Math 239 Fall 2014 Assignment 4 Solutions

1. Consider the following set of binary strings.

$$S = (\{1\}\ (\{0\}\{1\}^*\{0\})^*\ \{1\}\{0\}^*)^*$$

   (a) {2 marks} List all the binary strings in S of length at most 4.

   Solution. $\varepsilon, 11, 110, 1001, 1100, 1111$.

   (b) {3 marks} You are given that the expression for S is unambiguous. Using the length of a string as its weight, determine the generating series for the set S. Express it as a simplified rational expression.

   Solution. We see that

$$\Phi_{\{0\}\{1\}^*\{0\}}(x) = \frac{x^2}{1-x}.$$

   So then

$$\Phi_{(\{0\}\{1\}^*\{0\})^*}(x) = \frac{1}{1-\frac{x^2}{1-x}} = \frac{1-x}{1-x-x^2}.$$

   Then

$$\Phi_{\{1\}(\{0\}\{1\}^*\{0\})^*\{1\}\{0\}^*}(x) = x \cdot \frac{1-x}{1-x-x^2} \cdot \frac{x}{1-x} = \frac{x^2}{1-x-x^2}.$$

   Finally, this means that

$$\Phi_S(x) = \frac{1}{1-\frac{x^2}{1-x-x^2}} = \frac{1-x-x^2}{1-x-2x^2}.$$

   (c) {Extra credit: 2 marks} Give a simple description that characterizes all the strings that are in S. (No justification required.)

   Solution. These are binary representations of non-negative integers that are divisible by 3.

2. {4 marks} Explain why the following two decompositions are ambiguous.

   (a) $\{000, 00000\}^*$

   Solution. The string 00000000 can be decomposed in two different ways: 000/00000 and 00000/000.

   (b) $(\{1\}^*\{001, 0001\}^*)^*$

   Solution. The empty string ε is inside the main $*$, so it can be written as $\varepsilon\varepsilon$ and $\varepsilon\varepsilon/\varepsilon\varepsilon$.

3. For each of the following sets of binary strings, determine an unambiguous expression which generates every string in that set. Briefly justify your answers.

   (a) {2 marks} The set of binary strings where the length of each block of 0's is divisible by 5 and the length of each block of 1's is divisible by 3.

   Solution. $S = \{00000, 111\}^*$. Every time we have any 0's, we must use 5 of them, so any block of 0's has length divisible by 5. Similar with the 1's.

   (b) {3 marks} The set of binary strings where each block of 0's of length at least 3 must be followed by a block of 1's of even length.

   Solution. We use the block decomposition. The middle two-block combination is either at least 3 0's followed by even 1's, or 1 or 2 0's followed by any number of 1's. The last part must be either empty or a block of 0's of length 2 or less, for it does not have any block of 1's following it.

$$S = \{1\}^*(\{000\}\{0\}^*\{11\}\{11\}^* \cup \{0, 00\}\{1\}\{1\}^*)^*\{\varepsilon, 0, 00\}.$$

(c) {3 marks} The set of binary strings which contain 11000 as a substring. (Note: $\{0,1\}^*(11000)\{0,1\}^*$ is ambiguous.)

Solution. Take all the strings and remove those that do not contain 11000 as a substring. For such a string, any block of 1's of length at least 2 can be followed by a block of 0's of length at most 2, and any block of 1's of length 1 can be followed by any number of 0's.

$$S = \{0,1\}^* \setminus \{0\}^*(\{11\}\{1\}^*\{0,00\} \cup \{1\}\{0\}\{0\}^*)^*\{1\}^*.$$

4. {4 marks} On Martin's new game show "It Peis to Play", there is an unlimited supply of gold and silver coins buried inside a box of sand. As a contestant, you dig up one coin at a time, and Martin will offer you \$3 for each gold coin and \$2 for each silver coin. At any point, if you dig up 3 gold coins in a row, then the game ends and you win all the money you have earned. However, if you dig up 3 silver coins in a row, then the game ends and you lose everything. For any $n \in \mathbb{N}$, how many ways can you win exactly $\$n$ at the end of the game? (Express your answer as the coefficient of some power series, which you do not need to simplify.)

Solution. We set this up as a binary string problem, where 0 represents a silver coin and 1 represents a gold coin. For a string s, we define the weight function $w(s)$ to be 2 times the number of 0's plus 3 times the number of 1's in s.

In this problem, we need the set of all binary strings with no 3 consecutive 0's and no 3 consecutive 1's, except at the very end where it must be 3 consecutive 1's. This can be expressed as the following unambiguous expression:

$$\{\varepsilon, 0, 00\}(\{1,11\}\{0,00\})^*\{111\}.$$

Using the new weight function, we see that the generating series is

$$(1+x^2+x^4)\frac{1}{1-(x^3+x^6)(x^2+x^4)}x^9 = \frac{x^9+x^{11}+x^{13}}{1-x^5-x^7-x^8-x^{10}}.$$

The answer is then the coefficient of $[x^n]$ in this generating series.

5. A *switchback* in a binary string is a substring 101 or 010. Let S be the set of all binary strings with no switchbacks. For example, 100110, 011001110 are both in S, but 0100111, 1110100 are not. We wish to determine the generating series for S with respect to the lengths of the strings. We will find it with the help of the following two sets: Let T_0 be the set of all strings starting with 0 that have no switchbacks; let T_1 be the set of all strings starting with 1 that have no switchbacks. Then it is easy to see that

$$S = \{\varepsilon\} \cup T_0 \cup T_1.$$

(a) {3 marks} Prove that $T_0 = \{0, 01\} \cup \{0\}T_0 \cup \{01\}T_1$.

Solution. ($\subseteq$) Let $\sigma \in T_0$. Then $\sigma = 0\sigma'$. Note that σ' does not have any switchbacks. If $\sigma' = \varepsilon$, then $\sigma = 0$. If σ' starts with a 0, then $\sigma' \in T_0$, so $\sigma \in \{0\}T_0$. Otherwise $\sigma' = 1\sigma''$. Note that σ'' cannot start with 0, for otherwise we would have 010 in σ. So $\sigma'' = \varepsilon$ (in which case $\sigma = 01$), or σ'' starts with a 1 and $\sigma'' \in T_1$ (in which case $\sigma \in \{01\}T_1$).

($\supseteq$) Certainly 0, 01 start with 0 and do not have switchbacks, so they are in T_0. Any string $\omega \in T_0$ does not have switchbacks, and since it starts with 0, we cannot create a switchback in 0ω. So $0\omega \in T_0$. Any string in $\omega' \in T_1$ does not have switchbacks, and since it starts with 1, we cannot create a switchback in $01\omega''$. So $01\omega' \in T_0$.

(b) {3 marks} A similar proof would give $T_1 = \{1, 10\} \cup \{1\}T_1 \cup \{10\}T_0$. Using the three given equations in this question, determine the generating series for S. (Hint: Add the two equations derived from parts (a) and (b).)

Solution. The two equations derived from parts (a) and (b) are

$$\Phi_{T_0}(x) = (x+x^2) + x\Phi_{T_0}(x) + x^2\Phi_{T_1}(x), \quad \Phi_{T_1}(x) = (x+x^2) + x\Phi_{T_1}(x) + x^2\Phi_{T_0}(x).$$

Add the two equations to get

$$(\Phi_{T_0}(x) + \Phi_{T_1}(x)) = 2(x+x^2) + x(\Phi_{T_0}(x) + \Phi_{T_1}(x)) + x^2(\Phi_{T_1}(x) + \Phi_{T_0}(x)).$$

This means that

$$\Phi_{T_0}(x) + \Phi_{T_1}(x) = \frac{2x + 2x^2}{1 - x - x^2}.$$

From the equation given in the question,

$$\Phi_S(x) = 1 + \Phi_{T_0}(x) + \Phi_{T_1}(x) = \frac{1 + x + x^2}{1 - x - x^2}.$$

(c) {2 marks} The Fibonacci sequence $\{f_n\}$ is defined by $f_0 = 0, f_1 = 1$, and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. Prove that for $n \geq 1$, the number of binary strings of length n with no switchbacks is $2f_{n+1}$.

Solution. Let a_n be the number of strings of length n with no switchbacks. We know from part (b) that $a_n = [x^n]\frac{1+x+x^2}{1-x-x^2}$. So $\sum_{n\geq 0} a_n x^n = \frac{1+x+x^2}{1-x-x^2}$. Multiply both sides by $1 - x - x^2$ to get

$$a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + \sum_{n\geq 3}(a_n - a_{n-1} - a_{n-2}) = 1 + x + x^2.$$

By comparing coefficients, we get

$$a_0 = 1, a_1 = 2, a_2 = 4, a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 3.$$

Here we see that $a_1 = 2 = 2f_2$, and $a_2 = 4 = 2f_3$. Using induction, assuming that $a_k = 2f_{k+1}$ for all $1 \leq k \leq n - 1$, we see that

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2} && \text{from the recurrence we derived} \\ &= 2f_n + 2f_{n-1} && \text{by induction hypothesis} \\ &= 2f_{n+1} && \text{by the fibonacci recurrence.} \end{aligned}$$