CS 341: Algorithms
Module 7: Graph Algorithms

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Based on lecture notes by many previous CS 341 instructors

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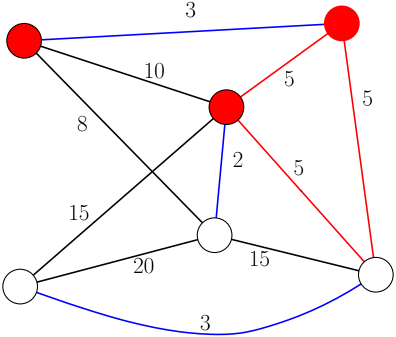
Minimum spanning tree

- Problem: In an undirected graph with non-negative weights on edges, find a spanning tree of minimum total weight
- Motivation: cheapest interconnection (electrical circuit, computer network, highway system)
- Important subroutine for network optimization problems

Example of MST



Blue edges with any two of red edges give an MST

MST approaches

- Solution is not necessarily unique
- There are many algorithms
- General idea of greedy algorithms:

$A \leftarrow \emptyset$

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while A is not a spanning tree
  find edge e of least weight with
  "certain properties"
  add e to A
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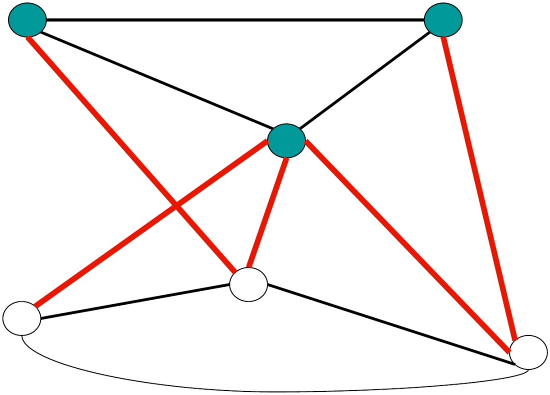
Finding an edge to add

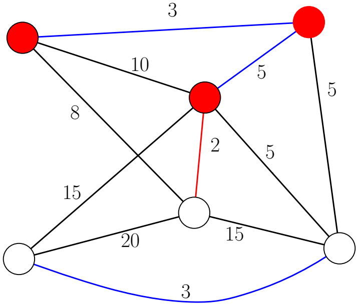
- Cut: partition of $V = (S, V - S)$
- Crossing edge: has endpoint in each set



Correctness of MST algorithm

Theorem 1

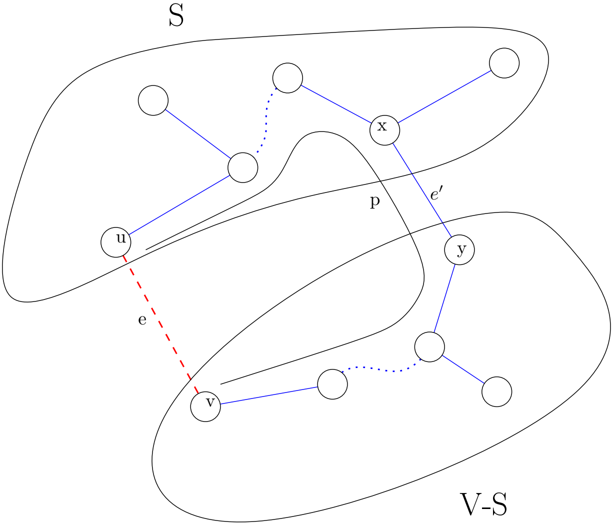
If A can be extended to a MST, and no edge in A crosses $(S, V - S)$, then the edge of minimum weight crossing this cut can be added to A.



Proof of Theorem 1

- Suppose we can find A, S, and edge e contradicting this. Then $A \bigcup \{e\}$ cannot be extended to a MST.
- Let T be a MST extending A.
- Adding e to T creates a unique cycle.

Proof of Theorem 1

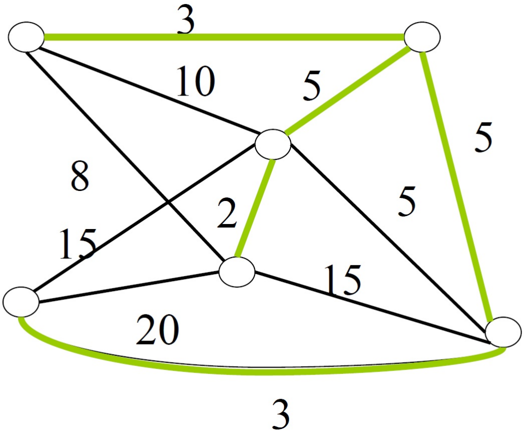


Proof of Theorem 1

- Adding e to T creates unique cycle
- Some other edge e' in this cycle also crosses the cut defined by S
- e' is not in A since it crosses the cut
- Weight of e' is at least weight of e
- $T \bigcup \{e\} - \{e'\}$ is also a spanning tree and must have weight no greater than T
- This is a MST extending $A \bigcup \{e\}$.

Kruskal's algorithm

- "certain property" $=$ can be added to A without forming a cycle



Correctness of Kruskal's algorithm

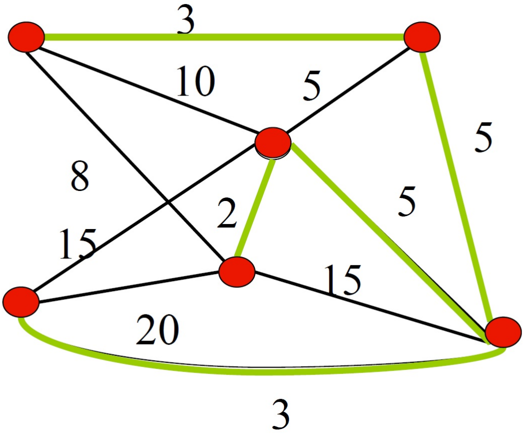
- During the algorithm, A is a forest of trees (stops when it is a single tree)
- What is the cut we can use in Theorem 1?
- $e = (u, v)$ is lightest edge that can be added without forming a cycle
- Let S be the vertices in the tree in A containing u
- v must be in $V - S$ (or e would form cycle)
- e must be lightest edge crossing this cut

Running time of Kruskal's algorithm

- Maintain components of A
- Presort edges, run through them in order
- Given $e = (u, v)$, it can be added to A if and only if u, v are in different components
- Adding e to A merges these components
- Need union-find data structure
- Loop executed $n - 1$ times
- Sequence of $n - 1$ unions and $2m$ finds can be done in $O(m \log n)$ time
- Algorithm takes $\Theta(m \log n)$ time

Prim's algorithm

- "certain property" = one endpoint shared with edge in A, one is not (i.e. leaves A)



Correctness of Prim's algorithm

- A is always a single tree (stops when it is a spanning tree)
- What is the cut we can use in Thm 1?
- $S =$ endpoints of edges in A (or starting vertex s if A is empty)
- Implementation is not so obvious: how do we find lightest edge crossing cut $(S, V - S)$?

Implementation of Prim's algorithm

- For each vertex v in $V - S$, maintain $near[v] = a \in S$ such that edge (v, a) is lightest edge from v to S
- Initially $near[v] = s$
- Add to S the vertex w minimizing weight of $(near[w], w)$ [takes $\Theta(n)$ time]
- When w added to S, update $near[v]$ if (w, v) is lighter than $(near[v], v)$ [takes $\Theta(n)$ time]
- Total running time $\Theta(n^2)$

Better implementation

- Keep vertices not in S in heap, ordered by near values
- Removing min or updating single near value takes takes $\Theta(\log n)$ time
- $n - 1$ removals, m updates
- Running time is $\Theta(m \log n)$
- Even better improvement uses Fibonacci heaps to get time of $\Theta(m + n \log n)$.

Single-source shortest path

- Think of weights as lengths of edges
- Given a weighted graph and source s, $\delta(s, v)$ = length of shortest $s - v$ path
- We wish to compute all $\delta(s, v)$ [and the corresponding paths]
- If edge weights are nonnegative, a greedy algorithm will work (Dijkstra's algorithm)
- General proof in book simplified here

Dijkstra's algorithm

- Looks similar to Prim's MST algorithm
- Start with source s in set S
- For vertices v not in S, maintain quantities
  - $\pi[v]$, a vertex in S
  - $d[v]$ which is $\delta(s, \pi[v]) + w(\pi[v], v)$
- Intuition: $d[v]$ is the length of the shortest path to v using vertices in S only (call this an S-internal path), and $\pi[v]$ is the last vertex in S on this path

Dijkstra's algorithm

- Initially $S \leftarrow \{s\}$, $d[s] \leftarrow 0$ and for all v not in S, if $(s, v) \in E$ then $\pi[v] \leftarrow s$ and $d[v] \leftarrow w(s, v)$ otherwise $\pi[v] \leftarrow nil$ and $d[v] \leftarrow \infty$
- To choose a vertex u to add to S, pick one with smallest d-value
- Update other d-values with

$$d[v] \leftarrow min\{d[v], d[u] + w(u, v)\}$$

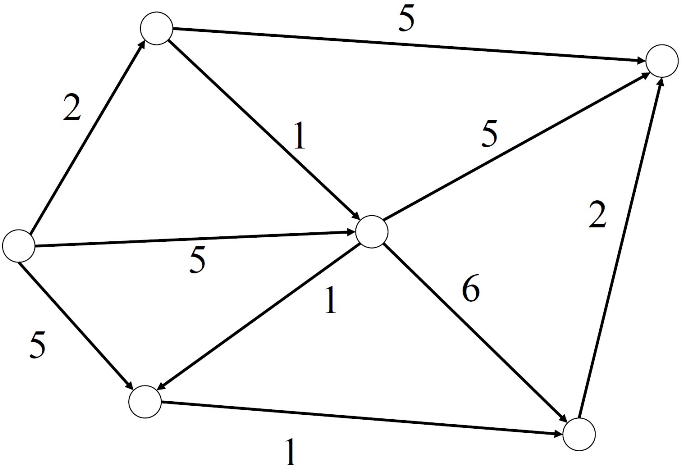
- We prove this works by induction on the size of S

Pseudocode for Dijkstra's algorithm

```
Initialize S, d, π
while S ≠ V
    u ← v ∉ S minimizing d[v]
    add u to S
    for v ∉ S
        d[v] ← min{d[v], d[u] + w(u, v)}
        (if d[v] changes, π[v] ← u)
```

- Running time of algorithm is $\Theta(n^2)$

Example of Dijkstra's alg'm



Proof of Dijkstra's algorithm

- Prove by induction on $|S|$ that
  1. For all $v \in S$, $d[v] = \delta(s, v)$
  2. For all $w \notin S$, $d[w]$= length of minimum S- internal $s - w$ path (so $d[w] \geq \delta(s, w)$) and $\pi[w] =$ last vertex on such a path
  3. For any $v \in S$, $w \notin S$, $d[v] \leq d[w]$
- Base case: $|S| = 1$
  - Since $d[s] = 0$ and for all v not in S, $\pi[v] = s$ and $d[v] = w(s, v)$, these are trivially true

Proving statement 1

- Assume statements true for $|S| = k - 1$
- When k^{th} vertex u chosen to be added to S, $d[u]$= length of a shortest S-internal path to u (by inductive hypothesis 2)
- Suppose $d[u] > \delta(s, u)$
- Choose any shortest $s - u$ path P
- It leaves S for the first time by some edge (x, y) and by ind.hyp. 1, $d[x] = \delta(s, x)$
- The segment of P from s to y has length $d[y]$ (by ind.hyp 2) so $d[y] \leq \delta(s, u) < d[u]$, contradicting the choice of u; so statement 1 is true

Proving statement 2

- Thus when k^{th} vertex u added to S, $d[u] = \delta(s, u)$, as required
- After u added, what do shortest S-internal path to $v \notin S$ look like?
- If one does not use u, then it must be the shortest $(S - \{u\})$-internal path to v, and this path has length $d[v] \leq d[u] + w(u, v)$, so the algorithm does not change anything
- If one uses u and (u, v) is the last edge, the path has length $\delta(s, u) + w(u, v)$, so the algorithm updates correctly

Proving statements 2 and 3

- If one uses u but some (y, v) is the last edge
  - $d[y] \leq d[u]$ (ind. hyp. 3)
  - u was just added, so the shortest $s - y$ path doesn't use u
  - Adding (y, v) gives a shortest S-internal path to v avoiding u
  - The algorithm does not change anything
- Thus statement 2 is proved
- Statement 3 follows because of the choice of u minimizing $d[u]$
- Where did we use non-negativity?