$$A(x) = \sum a_n x^n = \frac{f(x)}{g(x)}$$

$$g(x) = (1 - r_1 x)^{e_1} \cdots (1 - r_h)^{e_h}$$

$$\Rightarrow a_n = P(n)\, r_1^n + \cdots + P_h(n)\, r_h^b$$

$P_1(n)$: poly deg $e_1 - 1$

Charastic polynomial of $g(x)$ is

$$g^*(x) = (x - r_1)^{e_1} \cdots (x - r_h)^{e_k}$$

Then $r_1, \ldots, r_h$ are roots of the char poly w/ multipliesr $e_1, \ldots, e_k$

Ex: $g(x) = 1 - 7x + 10x^2 = (1-2x)(1-5x)$

$$g^*(x) = (x-2)(x-5) = x^2 - 7x + 10$$

In general, if:

$$g(x) = 1 + q_1 x + q_2 x^2 + \dots + q_k x^k$$

M's characteristic poly is

$$g^*(x) = x^k + q_1 x^{k-1} + q_2 x^{k-2} + \dots + q_k$$

(same as $g(x)$ except i is swapped with $k-i$)

Solving reccurences

Ex: $\{a_n\}$ satifies

$a_0 = 1, a_1 = 4$

$a_n - 3a_{n-1} + 2a_{n-2} = 0$

for $n \geq 2$

Find an explicit formula for a_n

Let $A(x) = \sum_{n \geq 0} a_n x^n$

(Previously: $A(x) = \dfrac{f(x)}{1 - 3x - 4x^2}$)

$a_n - 3a_{n-1} + 2a_{n-2} = 0$ (can convert back & forth)

(For each equation of the recurrence, multiply by x^n and sum them)

$(\cdot x^2)$ $a_2 x^{\boxed{2}} - 3a_{\boxed{1}} x^{\boxed{2}} + 2a_{\boxed{0}} x^{\boxed{2}} = 0$ $(n=2)$

x^3 $a_{\boxed{3}} x^{\boxed{3}} - 3a_{\boxed{2}} x^{\boxed{3}} + 2a_{\boxed{1}} x^{\boxed{3}} = 0$ $(n=3)$

$a_n x^n - 3a_{\boxed{n-1}} x^{\boxed{n}} + 2a_{\boxed{n-2}} x^{\boxed{n}} = 0$

$\sum_{n \geq} a_n x^{\boxed{n}} - 3\sum a_n x^{\boxed{n+1}} + 2\sum_{n \geq 0} a_n x^{\boxed{n+2}} = 0$

$$\sum_{n\geq 2} a_n x^n - 3x \sum_{n\geq 1} a_n x^n + 2x^2 \sum_{n\geq 0} a_n x^n = 0$$

$$(A(x) - a_1 x - a_0) - 3x(A(x) - a_0) + 2x^2 A(x) = 0$$

$$A(x) = \frac{1+x}{1-3x+2x^2}$$

Char poly is $x^2 - 3x + 2 = (x-2)(x-1)$

Roots $2, 1$ so

$$a_n = A 2^n + B 1^n$$

Use initial condition to solve A, B

$A = 3$

$$a_0: \quad A + B = 1$$
$$2A + B = 4 \Bigg\} \quad B = 2$$

shortcut

In general:

if the recurrence is

$$a_n + c_{n-1} a_{n-1} + \dots + c_{n-k} a_{n-k} = 0$$

its char poly is

$$x^k + c_{n-1} x^{k-1} + c_{n-2} x^{k-2} + \dots + c_{n-k}$$

Ex: $a_0 = 1,\ a_2 = a_2 = 1$

$$a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 0 \text{ for } n \geq 3$$

Char poly $x^3 - 3x^2 + 3x - 1 = (x-1)^3$

One root 1 w/ multiplicity 3

So $a_n = (A + Bn + Cn^2)(1)^n$

a_0: $A = 1$

a_1: $A + B + C = 2$

a_2: $A + 2B + 4C = 1$

$$\begin{cases} A = 1 \\ B = 2 \\ C = -1 \end{cases}$$

$\Rightarrow a_n = 1 + 2n$

Ex: $a_0 = 7$ $a_1 = 10$ $a_2 = 13$

$a_n - 7a_{n-1} - 15a_{n-2} - 9a_{n-3} = 0$

For $n \geq 3$

Char poly : $x^3 - 7x^2 + 15x - 9$

$= (x-3)^2 (x-1)$

Root 3 has multiplicity $\boxed{2}$

Root 1 has multiplicity $\boxed{1}$

$a_n = \underbrace{(A + Bn)}_{2} 3^n \underbrace{(C)}_{1} 1^n$

$a_0 = A + C = 7 \qquad A = 3$

$B = -1$

a_1 $3A + 3B + C = 10$ $\quad C = 4$

a_2 $9A + 18B + C = 13$

So $a_n = (3 - n) \cdot 3^n + 4$

Q: $f_0 = 0$ $\quad f_1 = 1$ $\quad f_n - f_{n-1} - f_{n-2} = 0$
for $n \geq 2$

Char poly: $x^2 - x - 1$ for $n \geq 2$

$x = \frac{1 \pm \sqrt{5}}{2}$ $\quad \frac{-b \pm \sqrt{4a - b^2}}{}$ whatever the quadratic formula is

So $a_n = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) n$

$$a_0 : A + B = 0$$

$$a_1 : A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$A = \frac{1}{\sqrt{5}} \qquad B = -\frac{1}{\sqrt{5}}$$

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$