MATH 239 Additional Notes: Euler tours

Definition. *An* Euler tour *is a closed walk that contains every edge exactly once.*

Theorem. *Let G be a connected graph. Then G has an Euler tour if and only if every vertex has even degree.*

Proof. ($\Rightarrow$) A closed walk contributes 2 to the degree of a vertex for each visit. Since an Euler tour uses each edge of the graph exactly once, each vertex in the graph must have even degree.

($\Leftarrow$) Suppose by way of contradiction that this is false, namely there exist connected graphs where every vertex has even degree, but do not have Euler tours. Let G be such a counterexample with the fewest number of edges. We may assume that G is not empty and $|V(G)| \geq 2$, hence every vertex has degree at least 2. This means that G contains at least one cycle.

Among all subgraphs of G that have Euler tours, let H be one with the largest number of edges. Such a subgraph has at least one edge since G contains at least one cycle, which has an Euler tour. Suppose a tour in H is $T_1 = (v_0, v_1, v_2, \ldots, v_k, v_0)$. By assumption, $H \neq G$.

Let G' be the graph obtained by removing the edges of H from G. Note that each vertex has even degree in both G and H, so it still has even degree in G'. Since $H \neq G$, there exists at least one edge in G', so there exists a component C in G' that has at least one edge. Since H has at least one edge, C has fewer edges than G. Also, every vertex in C has even degree, so C is not a counterexample to our statement. Hence C contains an Euler tour, say $T_2 = (w_0, w_1, \ldots, w_l, w_0)$.

Since G is connected, there exists a vertex v that is in both C and in H. We may adjust the tours in H and in C so that they both start and end at the vertex v, i.e. $v = v_0 = w_0$. Then by joining these two tours at v, we obtain $T = (v_0, v_1, \ldots, v_k, v_0 = w_0, w_1, \ldots, w_l, w_0 = v_0)$, which is an Euler tour for the subgraph of G containing the edges of H and C. This subgraph has more edges than H, which is a contradiction. Therefore, G has an Euler tour. $\square$