Math 239 Spring 2015 Tutorial Problems 2

The TA will cover a subset of these problems during the tutorial. Solutions will not be provided outside of the tutorial.

1. Let S_n be the set of all binary strings of length n. For each string σ, we define its weight $w(\sigma)$ to be the number of times 11 appears in the string, and let $\Phi_{S_n}(x)$ be the generating series of S_n with respect to w. For example, $w(1011) = 1, w(1111) = 3$.

   (a) Determine $\Phi_{S_4}(x)$.

   (b) Define w^* to be the weight function where $w(\sigma)$ is the number of times 00 appears in the string, and let $\Phi^*_{S_n}(x)$ be the generating series of S_n with respect to w^*. Prove that $\Phi_{S_n}(x) = \Phi^*_{S_n}(x)$ for all n.

2. Consider the following two power series:

$$f(x) = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{i \geq 0} x^i,$$
$$g(x) = 1 - x + x^2 - x^3 + x^4 - \cdots = \sum_{i \geq 0} (-x)^i$$

   Determine the coefficient of x^{2015} in $(f(x))^2$ and $f(x)g(x)$.

3. Prove the distributive property of power series: If $A(x), B(x), C(x)$ are formal power series, then $A(x)(B(x) + C(x)) = A(x)B(x) + A(x)C(x)$.

4. Determine a simplified rational expression for each of the following power series, or explain why it is not a power series.

   (a)
$$f(x) = \sum_{i=0}^{141} (-3x)^i.$$

   (b)
$$g(x) = \sum_{i \geq 1} \left(\frac{x}{1 - x^2} \right)^i$$

   (c)
$$g(f(x))$$

5. Determine the following coefficient.

$$[x^{11}]x^2(1 - x^3)^{-5}(1 - 3x^2)^{-1}$$