Module 4: Dictionaries and Balanced Search Trees

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Dictionary ADT

A *dictionary* is a collection of *items*,
each of which contains a *key* and some *data*
and is called a *key-value pair* (KVP).
Keys can be compared and are (typically) unique.

Operations:

- *search*(k)
- *insert*(k, v)
- *delete*(k)
- optional: *join*, *isEmpty*, *size*, *etc.*

Examples: symbol table, license plate database

Elementary Implementations

Common assumptions:

- Dictionary has n KVPs
- Each KVP uses constant space
  (if not, the "value" could be a pointer)
- Comparing keys takes constant time

Unordered array or linked list

search $\Theta(n)$

insert $\Theta(1)$

delete $\Theta(n)$ (need to search)

Ordered array

search $\Theta(\log n)$

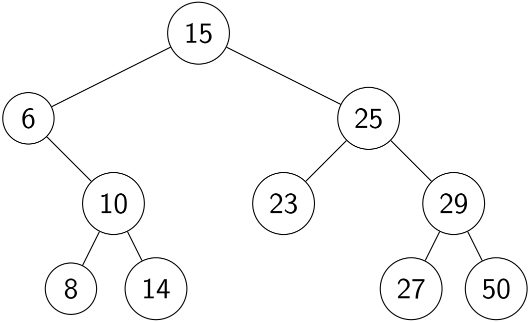
insert $\Theta(n)$

delete $\Theta(n)$

Binary Search Trees (review)

Structure A BST is either empty or contains a KVP, left child BST, and right child BST.

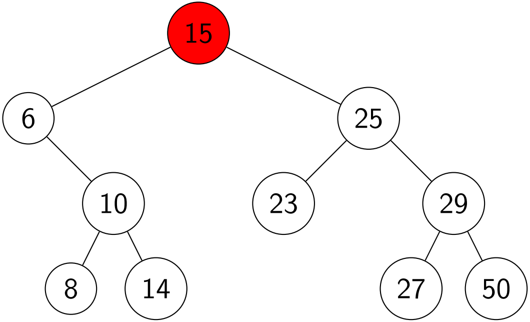
Ordering Every key k in $T.left$ is less than the root key.
Every key k in $T.right$ is greater than the root key.



BST Search and Insert

search(k) Compare k to current node, stop if found, else recurse on subtree unless it's empty

Example: *search*(24)



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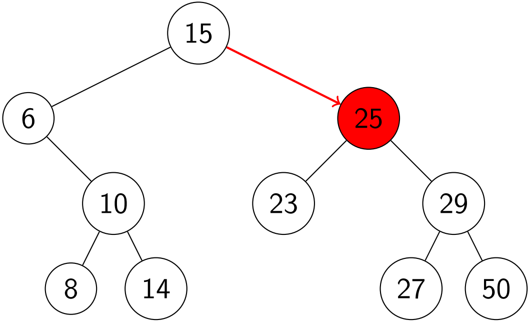

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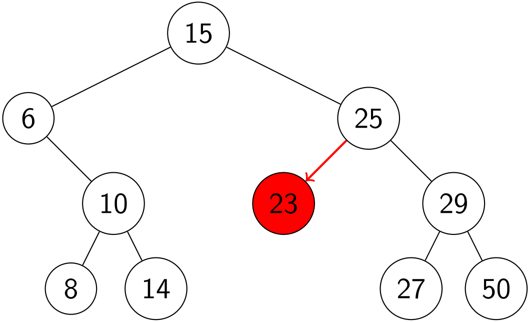

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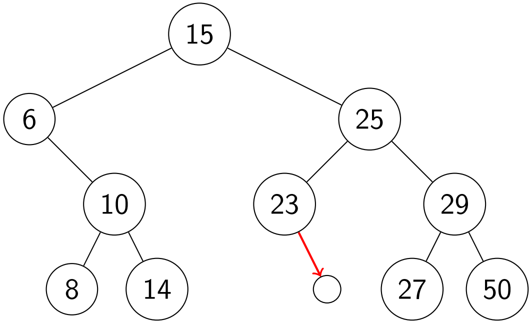
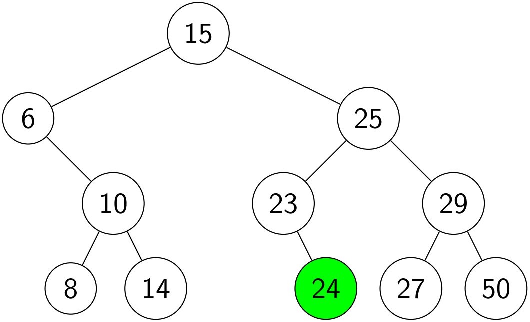


BST Search and Insert

search(k) Compare k to current node, stop if found, else recurse on subtree unless it's empty

insert(k, v) Search for k, then insert (k, v) as new node

Example: *insert*$(24, \ldots)$



BST Delete

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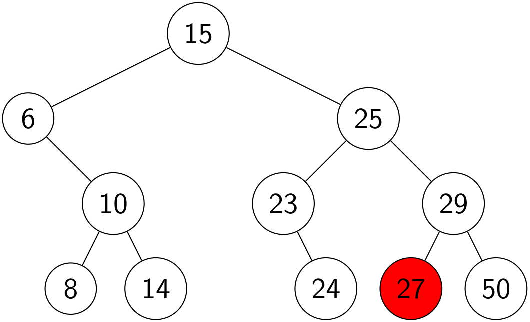

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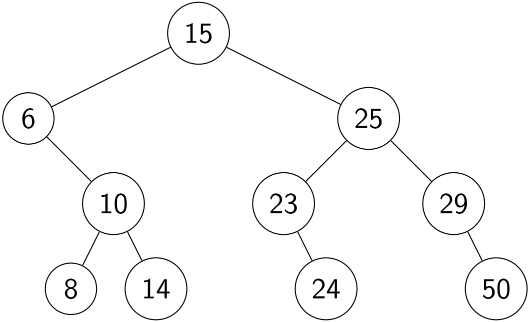
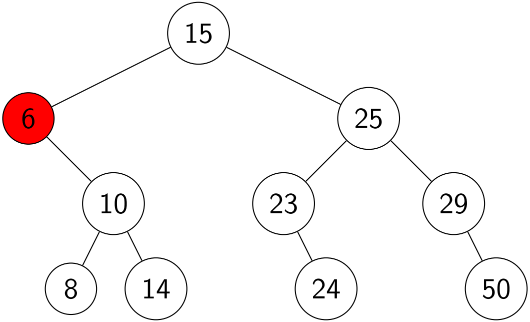


BST Delete

- If node is a leaf, just delete it.
- If node has one child, move child up



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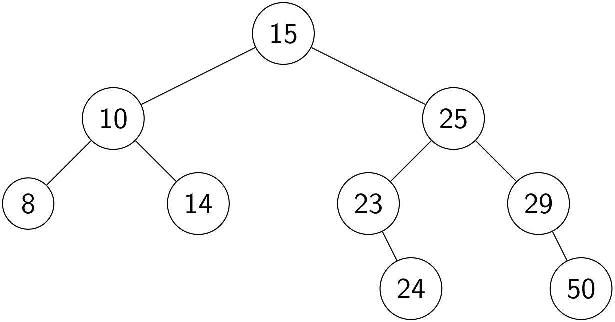

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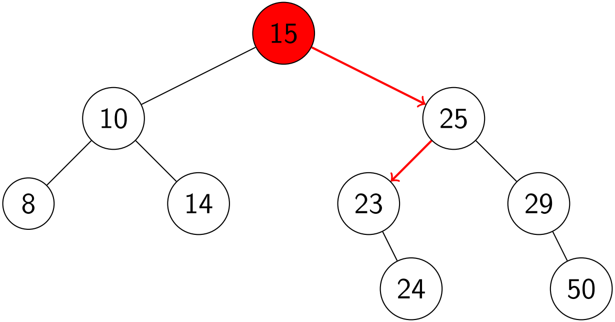
- Else, swap with *successor* or *predecessor* node and then delete



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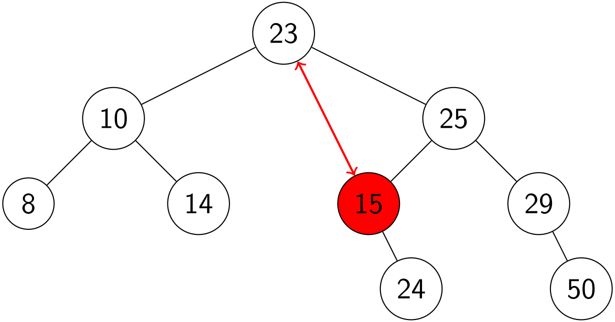

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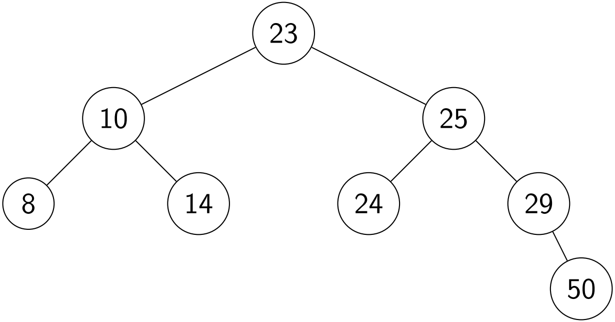

Height of a BST

search, *insert*, *delete* all have cost $\Theta(h)$, where
$h =$ height of the tree $=$ max. path length from root to leaf

If n items are *insert*ed one-at-a-time, how big is h?

- Worst-case:

Height of a BST

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- Worst-case: $n - 1 = \Theta(n)$
- Best-case:

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- Average-case:

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- Worst-case: $n - 1 = \Theta(n)$
- Best-case: $\lfloor \lg(n) \rfloor = \Theta(\log n)$
- Average-case: $\Theta(\log n)$
  (just like recursion depth in *quick-sort1*)

AVL Trees

Introduced by Adel'son-Vel'skiĭ and Landis in 1962,
an *AVL Tree* is a BST with an additional structural property:
The heights of the left and right subtree differ by at most 1.

(The height of an empty tree is defined to be -1.)

At each non-empty node, we store $height(R) - height(L) \in \{-1, 0, 1\}$:

- -1 means the tree is *left-heavy*
- 0 means the tree is *balanced*
- 1 means the tree is *right-heavy*

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-1 means the tree is *left-heavy*

0 means the tree is *balanced*

1 means the tree is *right-heavy*

- We could store the actual height, but storing balances is simpler and more convenient.

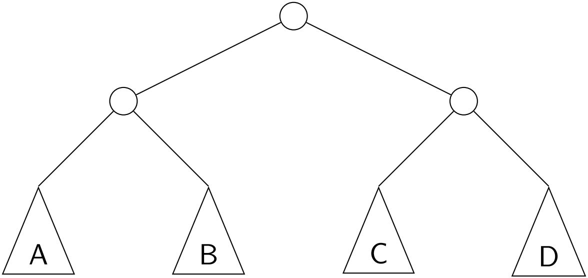
AVL insertion

To perform *insert*(T, k, v):

- First, insert (k, v) into T using usual BST insertion
- Then, move up the tree from the new leaf, updating balance factors.
- If the balance factor is -1, 0, or 1, then keep going.
- If the balance factor is ± 2, then call the *fix* algorithm
  to "rebalance" at that node.

How to “fix” an unbalanced AVL tree

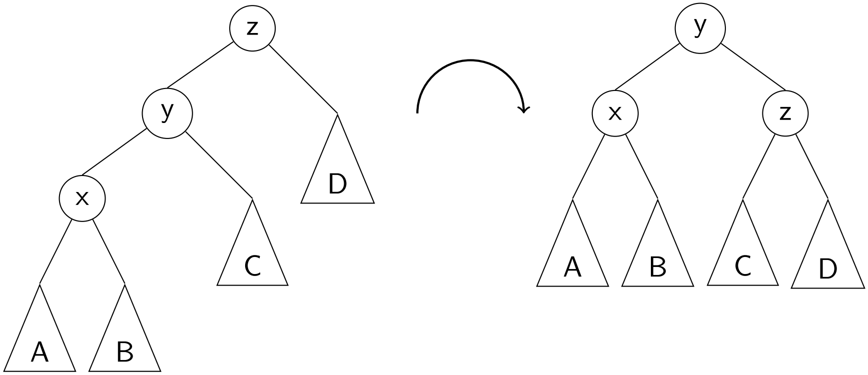
Goal: change the *structure* without changing the *order*



Notice that if heights of A, B, C, D differ by at most 1, then the tree is a proper AVL tree.

Right Rotation

This is a *right rotation* on node z:



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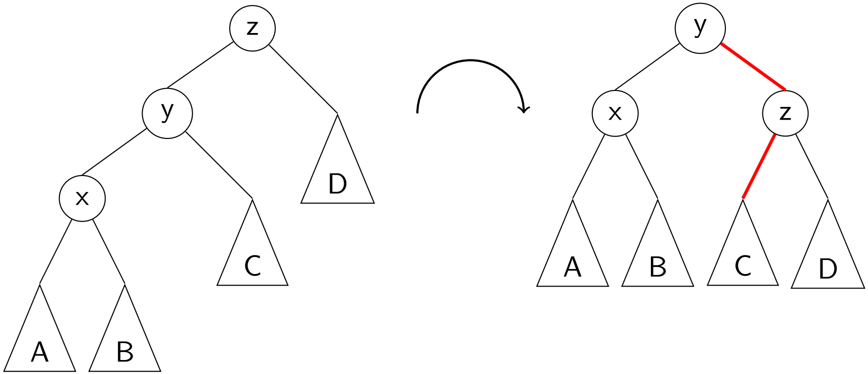
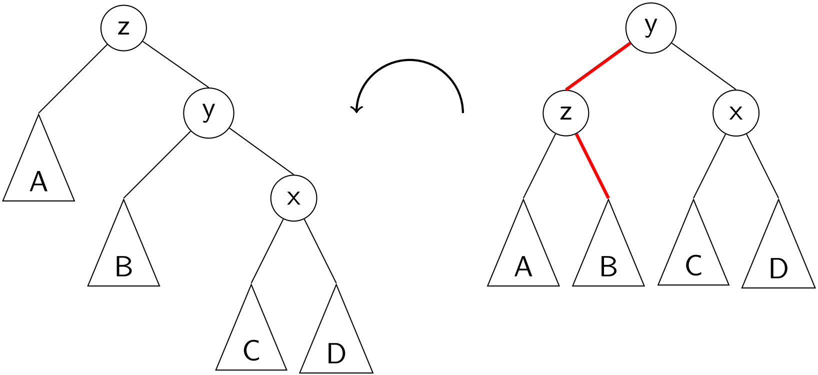


Note: Only two edges need to be moved, and two balances updated.

Left Rotation

This is a *left rotation* on node z:



Again, only two edges need to be moved and two balances updated.

Pseudocode for rotations

rotate-right(T)
T: AVL tree
returns rotated AVL tree

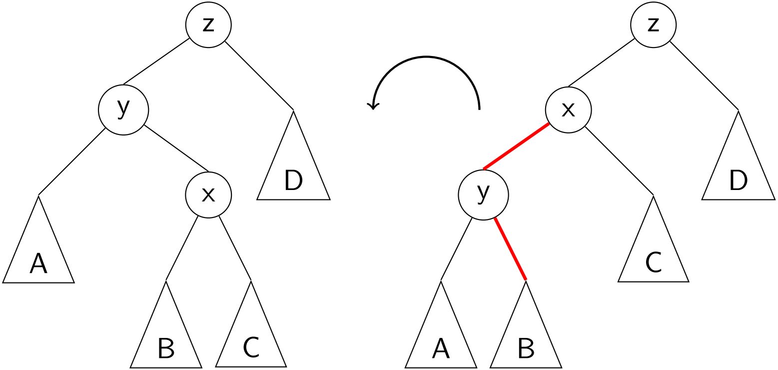
1. $newroot \leftarrow T.left$
2. $T.left \leftarrow newroot.right$
3. $newroot.right \leftarrow T$
4. **return** $newroot$

rotate-left(T)
T: AVL tree
returns rotated AVL tree

1. $newroot \leftarrow T.right$
2. $T.right \leftarrow newroot.left$
3. $newroot.left \leftarrow T$
4. **return** $newroot$

Double Right Rotation

This is a *double right rotation* on node z:



First, a left rotation on the left subtree (y).

Double Right Rotation

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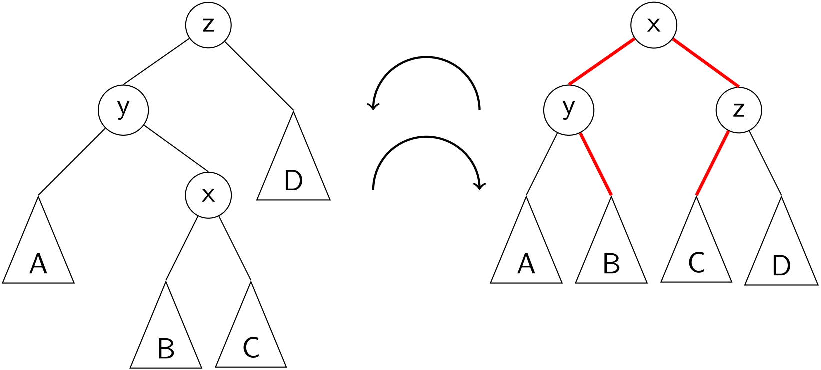
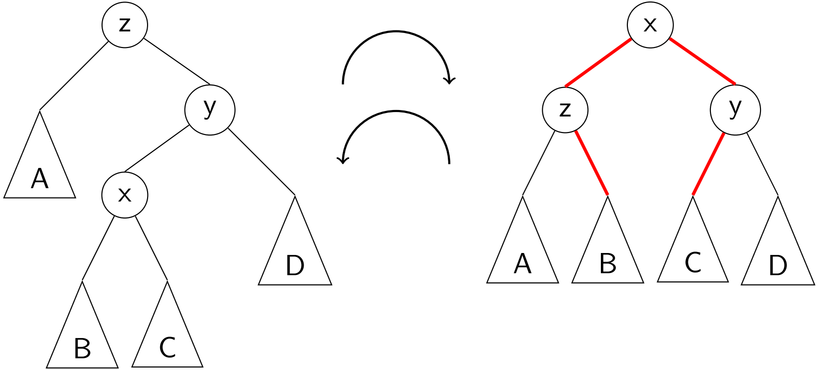


First, a left rotation on the left subtree (y).
Second, a right rotation on the whole tree (z).

Double Left Rotation

This is a *double left rotation* on node z:



Right rotation on right subtree (y),
followed by left rotation on the whole tree (z).

Fixing a slightly-unbalanced AVL tree

Idea: Identify one of the previous 4 situations, apply rotations

fix(T)
T: AVL tree with $T.balance = \pm 2$
returns a balanced AVL tree

```
if T.balance = −2 then
    if T.left.balance = 1 then
        T.left ← rotate-left(T.left)
    return rotate-right(T)
else if T.balance = 2 then
    if T.right.balance = −1 then
        T.right ← rotate-right(T.right)
    return rotate-left(T)
```

AVL Tree Operations

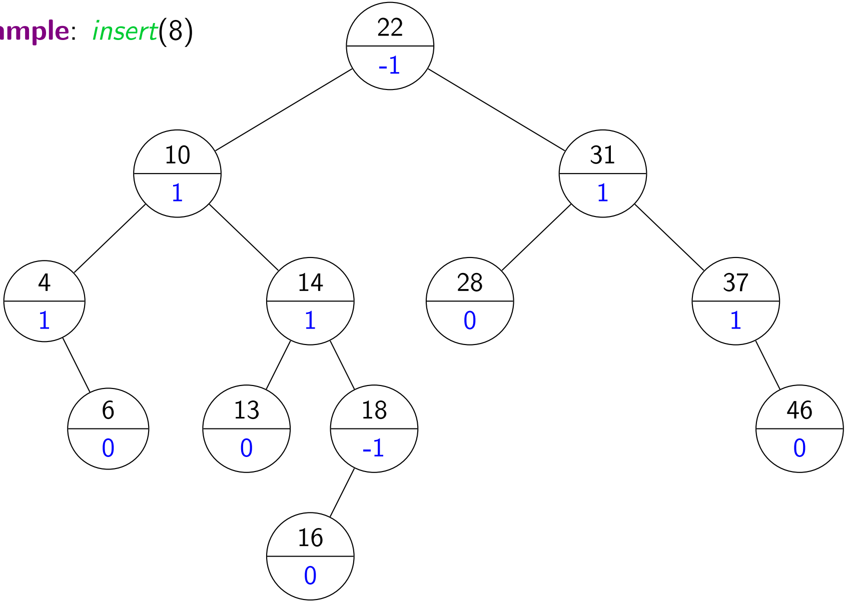
search: Just like in BSTs, costs $\Theta(height)$

insert: Shown already, total cost $\Theta(height)$
fix will be called *at most once*.

delete: First search, then swap with successor (as with BSTs), then move up the tree and apply *fix* (as with *insert*).
fix may be called $\Theta(height)$ times.
Total cost is $\Theta(height)$.

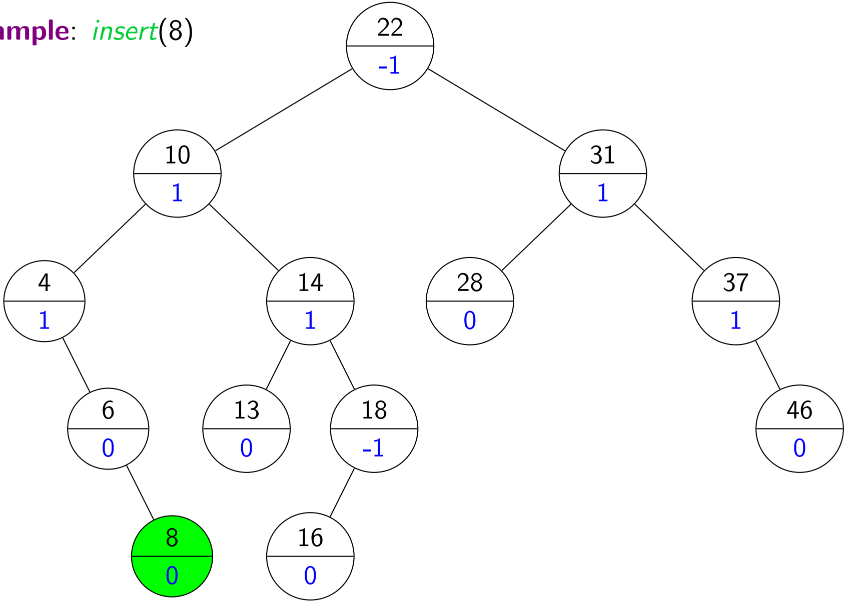
AVL tree examples

Example: *insert*(8)



AVL tree examples

Example: *insert*(8)



AVL tree examples

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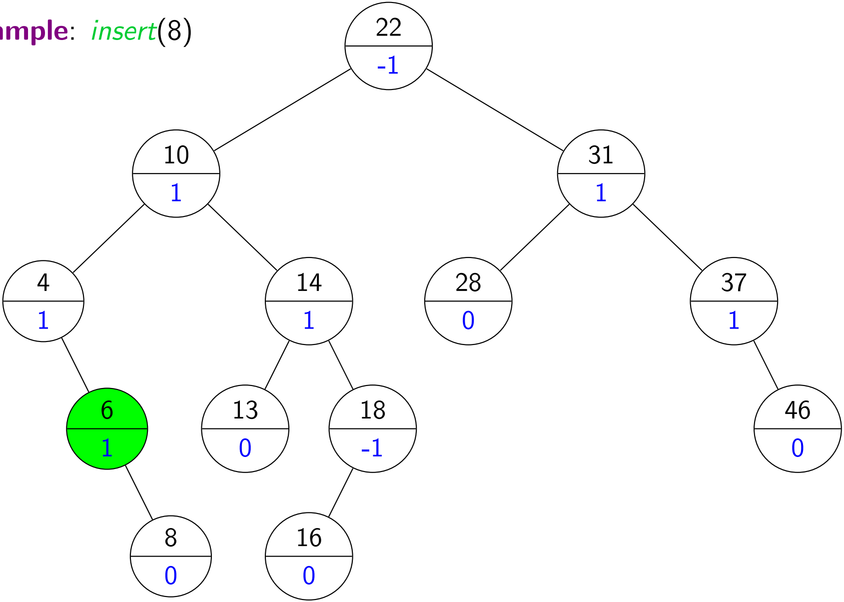

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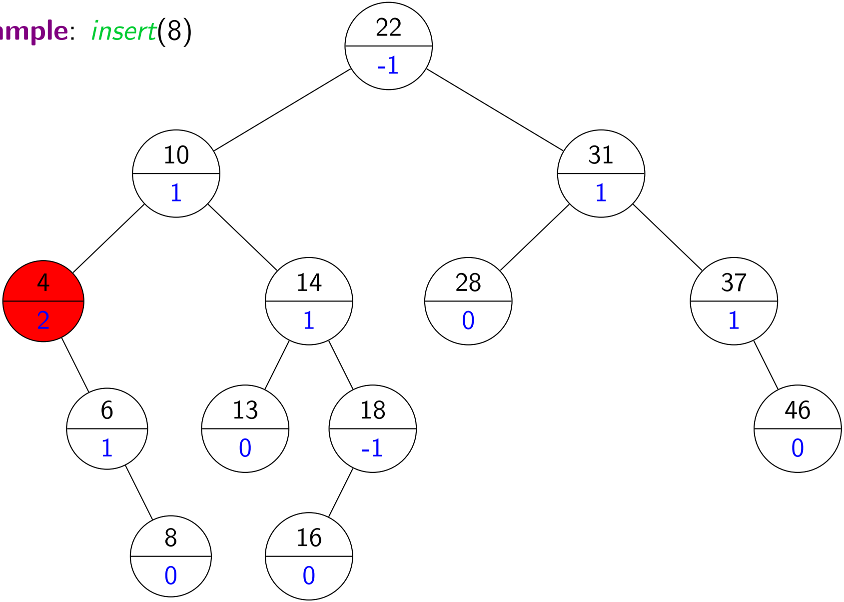

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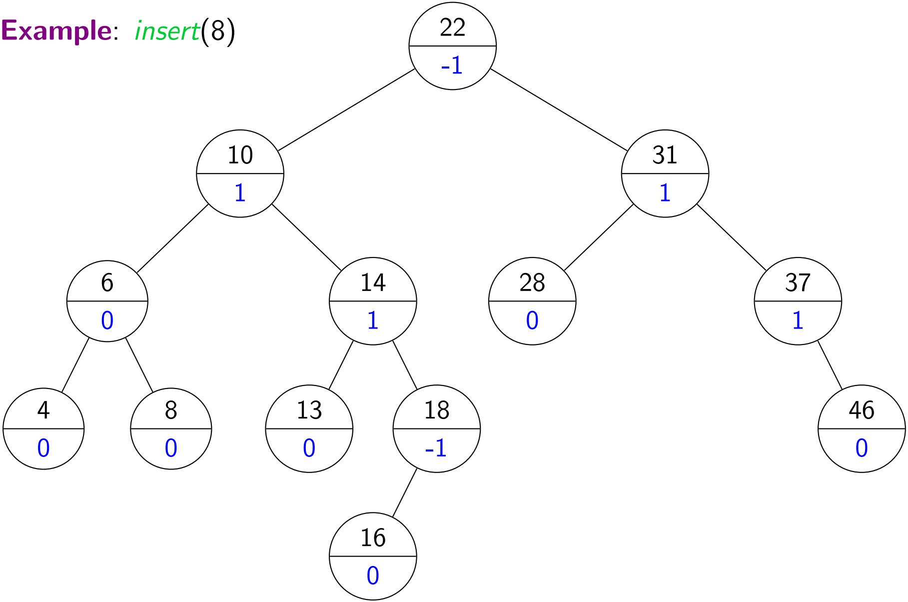
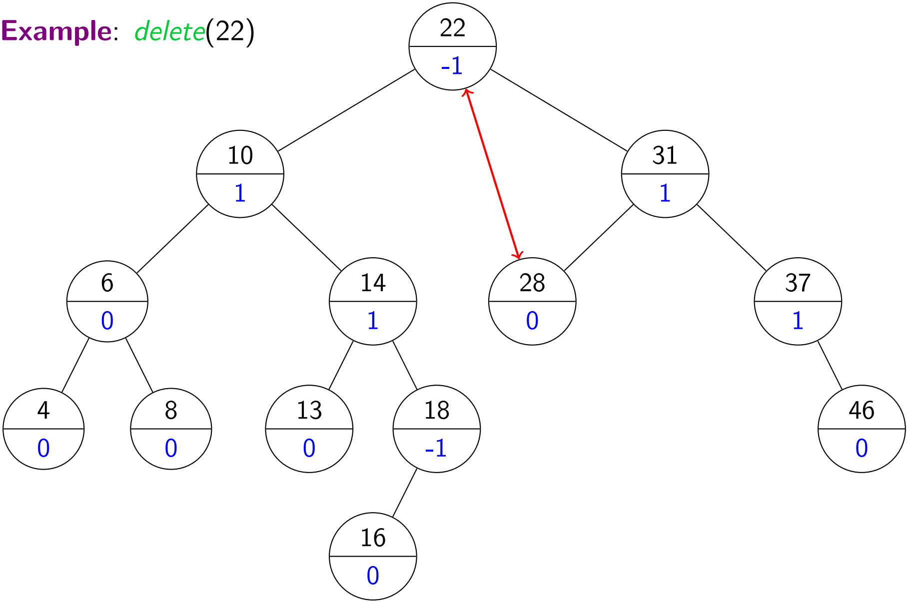


AVL tree examples

Example: *delete*(22)



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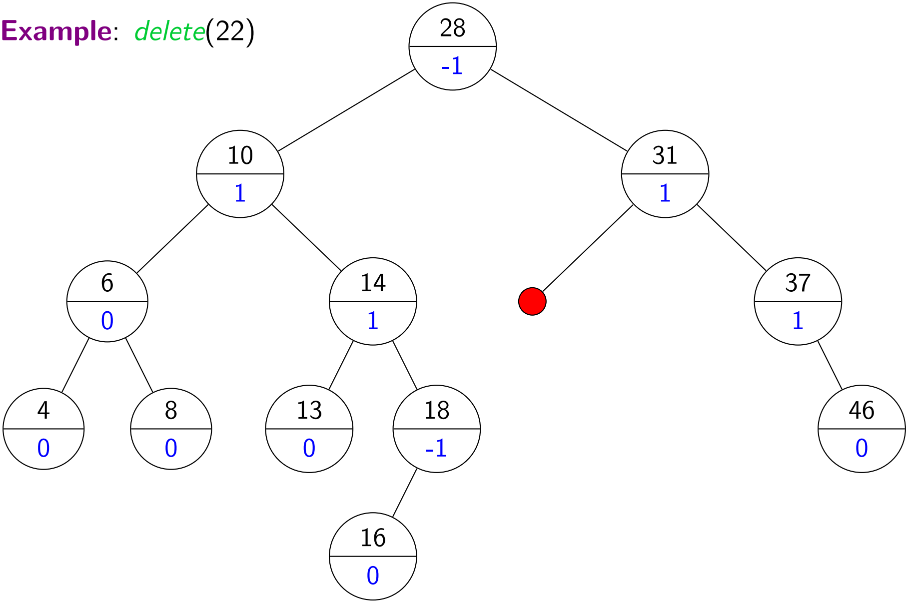

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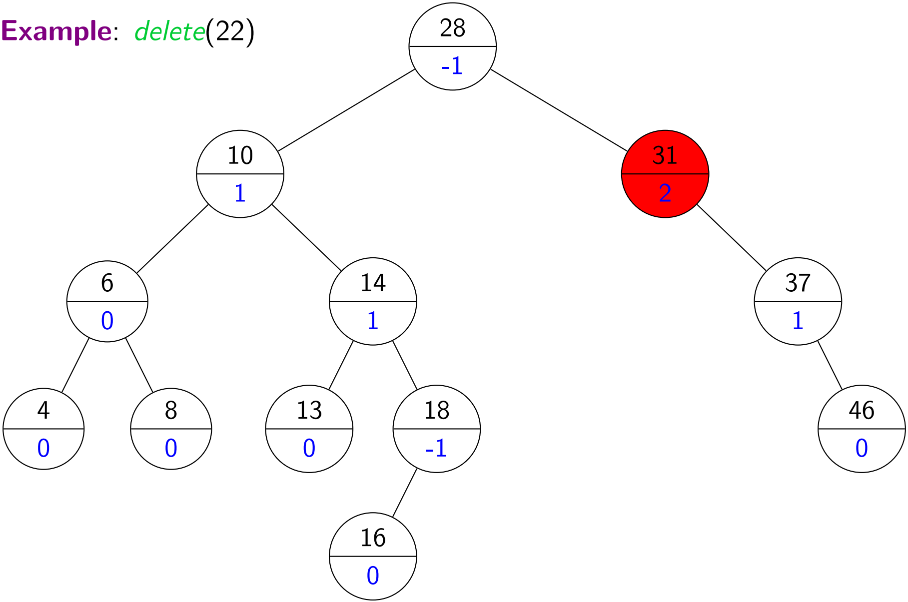

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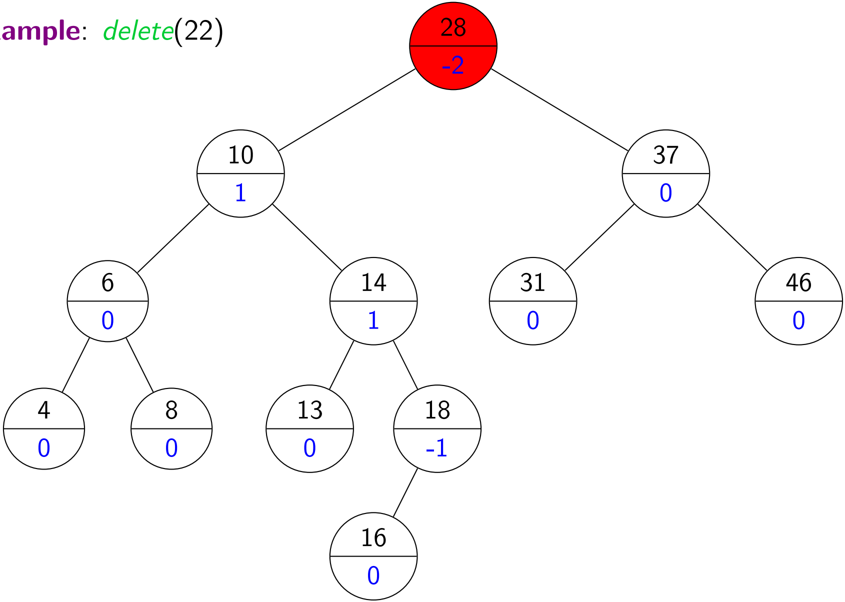

AVL tree examples

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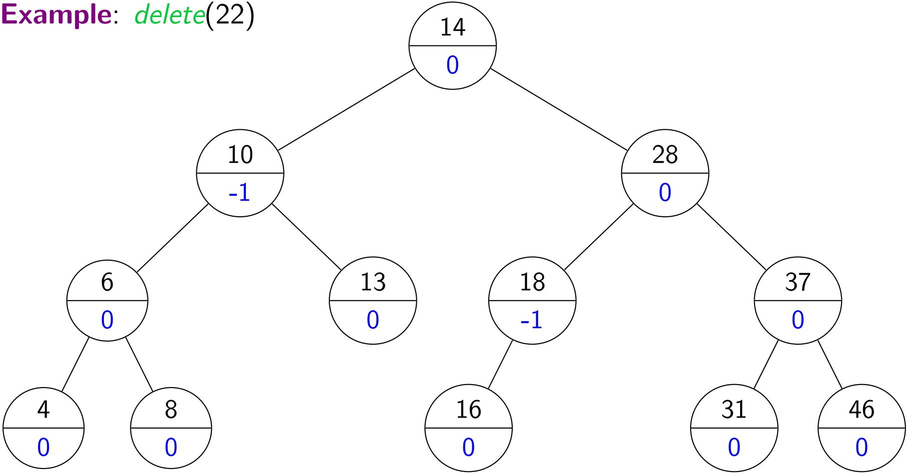

Height of an AVL tree

Define $N(h)$ to be the *least* number of nodes in a height-h AVL tree.

One subtree must have height at least $h-1$, the other at least $h-2$:

$$N(h) = \begin{cases} 1 + N(h-1) + N(h-2), & h \geq 1 \\ 1, & h = 0 \\ 0, & h = -1 \end{cases}$$

What sequence does this look like?

Height of an AVL tree

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What sequence does this look like? The Fibonacci sequence!

$$N(h) = F_{h+3} - 1 = \left\lceil \frac{\varphi^{h+3}}{\sqrt{5}} \right\rfloor - 1, \text{ where } \varphi = \frac{1+\sqrt{5}}{2}$$

AVL Tree Analysis

Easier lower bound on $N(h)$:

$$N(h) > 2N(h-2) > 4N(h-4) > 8N(h-6) > \cdots > 2^i N(h-2i) \geq 2^{\lfloor h/2 \rfloor}$$

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Since $n > 2^{\lfloor h/2 \rfloor}$, $h \leq 2 \lg n$,
and thus an AVL tree with n nodes has height $O(\log n)$.
Also, $n \leq 2^{h+1} - 1$, so the height is $\Theta(\log n)$.

$\Rightarrow$ *search*, *insert*, *delete* all cost $\Theta(\log n)$.

2-3 Trees

A 2-3 Tree is like a BST with additional structual properties:

- Every internal node either contains *one KVP* and *two children*, or *two KVPs* and *three children*.
- The leaves are *NIL* (do not store keys)
- All the leaves are at the same level.

Searching through a 1-node is just like in a BST.
For a 2-node, we must examine both keys and follow the appropriate path.

Insertion in a 2-3 tree

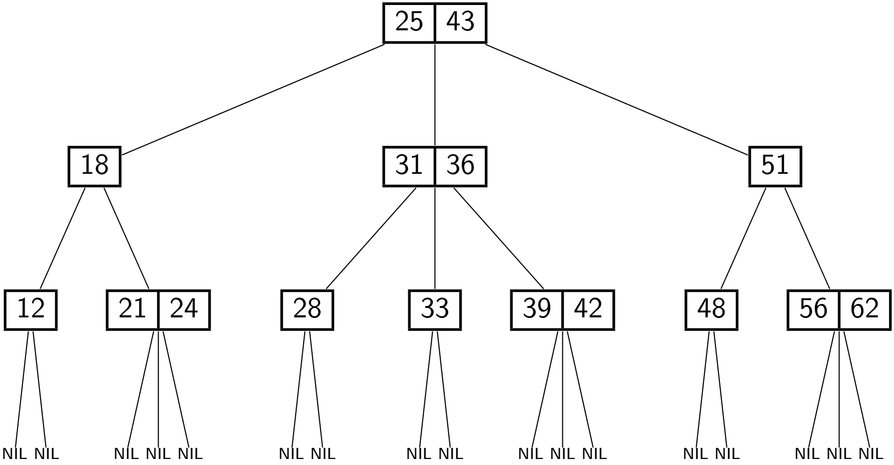
First, we search to find the lowest internal node where the new key belongs.

If the node has only 1 KVP, just add the new one to make a 2-node.

Otherwise, order the three keys as $a < b < c$.
Split the node into two 1-nodes, containing a and c,
and (recursively) insert b into the parent along with the new link.

2-3 Tree Insertion

Example: *insert*(19)



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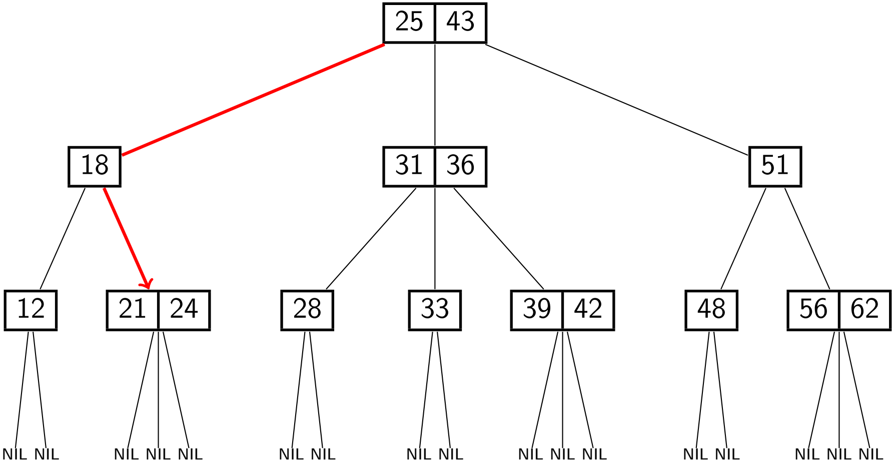

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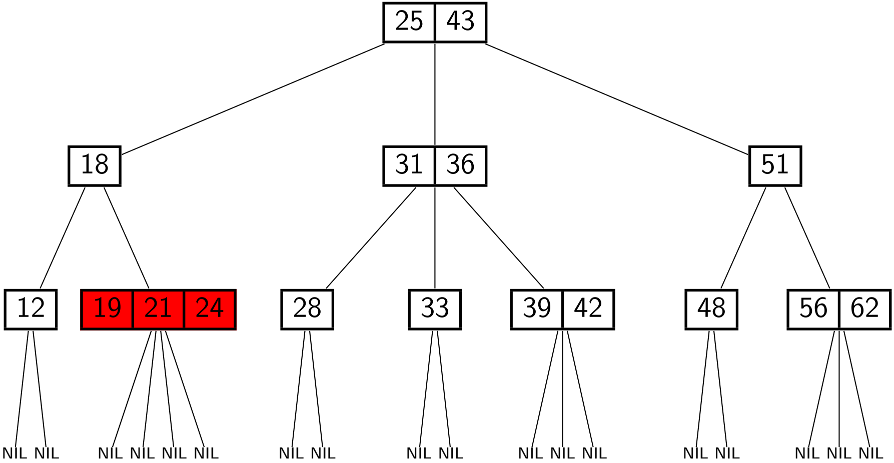

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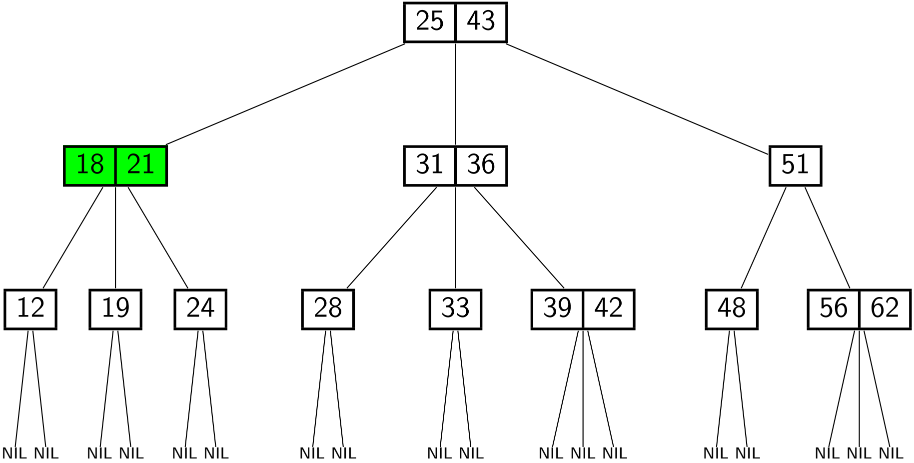
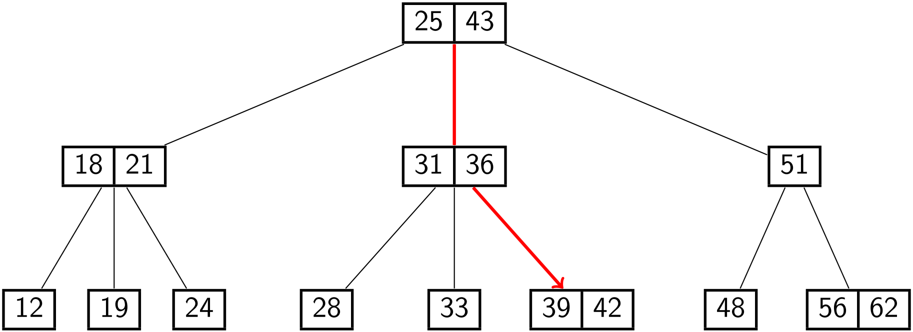


2-3 Tree Insertion

Example: *insert*(41)



(NIL-leaves not shown to simplify picture)

2-3 Tree Insertion

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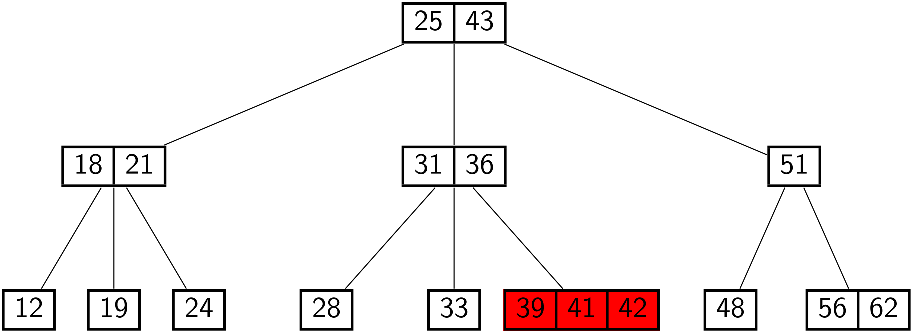

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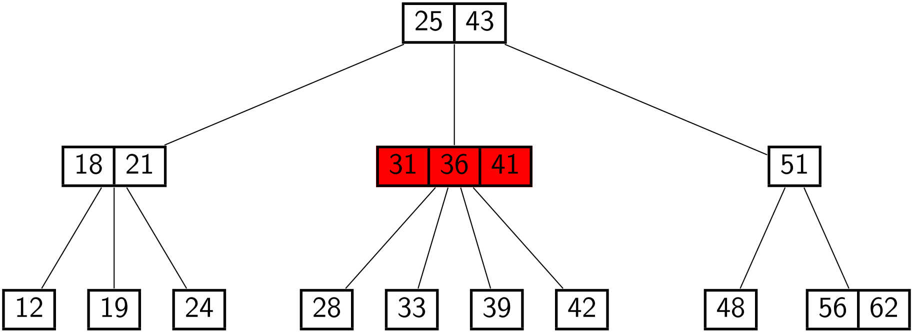

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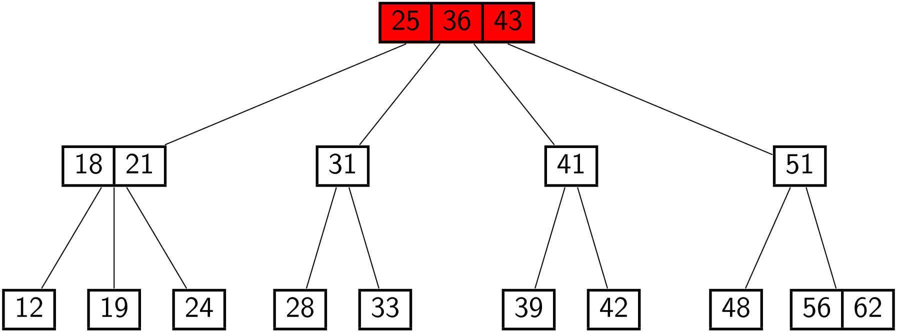

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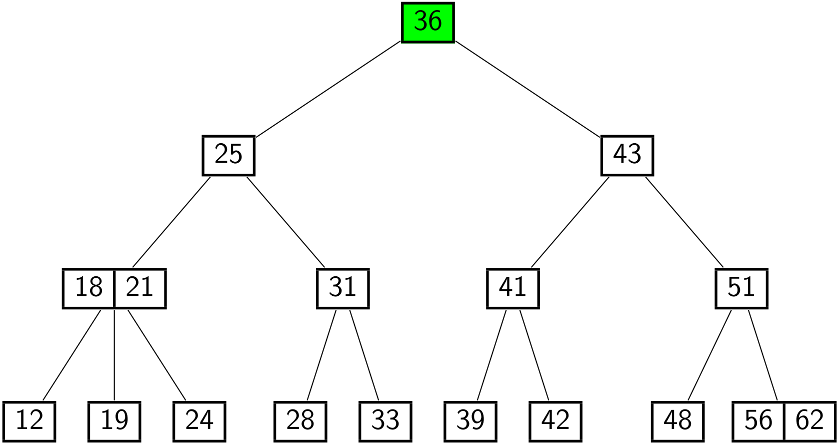


Deletion from a 2-3 Tree

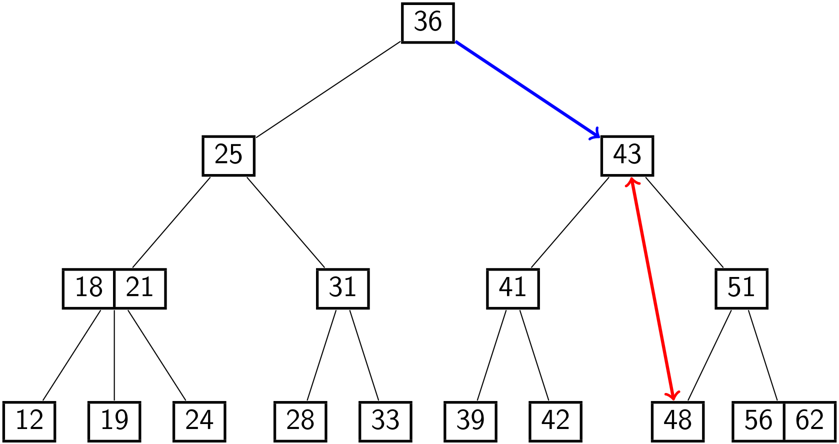
As with BSTs and AVL trees, we first swap the KVP with its successor, so that we always delete from a leaf.

Say we're deleting KVP x from a node V:

- If V is a 2-node, just delete x.
- ElseIf V has a 2-node *immediate* sibling U, perform a *transfer*: Put the "intermediate" KVP in the parent between V and U into V, and replace it with the adjacent KVP from U.
- Otherwise, we *merge* V and a 1-node sibling U: Remove V and (recursively) delete the "intermediate" KVP from the parent, adding it to U.

2-3 Tree Deletion

Example: *delete*(43)



2-3 Tree Deletion

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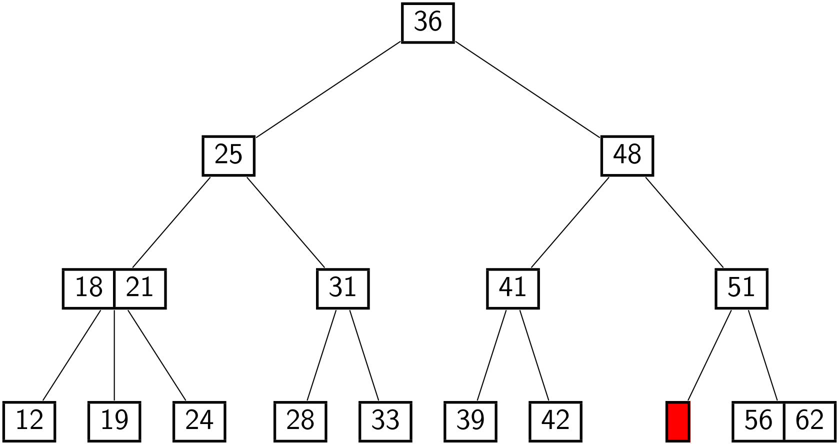

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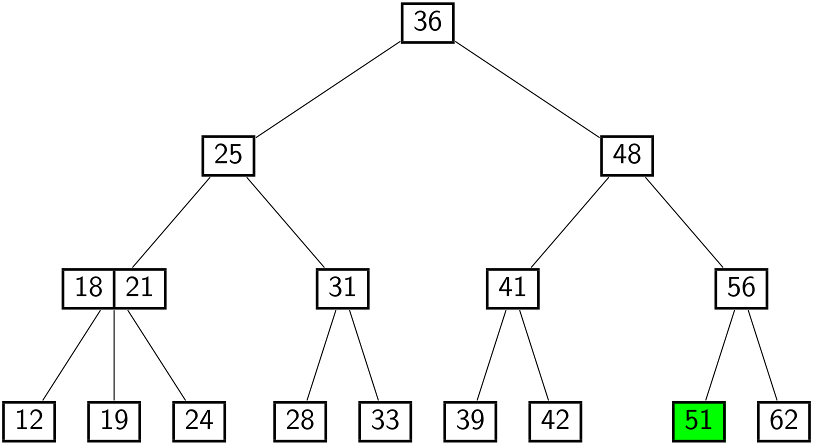
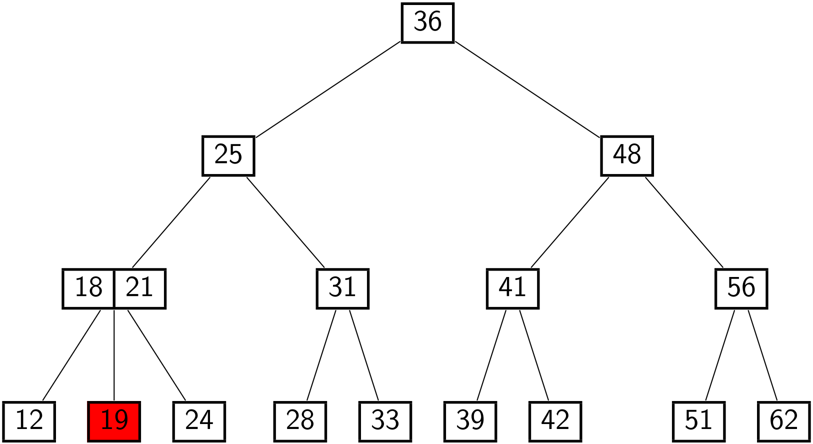


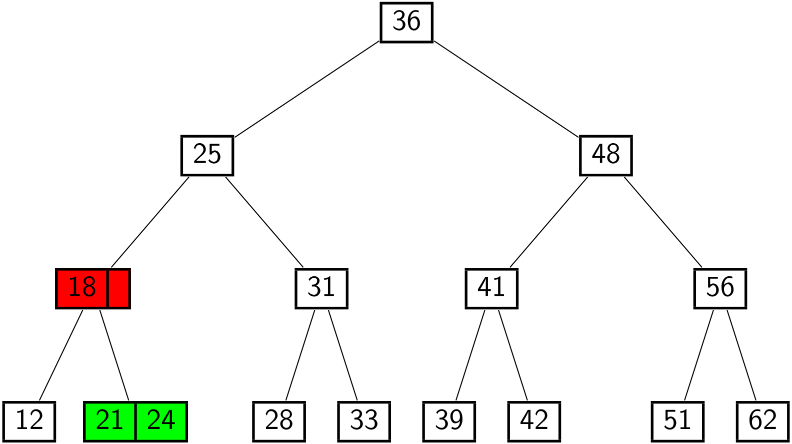
2-3 Tree Deletion

Example: *delete*(19)



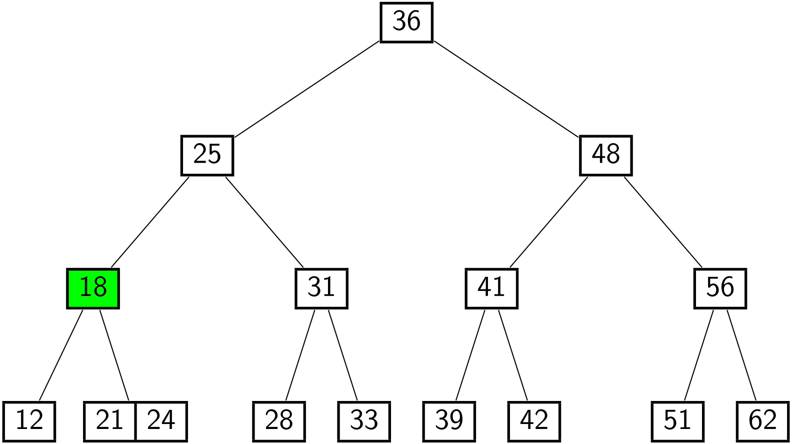
2-3 Tree Deletion

Example: *delete*(19)



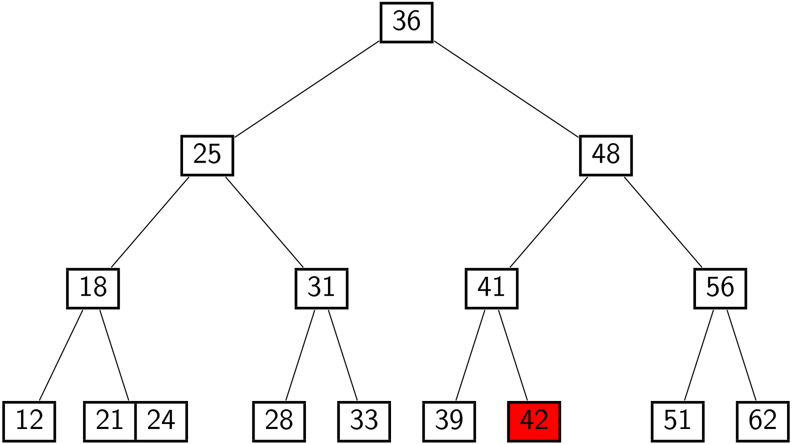
2-3 Tree Deletion

Example: *delete*(19)



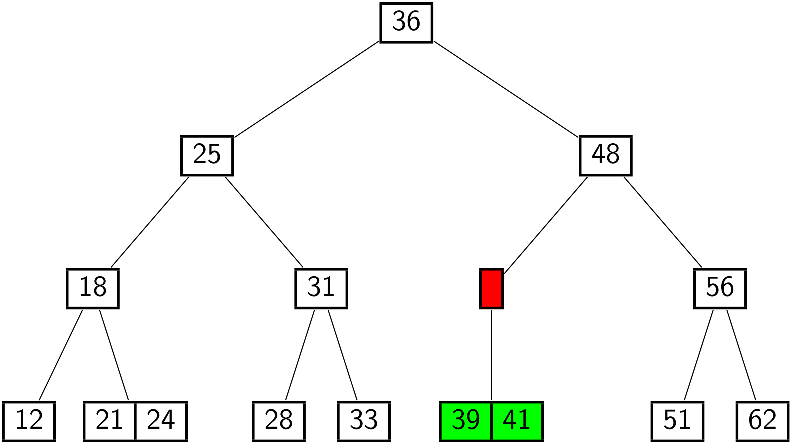
2-3 Tree Deletion

Example: *delete*(42)



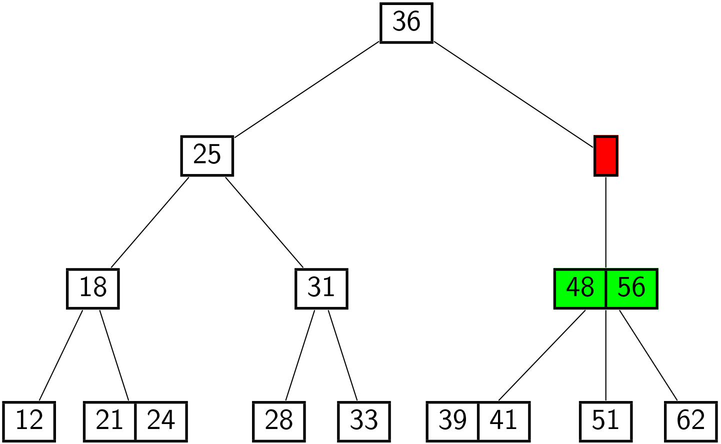
2-3 Tree Deletion

Example: *delete*(42)



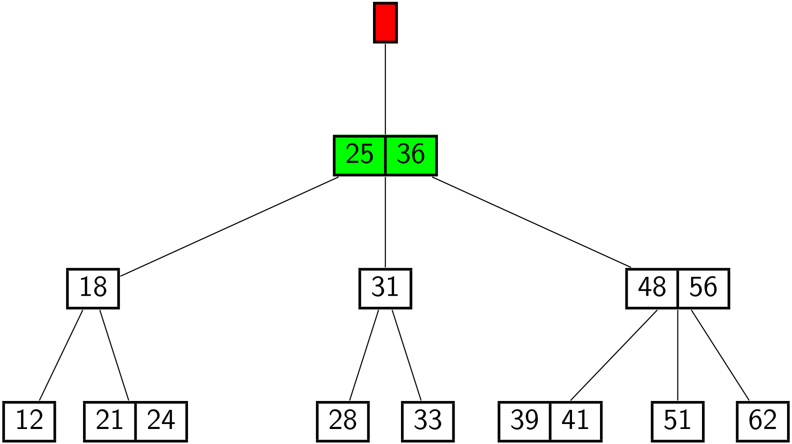
2-3 Tree Deletion

Example: *delete*(42)



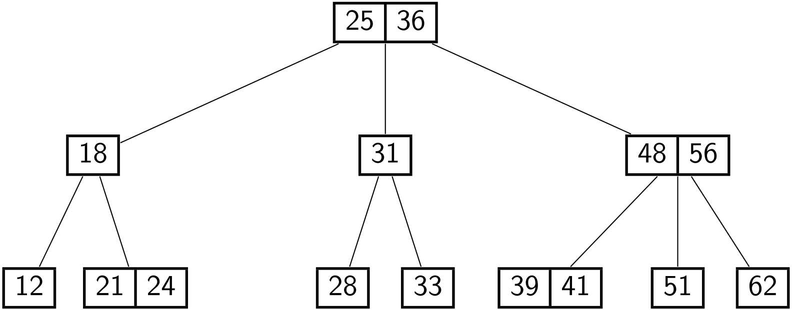
2-3 Tree Deletion

Example: *delete*(42)



2-3 Tree Deletion

Example: *delete*(42)



B-Trees

The 2-3 Tree is a specific type of (a, b)-tree:

An *(a, b)-tree of order M* is a search tree satisfying:

- Each internal node has at least a children, unless it is the root. The root has at least 2 children.
- Each internal node has at most b children.
- If a node has k children, then it stores $k - 1$ key-value pairs (KVPs).
- Leaves store no keys and are at the same level.

A *B-tree of order M* is a $(\lceil M/2 \rceil, M)$-tree.
A 2-3 tree has $M = 3$.

search, *insert*, *delete* work just like for 2-3 trees.

Height of a B-tree

What is the least number of KVPs in a height-h B-tree?
(Height $=$ # levels **not** counting the dummy-level - 1)

Level	Nodes	Links/node	KVP/node	KVPs on level
0	1	2	1	1
1	2	$M/2$	$M/2-1$	$2(M/2-1)$
2	$2(M/2)$	$M/2$	$M/2-1$	$2(M/2)(M/2-1)$
3	$2(M/2)^2$	$M/2$	$M/2-1$	$2(M/2)^2(M/2-1)$
...	...	...	...	...
h	$2(M/2)^{h-1}$	$M/2$	$M/2-1$	$2(M/2)^{h-1}(M/2-1)$

$$\text{Total: } n \geq 1 + 2\sum_{i=0}^{h-1}(M/2)^i(M/2-1) = 2(M/2)^h - 1$$

Therefore height of tree with n nodes is $\Theta\big((\log n)/(\log M)\big)$.

Analysis of B-tree operations

Assume each node stores its KVPs and child-pointers in a dictionary that supports $O(\log M)$ search, insert, and delete.

Then *search*, *insert*, and *delete* work just like for 2-3 trees, and each require $\Theta(\mathit{height})$ node operations.

Total cost is $O\left(\frac{\log n}{\log M} \cdot (\log M)\right) = O(\log n)$.

Dictionaries in external memory

Tree-based data structures have poor *memory locality*:
If an operation accesses m nodes, then it must access
m spaced-out memory locations.

Observation: Accessing a single location in *external memory*
(e.g. hard disk) automatically loads a whole block (or "page").

In an AVL tree or 2-3 tree, $\Theta(\log n)$ pages are loaded in the worst case.

If M is small enough so an M-node fits into a single page,
then a B-tree of order M only loads $\Theta\big((\log n)/(\log M)\big)$ pages.

This can result in a *huge* savings:
memory access is often the largest time cost in a computation.

B-tree variations

Max size M: Permitting one additional KVP in each node allows *insert* and *delete* to avoid *backtracking* via *pre-emptive splitting* and *pre-emptive merging*.

Red-black trees: Identical to a B-tree with minsize 1 and maxsize 3, but each 2-node or 3-node is represented by 2 or 3 binary nodes, and each node holds a "color" value of red or black.

B^+-trees: All KVPs are stored at the leaves (interior nodes just have keys), and the leaves are linked sequentially.