CS 341: Algorithms
Module 8: Intractability and Undecidability

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Based on lecture notes by many previous CS 341 instructors

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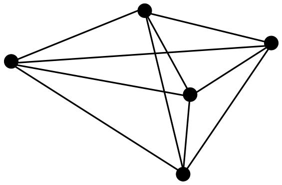
What to do with NP-hard optimization problems

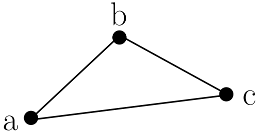
- Efficient exhaustive search (backtracking, branch & bound) $\rightarrow$ exponential time.
- Heuristics

  local search: start with some solution and try to improve it via small " local " changes.
  "simulated annealing " overcomes local optima.
- Approximation algorithms.

Example: TSP for points in the plane with Euclidean distances.



Triangle inequality:



$$w(a, c) \leq w(a, b) + w(b, c)$$

Algorithm 1: Approx. Alg.

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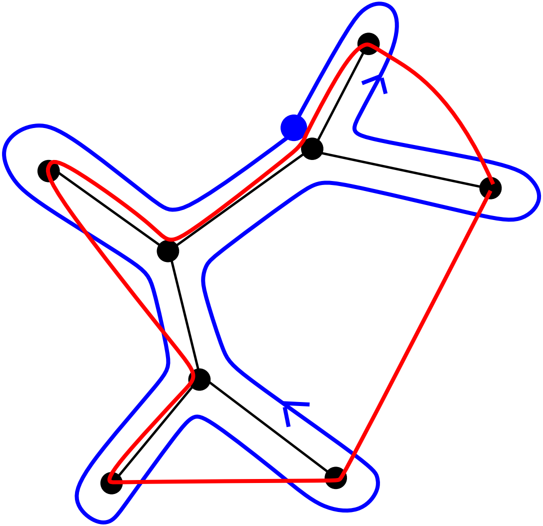
1. Compute MST
2. Take a tour by walking around it. (we visit every vertex but maybe more than once)
3. Take short cuts to avoid revisiting.

Note: Triangle inequality $\Rightarrow$ short cuts, shorter.

This can be done in polynomial time.

Example



Let ℓ be the length of the resulting tour

ℓ_{TSP} = length of min TSP tour

claim: $\ell_{\mathrm{TSP}} \leq \ell \leq 2\ell_{\mathrm{TSP}}$.

proof: ℓ_{MST} = length of MST.

$\ell_{\mathrm{MST}} \leq \ell_{\mathrm{TSP}}$, since deleting one edge of TSP gives a spanning tree.

$\ell \leq 2\ell_{\mathrm{MST}}$, since we use every MST edge twice, then take short cuts (use triangle inequality)

Putting these together:

$$\ell \leq 2\ell_{\mathrm{TSP}}.$$

So in polynomial time we find a tour within $2\times$ optimum. We say this algorithm has approximation factor 2.

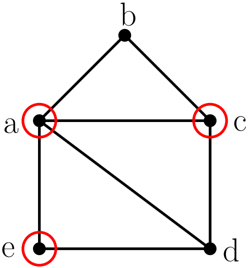
Vertex Cover

For a given graph $G = (V, E)$ find $C \subseteq V$ s.t.

$$(u, v) \in E \Rightarrow u \in C \text{ or } v \in C$$

and $|C|$ is minimum.

Example:



Algorithm 2: Greedy approximation algorithm

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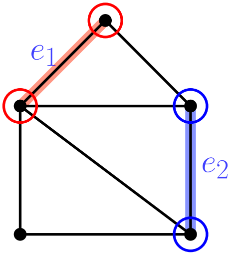
C ← ∅
F ← E          // F is uncovered edges.
while F ≠ ∅
        pick (u, v) from F
        add u and v to C
        remove edges incident to u from F
        remove edges incident to v from F
```

Example:



Note that the algorithm takes polynomial time.
Let C =Vertex cover found by the algorithm.
C_{OPT} =a minimum vertex cover.
Claim: $|C| \leq 2 \cdot |C_{\mathrm{OPT}}|$
Proof: The set of edges you pick forms a matching M. Any vertex cover must have at least one vertex from each edge in a matching. $|M| \leq |C_{\mathrm{OPT}}|$. Thus $|C| \leq 2 \cdot |C_{\mathrm{OPT}}|$. This algorithm has approximation factor 2.

General TSP cannot be approximated to within constant factor in polynomial time (unless $P = NP$)

Suppose we have a polynomial time algorithm for TSP that guaranties a tour of length $\leq k \cdot \ell_{\mathrm{TSP}}$.

Claim: We can make a polynomial time algorithm for hamiltonian cycle. Hence $P = NP$.

Algorithm 3: Algorithm for hamiltonian cycle

Input: $G = (V, E)$, $|V| = n$

construct $G' = (V, E' = \{(u, v) : u, v \in V, u \neq v\})$

$$\text{for } e \in E',\ w(e) = \begin{cases} 1 & e \in E \\ k \cdot n & otherwise \end{cases}$$

Run approximation TSP algorithm on G' to get a tour of length ℓ

if $\ell \leq k \cdot n$ output YES

else output NO

Correctness:

In G', a tour that only uses edges of G has length n.
A tour that uses at least one edge not in G has *length* $\geq (n-1) + k \cdot n > k \cdot n$ (assuming $n > 1$).
Claim: $\ell \leq k \cdot n$ iff G has hamiltonian cycle.
Proof: $(\Rightarrow)$ $\ell \leq k \cdot n \Rightarrow \ell = n$ so G has hamiltonian cycle.
$(\Leftarrow)$ G has hamiltonian cycle $\Rightarrow$ G' has a tour of length $n \Rightarrow k-$approx has length $\leq k \cdot n$.