Math 239 Fall 2014 Assignment 2 Solutions

1. For each of the following, determine the generating series of the set with respect to the weight function, and express the series as a rational expression. In addition, determine whether or not the series is invertible.

   (a) {4 marks} Set: $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Weight function: $w(a) = \begin{cases} a/3+1 & a \equiv 0 \pmod 3 \\ a & a \equiv 1 \pmod 3 \\ 2a & a \equiv 2 \pmod 3 \end{cases}$.

   Solution. We partition $\mathbb{N}_0$ into three sets: A, B, C where

$$A = \{3k \mid k \in \mathbb{N}_0\}, B = \{3k+1 \mid k \in \mathbb{N}_0\}, C = \{3k+2 \mid k \in \mathbb{N}_0\}.$$

   Using the given weight function,

$$\begin{aligned}
\Phi_A(x) &= \sum_{k\geq 0} x^{w(3k)} = \sum_{k\geq 0} x^{k+1} = x\sum_{k\geq 0} x^k = \frac{x}{1-x} \\
\Phi_B(x) &= \sum_{k\geq 0} x^{w(3k+1)} = \sum_{k\geq 0} x^{3k+1} = x\sum_{k\geq 0} x^{3k} = \frac{x}{1-x^3} \\
\Phi_C(x) &= \sum_{k\geq 0} x^{w(3k+2)} = \sum_{k\geq 0} x^{6k+4} = x^4\sum_{k\geq 0} x^{6k} = \frac{x^4}{1-x^6}
\end{aligned}$$

   Since $\mathbb{N}_0 = A \cup B \cup C$ and these are disjoint sets, by sum lemma,

$$\Phi_{\mathbb{N}_0}(x) = \Phi_A(x) + \Phi_B(x) + \Phi_C(x) = \frac{x}{1-x} + \frac{x}{1-x^3} + \frac{x^4}{1-x^6} = \frac{2x - x^2 - x^5 - 3x^7 + 2x^8 + x^{10}}{(1-x)(1-x^3)(1-x^6)}$$

   This series is not invertible since the constant term is 0.

   (b) {4 marks} Set: $\mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0$. Weight function: $w(a,b,c) = a + 2b + 3c$.

   Solution. Define the weight functions α, β, γ for each of the $\mathbb{N}_0$'s where $\alpha(a) = a, \beta(b) = 2b, \gamma(c) = 3c$. Let the respective generating series be $\Phi^{\alpha}_{\mathbb{N}_0}(x), \Phi^{\beta}_{\mathbb{N}_0}(x), \Phi^{\gamma}_{\mathbb{N}_0}(x)$. Then

$$\begin{aligned}
\Phi^{\alpha}_{\mathbb{N}_0}(x) &= 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \\
\Phi^{\beta}_{\mathbb{N}_0}(x) &= 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2} \\
\Phi^{\gamma}_{\mathbb{N}_0}(x) &= 1 + x^3 + x^6 + x^9 + \cdots = \frac{1}{1-x^3}
\end{aligned}$$

   Since $w(a,b,c) = \alpha(a) + \beta(b) + \gamma(c)$, by the product lemma,

$$\Phi_{\mathbb{N}_0\times\mathbb{N}_0\times\mathbb{N}_m}(x) = \Phi^{\alpha}_{\mathbb{N}_0}(x)\Phi^{\beta}_{\mathbb{N}_0}(x)\Phi^{\gamma}_{\mathbb{N}_0}(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)}.$$

   This series is invertible since the constant term is not zero.

2. {4 marks} Using mathematical induction on k, prove that for any integer $k \geq 1$,

$$(1-x)^{-k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} x^n.$$

   Solution. When $k = 1$, $\binom{n+k-1}{k-1} = \binom{n}{0} = 1$. So $(1-x)^{-1} = \sum_{n\geq 0} x^n = \sum_{n\geq 0} \binom{n+1-1}{k-1} x^n$. So the base case holds.

   Assume that for some positive integer m, $(1-x)^{-m} = \sum_{n\geq 0} \binom{n+m-1}{m-1} x^n$.

   We need to prove the equation for $m+1$. We see that

$$(1-x)^{-(m+1)} = (1-x)^{-m}(1-x)^{-1}.$$

By induction hypothesis, $[x^i](1-x)^{-m} = \binom{i+m-1}{m-1}$. Also, we know that $[x^i](1-x)^{-1} = 1$. Using rules of multiplication of power series, we get

$$[x^n](1-x)^{-(m+1)} = \sum_{i=0}^{n}([x^i](1-x)^{-m})([x^{n-i}](1-x)^{-1}) = \sum_{i=0}^{n}\binom{i+m-1}{m-1} = \binom{n+m}{m}$$

where the final step uses an identity from class. Therefore,

$$(1-x)^{-(m+1)} = \sum_{n\geq 0}\binom{n+m}{m}x^n.$$

Therefore, by induction, the result holds.

3. {4 marks} Determine the value of the following coefficient. (You do not need to evaluate large powers and binomial coefficients.)

$$[x^{21}](x^2+x^5)(1-3x^4)^{-31}(1+x^6)^{-41}.$$

Solution. We see that

$$[x^{21}](x^2+x^5)(1-3x^4)^{-31}(1+x^6)^{-41} = [x^{19}](1-3x^4)^{-31}(1+x^6)^{-41} + [x^{16}](1-3x^4)^{-31}(1+x^6)^{-41}.$$

Note that in the expansion of $(1-3x^4)^{-31}(1+x^6)^{-41}$, the exponents of x are integer combinations of 4's and 6's. Such exponents are multiples of 2, so the coefficient of x^{19} is 0. The required coefficient is then equal to the coefficient of x^{16} in $(1-3x^4)^{-31}(1+x^6)^{-41}$.

There are 2 ways to get x^{16} in this multiplication: $[x^{16}](1-3x^4)^{-31}[x^0](1+x^6)^{-41}$ and $[x^4](1-3x^4)^{-31}[x^{12}](1+x^6)^{-41}$. These correspond to the numbers $3^4\binom{4+31-1}{31-1}$ and $3^1\binom{1+31-1}{31-1}\cdot(-1)^2\binom{2+41-1}{41-1}$. So the required coefficient is $3^4\binom{34}{30}+3\binom{31}{30}\binom{42}{40} = 3836529$.

4. {4 marks} Let $\{a_n\}_{n\geq 0}$ be a sequence whose corresponding power series $A(x) = \sum_{n\geq 0} a_n x^n$ is

$$A(x) = \frac{-6-34x}{1+2x-3x^2}.$$

Determine a recurrence relation that $\{a_n\}$ satisfies, with sufficient initial conditions to uniquely specify $\{a_n\}$. Use this recurrence relation to find a_4.

Solution. We see that

$$(1+2x-3x^2)A(x) = -6-34x.$$

So

$$\begin{aligned}-6-34x &= (1+2x-3x^2)(a_0+a_1x+a_2x^2+a_3x^3+\cdots)\\ &= a_0+(a_1+2a_0)x+\sum_{n\geq 2}(a_n+2a_{n-1}-3a_{n-2})x^n\end{aligned}$$

By comparing the coefficients, we see that $a_0=-6$; $a_1+2a_0=-34$, so $a_1=-22$; and $a_n+2a_{n-1}-3a_{n-2}=0$ for $n\geq 2$. These are the initial conditions and the recurrence that $\{a_n\}$ satisfies. To get a_4, we apply the recurrence relation.

$$\begin{aligned}a_2 &= -2a_1+3a_0 = 26\\ a_3 &= -2a_2+3a_1 = -118\\ a_4 &= -2a_3+3a_2 = 314\end{aligned}$$

5. Let $n\in\mathbb{N}$. A permutation of $[n]$ is a rearrangement of the elements of $[n]$. We may think of a permutation as a bijection $\sigma:[n]\to[n]$. For example, the permutation (213) of [3] can be represented by $\sigma:[3]\to[3]$ such that $\sigma(1)=2, \sigma(2)=1, \sigma(3)=3$ (first position is 2, second position is 1, third position is 3).

A pair of integers (i,j) is called an *inversion* of σ if $i<j$ and $\sigma(i)>\sigma(j)$. For example, in the permutation (32415) on [5], $(1,2)$ is an inversion since $1<2$ and $\sigma(1)=3>2=\sigma(2)$. This permutation has 4 inversions: $(1,2),(1,4),(2,4),(3,4)$.

Define the weight function w on a permutation σ to be the number of inversions in σ. Let S_n be the set of all permutations of $[n]$. To check that you have understood the definitions, the generating series for S_1, S_2, S_3 with respect to w are $1, 1+x, 1+2x+2x^2+x^3$ respectively.

(a) {2 marks} For $1 \leq k \leq n$, let $T_{n,k}$ be the set of all permutations of $[n]$ where $\sigma(k) = n$ (i.e. the element n is at the k-th position). Describe a bijection between S_{n-1} and $T_{n,k}$, and describe its inverse.

Solution. For $k < n$, define $f : S_{n-1} \to T_{n,k}$ where for each permutation $\sigma \in S_{n-1}$, $f(\sigma)$ is the permutation of $[n]$ where n is inserted to the left of the k-th position in σ. When $k = n$, we define $f(\sigma)$ to be the permutation of $[n]$ where n is inserted to the right of every entry. (For example, when $n = 5, k = 3$, the permutation (2143) is being mapped to (21543).)

We can define the inverse $f^{-1} : T_{n,k} \to S_{n-1}$ so that for each $\omega \in T_{n,k}$, $f^{-1}(\omega)$ is the permutation of $[n-1]$ where n is removed from ω.

(b) {2 marks} Using w as the weight function, prove that $\Phi_{T_{n,k}}(x) = x^{n-k}\Phi_{S_{n-1}}(x)$.

Solution. Let $\sigma \in S_{n-1}$, and consider $f(\sigma)$. The element n is inserted at position k. Since n is the largest element, we have created new inversions with all entries to the right of n. Existing inversions in σ remain inversions. Since there are $n - k$ entries to the right of the k-th position, the number of inversions of $f(\sigma)$ is $n - k$ more than the number of inversions of σ. So $w(f(\sigma)) = w(\sigma) + (n - k)$. Since f is a bijection, we can say that

$$\Phi_{T_{n,k}} = \sum_{\omega \in T_{n,k}} x^{w(\omega)} = \sum_{\sigma \in S_{n-1}} x^{w(f(\sigma))} = \sum_{\sigma \in S_{n-1}} x^{w(\sigma)+(n-k)} = x^{n-k} \sum_{\sigma \in S_{n-1}} x^{w(\sigma)} = x^{n-k}\Phi_{S_{n-1}}(x).$$

(In other words, since the weight of each permutation increases by $n-k$, the generating series must multiply by x^{n-k}.)

(c) {2 marks} Prove that for $n \geq 2$, $\Phi_{S_n}(x) = (1 + x + \cdots + x^{n-1})\Phi_{S_{n-1}}(x)$.

Solution. We split S_n into n sets according to the location of the element n in the permutation.

$$S_n = T_{n,1} \cup T_{n,2} \cup \cdots \cup T_{n,n}.$$

From part (b), we have $\Phi_{T_{n,k}}(x) = x^{n-k}\Phi_{S_{n-1}}(x)$. So using the sum lemma, we get

$$\Phi_{S_n}(x) = \sum_{k=1}^{n} \Phi_{T_{n,k}}(x) = \sum_{k=1}^{n} x^{n-k}\Phi_{S_{n-1}}(x) = (1 + x + x^2 + \cdots + x^{n-1})\Phi_{S_{n-1}}(x).$$

(d) {2 marks} Prove that the number of permutations of $[n]$ with k inversions is

$$[x^k]\frac{\prod_{i=1}^{n}(1-x^i)}{(1-x)^n}.$$

Solution. We see that $\Phi_{S_1}(x) = 1$ from the question, so this is satisfied for $n = 1$. Using induction, we see that

$$\begin{aligned}
\Phi_{S_n}(x) &= (1 + x + \cdots + x^{n-1})\Phi_{S_{n-1}}(x) \text{ by part (c)} \\
&= \frac{1-x^n}{1-x}\Phi_{S_{n-1}}(x) \\
&= \frac{1-x^n}{1-x}\frac{\prod_{i=1}^{n-1}(1-x^i)}{(1-x)^{n-1}} \text{ by ind hyp} \\
&= \frac{\prod_{i=1}^{n}(1-x^i)}{(1-x)^n}
\end{aligned}$$