Math 239 Fall 2014 Assignment 5 Solutions

1. {5 marks} Let $\{a_n\}$ be the sequence which satisfies

$$a_n - a_{n-1} - 8a_{n-2} + 12a_{n-3} = 0$$

for $n \geq 3$ with initial conditions $a_0 = 1, a_1 = 13, a_2 = 23$. Determine an explicit formula for a_n.

Solution. The characteristic polynomial is

$$x^3 - x^2 - 8x + 12 = (x-2)^2(x+3).$$

The root 2 has multiplicity 2, and the root -3 has multiplicity 1. So

$$a_n = (A + Bn) \cdot 2^n + C \cdot (-3)^n$$

for some constants A, B, C. Plugging in the initial conditions, we get

$$\begin{aligned} 1 &= A + C \\ 13 &= 2A + 2B - 3C \\ 23 &= 4A + 8B + 9C \end{aligned}$$

Solving this gives us $A = 2, B = 3, C = -1$. So an explicit formula for a_n is

$$a_n = (2 + 3n) \cdot 2^n - (-3)^n.$$

2. {5 marks} Let $\{b_n\}$ be the sequence which satisfies

$$b_n - b_{n-1} - 8b_{n-2} + 12b_{n-3} = 20$$

for $n \geq 3$ with initial conditions $b_0 = 1, b_1 = 26, b_2 = 30$. Determine an explicit formula for b_n. (Note: This recurrence is similar to the one in question 1.)

Solution. We suppose that $c_n = \alpha$ is a specific solution to the recurrence. Then

$$c_n - c_{n-1} - 8c_{n-2} + 12c_{n-3} = \alpha - \alpha - 8\alpha + 12\alpha = 4\alpha.$$

This equals 20, so $\alpha = 5$. A specific solution is then $c_n = 5$.
The characteristic polynomial is the same as part (a), so

$$b_n = (A + Bn) \cdot 2^n + C \cdot (-3)^n + 5$$

for some constants A, B, C. Plugging in the initial conditions, we get

$$\begin{aligned} 1 &= A + C + 5 \\ 26 &= 2A + 2B - 3C + 5 \\ 30 &= 4A + 8B + 9C + 5 \end{aligned}$$

Solving this gives us $A = -1, B = 7, C = -3$. So an explicit formula for b_n is

$$b_n = (-1 + 7n) \cdot 2^n - 3 \cdot (-3)^n + 5.$$

3. Consider the sequence $\{a_n\}$ where for each integer $n \geq 0$,

$$a_n = \sqrt{5}\left(\frac{3+\sqrt{5}}{2}\right)^n - \sqrt{5}\left(\frac{3-\sqrt{5}}{2}\right)^n.$$

(a) {3 marks} Derive a simplified rational expression for $A(x) = \sum_{n\geq 0} a_n x^n$. **Solution.**

$$\begin{aligned}
\sum_{n\geq 0} a_n x^n &= \sum_{n\geq 0}\left(\sqrt{5}\left(\frac{3+\sqrt{5}}{2}\right)^n - \sqrt{5}\left(\frac{3-\sqrt{5}}{2}\right)^n\right)x^n \\
&= \sqrt{5}\left(\sum_{n\geq 0}\left(\frac{3+\sqrt{5}}{2}\right)^n x^n - \sum_{n\geq 0}\left(\frac{3-\sqrt{5}}{2}\right)^n x^n\right) \\
&= \sqrt{5}\left(\frac{1}{1-\frac{3+\sqrt{5}}{2}x} - \frac{1}{1-\frac{3-\sqrt{5}}{2}x}\right) \\
&= \sqrt{5}\left(\frac{1-\frac{3-\sqrt{5}}{2}x - 1 + \frac{3+\sqrt{5}}{2}x}{\left(1-\frac{3+\sqrt{5}}{2}x\right)\left(1-\frac{3-\sqrt{5}}{2}x\right)}\right) \\
&= \sqrt{5}\left(\frac{\sqrt{5}x}{1-3x+x^2}\right) \\
&= \frac{5x}{1-3x+x^2}.
\end{aligned}$$

(b) {3 marks} Use part (a) to prove that a_n is an integer for all $n \geq 0$.

Solution. From the power series in part (a), we see that a_n satisfies the recurrence $a_n - 3a_{n-1} + a_{n-2} = 0$ for $n \geq 2$ with initial conditions $a_0 = 0$ and $a_1 = 5$. We see that both a_0 and a_1 are integers. Using strong induction, we see that for each $n \geq 2$, $a_n = 3a_{n-1} - a_{n-2}$, which is a difference of two integers (by induction hypothesis). So a_n is an integer for all $n \geq 0$.

4. {4 marks} From assignment 4, the set of binary strings whose integer representations are multiples of 3 can be expressed as $S = (\{1\}(\{0\}\{1\}^*\{0\})^*\{1\}\{0\}^*)^*$. You have found that the generating series for S is

$$\Phi_S(x) = \frac{1-x-x^2}{1-x-2x^2}.$$

Let $a_n = [x^n]\Phi_S(x)$, which represents the number of strings in S of length n. Use partial fraction expansion to determine an explicit formula for a_n for all integers $n \geq 0$. (Your solution should make sense in some way.)

Solution. We first note that this rational function is improper, the degree of the numerator is not strictly less than the denominator. We can apply the division algorithm first to get

$$\frac{1-x-x^2}{1-x-2x^2} = \frac{1}{2} + \frac{\frac{1}{2}-\frac{1}{2}x}{1-x-2x^2}.$$

Now $(1-x-2x^2) = (1-2x)(1+x)$. Using partial fractions, we see that there exists constants A, B such that

$$\frac{\frac{1}{2}-\frac{1}{2}x}{1-x-2x^2} = \frac{A}{1-2x} + \frac{B}{1+x}.$$

Solving this gives $A = \frac{1}{6}, B = \frac{1}{3}$. So

$$[x^n]\frac{\frac{1}{2}-\frac{1}{2}x}{1-x-2x^2} = \frac{1}{6}2^n + \frac{1}{3}(-1)^n = \frac{1}{3}(2^{n-1} + (-1)^n).$$

The constant term here is $\frac{1}{2}$, and added to the $\frac{1}{2}$ in the first equation, we get that the constant term is 1. So overall,

$$a_n = \begin{cases} \frac{1}{3}(2^{n-1} + (-1)^n) & n \geq 1 \\ 1 & n = 0 \end{cases}$$

(Note: This formula makes sense since there are 2^{n-1} binary strings of length n that start with 1, and among them, about a third are multiples of 3.)

5. (a) {2 marks} Let $n \in \mathbb{N}$, and let S_n be the set of all compositions of n. We know that $|S_n| = 2^{n-1}$. In particular, $|S_n| = 2|S_{n-1}|$ for $n \geq 2$. We can interpret this combinatorially as follows: Let $S_n = A \cup B$ where A consists of compositions of n whose last part is 1, and B consists of compositions of n whose last part is greater than 1. Write down two bijections $f : A \to S_{n-1}$ and $g : B \to S_{n-1}$.

Solution. Define $f : A \to S_{n-1}$ by

$$f(a_1, \ldots, a_{k-1}, 1) = (a_1, \ldots, a_k).$$

The inverse is

$$f^{-1}(b_1, \ldots, b_l) = (b_1, \ldots, b_l, 1).$$

Define $g : B \to S_{n-1}$ by

$$g(a_1, \ldots, a_k) = (a_1, \ldots, a_{k-1}, a_k - 1).$$

The inverse is

$$g^{-1}(b_1, \ldots, b_l) = (b_1, \ldots, b_{l-1}, b_l + 1).$$

(b) {2 marks} Let a_n denote the total number of parts of all possible composition of n. For example, compositions of 3 are $(3), (1, 2), (2, 1), (1, 1, 1)$, so $a_3 = 1 + 2 + 2 + 3 = 8$. Using the bijections you have defined in part (a), prove that for $n \geq 2$,

$$a_n = 2a_{n-1} + 2^{n-2}.$$

Solution. In the bijections in part (a), each composition in A has one more part than its corresponding composition in S_{n-1}. There are a_{n-1} parts among all compositions in S_{n-1}, and there are 2^{n-2} compositions in S_{n-1}, so the total number of parts over all compositions in A is $a_{n-1} + 2^{n-2}$.

Each composition in B has the same number of parts as its corresponding composition in S_{n-1}. There are a_{n-1} parts among all compositions in S_{n-1}, so there are a_{n-1} parts among all compositions in B.

Therefore, the total number of parts in S_n is the sum of the parts in A and B, hence $a_n = 2a_{n-1} + 2^{n-2}$.

(c) {3 marks} Using the initial condition $a_1 = 1$, determine an explicit formula for a_n when $n \geq 1$.

Solution. The characteristic polynomial is $x - 2$, so it has one root 2. The homogeneous part of the recurrence has the form $A \cdot 2^n$.

To find a specific solution to the nonhomogeneous recurrence, we cannot use $b_n = \alpha 2^n$, as 2 is a root of the characteristic polynomial. So we try $b_n = \alpha n 2^n$. Then

$$b_n - 2b_{n-1} = \alpha(n2^n - 2(n-1)2^{n-1}) = \alpha(n2^n(n-1)2^n) = \alpha 2^n.$$

This equals 2^{n-2}, so $\alpha = 1/4$. So a specific solution is then $b_n = n2^{n-2}$. For a_n, an explicit formula has the form

$$a_n = A \cdot 2^n + n2^{n-2}.$$

Using $a_1 = 1$, we get $A = 1/4$. So an explicit formula is

$$a_n = (n+1)2^{n-2}.$$