CS 341: Algorithms
Module 4: Divide and Conquer

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Based on lecture notes by many previous CS 341 instructors

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Multiprecision Multiplication

Problem

Instance: Two k-bit positive integers, X and Y, having binary representations

$$X = (X[k-1], ..., X[0])$$

and

$$Y = (Y[k-1], ..., Y[0]).$$

Question: Compute the $2k$-bit positive integer $Z = XY$, where

$$Z = (Z[2k-1], ..., Z[0]).$$

We are interested in the *bit complexity* of algorithms that solve *Multiprecision Multiplication*, which means that the complexity is expressed as a function of k.

Consider

$$X = X_L 2^{k/2} + X_R, \quad , Y = Y_L 2^{k/2} + Y_R$$

and write $X \cdot Y$ as

$$X_L \cdot Y_L \cdot 2^k + (X_L \cdot Y_R + X_R \cdot Y_L) 2^{k/2} + X_R \cdot Y_R.$$

Not-So-Fast D&C Multiprecision Multiplication

Algorithm 1: NotSoFastMultiply (X, Y, k)

if $k = 1$ **then** $Z \leftarrow X[0] \times Y[0]$ **else**
    $Z_1 \leftarrow \text{NotSoFastMultiply}(X_L, Y_L, k/2)$
    $Z_2 \leftarrow \text{NotSoFastMultiply}(X_R, Y_R, k/2)$
    $Z_3 \leftarrow \text{NotSoFastMultiply}(X_L, Y_R, k/2)$
    $Z_4 \leftarrow \text{NotSoFastMultiply}(X_R, Y_L, k/2)$
    $Z \leftarrow \text{LeftShift}(Z_1, k) + Z_2 + \text{LeftShift}(Z_3 + Z_4, k/2)$
return (Z)

Complexity?

Consider again

$$X = X_L 2^{k/2} + X_R, \quad , Y = Y_L 2^{k/2} + Y_R$$

and write $X \cdot Y$ as

$$X_L \cdot Y_L \cdot 2^k + ((X_L + X_R)(Y_L + Y_R) - X_L \cdot Y_L - X_R \cdot Y_R) \cdot 2^{k/2} + X_R \cdot Y_R.$$

Fast D&C Multiprecision Multiplication

Algorithm 2: FastMultiply(X,Y,k)

if $k = 1$ **then** $Z \leftarrow X[0] \times Y[0]$
else
  $X_T \leftarrow X_L + X_R$
  $Y_T \leftarrow Y_L + Y_R$
  $Z_1 \leftarrow \text{FastMultiply}(X_L, Y_L, k/2)$
  $Z_2 \leftarrow \text{FastMultiply}(X_R, Y_R, k/2)$
  $Z_3 \leftarrow \text{FastMultiply}(X_T, Y_T, k/2)$,
  $Z \leftarrow \text{LeftShift}(Z_1, k) + Z_2 + \text{LeftShift}(Z_3 - Z_1 - Z_2, k/2)$
return (Z)

Complexity? Technical details?

Matrix Multiplication

Problem

Instance: Two n by n matrices, X and Y.
Question: Compute the n by n matrix product $Z = XY$.

The naive algorithm for *Matrix Multiplication* has complexity $\Theta(n^3)$.

Matrix Multiplication: Problem Decomposition

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \quad C = AB = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

If A, B are n by n matrices, then $a, b, ..., h, r, s, t, u$ are $\frac{n}{2}$ by $\frac{n}{2}$ matrices, where

$$\begin{aligned} r &= a\,e + b\,g & \qquad s &= a\,f + b\,h \\ t &= c\,e + d\,g & \qquad u &= c\,f + d\,h \end{aligned}$$

We require 8 multiplications of $\frac{n}{2}$ by $\frac{n}{2}$ matrices in order to compute $C = AB$.

Efficient D&C Matrix Multiplication

Define

$$\begin{aligned} P_1 &= a(f-h) & P_2 &= (a+b)h \\ P_3 &= (c+d)e & P_4 &= d(g-e) \\ P_5 &= (a+d)(e+h) & P_6 &= (b-d)(g+h) \\ P_7 &= (a-c)(e+f). \end{aligned}$$

Then, compute

$$\begin{aligned} r &= P_5 + P_4 - P_2 + P_6 & s &= P_1 + P_2 \\ t &= P_3 + P_4 & u &= P_5 + P_1 - P_3 - P_7. \end{aligned}$$

We now require only 7 multiplications of $\frac{n}{2}$ by $\frac{n}{2}$ matrices in order to compute $C = AB$.

Convex hull

Convex set

A set of points in the plane is called convex if for any two points p and q in the set, the entire line segment at p and q belongs to the set.

Convex hull

The convex hull of a set S of points is the smallest (with respect to inclusion) convex set containing S.

Theorem

The convex hull of any set S of $n > 2$ points, not all on the same line, is a convex polygon with vertices at some of the points of S. (If all the points lie on the same line, the polygon degenerates to a line segment but still with the endpoints at two points of S.)

Convex hull

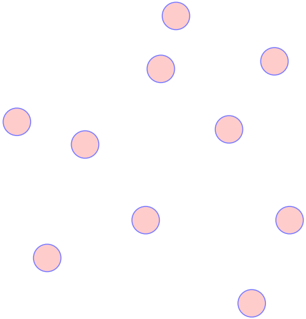
Convex hull problem

For a given set S of n points, construct the convex hull of S.

Solution

Find the points that will serve as the vertices of the polygon in question and list them in some regular order.

Example:



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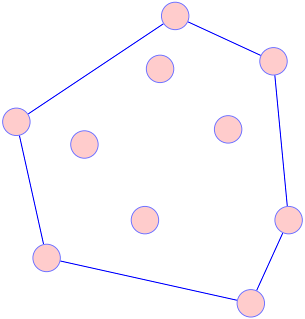
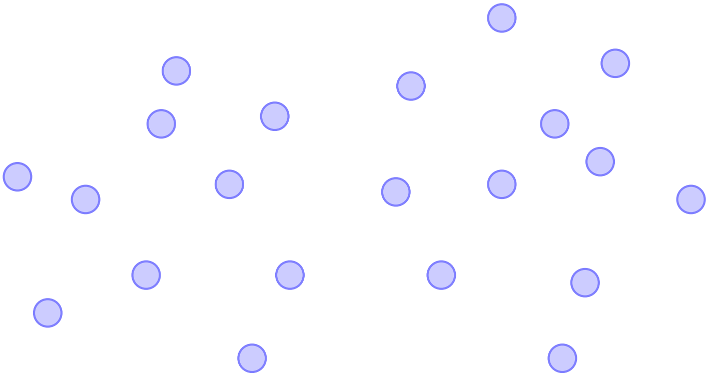
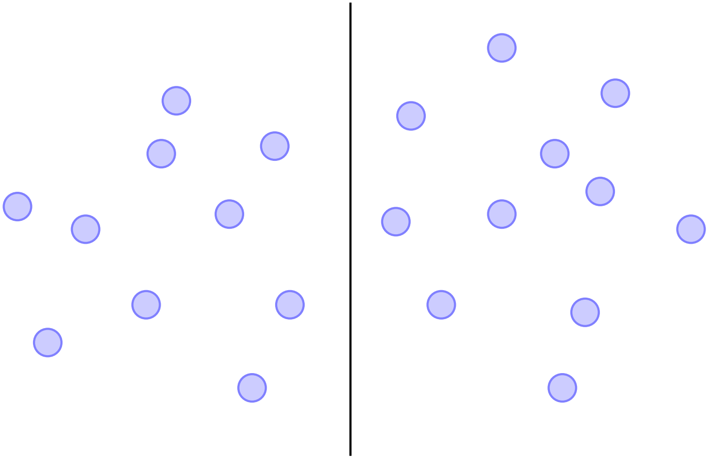


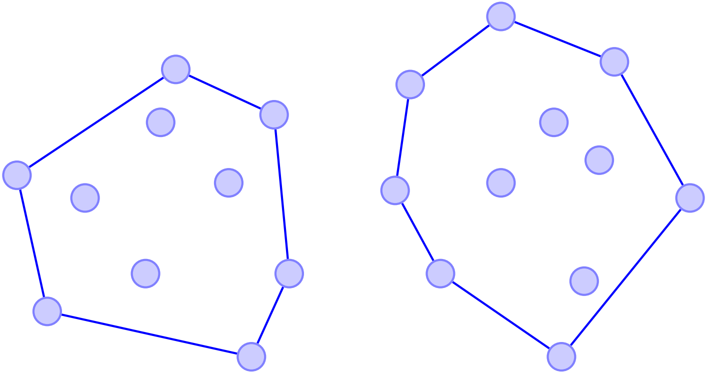
Convex hull



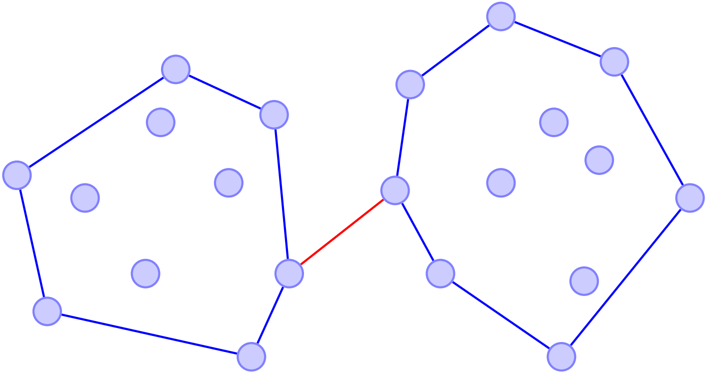
Convex hull



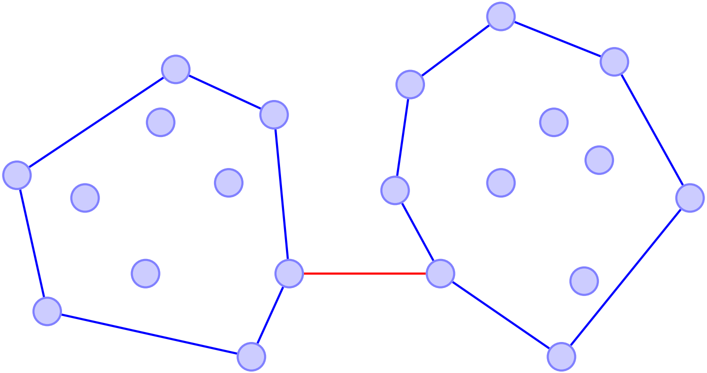
Convex hull



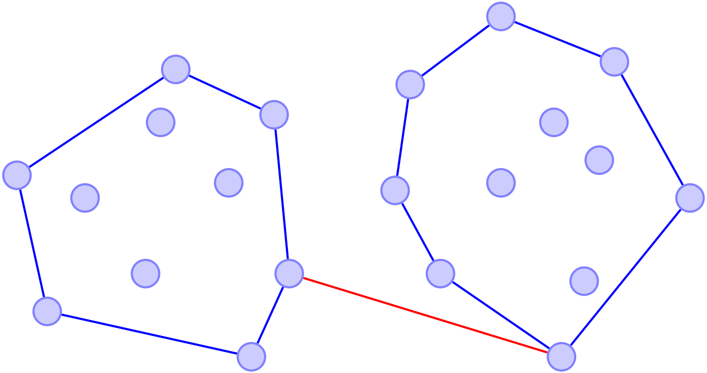
Convex hull



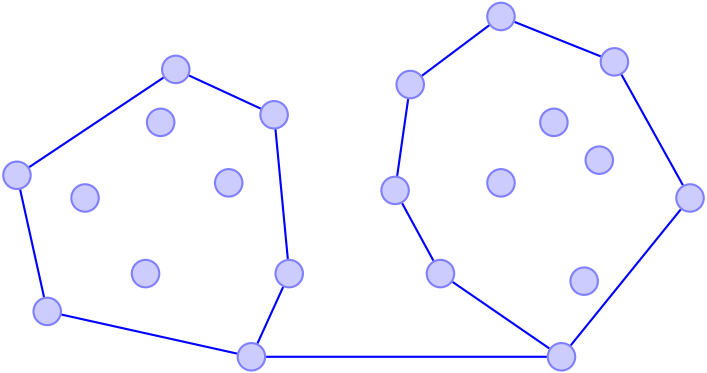
Convex hull



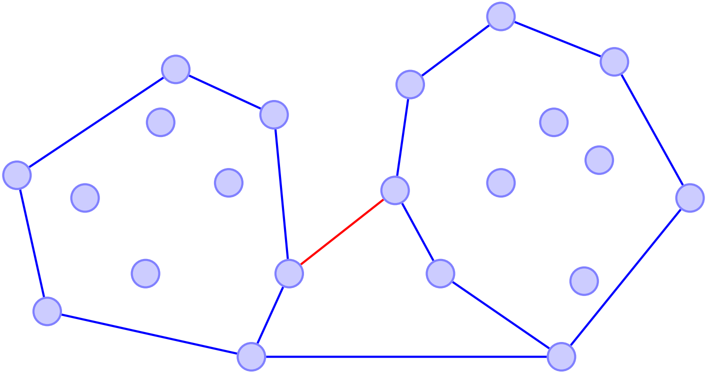
Convex hull



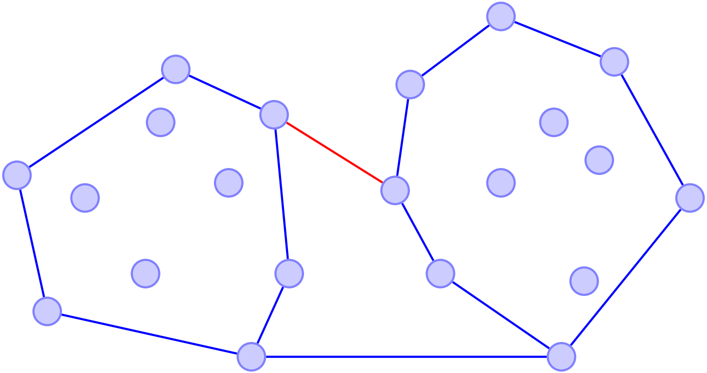
Convex hull



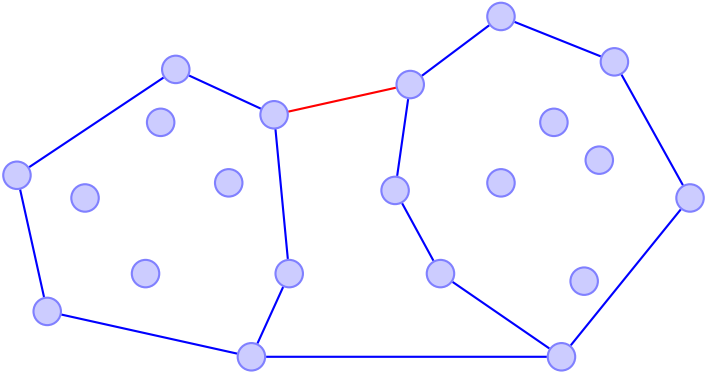
Convex hull



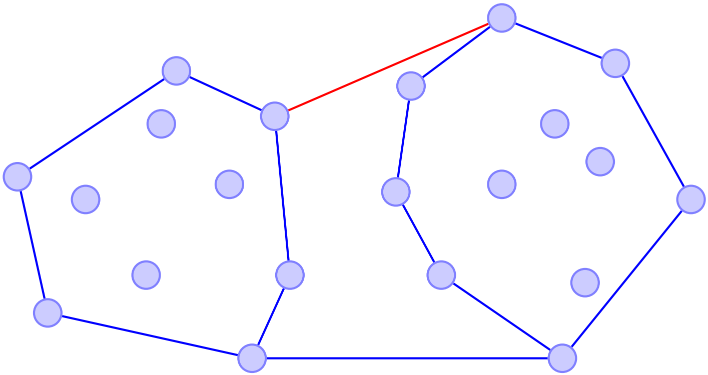
Convex hull



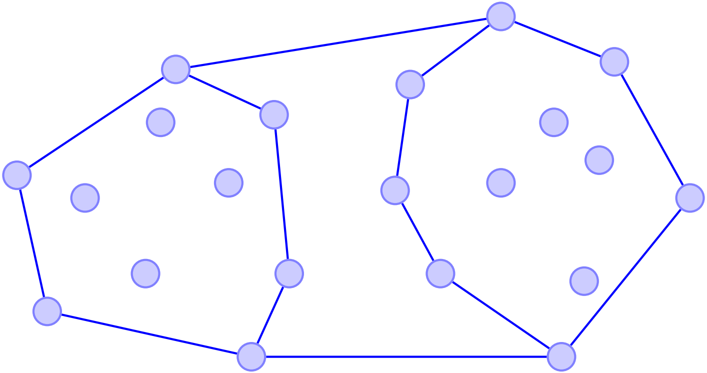
Convex hull



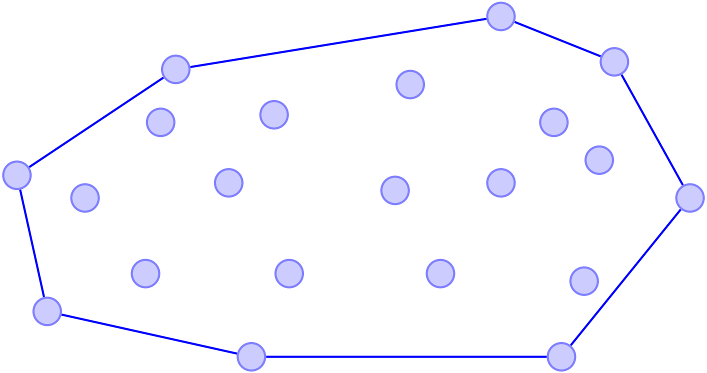
Convex hull



Convex hull



Convex hull



Convex hull

Assume that hulls are defined in ccw order.

Algorithm 3: ConvexHullMerge(L, R)

$A \leftarrow$ rightmost point of L
$B \leftarrow$ leftmost point of R
while $T = AB$ is not the lower tangent to both L and R
    while T is not lower tangent to L
        $A \leftarrow A - 1$
    while T is not lower tangent to R
        $B \leftarrow B + 1$
$\vdots$

Analytic Geometry formulas

Assume $q_1(x_1, y_1), q_2(x_2, y_2), q_3(x_3, y_3) \in \mathbb{R}^2$.

The area of the triangle $\triangle q_1 q_2 q_3$ is equal to one half of the magnitude of the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = x_1 y_2 + x_3 y_1 + x_2 y_3 - x_3 y_2 - x_2 y_1 - x_1 y_3,$$

while the sign of this expression is positive if and only if the point q_3 is to the left of the line $q_1 q_2$.