Math 239 Theorems and Definitions

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1 Combinatorial Analysis

1.3 Binomial Coefficients

1.3.1 Theorem: For non-negative integers n and k, the number of k-element subsets of an n-element set is:

$$\frac{n(n-1)...(n-k+1)}{k!} = \binom{n}{k} = \binom{n}{n-k}$$

1.3.2 Theorem: For any non-negative integer n,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

1.3.3 Problem: For any non-negative integers n and k:

$$\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$$

1.4 Generating Series

1.4.2 Definition: Let S be a set of configurations with a weight function w. The generating series for S with respect to w is defined by:

$$\begin{aligned}\Phi_S(x) &= \sum_{\sigma \in S} x^{w(\sigma)} \\ &= \sum_{k \geq 0} a_k x^k\end{aligned}$$

1.4.3 Theorem: Let $\Phi_S(x)$ be the generating series for a finit set S with respect to a weight function w. Then,

- $\Phi_S(1) = |S|$
- the sum of the weights of the elements in S is $\Phi'_S(1)$, and
- the average weight of an element in S is $\Phi'_S(1)/\Phi_S(1)$

1.5 Formal Power Series

1.5.0 Definition: For a sequence of $(a_0, a_1, a_2...)$ which are rational numbers, then A(x) = $a_0 + a_1x + a_2x^2 + ...$ is called the formal power series. We say that a_n is the coefficient of x^n and we write $a_n = [x^n]A(x)$.
Also:

$$A(x) + B(x) = \sum_{n \geq 0} (a_n + b_n)x^n$$

$$A(x)B(x) = \sum_{n \geq 0} (\sum_{k=0}^{n} a_k b_{n-k})x^n$$

1.5.2 Theorem: Let $A(x) = a_0 + a_1x + a_2x^2 + ...$, $P(x) = p_0 + p_1x + p_2x^2 + ...$ and $Q(x) = 1 - q_1x - q_2x^2 - ...$ be formal power series. Then:

$$Q(x)A(x) = P(x)$$

if and only if for each n ≥ 0

$$a_n = p_n + q_1a_{n-1} + q_2a_{n-2} + ... + q_na_0$$

1.5.3 Corollary: Let P(x) and Q(x) be formal power series. If the constant term of Q(x) is non-zero, then there is a formal power series A(x) satisfying:

$$Q(x)A(x) = P(x)$$

Moreover, the solution A(X) is unique