Math 239 Lecture 8

Graham Cooper

May 22th, 2015

Product Lemma

Recall:

Sets A,B with weight α, β
Set A $\times$ B, with weight w(a,b) = $\alpha(a) + \beta(b)$
Then $\Phi_{A \times B}(x) \cdot \Phi_B(x)$

Proof of the Product Lemma

$$\Phi_A(x) \cdot \Phi_B(x) = (\sum_{a \in A} x^{\alpha(a)})(\sum_{b \in B} x^{\beta(b)})$$
$$= \sum_{a \in A} \sum_{b \in B} x^{\alpha(a)} x^{\beta(b)}$$
$$\sum_{(a,b) \in A \times B} x^{\alpha(a)+\beta(b)}$$
$$= \sum_{(a,b) \in A \times B} x^{w(a,b)}$$
$$= \Phi_{A \times B}(x)$$

<u>**Example:**</u> Let $N_0 = \{0, 1, 2, 3, ...\}$ w(a) = a. Then:

$$\Phi_{N_0}(x) = 1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$$

$\frac{1}{(1-x)^k}$ is the generating series for $N_0 \times N_0 ... \times N_0 = N_0^k$
Where $w(a_1, a_2 ... a_k) = a_1 + a_2 + ... + a_k$ by product lemma.

So $[x^n]\frac{1}{(1-x)^k}$ is the number of k tuples $(a_1 ... a_k) \in N_0^k$ where they sum to n.
$\iff$ the number of non-negative integer solutions to $a_1 + a_2 +,,, + a_k = n$

In general, any solution $(a_1, ... a_k)$ corresponds to an arrangement of n 0's and k-1 1's
$0^{a_1}|0^{a_2}|...|0^{a_k}$

So there are $\binom{n+k-1}{k-1}$ of them. So $[x^n]\frac{1}{(1-x)^k} = \binom{n+k-1}{k-1}$

mytExample: How many ways can n identical pieces of sushi be eaten so Al eats at most 5, Bob eats at least 3 and Cam eats an even number?

Model the problem as $(a,b,c) \in A \times B \times C$ where:

$$A = \{0,1,2,3,4,5\}$$
$$B = \{3,4,5,6...\}$$
$$C = \{0,2,4,6...\}$$

Define w(a,b,c) = a + b + c. Using $\alpha(a) = a$ for all A,B,C we can apply the product lemma.
Then

$$\Phi_A(x) = 1 + x + x^2 + x^3 + x^4 + x^5 = \frac{1-x^6}{1-x}$$
$$\Phi_B(x) = x^3 + x^4 + x^5 + x^6 ... = \frac{x^3}{1-x}$$
$$\Phi_C(x) = 1 + x^2 + x^6 + ... = \frac{1}{1-x^2}$$

So

$$\Phi_{A\times B\times C}(x) = \Phi_A(x)\Phi_B(x)\Phi_C(x) = \frac{x^3(1-x^6)}{(1-x)^2(1-x^2)}$$

The number of ways is $[x^n]\frac{x^3(1-x^6)}{(1-x)^2(1-x^2)}$

Integer Compositions

0.1 Definition

A k-tuple $(a_1, ...a_k)$ of positive integers is a composition of n if $n = a_1 + ...a_k$
Such a composition is said to have k parts.

example Compositions of 5 include (1,3,1), (2,3), (1,1,1,2), (2,1,1,1), (5)

notes

1. Each part is at least 1
2. Order of the parts matter
3. The number 0 has 1 composition which has 0 parts ()

example: How many compositions of n have k parts?
$[n = 4, k = 3 : (1,1,2),(1,2,1),(2,1,1)]$

The set of all compositions of k parts (disregardig n) is $N \times N \times N ... N = N^k$, one composition has the form $(a_1, ...a_k)$

Define $w(a_1, ...a_k) = a_1 + ...a_k$
Use w(a) = a for each N.

$$\Phi_N(x) = x^1 + x^2 + x^3 + ... = \frac{x}{1-x}$$

By Product Lemma

$$\Phi_{N^k}(x) = (\Phi_N(x))^k = \frac{x^k}{(1-x)^k}$$

Our answer is:

$$[x^n]\frac{x^n}{(1-x)^n}$$