \in NP$ if the decision problem Π is in the complexity class NP.

Finding Certificates vs Verifying Certificates: It is not required to be able to find a certificate C for a yes-instance in polynomial time in order to say that a decision problem $\Pi \in NP$.

Important Fact: $P \subseteq NP$.

Certificate verification algorithm for subset sum.
Recall subset sum problem has as input an array of integers $a_1, a_2, \ldots, a_n$ and an integer S (target value).
The question is whether or not there is subset of integers $a_1, a_2, \ldots, a_n$ that sums to S.
A certificate consists of an array $B = [b_1, \ldots, b_n]$, $b_i \in \{0, 1\}$.

```
SubS-D_verifier(a,S,B) {
   for i  = 1 to n do
     S = S - a[i]*b[i]
   od
  return (S==0)
}
```

The verification algorithm takes time $\Theta(n)$, so it is polynomial time in size of the instance. Thus, **SubS-D** $\in NP$.

Recall **CLIQUE-D** decision problem: given a graph $G = (V, E)$, and an integer k, decide if graph G has a cliques of size $\geq k$.
A certificate is subset $C \subseteq V$ that might be a clique of size $\geq k$.

```
CLIQUE-D_verifier(V,E,k,C) {
  m = |C|;
  if m<k return FALSE;
  for i = 1 to |C|-1 do
    for j = i+1 to |C| do
      if E[C[i],C[j]]==0
\\      edge (C[i],C[j]) is not in E
        then return FALSE;
    od
  od
  return TRUE;
}
```

The verification algorithm takes time $O(|V|^2)$, so it is polynomial time in size of the instance. Thus **CLIQUE-D** $\in NP$.

Certificate Verification Algorithm for Hamiltonian Cycle

A certificate consists of an n-tuple, $X = [x_1, \ldots, x_n]$, that might be a hamiltonian cycle for a given graph $G = (V, E)$ (where $n = |V|$).

Algorithm 1: Hamiltonian Cycle Certificate Verification(G, X)

$flag \leftarrow$ **true**
$Used \leftarrow \{x_1\}$
$j \leftarrow 2$
while $(j \leq n)$ **and** $flag$ **do**
    $flag \leftarrow (x_j \notin Used)$ **and** $(\{x_{j-1}, x_j\} \in E)$
    If $(j = n)$ **then** $flag \leftarrow flag$ **and** $(\{x_n, x_1\} \in E)$
    $Used \leftarrow Used \cup \{x_j\}$
    $j \leftarrow j + 1$
return $(flag)$

Polynomial Transformations

For a decision problem Π, let $\mathcal{I}(\Pi)$ denote the set of all instances of Π. Let $\mathcal{I}_{yes}(\Pi)$ and $\mathcal{I}_{no}(\Pi)$ denote the set of all yes-instances and no-instances (respectively) of Π.
Suppose that $\Pi_1$and Π_2 are decision problems. We say that there is a polynomial transformation from Π_1 to Π_2 (denoted $\Pi_1 \leq_P \Pi_2$) if there exists a function $f : \mathcal{I}(\Pi_1) \rightarrow \mathcal{I}(\Pi_2)$ such that the following properties are satisfied:

- $f(\mathrm{I})$ is computable in polynomial time (as a function of size(I), where $\mathrm{I} \in \mathcal{I}(\Pi_1)$)
- if $\mathrm{I} \in \mathcal{I}_{yes}(\Pi_1)$, then $f(\mathrm{I}) \in \mathcal{I}_{yes}(\Pi_2)$
- if $\mathrm{I} \in \mathcal{I}_{no}(\Pi_1)$, then $f(\mathrm{I}) \in \mathcal{I}_{no}(\Pi_2)$

Polynomial Transformations (cont.)

Polynomial transformations are also known as Karp reductions or many-one reductions.

A polynomial transformation can be thought of as a (simple) special case of a polynomial-time Turing reduction, i.e., if $\Pi_1 \leq_P \Pi_2$, then $\Pi_1 \leq_P^T \Pi_2$. Given a polynomial transformation f from Π_1 to Π_2 , the corresponding Turing reduction is as follows:

- Given $\mathrm{I} \in \mathcal{I}(\Pi_1)$, construct $f(\mathrm{I}) \in \mathcal{I}(\Pi_2)$.
- Given an oracle for Π_2 , say A, run $A(f(\mathrm{I}))$.

We transform the instance, and then make a single call to the oracle.

Very important point: We do not know whether I is a yes-instance or a no-instance of Π_1 when we transform it to an instance $f(\mathrm{I})$ of Π_2 .

To prove the implication "if $\mathrm{I} \in \mathcal{I}_{no}(\Pi_1)$, then $f(\mathrm{I}) \in \mathcal{I}_{no}(\Pi_2)$", we usually prove the contrapositive statement "if $f(\mathrm{I}) \in \mathcal{I}_{yes}(\Pi_2)$, then $\mathrm{I} \in \mathcal{I}_{yes}(\Pi_1)$.

Two Graph Theory Decision Problems

Clique Problem

Instance: An undirected graph $G = (V, E)$ and an integer k, where $1 \leq k \leq |V|$.
Question: Does G contain a clique of size $\geq k$? (A clique is a subset of vertices $W \subseteq V$ such that $uv \in E$ for all $u, v \in W, u \neq v$.)

Vertex Cover Problem

Instance: An undirected graph $G = (V, E)$ and an integer k, where $1 \leq k \leq |V|$.
Question: Does G contain a vertex cover of size $\leq k$? (A vertex cover is a subset of vertices $W \subseteq V$ such that $\{u, v\} \cap W \neq \emptyset$ for all edges $uv \in E$.)

Clique $\leq_P$ Vertex-Cover

Suppose that $\mathrm{I} = (G, k)$ is an instance of Clique, where $G = (V, E)$, $V = \{v_1, \ldots, v_n\}$ and $1 \leq k \leq n$.
Construct an instance $f(\mathrm{I}) = (H, \ell)$ of Vertex Cover, where $H = (V, F)$, $\ell = nk$ and

$$v_i v_j \in F \Leftrightarrow v_i v_j \notin E\ .$$

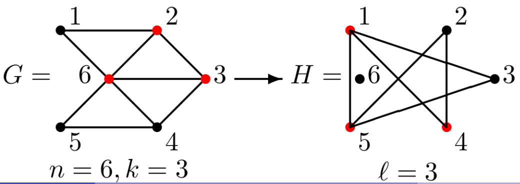
H is called the complement of G, because every edge of G is a non-edge of H and every non-edge of G is an edge of H.
We have *Size*$(\mathrm{I}) = n^2 + \log_2 k \in \Theta(n^2)$. Computing H takes time $\Theta(n^2)$ and computing ℓ takes time $\Theta(\log n)$, so $f(\mathrm{I})$ can be computed in time $\Theta($*Size*$(\mathrm{I}))$, which is polynomial time.

Clique $\leq_P$ Vertex-Cover (cont.)

Suppose I is a yes-instance of Clique. Therefore there exists a set of k vertices W such that $uv \in E$ for all $u, v \in W$. Define $W' = V \setminus W$.

Clearly $|W'| = n - k = \ell$. We claim that W' is a vertex cover of H. Suppose $uv \in F$ (so $uv \notin E$). If $\{u, v\} \cap W' \neq \emptyset$, we are done, so assume $u, v \notin W'$. Therefore $u, v \in W$. But $uv \notin E$, so W is not a clique. This is a contradiction and hence $f(\mathrm{I})$ is a yes-instance of Vertex Cover.

Suppose $f(\mathrm{I})$ is a yes-instance of Vertex Cover. Therefore there exists a set of $\ell = n - k$ vertices W' that is a vertex cover of H. Define $W = V \setminus W'$. Clearly $|W| = k$. We claim that W is a clique in G . . . .



Properties of Polynomial-time Transformations

Theorem 1

If Π_1 and Π_2 are decision problems, $\Pi_1 \leq_P \Pi_2$ and $\Pi_2 \in P$, then $\Pi_1 \in P$.

Proof.

Suppose A is a poly-time algorithm for Π_2 , having complexity $O(m^\ell)$ on an instance of size m. Suppose f is a transformation from Π_1 to Π_2 having complexity $O(n^k)$ on an instance of size n. We solve Π_1 as follows:

1. Given $\mathrm{I} \in \mathcal{I}(\Pi_1)$, construct $f(\mathrm{I}) \in \mathcal{I}(\Pi_2)$.
2. Run $A(f(\mathrm{I}))$.

It is clear that this yields the correct answer. We need to show that these two steps can be carried out in polynomial time as a function of $n = \mathit{Size}(\mathrm{I})$. Step (1) can be executed in time $O(n^k)$ and it yields an instance $f(\mathrm{I})$ having size $m \in O(n^k)$. Step (2) takes time $O(m^\ell)$. Since $m \in O(n^k)$, the time for step (2) is $O(n^{k\ell})$, as is the total time to execute both steps.

Properties of Polynomial-time Transformations (cont.)

Theorem 2

Suppose that Π_1, Π_2 and Π_3 are decision problems. If $\Pi_1 \leq_P \Pi_2$ and $\Pi_2 \leq_P \Pi_3$, then $\Pi_1 \leq_P \Pi_3$.

Proof

We have a polynomial transformation f from Π_1 to Π_2 , and another polynomial transformation$g$ from Π_2 to Π_3 . We define $h = f \circ g$, i.e., $h(\mathrm{I}) = g(f(\mathrm{I}))$ for all instances I of Π_1 . (Exercise: fill in the details.)

The Complexity Class NPC

The complexity class NPC denotes the set of all decision problems Π that satisfy the following two properties:

- $\Pi \in NP$
- For all $\Pi' \in NP, \Pi' \leq_P \Pi$.

NPC is an abbreviation for NP-complete.

Note that the definition does not imply that NP-complete problems exist!