Math 239 Spring 2015 Tutorial Problems 1

The TA will cover a subset of these problems during the tutorial. Solutions will not be provided outside of the tutorial.

1. For some $0 \leq r \leq k \leq n$, how many subsets of $[n]$ have r elements in common with the set $\{1, \ldots, k\}$? Describe two sets S and T such that the answer to our question is the cardinality of the cartesian product $S \times T$, then determine what is this answer.

2. Let $n \in \mathbb{N}$. Define E_n to be the set of all subsets of $[n]$ of even cardinality, and define O_n to be the set of all subsets of $[n]$ of odd cardinality.

   (a) Define a bijection $f : E_n \to O_n$.

   (b) Illustrate your bijection by pairing up each element X of E_4 with its image $f(X)$ of O_4.

   (c) Determine the cardinalities of E_n and O_n.

   (d) Use the results in this question to give a combinatorial proof of the following identity:

   $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

   (e) Give an algebraic proof of the identity in part (d).

3. For any integers n, k, r where $n \geq k \geq r \geq 0$, give a combinatorial proof of the following identity.

   $$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}.$$

4. Give a combinatorial proof of the following identity for any positive integer n.

   $$\sum_{i=0}^{n} \binom{n}{i} i = n2^{n-1}.$$

5. Consider the k-tuples $(T_1, \ldots, T_k)$ where each $T_i \subseteq [n]$. In other words, if P is the set of all subsets of $[n]$, then such a k-tuple is in the cartesian product P^k. We define the following two subsets of P^k:

   (a) S is all such k-tuples where $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k$.

   (b) T is all such k-tuples that are mutually disjoint, i.e. $T_i \cap T_j = \emptyset$ for any $i \neq j$.

   Find a bijection between S and T, which proves that $|S| = |T|$. What is this cardinality?