Math 239 Fall 2014 Assignment 1 Solutions

1. Let $n \in \mathbb{N}$. Define E_n to be the set of all subsets of $[n]$ of even cardinality, and define O_n to be the set of all subsets of $[n]$ of odd cardinality.

   (a) {3 marks} Define a bijection $f : E_n \to O_n$. Prove that for any $X \in E_n$, $f(X) \in O_n$. Provide the inverse of f. (Note: The mapping $f(A) = [n] \setminus A$ only works when n is odd. We would like you to come up with a bijection that works for all n.)

   Solution. One mapping is $f : E_n \to O_n$ where for any $X \in E_n$,

$$f(X) = \begin{cases} X \setminus \{1\} & \text{when } 1 \in X \\ X \cup \{1\} & \text{when } 1 \notin X \end{cases}$$

   We note that X has even size, and $f(X)$ either removes 1 element from X or adds 1 element to X. So $f(X)$ must have odd size, hence $f(X) \in O_n$.
   The inverse function is $f^{-1} : O_n \to E_n$ where for any $Y \in O_n$,

$$f^{-1}(Y) = \begin{cases} Y \setminus \{1\} & \text{when } 1 \in Y \\ Y \cup \{1\} & \text{when } 1 \notin Y \end{cases}$$

   (b) {2 marks} Illustrate your bijection by pairing up each element X of E_4 with its image $f(X)$ of O_4.

E_4 :	$\emptyset$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	$\{1,2,3,4\}$
O_4 :	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$

   (c) {2 marks} Determine (with proof) the cardinalities of E_n and O_n.

   Solution. Since f is a bijection between E_n and O_n, we can conclude that $|E_n| = |O_n|$. Also, if S is the set of all subsets of $[n]$, then $S = E_n \cup O_n$ is a disjoint union of sets. Therefore, $|S| = |E_n| + |O_n| = 2|E_n|$. Since we know that $|S| = 2^n$, we get $|E_n| = 2^{n-1}$ and $|O_n| = 2^{n-1}$.

   (d) {3 marks} Use the results in this question to give a combinatorial proof of the following identity:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

   Solution. Notice that $(-1)^k = 1$ when k is even, and $(-1)^k = -1$ when k is odd. By moving all the terms for odd k to the right hand side, we can rewrite the identity as

$$\sum_{\substack{k \text{ is even} \\ 0 \le k \le n}} \binom{n}{k} = \sum_{\substack{k \text{ is odd} \\ 0 \le k \le n}} \binom{n}{k}.$$

   We can prove this identity combinatorially as follows. Let S be the set of all subsets of $[n]$, and let S_k be the set of all subsets of $[n]$ of size k. We see that $E_n = S_0 \cup S_2 \cup \cdots \cup S_{2\lfloor n/2 \rfloor}$ and $O_n = S_1 \cup S_3 \cup \cdots \cup S_{2\lfloor n/2 \rfloor + 1}$. Since these are disjoint unions and $|E_n| = |O_n|$, we see that

$$\sum_{\substack{k \text{ is even} \\ 0 \le k \le n}} |S_k| = \sum_{\substack{k \text{ is odd} \\ 0 \le k \le n}} |S_k|.$$

   Since $|S_k| = \binom{n}{k}$, the result follows.

(e) {2 marks} Give an algebraic proof of the identity in part (d).

Solution. Recall that the Binomial theorem is $(1+x)^n = \sum_{k\geq 0}\binom{n}{k}x^k$. By substituting $x=-1$, we get

$$\sum_{k\geq 0}\binom{n}{k}(-1)^k = (1+(-1))^n = 0.$$

2. {4 marks} Let x, y, z, n be positive integers such that $x \leq y \leq z \leq n$. Consider the following set.

$$S = \{(X, Y, Z) \mid X \subseteq Y \subseteq Z \subseteq [n], |X| = x, |Y| = y, |Z| = z\}.$$

By counting S in two different ways, prove that

$$\binom{n}{x}\binom{n-x}{y-x}\binom{n-y}{z-y} = \binom{n}{z}\binom{z}{y}\binom{y}{x}.$$

Solution. We will count the set of all triples in S in two ways.

In the first method, we will choose X first. Since X is an x-subset of $[n]$, there are $\binom{n}{x}$ ways to pick X. After we have picked X, we note that Y must include X, so out of the y elements of Y, x of them are chosen. So we need to choose $y-x$ elements out of $[n]\setminus X$, which has size $n-x$. So for every X, there are $\binom{n-x}{y-x}$ possible Y. After we have picked X, Y, we note that Z must include Y, so out of the z elements of Z, y of them are chosen. So we need to choose $z-y$ elements out of $[n]\setminus Y$, which has size $n-y$. So for every choice of X and Y, there are $\binom{n-y}{z-y}$ ways to choose Z. In total, $|S| = \binom{n}{x}\binom{n-x}{y-x}\binom{n-y}{z-y}$.

In the second method, we will choose Z first. Since Z is a z-subset of $[n]$, there are $\binom{n}{z}$ ways to pick Z. After we have picked Z, we see that Y is a y-subset of Z, so there are $\binom{z}{y}$ ways to choose Y. After we have picked Y, we see that X is an x-subset of Y, so there are $\binom{y}{x}$ ways to choose X. In total, $|S| = \binom{n}{z}\binom{z}{y}\binom{y}{x}$, hence the identity holds.

3. {4 marks} Give a combinatorial proof of the following identity.

$$3^n = \sum_{i=0}^{n}\binom{n}{i}2^{n-i}.$$

(Hint: Start by finding a set whose cardinality is 3^n.)

Solution. Consider the set $S = \{1,2,3\}^n$, which consist of all n-tuples $(a_1,\ldots,a_n)$ where each $a_i \in \{1,2,3\}$. Clearly $|S| = 3^n$.

We partition S into $n+1$ sets $S_0,\ldots,S_n$ where for each $i = 0,\ldots,n$, S_i is the set of elements of S that contains exactly i 1's. We can count S_i by first deciding which i of the n spots are 1's, then fill in the remaining $n-i$ spots with either 2 or 3. There are $\binom{n}{i}$ ways to choose the i spots, and 2^{n-i} ways to fill in the remaining spots. So $|S_i| = \binom{n}{i}2^{n-i}$.

Since $S = S_0 \cup \cdots \cup S_n$ is a disjoint union,

$$3^n = \sum_{i=0}^{n}\binom{n}{i}2^{n-i}.$$

4. Let S be the set of all subsets of $[3]$.

(a) {2 marks} Let w be the weight function on S such that $w(\emptyset) = 0$, and for any nonempty set $A \in S$, $w(A)$ is the largest element in A. Determine the generating series $\Phi_S(x)$ with respect to w.

Solution. We can check the weight of each subset of $[3]$:

$$w(\emptyset) = 0, \quad w(\{1\}) = 1, \quad w(\{2\}) = 2, \quad w(\{3\}) = 3,$$

$$w(\{1,2\}) = 2, \quad w(\{1,3\}) = 3, \quad w(\{2,3\}) = 3, \quad w(\{1,2,3\}) = 3.$$

Using these weights, we see that

$$\Phi_S(x) = 1 + x + 2x^2 + 4x^3.$$

(b) {2 marks} Let w^* be the weight function on S such that for any $A \in S$, $w^*(A) = 3w(A) + 1$. Determine the generating series $\Phi_S^*(x)$ with respect to w^*.

Solution. The new weights are

$$w(\emptyset) = 1, \quad w(\{1\}) = 4, \quad w(\{2\}) = 7, \quad w(\{3\}) = 10,$$

$$w(\{1,2\}) = 7, \quad w(\{1,3\}) = 10, \quad w(\{2,3\}) = 10, \quad w(\{1,2,3\}) = 10.$$

Using these weights, we see that

$$\Phi_S(x) = x + x^4 + 2x^7 + 4x^{10}.$$

(c) {3 marks} In general, let T be a set and let w be a weight function on T. Let $\Phi_T(x)$ be the generating series with respect to w. For positive integers k, m, define w^* to be the weight function on T where for any $a \in T$, $w^*(a) = k \cdot w(a) + m$. Let $\Phi_T^*(x)$ be the generating series with respect to w^*. Use the definition of generating series to determine a relationship between $\Phi_T(x)$ and $\Phi_T^*(x)$.

Solution. Using the definition of generating series, we have

$$\begin{aligned}
\Phi_T^*(x) &= \sum_{\sigma \in T} x^{w^*(\sigma)} \\
&= \sum_{\sigma \in T} x^{k \cdot w(\sigma) + m} \\
&= \sum_{\sigma \in T} x^m \cdot (x^k)^{w(\sigma)} \\
&= x^m \Phi_T(x^k).
\end{aligned}$$