

# CS240 Tutorial 2

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May 13th, 2015

## Topics:

- small o proof
- loop analysis
- asymptotic proofs

**Recall:** We say that  $f(n) \in o(g(n)) \iff \forall \epsilon > 0, \exists n_0 > 0$  such that  $f(n) \leq \epsilon g(n)$ .

## Q1)

$2^{\sqrt{\log n}} \in o(n)$  : Show from first principals.

Goal:  $2^{\sqrt{\log n}} < cn$

Observe that  $2^{\sqrt{\log n}} \underset{1}{<} 2^{\log_4 n} \underset{2}{<} cn$

What values of n does 1 hold?

$$\sqrt{\log n} < \log_4 n$$

$$\sqrt{\log n} < 1/2 \log(n)$$

$$2 < \sqrt{\log n}$$

$$4 < \log(n)$$

$$16 < n$$

$$2^{\log_4 n} < cn$$

$$\sqrt{n} < cn$$

$$1/c < \sqrt{n}$$

$$1/c^2 < n$$

Pick  $n_0 = \max(16, 1/c^2)$

**Q2)**

Analyze the following:

```

for i = 1 to n {
  for j = 1 to i {
    k = j
    while k > 1 {
      k = k / 2
    }
  }
}

```

$$\begin{aligned}
 T(n) &= \sum_{i=1}^n \sum_{j=1}^i \log j \\
 &= \sum_{i=1}^n \log i!
 \end{aligned}$$

$$\begin{aligned}
 \text{Upper bound: } &\leq \sum_{i=1}^n \log n! \\
 &\leq \sum_{i=1}^n cn \log n
 \end{aligned}$$

Recall that

$$\log n! \in O(n \log n) \implies \sum_{i=1}^n cn \log n = cn^2 \log n \in O(n^2 \log n)$$

Lower Bound:

$$\begin{aligned}
 &\sum_{i=1}^n \log i! \\
 &\geq \sum_{i=n/2+1}^n \log i! \\
 &\geq \sum_{i=n/2+1}^{\log n/2!} \log i! \\
 &n/2(c(n/2) \log n/2) = n/2 \log n/2! \\
 &\geq c(n/4)(\log n/2) \\
 &\in \Omega(n^2 \log(n))
 \end{aligned}$$

**Q3)**

```
for i = 1 to n
  j = i
  while j ≤ n
    j=2j
```

Aside:

After  $t$  iterations(of the inner loop),  $j = i2^t$

$i2^{t-1} < n < i2^t$

$t = \log n/i$

Answer:

$$\begin{aligned} T(n) &= \sum_{i=1}^n \log n/i \\ &= \sum_{i=1}^n n \log n - \log i \\ &= n \log n - \sum_{i=1}^n \log i \\ &= n \log n - \log n! \\ &\approx n \log n - \log \left(\left[\frac{n}{e}\right]^n\right) \\ &= n \log n - n \log \left(\frac{n}{e}\right) \\ &= n \log n - (n \log n - n \log e) \\ &= n \log e \in \Theta(n) \end{aligned}$$

**ASIDE: Stirling's Approximation:**

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

or

$$n! \approx \left(\frac{n}{e}\right)^n$$

**Q4**

Prove or disprove:

If  $f(n) \in \Theta(g(n))$  is  $f(n) - g(n) \in \Theta(1)$

This is false, as shown above

$$f(n) = n^2, g(n) = n^2 + n$$

$$f(n) \in \Theta g(n)$$

$$f(n) - g(n) = n \notin \Theta(1)$$

**Q5**

If  $f(n) \in O(g(n))$ , and  $g(n) \in O(h(n))$ , is  $f(n) \in O(h(n))$

True  $\rightarrow f(n) \leq cg(n), \forall n \geq n_0$

$g(n) \leq c'h(n), \forall n \geq n'_0$

$f(n) \leq cg(n) \leq cc'h(n)$

Pick  $c' = cc'$

$f(n) \leq c''h(n), \forall n \geq \max(n_0, n'_0)$

**Q6**

If  $f(n) \in O(g(n))$ , and  $g(n) \in \Omega(h(n))$ , is  $f(n) \in \Omega(h(n))$

Consider  $f(n) = n$  and  $g(n) = n^3$  and  $h(n) = n^2$

check:  $f(n) \in O(g(n))$  - True

$g(n) \in \Omega(h(n))$  - True

$f(n) \in \Omega(h(n))$  - False