# Math 239 Lecture 33

Graham Cooper

July 28th, 2015

## Hall's Theorem

**Hall's Theorem**: G bipartite (A,B). There is a matching that saturates A if and only if $\forall x \subseteq A$, $\underbrace{|N(x)| \geq |x|}_{Hall's Condition}$

**Proof:**
$\Longrightarrow$ (Done last class, whoops!)
$\Longleftarrow$ (contrapositive if no matchign saturates A, then $\exists x \subseteq A$ where $|N(X) < |X|$)

Let M be a maximum matching. By assumption, $|M| < |A|$
Let C be a minumum cover. By Konig's theorem $|C| = |M| < |A|$

Note that no edge joins A/C to B/C since none of these vertices are in the cover. So M(A/C) $\subseteq B \cap C$
So

$$|N(A/C)| \leq |B \cap C|$$
$$= |C| - |A \cap C|$$
$$< |A| - |A \cap C|$$

(since $|C| < |A|$)

$$= |A/C|$$

So $|N(A/C)| < |A/C|$ meaning A/C violates Hall's Condition

**Corollary:** If G is a k-regular bipartite graph with k $\geq$ 1 then G has a perfect matching

**Proof:** Suppose G has bipartition (A,B). We claim that $|A| = |B|$ : from assignment,

$$\sum_{v \in A} deg(v) = \sum_{v \in B} deg(v)$$
$$\sum_{v \in A} k \in \sum_{v \in B} k$$

So k$|A| = k|B|$ since $k \neq 0$, $|A| = |B|$

Let $x \subseteq A$. Any edge with one end in X must have the other end in N(x).
So $\sum_{v \in N(x)} deg(v) \geq \sum_{v \in X} deg(v)$
Since G is k-regular, $k|N(x)| \geq k|x|$

Since $k >= 1$, $|N(x)| \geq |X|$ so Hall's condition holds for all $X \leq A$ by Hall's Theorem, there is a matching. Since $|A| = |B|$, this matching is a perfect matching.

# Final Exam

Aug 5, 12:30 PAC
Covers everything
Look at assignment 12
Practice final
Posted theorems