Math 239 - Tutorial 2

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Tutorial Problems

Problem 1

a)

Find $\Phi_{s_4}(x)$

σ	$w(\sigma)$
0000	0
0001	0
0010	0
0011	1
0100	0
0101	0
0110	1
0111	2
1000	0
1001	0
1010	0
1011	1
1100	1
1110	2
1111	3
1101	1

$\Phi_{s_4}(x) = \sum_{\delta \in s_4} x^{w(\sigma)} = 8x^0 + 5x + 2x^2 + x^3$

b)

Prove that for all $n \in N \Phi_{s_n}(x) = \Phi^*_{s_n}(x)$

solution: Define

$$f : S_n \implies S_n$$
$$x_1 \cdot x_2 ... x_n \implies (1 - x_1)(1 - x_2)...(1 - x_n)$$

Notice that f is a permutation of S_n
Also notice that $w(x) = w^*(f(x)) \forall x \in S_n$

$$\Phi_{S_n}(x) = \sum_{\sigma \in S_n} x^{w(\sigma)}$$
$$= \sum_{\sigma \in S_n} x^{w^*(f(\sigma))}$$
$$= \sum_{\sigma' \in S_n} x^{w^*(\sigma')}$$
$$= \Phi^*_{S_n}(x)$$

Problem 2

Notice that:

$$f(x) = \frac{1}{1-x}$$
$$g(x) = \frac{1}{1+x}$$

Then

$$f(x)^2 = \frac{1}{(1-x)^2}$$
$$= \sum_{i=0}^{\infty} i + 1 choose 1 x^i$$
$$= \sum_{i=0}^{\infty} (i+1)x^i$$
$$[x^{2015}]f(x)^2 = 2015 + 1 = 2016$$

As well,

$$f(x)g(x) = \frac{1}{(1-x)(1+x)}$$
$$= \frac{1}{1-x^2}$$
$$= \sum_{i=0}^{\infty} (x^2)^i = \sum_{i=0}^{\infty} x^{2i}$$
$$[x^{2015}]f(x)g(x) = 0$$

Problem 3

Solution: By theorem of uniqueness of power series representation. We only need to prove equality of coefficients.

$$\begin{aligned}
&[x^n]A(x)(B(x)+C(x)) \\
&= \sum_{i=0}^{n}[x^i]A(x)[x^{n-i}](B(x)+C(x)) \\
&= \sum_{i=0}^{n}[x^i]A(x)[[x^{n-i}]B(x)+[x^{n-i}]C(x)] \\
&= \sum_{i=0}^{n}([x^i]A(x)[x^{n-i}]B(x)+[x^i]A(x)[x^{n-i}]C(n)) \\
&= \sum_{i=0}^{n}[x^i]A(x)[x^{n-i}]B(x)+\sum_{i=0}^{n}[x^i]A(x)[x^{n-i}]C(x) \\
&= [x^n]A(x)B(x)+[x^n]A(x)C(x) \\
&= [x^n](A(x)B(x)+A(x)C(x))
\end{aligned}$$

Problem 4

a)

$$\begin{aligned}
f(x) &= \sum_{i=0}^{\infty}(-3x)^i - \sum_{n=142}^{\infty}(-3x)^i \\
&= \frac{1-3^{142}x^{142}}{1+3x}
\end{aligned}$$

b)

$$h(x) = \sum_{i=1}\infty x^i$$

Notice that $g(x) = h(\frac{x}{1-x^2})$. This power series composition is well-defined because $[x^0](\frac{x}{1-x^2}) = 0$

$$g(x) = \sum i = 1^{\infty}(\frac{x}{1-x^2})^i$$
$$= (\frac{x}{1-x^2})\sum_{i=0}^{\infty}(\frac{x}{1-x^2})^i$$
$$= (\frac{x}{1-x^2})(\frac{1}{1-\frac{x}{1-x^2}})$$
$$= \frac{x}{1-x-x^2}$$

c)

g(f(x)) is not defined because g has some power series representation but the constant term of g(x) is non-zero.

Problem 5

$$\frac{1}{(1-x^3)^5} = \sum_{n=0}^{\infty}\binom{n+4}{4}x^{3n}$$
$$\frac{1}{1-3x^2} = \sum_{m=0}^{\infty}3^m x^{2m}$$

Where did the n go, why is there an n?

We get x^{11} from this product whenever we shoose m,n $\in$ N $\cup$ $\{0\}$ such that $x^{2+3n+2m} = x^11$
If and only if 2 + 3n + 2m = 11 if and only if 3n + 2m = 9
The solutions are n=3, m=0, n=1, m=3
$\therefore$ the coefficient of x^11 in $x^2(1-x^3)^{-5}(1-3x^2)^{-1}$ is $\binom{7}{4} + \binom{5}{4} \cdot 3^3$

Alternate Problem 2 Solution

$$f(x)^2 = (1 + x + x^2 + ...)(1 + x + x^2 + ...)$$

You'll get a contribution of 1 towarsd the coefficient of x^{2015} for each solution to $i + j = 2015$, $i, j \in N \cup \{0\}$.

It's not hard to see that there are 2016 such pairs: $\{(i, 2015-i)\}$ for $i = 0$ to 2015