Math 239 - Lecture 5

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Generating Series

Recal: Set S, weight w:

$$\Phi_s(x) = \sum_{\delta \in S} x^{w(\delta)} = \sum_{k \geq 0} a_k x^k$$

a_x = number of things in S of weight k

Example: How many ways can we throw 2 ∞ sided dice to get a sum of k?
$(a, b) \in N \times N$
Let $S = N \times N$ Define w(a,b) = a+b.
Coeff of x^k in $\Phi_s(x)$ is k-1
So $\Phi_s(x) = x^2 + 2x^3 + 3x^5 + 4x^5 + ... = \sum_{k \geq 2}(k - 1x^k)$
$= \frac{x^2}{(1-x)^2}$

Notes on Generating Series:

1. General Steps (given a counting problem):
   (a) Define a set of objects S
   (b) Define Weight function w related to our problem
   (c) Find Generating series of S with respect to w, $\Phi_s(x)$
   (d) The answer is in some coeff of $\Phi_s(x)$
2. X is a literal, we doo not put values into x. The coeff of x^k keeps track of answers to counting problems.
3. Later problems involve finding generating seriesfirst then possibly finding the coefficient.

Q: How many binary strings have no 3 consecutive 1's?
A: The answer is the coeff of x^n in $\frac{1+x+x^2}{1-x-x^2-x^3}$

Formal Power Series

Definition: Let $(a_0, a_1, a_2, ...)$ be a sequence of numbers. The formal power series associated with $\{a_n\}_{n\geq 0}$ is:

$$A(x) = a_0 + a_1x + a_2x^2 + ...$$
$$= \sum_{k\geq 0} a_kx^k$$

We say a_k is the coeff of x^k, denoted $[x^k]A(x)]$

Example: $A(x) = 1 + 3x + 5x^2$.
Then $[x^2]A(x) = 5$
Let $A(x) = \sum_{k\geq 0} a_kx^k andB(x) = \sum_{k\geq 0} b_kx^k$
$A(x) = B(x)$ if and only if $[x^k]A(x) = [x^k]B(x)$ for all $k \geq 0$

Two Operations:

1. Addition: $A(x) + B(x) = \sum_{k\geq 0}(a_k + b_x)x^k$
2. Multiplication:
   Example: $(1 + 2x + 3x^2)(1 - 3x + 5x^2)$
   Coeff of x^2 comes from
   $1 \cdot 5x^2, 2x(-3x), 3x^2 \cdot 1$
   $\implies 5x^2 - 6x^2 + 3x^2 = 2x^2$
   In Genera:

$$A(x)B(x) = (\sum_{i\geq 0} a_ix^i)(\sum_{j\geq 0} b_jx^k)$$
$$= \sum_{n\geq 0}(a_0b_n + a_1b_{n-1} + ... + a_nb_0)$$
$$= \sum_{n\geq 0}(\sum_{i=0}^{n} a_ib_n - i)$$
$$= \sum_{i\geq 0}\sum_{j\geq 0} a_ib_jx^{i+j}$$

Tool:

$$[x^n]x^kA(x)$$

$$A(x) = 1 + 2x + 3x^2$$

$$[x^5]x^4A(x) = [x^5](x^4 + 2x^5 + 3x^6) = 2 = [x^1]A(x)$$

$[x^n]x^kA(x) = [x^{n-k}A(x)]n \geq k$
$0 n < k$

Example: Let $A(x) = \sum_{i \geq 0} x^i$ $B(x) = \sum_{j \geq 0}(j + 1)x^j$

$$A(x) = 1 + x + x^2 + x^3 ...$$

$$B(x) = 1 + 2x + 3x^2 + 4x^3 + ...$$

$[x^n]A(x)B(x) = 1 \cdot (n + 1) + 1 \cdot n + 1 \cdot (n - 1) + ... + 1 \cdot 1$
$= (n + 1) + n + ... + 1$
$= \frac{(n+1)(n+2)}{2}$

$[x^n]A(x)B(x) = \sum_{i=0} n[x^i]A(x)[x^{n-i}]B(x)$
$\sum_{i=0}^{n} 1 \cdot (n - i + 1)$
$= \frac{(n+1)(n+2)}{2}$