# CS 341: Algorithms
# Module 7: Graph Algorithms


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Based on lecture notes by many previous CS 341 instructors

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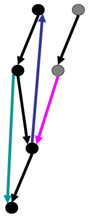
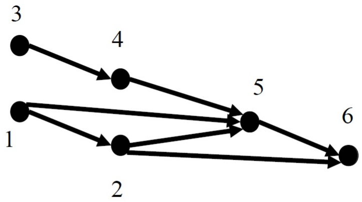
# DFS on directed graphs

- Interpret "$w$ adjacent to $v$" as finding directed edge $(v, w)$
- Edges $(v, w)$ grouped into four types:
  - Tree edge
    - White $w$ discovered from gray $v$
    - Actually a set of trees, or forest
  - Back edges
    - $w$ ancestor of $v$ or on stack when $w$ visited ($w$ gray)
  - Forward edges
    - $w$ descendant of $v$ ($w$ black, $d[v] < d[w]$)
  - Cross-edges
    - All others ($w$ black, $d[v] > d[w]$)



# Topological sort

- A linear ordering of vertices of a Directed Acyclic Graph (DAG)
- For any directed edge $(u, v)$, $u$ precedes $v$ in ordering



# Use of topological sort

- Application: nodes are tasks, edges are "precedences" (e.g. one task must be done before another can be started)
- A topological sort gives an order in which to do tasks
- Naive algorithm: look for a source (no incoming edges), choose and delete it
- This is $\Theta(n(n+m))$

# Using DFS

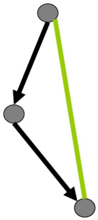
- The finishing times $f[u]$ give a topological ordering (taken in decreasing order).
- Equivalently, in postprocessing (when vertex coloured black), put it on front of a linked list; resulting list is topologically ordered.
- Why does this work? Intuitively OK
- Need to show that for any directed edge $(u, v)$, $f[u] > f[v]$; this is not obvious.

# Proof of topological sort

**Lemma**

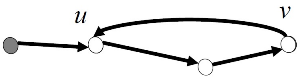
A graph is acyclic iff there are no back edges in a DFS of the graph

Proof: ($\Rightarrow$) If there is a back edge, that edge plus the tree path forward gives a cycle.



# Proof of topological sort

($\Leftarrow$) If there is a cycle, let $u$ be the first discovered cycle vertex in DFS, and let $(v, u)$ be a cycle edge.




The white-path theorem applied to $v, u$ says that $v$ is a descendant of $u$, so $(v, u)$ is a back edge.

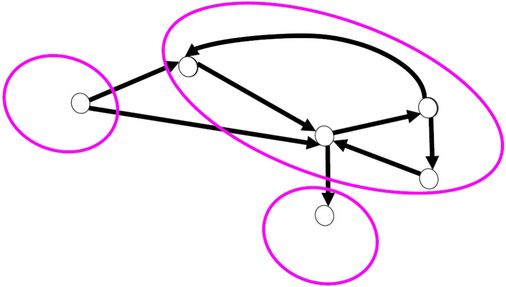
# Proof of topological sort

Apply DFS to a DAG, and consider directed edge $(u, v)$; must show $f[v] < f[u]$.

- When $(u, v)$ explored, $v$ can not be gray, because $(u, v)$ would be a back edge.
- If $v$ is white, it becomes descendant of $u$, so $f[v] < f[u]$ by parenthesis theorem.
- If $v$ is black, $f[v]$ already set; $f[u]$ must be bigger when it is set.

# Strongly connected components

- A strongly connected component is a maximal set of vertices $C \subseteq V$ such that for any $u, v$ in $C$, there are directed paths from one to the other.



# A naive algorithm for SCC

- Run DFS-visit from each node $u$ to get $reach(u) =$ vertices reachable from $u$.
- $S \leftarrow reach(u)$; for every $v$ in $S$, if $u \notin reach(v)$, delete $v$ from $S$.
- What is left is a strongly connected component
- This takes $\Theta(n(n+m))$ time just to get one strongly connected component

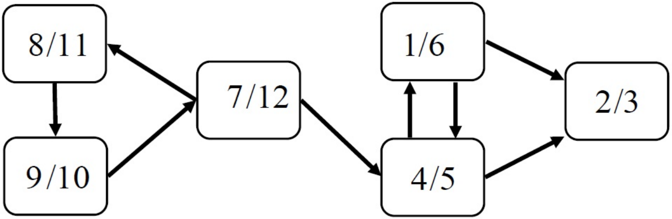
# Better use of DFS for SCC

- Let $G^T$ be $G$ with all edges reversed.
- $G$ and $G^T$ have the same strongly connected components.
- Can create $G^T$ in $O(n + m)$ time.

**Strongly-Connected-Components(G)**

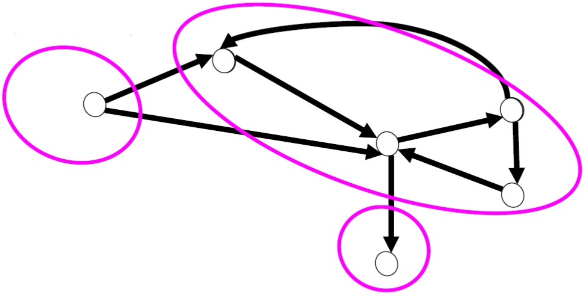
1. Call a DFS on $G$, recording finishing times.
2. Compute $G^T$.
3. Call a DFS on $G^T$, choosing roots in order of decreasing finishing time in first DFS (step 1).
4. Vertices of each tree in the depth-first forest is a strongly connected component.

# SCC algorithm example



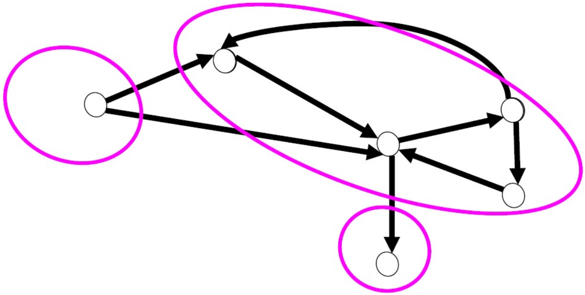
# Intuition: the component graph

- Define a graph $G^{SCC}$: Each vertex is a strongly connected component of $G$.
- $(u, v)$ is an edge in $G^{SCC}$ iff there is an edge in $G$ from a vertex in the component $u$ to the component $v$.



# The component graph

- $G^{SCC}$ is a directed acyclic graph (DAG).
- The second DFS on $G^T$ basically visits the vertices of $(G^T)^{SCC}$ in **reverse** topological order (or of $G^{SCC}$ in topological order).



# Proof of SCC algorithm

Extend definition of $d$ and $f$ (discovery time and finishing times) to sets:

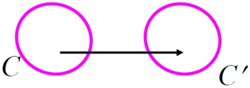
For $U \subseteq V, d(U) = min_{u \in U} d[u]$ and $f(U) = max_{u \in U} f[u]$

**Lemma 4**

For two components $C$ and $C'$, if there is an edge from $C$ to $C'$, then $f(C) > f(C')$.

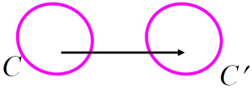
Proof:

If $d(C) < d(C')$, then when the first vertex $x$ was discovered in $C$, there was a white path from $x$ to all vertices in $C$ and $C'$; the white-path and parenthesis theorems show $f[x] = f(C) > f(C')$.



# Proof of lemma 4

- If $d(C) > d(C')$, when first vertex $y$ discovered in $C'$, all other vertices in $C'$ are white, and as before $f[y] = f(C')$.
- Vertices of $C$ are also white, and because of edge $(u, v)$ from $C$ to $C$, no vertices of $C$ are reachable from $y$, so their discovery times and finishing times are $> f[y]$.
- Thus $f(C) > f(C')$.



# Proof of SCC algorithm (ctd.)

**Corollary 5**

For two components $C$, $C'$, if there is an edge from $C'$ to $C$ in $G^T$, then $f(C) > f(C')$.

Thus the component first visited in the DFS search on $G^T$ has no edge to any other component.

# Conclusion of proof

Can now use induction on the number of trees visited in second DFS to show each one is a separate component