Math 239 Tutorial 3

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May 26th, 2015

1)

a)

$$\frac{6-x+5x^2}{1-3x^2-3x^3} = \sum_{n=0}^{\infty} a_n x^n$$

$$\iff 6-x+5x^2 = (1-3x^2-2x^3)\sum_{n=0}^{\infty} a_n x^n$$

$$\iff 6-x+5x^2 = a_0 + a_1 x + (a_2 - 3a_0)x^2 + \sum_{n=3} \infty (a_n - 3a_{n-2} - 2a_{n-3})x^n$$

Equating Coefficients

$a_0 = 6$
$a_1 = -1$
$a_2 - 3a_0 = 5$
$a_2 = 5 + 18 = 23$
$a_n - 3a_{n-2} - 2a_{n-3} = 0$

Taking n = 5
$a_5 = 3a_3 + 2a_2$
$\implies a_5 = 9a_1 + 6a_0 + 2a_2 = 73$

b)

Observe that $1 - 3x^2 - 2x^3 = (1-2x)(1+x)^2$

And hence

$$A(x) = \frac{6-x+5x^2}{(1-2x)(1+x)^2} = \frac{C_1}{1-2x} + \frac{C_2}{1-x} + \frac{C_3}{(1+x)^2}$$

Find C_1, C_2, C_3 and use them to find an explicit formula for a_n

Solution:

$$\frac{6 - x + 5x^2}{(1-2x)(1+x)^2} = \frac{C_1}{1-2x} + \frac{C_2}{1-x}$$
$$= \frac{C_1(1+x)^2 + C_2(1-2x)(1+x) + C_3(1-2x)}{(1-2x)(1+x^2)}$$

after expanding

$$\frac{(C_1 + C_2 + C_3) + (2C_1 - C_2 - 2C_3)x + (C_1 - 2C_2)x^2}{(1-2x)(1+x)^2}$$

Equating Coefficients:

$$C_1 + C_2 + C_3 = 6$$
$$2C_1 - C_2 - 2C_3 = -1$$
$$C_1 - 2C_2 = 5$$
$$C_1 = 3$$
$$C_2 = -1$$
$$C_3 = 4$$

So

$$A(x) = \frac{3}{1-2x} + \frac{-1}{1-x} + \frac{4}{(1+x)^2}$$

$$= 3\sum_{n=1} \infty 2^n x^n - \sum_{n=0}^{\infty}(-1)^n x^n + 4\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$$
$$= \sum_{n=0}^{\infty}[3 \cdot 2^n + (4n+3)(-1)^n]x^n$$
$$a_n = 3 \cdot 2^n + (4n+3)(-1)^n$$

Taking n = 5

$$a_5 = 3 \cdot 32 + 23 - 1$$
$$= 73$$

2)

By Binomial Theorem:

$$[x^n](1-x)^k = (-1)^n\binom{-k}{n}$$

$$\binom{-k}{n} = \frac{(-k)(-k-1)...(-k-n+1)}{n!}$$
$$= \frac{(-1)^n k(k+1)...(n+k-1)}{n!}$$
$$= (-1)^k\frac{(n-k-1)!}{n!(k-1)!}$$
$$= (-1)^n\binom{n+k-1}{k-1}$$

Then

$$[x^n](1-x)^{-k}$$
$$= (-1)^n\binom{-k}{n}$$
$$= (-1)^n(-1)^n\binom{n+k-1}{k-1}$$
$$= (-1)^{2n}\binom{n+k-1}{k-1}$$
$$= \binom{n+k-1}{k-1}$$

3)

a)

Define α: {0,1} *rightarrow*
len $\rightarrow$ len + 1

Observe that if $\sigma = \sigma_1\sigma_2...\sigma_n \in S_n$ then
w(σ) = $\sum_{i=1}^{n}\alpha(\sigma_i)$

Then $S_n = \{0,1\}^n$ and so by the product lemma

$$= \Phi_{S_n}(x) = \Pi_{i=1}^{n}\Phi_{\{0,1\}}(x)$$
$$= (\Phi_{\{0,1\}}(x))^n$$
$$= (x + x^2)^n$$

b)

Observe that i) $S_i \cap S_j = \theta for i \neq j$
ii) $T = \bigcap_{i=0}^{\infty} S_i$

$\therefore$ by the sum lemma $\therefore$ $\Phi_T(x) = \frac{1}{1-(x+x^2)}$

$$\Phi_T(x) = \sum_{n=0} \infty \Phi_{S_n}(x)$$
$$= \sum_{n=0} \infty (x + x^2)^n$$

B/c $[x^0](x + x^2) = 0$
We can convert it to the power series above

4)

a)

We can think of S as $\{0,...9\}^6$ by appending leading zeros to "too short" integers."
Define $\alpha = \{0...9\} \to N$ $x \to x$
Then if $\sigma \in S$ we write $\sigma = \sigma_1\sigma_2...\sigma_6$ and note that w(σ) = $\sum_{i=1}^{6} \alpha(\sigma_i)$

$\therefore$ by the product lemma

$$\Phi_S(x) = \Pi^6\Phi_{\{0...9\}}(x)$$
$$= (\Phi_{\{0...9\}}(x))^6$$

It is easy to see that:

$$\Phi_{\{0..9\}}(x) = 1 + x + x^2 + ... + x^9$$

$$\Phi_S(x) = (1 + x + x^2 + ...x^9)^6$$
$$= (\frac{1 - x^{10}}{1 - x})^6$$

$\therefore$ the nuber of integers in S whose digits sum to k should be:

$$[x^k](\frac{1 - x^{10}}{1 - x})^6$$

b)

For each length l between 2 and 6 the string looks like this:
$(\sigma, \sigma_2, ...\sigma_{l-1}, \sigma_{l+1})$
For each l define T_l to be the set of all strings of length l within the desired propery.
Then:

$$F_T(x) = (x^3 + x^5 + ...x^{17})(1 + x + ...x^9)^{l-2}$$

Then:

$$\Phi_T(x) = \sum_{l=2}^{6} \Phi_{T_l}(x)$$
$$= \sum_{l=2}^{6} x^3(\frac{1 - x^{10}}{1 - x^2})(\frac{1 - x^{10}}{1 - x})^l$$

The number we need is $[x^k]\Phi_T(x)$