Math 239 Fall 2014 Assignment 3 Solutions

1. Let $n \in \mathbb{N}$. Suppose we have an unlimited supply of Canadian loonies and toonies (they are worth $\$1, \$2$ each, respectively). Note that all loonies are identical, and all toonies are identical. You answers below can be expressed as certain coefficients of simplified power series.

   (a) {3 marks} How many ways can we make n dollars using loonies and toonies?

   Solution. We can model our set as $L \times T$ where $L = \{0, 1, 2, 3, \ldots\}$ and $T = \{0, 2, 4, 6, \ldots\}$. Then each collection of coins can be represented by $(a, b) \in T \times L$, and we define the weight to be the value of this collection, which is $w(a, b) = a + b$. Using $\alpha(a) = a$ for both L and T, we find that

   $$\Phi_L(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$
   $$\Phi_T(x) = 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2}.$$

   Using the product lemma,

   $$\Phi_{L\times T}(x) = \Phi_L(x) \cdot \Phi_T(x) = \frac{1}{(1-x)(1-x^2)}.$$

   The answer to our question is then $[x^n]\frac{1}{(1-x)(1-x^2)}$.

   (b) {3 marks} How many ways can we make n dollars using loonies and toonies such that the number of loonies is different from the number of toonies?

   Solution. We will partition all possibilities into different cases depending on the number of loonies used. Let $T_k = T \setminus \{2k\}$. The set we are enumerating is then

   $$S = (\{0\} \times T_0) \cup (\{1\} \times T_1) \cup (\{2\} \times T_2) \cup \cdots = \bigcup_{k\geq 0} (\{k\} \times T_k).$$

   Now the generating series for $\{k\} \times T_k$ is

   $$\Phi_{\{k\}\times T_k}(x) = x^k \cdot \left(\frac{1}{1-x^2} - x^{2k}\right) = \frac{x^k}{1-x^2} - x^{3k}.$$

   So by sum lemma,

   $$\Phi_S(x) = \sum_{k\geq 0}\left(\frac{x^k}{1-x^2} - x^{3k}\right) = \sum_{k\geq 0}\frac{x^k}{1-x^2} - \sum_{k\geq 0} x^{3k} = \frac{1}{1-x^2}\frac{1}{1-x} - \frac{1}{1-x^3} = \frac{x + x^2 - 2x^3}{(1-x)(1-x^2)(1-x^3)}.$$

   The answer is the coefficient of x^n in this series.

2. {2 marks} Which result from number theory implies the following identity?

   $$\frac{1}{1-x} = \prod_{k\geq 0}(1 + x^{2^k})$$

   Solution. The right hand side represents the generating series for the set

   $$S = \{0, 1\} \times \{0, 2\} \times \{0, 4\} \times \{0, 8\} \times \cdots$$

   with the weight function $w(a_1, a_2, \ldots) = a_1 + a_2 + \cdots$. This is a sum of distinct powers of 2. The coefficient of x^n is then the number of ways that n can be expressed as a sum of distinct powers of 2. Since this coefficient is 1 on the left hand side, it represents the result that "every non-negative integer can be uniquely expressed as distinct powers of 2." In other words, there is only one possible binary representation for each non-negative integer.

3. {5 marks} Let $n \in \mathbb{N}$. How many compositions of n are there with 6 or 7 parts, and every part is an odd integer at least 3? Find the appropriate generating series, and then give an explicit formula as your answer (you may need to break into two cases depending on the parity of n).

Solution. Let $A = \{3, 5, 7, 9, \ldots\}$ be the set of all odd integers at least 3. Then the set of all compositions satisfying our requirements is

$$S = A^6 \cup A^7.$$

Using the weight function $\alpha(a) = a$ for A, we see that

$$\Phi_A(x) = x^3 + x^5 + x^7 + x^9 + \cdots = \frac{x^3}{1-x^2}.$$

Using the weight function w on S where the weight of a composition is the sum of it parts, we see that

$$\Phi_S(x) = \Phi_{A^6}(x) + \Phi_{A^7}(x) = \left(\frac{x^3}{1-x^2}\right)^6 + \left(\frac{x^3}{1-x^2}\right)^7 = \frac{x^{18}}{(1-x^2)^6} + \frac{x^{21}}{(1-x^2)^7}.$$

Notice that only even powers of x at least 18 appear in $\frac{x^{18}}{(1-x^2)^6}$, and only odd powers of x at least 21 appear in $\frac{x^{21}}{(1-x^2)^7}$. So the answer to our question is

$$\begin{aligned}
[x^n]\Phi_S(x) &= \begin{cases} [x^n]\frac{x^{18}}{(1-x^2)^6} & n \text{ is even} \\ [x^n]\frac{x^{21}}{(1-x^2)^7} & n \text{ is odd} \end{cases} \\
&= \begin{cases} [x^{n-18}]\frac{1}{(1-x^2)^6} & n \text{ is even}, n \geq 18 \\ [x^{n-21}]\frac{1}{(1-x^2)^7} & n \text{ is odd}, n \geq 21 \end{cases} \\
&= \begin{cases} \binom{\frac{n-18}{2}+6-1}{6-1} & n \text{ is even}, n \geq 18 \\ \binom{\frac{n-21}{2}+7-1}{7-1} & n \text{ is odd}, n \geq 21 \end{cases} \\
&= \begin{cases} \binom{\frac{n}{2}-4}{5} & n \text{ is even}, n \geq 18 \\ \binom{\frac{n-1}{2}-4}{6} & n \text{ is odd}, n \geq 21 \end{cases}
\end{aligned}$$

4. {4 marks} By taking the difference of two generating series, prove that for $n \geq 2$, the number of compositions of n with even number of parts is equal to the number of compositions of n with odd number of parts.

Solution. We can enumerate the set of all compositions with odd number of parts by $O = \bigcup_{k \geq 0} \mathbb{N}^{2k+1}$. Also, we can enumerate the set of all compositions with even number of parts by $E = \bigcup_{k \geq 0} \mathbb{N}^{2k}$. If we look at the difference between the generating series for the two sets, we get

$$\begin{aligned}
\Phi_O(x) - \Phi_E(x) &= \sum_{k \geq 0} \left(\frac{x}{1-x}\right)^{2k+1} - \sum_{k \geq 0} \left(\frac{x}{1-x}\right)^{2k} \\
&= \left(\sum_{k \geq 0} \left(\frac{x}{1-x}\right)^{2k}\right)\left(\frac{x}{1-x} - 1\right) \\
&= \frac{1}{1 - \frac{x^2}{(1-x)^2}} \frac{-1+2x}{1-x} \\
&= \frac{(1-x)^2}{1-2x} \frac{-1+2x}{1-x} \\
&= x - 1
\end{aligned}$$

For $n \geq 2$, since the coefficient is 0, it means that $[x^n]\Phi_O(x) = [x^n]\Phi_E(x)$. Hence the number of compositions of n (for $n \geq 2$) with even number of parts is equal to those with odd number of parts.

5. Let $n \in \mathbb{N}$. Define S_n to be the set of compositions of n where no part is equal to 2, and let $a_n = |S_n|$.

(a) {3 marks} Prove that

$$a_n = [x^n]\frac{1-x}{1-2x+x^2-x^3}.$$

Solution. Let $A = \mathbb{N} \setminus \{2\}$. Then the set of all compositions where no part is 2 is

$$S = \bigcup_{k \geq 0} A^k.$$

Using the usual weight function for compositions, we get

$$\Phi_A(x) = \frac{x}{1-x} - x^2 = \frac{x - x^2 + x^3}{1-x},$$

$$\Phi_S(x) = \sum_{k\geq 0} \Phi_{A^k}(x) = \sum_{k \geq 0}\left(\frac{x-x^2+x^3}{1-x}\right) = \frac{1}{1-\frac{x-x^2+x^3}{1-x}} = \frac{1-x}{1-2x+x^2-x^3}.$$

Therefore, $a_n = [x^n]\frac{1-x}{1-2x+x^2-x^3}$.

(b) {4 marks} The power series gives the recurrence $a_n - 2a_{n-1} + a_{n-2} - a_{n-3} = 0$ for $n \geq 3$. Rearranging gives

$$a_n + a_{n-2} = 2a_{n-1} + a_{n-3}.$$

Give a combinatorial proof of this recurrence using bijections. (Hint: One way to do this is to partition $S_n \cup S_{n-2}$ into four sets, and find four different bijections. Relevant compositions for $n = 5$ are included on the next page.)

Solution. We partition S_n into three parts: let A be those compositions in S_n whose last part is 1; let B be those compositions in S_n whose last part is 3; and let C be those compositions in S_n whose last part is at least 4.

We will need two copies of S_{n-1}. For one copy, we will partition it into two parts: let X be those compositions in S_{n-1} whose last part is 1; let Y be those compositions in S_{n-1} whose last part is at least 2.

We will create four bijections, from A, B, C, S_{n-2} to S_{n-1}, X, Y, S_{n-3}.

- Define $f_1 : A \to S_{n-1}$ by $f_1(a_1, \ldots, a_{k-1}, a_k) = (a_1, \ldots, a_{k-1})$ (i.e. we remove the last part). Note that since the last part $a_k = 1$, removing this part results in a composition of $n-1$, and still does not have a 2. The inverse is $f_1^{-1} : S_{n-1} \to A$ where $f_1^{-1}(b_1, \ldots, b_l) = (b_1, \ldots, b_l, 1)$. (This is illustrated by the blue arrows in the diagram.)
- Define $f_2 : B \to S_{n-3}$ by $f_2(a_1, \ldots, a_{k-1}, a_k) = (a_1, \ldots, a_{k-1})$ (same function as f_1, we remove the last part). Note that since the last part $a_k = 3$, removing this part results in a composition of $n-3$, and still does not have a 2. The inverse is $f_2^{-1} : S_{n-3} \to B$ where $f_2^{-1}(b_1, \ldots, b_l) = (b_1, \ldots, b_l, 3)$. (This is illustrated by the green arrows in the diagram.)
- Define $f_3 : C \to Y$ by $f_3(a_1, \ldots, a_{k-1}, a_k) = (a_1, \ldots, a_{k-1}, a_k - 1)$ (i.e. we subtract 1 from the last part). Note that since the last part $a_k \geq 4$, this part is at least 3 after subtracting 1, so it is still a composition in Y. The inverse is $f_3^{-1} : Y \to C$ where $f_3^{-1}(b_1, \ldots, b_{l-1}, b_l) = (b_1, \ldots, b_{l-1}, b_l + 1)$. (This is illustrated by the red arrows in the diagram.)
- Define $f_4 : S_{n-2} \to X$ by $f_4(a_1, \ldots, a_k) = (a_1, \ldots, a_k, 1)$ (i.e. we added a part 1 at the end). This is a composition in S_{n-1} whose last part is 1, so it is in X. The inverse is $f_4^{-1} : X \to S_{n-2}$ where $f_4^{-1}(b_1, \ldots, b_{l-1}, b_l) = (b_1, \ldots, b_{l-1})$. (This is illustrated by the orange arrows in the diagram.)

Since these are bijections, $|A| = |S_{n-1}|$, $|B| = |S_{n-3}|$, $|C| = |Y|$, $|S_{n-2}| = |X|$. So

$$\begin{aligned} a_n + a_{n-2} &= |S_n| + |S_{n-2}| = |A| + |B| + |C| + |S_{n-2}| \\ &= |S_{n-1}| + |S_{n-3}| + |Y| + |X| = |S_{n-1}| + |S_{n-3}| + |S_{n-1}| \\ &= 2a_{n-2} + a_{n-3}. \end{aligned}$$

$$|S_n| + |S_{n-2}| = 2|S_{n-1}| + |S_{n-3}|$$