Math 239 Theorems and Definitions

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July 27th, 2015

1 Combinatorial Analysis

1.3 Binomial Coefficients

1.3.1 Theorem: For non-negative integers n and k, the number of k-element subsets of an n-element set is:

$$\frac{n(n-1)...(n-k+1)}{k!} = \binom{n}{k} = \binom{n}{n-k}$$

1.3.2 Theorem: For any non-negative integer n,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

1.3.3 Problem: For any non-negative integers n and k:

$$\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$$

1.4 Generating Series

1.4.2 Definition: Let S be a set of configurations with a weight function w. The generating series for S with respect to w is defined by:

$$\begin{aligned} \Phi_S(x) &= \sum_{\sigma \in S} x^{w(\sigma)} \\ &= \sum_{k \geq 0} a_k x^k \end{aligned}$$

1.4.3 Theorem: Let $\Phi_S(x)$ be the generating series for a finit set S with respect to a weight function w. Then,

- $\Phi_S(1) = |S|$
- the sum of the weights of the elements in S is $\Phi'_S(1)$, and
- the average weight of an element in S is $\Phi'_S(1)/\Phi_S(1)$

1.5 Formal Power Series

1.5.0 Definition: For a sequence of $(a_0, a_1, a_2...)$ which are rational numbers, then A(x) = $a_0 + a_1x + a_2x^2 + ...$ is called the formal power series. We say that a_n is the coefficient of x^n and we write $a_n = [x^n]A(x)$.
Also:

$$A(x) + B(x) = \sum_{n\geq 0}(a_n + b_n)x^n$$

$$A(x)B(x) = \sum_{n\geq 0}(\sum_{k=0}^{n} a_k b_{n-k})x^n$$

1.5.2 Theorem: Let $A(x) = a_0 + a_1x + a_2x^2 + ...$, $P(x) = p_0 + p_1x + p_2x^2 + ...$ and $Q(x) = 1 - q_1x - q_2x^2 - ...$ be formal power series. Then:

$$Q(x)A(x) = P(x)$$

if and only if for each n ≥ 0

$$a_n = p_n + q_1a_{n-1} + q_2a_{n-2} + ... + q_na_0$$

1.5.3 Corollary: Let P(x) and Q(x) be formal power series. If the constant term of Q(x) is non-zero, then there is a formal power series A(x) satisfying:

$$Q(x)A(x) = P(x)$$

Moreover, the solution A(X) is unique

1.5.4 Definition: We say that B(x) is the inverse of A(X) if

$$A(x)B(x) = 1$$

we denote this by B(x) = A(x)$^{-1}$ or by B(x) = $\frac{1}{A(x)}$

1.5.7 Theorem: A formal power series has an inverse if and only if it has a non-zero costant term. Moreover, if the constant term is non-zero, then the inverse is unique

1.5.8 Definition: The composition of formal power series A(x) = $a_0 + a_1x + a_2x^2 + ...$ B(x) is defined by:

$$A(B(x)) = a_0 a_1 B(x) + a_2(B(x))^2 + ...$$

However unlink for polynomials, this composition operation is not always well defined. Consider, for example, the case that A(X) = 1 + x + x^2 + ... and B(x) = (1+x). Then

$$A(B(x)) = 1 + (1 + x) + (1 + x)^2 + ...$$

The constant term of the right-hand side has non-zero contributions from an infinite number of terms, so A(B(x)) is not a formal power series. The following result shows that A(B(x)) is well-defined so long as B(x) has its constant term equal to zero (that is B(0) = 0).

1.6 The Sum and Product Lemma

1.6.1 (Sum Lemma) Theorem: Let (A,B) be a partition of a set S. (That is, A and B are disjoint sets whose union is S.) Then,

$$\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$$

1.6.2 (Product Lemma) Theorem: Let A and B be sets of configurations with weight functions α and β respectively. If $w(\sigma) = \alpha(a) + \beta(b)$ for each $\sigma = (a, b) \in A \times B$, then

$$\Phi_{A\times B}(x) = \Phi_A(x) \cdot \Phi_B(x)$$

1.6.5 Theorem: For any positive integer k and non-negative integer n,

$$(1 - x)^{-k} = \sum_{n\geq 0} \binom{n + k - 1}{k - 1} x^n$$

2 Copositions and Strings

2.1 Compositions of an Integer

2.1.1 Definition For non-negative integers n and k, a composition of n with k parts is an ordered list $(c_1, ...c_k)$ of positive integers $c_1, ...c_k$ such that

$c_1 + ... + c_k = n$. The positive integers $c_1 ... c_k$ are called the parts of the composition. There is one composition of 0, the empty composition, which is a composition with 0 parts.

2.3 Binary Strings

2.3.1 Definition: A binary string or {0,1}-string is a string of 0's and 1's, its length is the number of occurences of 0 and 1 in the string. We use ϵ to denote the empty string of length 0

2.4 Unambiguous Expressions

Definition: We say that the expression AB is ambiguous if there exists distinct pairs (a_1, b_1) and (a_2, b_2) in $A \times B$ with $a_1 b_1 = a_2 b_2$ - otherwise we say that AB is an unambigious expression.

If A and B are finite sets, then AB is unambiguous if and only if $|AB| = |A \times B|$

2.5 Decomposition Rules

Basic string decompositions, all are unambiguous

- Decompose a string after each 0 or 1

$$S = \{0, 1\}^*$$

- Decompose a string after each occurence of 0. Each piece in the decomposition will be from {0,10,110,...} = $\{1\}^*\{0\}$ except possibly for the last piece which may consist only of 1s. This gives rise to the expression:

$$S = (\{1\}^*\{0\})^*\{1\}^*$$

- Decompose a string after each block of 0s. Each piece in the decomposition, except possibly the first and last pieces, will constist of a block of 1s followed by a block of 0s. The first piece may consist only of a block of 0s and the last piece may consist only of a block of 1s. This gives rise to the expression:

$$S = \{0\}^*(\{1\}\{1\}^*\{0\}\{0\}^*)^*\{1\}^*$$

2.6 Sum and Product Rules for Strings

2.6.1 Theorem: Let A,B be a set of {0,1}-strings.

1. If $A \cap B = \emptyset$ then

$$Phi_{A \cup B}(x) = \Phi_A(x) + \Phi_B(x)$$

2. If the expression AB is unambiguous then:

$$\Phi_{AB}(X) = \Phi_A(x)\Phi_B(x)$$

3. If the expression A^* is unambiguous then:

$$\Phi_{A^*}(x) = (1 - \Phi_A(X))^{-1}$$

2.8 Recursive Decompositions of Binary Strings

All strings can be written as:

- The empty string is in S
- Any other element of S consists of a symbol (either 0 or 1) followed by an element of S

Recursive definition leads directly into the recusrive decomposition:

$$S = \{\epsilon\} \cup \{0,1\}S$$

3 Recurrences, Binary Trees and Sorting

3.1 Coefficients of Rational Functions

<u>3.1.1 Lemma:</u> If f(x) is a polynomial of degree less than r, then there is a polynomial P(x) with degree less than r such that

$$[x^n]\frac{f(x)}{(1-\theta x)^r} = P(n)\theta^n$$

<u>3.1.2 Lemma:</u> Suppose f and g are polynomials with deg(f) < deg(g). If g(x) = $g_1(x)g_2(x)$ where g_1, g_2 are coprime, there are polynomials f_1, f_2 such that $deg(f_i) < deg(g_i)$ for i = 1,2 and

$$\frac{f(x)}{g(x)} = \frac{f_1(x)}{g_1(x)} + \frac{f_2(x)}{g_2(x)}$$

<u>3.1.3 Theorem:</u> Suppose f and g are polynomials such that deg(f) < deg(g). If for i = 1...k there are complex numbers θ_i and positive integers m_i such that

$$g(x) = \prod_i (1-\theta_i x)^{mi}$$

then there are polynomials P_i such that deg(P_i) < m_i and

$$[x^n]\frac{f(x)}{g(x)} = \sum_{i=1}^{k} P_i(n)\theta_i^n$$

3.2 Solutions to Recurrence Equations

<u>3.2.1 Theorem:</u> Let $C(x) = \sum_{n\geq 0} c_n x^n$ where the coefficients c_n satisfy the recurence:

$$c_n + q_1 c_{n-1} + ... + q_k c_{n-k} = 0$$

IF

$$g(x) := 1 + q_1 x + ... + q_k x^k$$

there is a polynomial f(x) with degree less than k such that:

$$C(x) = \frac{f(x)}{g(x)}$$

3.2.2 theorem Suppose $(c_n)_{n\geq 0}$ satisfies the recurrence equation (3.2.1) if the characteristic polynomial of this recurrence has root β_i whith multiplicity m_i, for i = 1...j, then the general solution to (3.2.1) is:

$$c_n = P_1(n)\beta_1^n + ... + P_j(n)\beta_j^n$$

where $P_i(n)$ is a polynomial in n with degree less than m_i and these polynomials are determined by the $c_0...c_{k-1}$.

3.3 Nonhomogeneous Recurrence Equations

3.3.1 Theorem: Suppose that $a_0, a_1, ...$ is a solution to (3.3.1)(any solution without checking the initial conditions). Then the general solution to (3.3.1) is given by:

$$b_n = a_n + c_n$$

where c_n is given by theorem 3.2.2 and the k constants $b_{11}...b_{j,m_j}$ in c_n can be chosent to fit the initial conditions for b_n

3.6 Binary Trees

Definition A binary tree is either:

- the empty tree
- a tree with a fixed root vertex such that each verte has a left branch and a right branch (either of which may be empty)

3.6.2 Theorem: The number of binary trees with $n \geq 0$ vertices is:

$$\frac{1}{n+1}\binom{2n}{n}$$

3.7 The Binomial Series

3.7.1 Theorem the Binomial Series: For any rational number a:

$$(1+x)^a = \sum_{k\geq 0}\binom{a}{k}a^k$$

$$(1+x)^{-n} = \sum_{k\geq 0} \binom{-n}{n}(-x)^k = \sum_{k\geq 0} \binom{n+k-1}{n-1} x^k$$

3.7.2 Lemma:

$$(1-4x)^{\frac{1}{2}} = 1 - 2\sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n$$

4 Introduction to Graph Theory

4.1 Definitions

4.1.1 Definition: A graph G is a finite non-empty set, V(G) of objects, called vertices, together with a set E(G), of unordered pairs of distinct vertices. The elements of E(G) are called edges.
More definitions!

- Adjacent if an edge connects two vertices
- Incident is the edge between two adjacent vertices (or the edge joins)
- Degree of vertex u is the number of vertices adjacent with u

4.2 Isomorphism

4.2.1 Definition: Two graphs G_1 and G_2 are isomorphic if there exists a bijection f : V(G_1) $\rightarrow$ V(G_2) such that vertices f(u) and f(v) are adjacent in G_2 if and only if u and v are adjacent in G_1.

4.3 Degree

4.3.1 Theorem: For any graph G we have:

$$\sum_{v\in V(G)} deg(v) = 2|E(G)|$$

4.3.2 Corollary: The number of vertices of odd degree is even

4.3.3 Corollary: The average degree of a vertex in the graph G is:

$$\frac{2|E(G)|}{|V(G)|}$$

4.3.4 Definition: A complete graph is one in which all pairs of distinct vertices are adjacent. (Thus each vertex is joined to every other vertex). The complete graph with p vertices is denoted by K_p $p \geq 1$

4.4 Bipartite Graphs

Definition: A grpah in which the vertices can be partitioned into two sets A and B, so that all edges join a vertex in A to a vertex in B is called a bipartite graph.

4.4.1 Definition: For $n \geq 0$ the n-cube is the graph whose vertices are the {0,1}-strings of length n, and two strings are adjacent if and only if they differe in exactly one position.

4.5.1 Defintion Adjacency Matrix (I DONT HTINK WE DID THIS)

4.5.2 Definition Incidence Matrix (I DONT THINK WE DID THIS??)

4.6 Paths and Cycles

4.6.1 Definition: A subgraph of a graph G is a graph whose vertex set is a subset U of V(G) and whose edge set is a subset of those edges in G that have both vertices in U

Definitions!

- Spanning graph of G if a subgraph has all of the vertexes in G
- A proper-subgraph of G if the subgraph is not equal to G
- A walk in a graph G from v_0 to v_n $n \geq 0$ is an alternating sequence of vertices and edges in G. The length of a walk is the number of edges in it.

- A closed walk is when the walk starts and ends at the same point
- A path is a walk in which all vertices are distinct

4.6.2 Theorem: If there is a walk from vertex x to vertex y in Gm then there is a path from x to y in G.

4.6.3 Corollary: Let x,y,z be vertices of G. If there is a path from x to y in G and a path from y to z in G then there is a path from x to z in G.

Hamilton Cycle: A spanning cycle in graph.

4.8 Connectedness

4.8.1 Definition: A graph G is connected if for each two vertices x and y there is a path from x to y.

4.8.2 Theorem Let G be a graph and let v be a vertex in G. If for each vertex w in G there is a path from v to w in G, then G is connected

4.8.4 Definition A component of G is a subgraph C of G such that

- C is connected
- No subgraph of G that properly contains C is connected

4.8.5 Theorem A graph G is not connected if and only if there exists a proper nonempty subset X of V(G) such that the cut induced by X is empty.

4.9 Bridges

4.9.1 Definition: An edge e of G is a bridge if G-e has more components than G.

4.9.2 Lemma: If e = {x,y} is a bridge of a connected graph G, then G-e has precisely two components; furthermore, x and y are in different components.

4.9.3 Theorem: An edge e is a bridge of a graph G if and only if it is not contained in any cycle of G.

4.9.4 Corollary: If there are two distinct paths from vertex u to vertex v in G then G contains a cycle

4.10 (NOT IN BOOK) Euler Tours