Math 239 Spring 2015 Midterm Solutions

1. {2 marks} How many binary strings of length 239 have exactly 135 ones?

   Solution. $\binom{239}{135}$.

2. Determine the following coefficients.

   (a) {3 marks}
   $$[x^{50}]\frac{1}{(1-x^{30})(1-2x^{20})^{10}}$$

   Solution. We see that
   $$\frac{1}{(1-x^{30})} = \sum_{i\geq 0} x^{30i}, \quad \frac{1}{(1-2x^{20})^{10}} = \sum_{j\geq 0}\binom{j+9}{9}2^j x^{20j}.$$

   There is only one way to get the coefficient of x^{50} in the multiplication of these two series, i.e. from x^{30} of the first series and x^{20} of the second series. So
   $$[x^{50}]\frac{1}{(1-x^{30})(1-2x^{20})^{10}} = \left([x^{30}]\frac{1}{(1-x^{30})}\right)\left([x^{20}]\frac{1}{(1-2x^{20})^{10}}\right) = 1\cdot\binom{1+9}{9}2^1 = 10\cdot 2 = 20.$$

   (b) {2 marks}
   $$[x^3](1+3x)^{2/3}$$

   Solution. Using the binomial theorem, we have
   $$[x^3](1+3x)^{2/3} = \binom{2/3}{3}\cdot 3^3 = \frac{\frac{2}{3}\cdot(\frac{-1}{3})\cdot(\frac{-4}{3})}{3!}\cdot 3^3 = \frac{4}{3}.$$

3. {3 marks} Let $S = \{1,2,3,4,5\}$, and define the weight function w for S where $w(a) = 3a-1$ for each $a\in S$. Determine the generating series of S with respect to w.

   Solution. Using the given weight function, the weights of $1,2,3,4,5$ are $2,5,8,11,14$ respectively. So using the definition of generating series,
   $$\Phi_S(x) = x^2+x^5+x^8+x^{11}+x^{14}.$$

4. {5 marks} Give a combinatorial proof of the following identity for any integer $n\geq 1$.
   $$\sum_{k=0}^{n}\binom{n}{k}\cdot k = n\cdot 2^{n-1}.$$

   (Hint: Consider counting the set $\{(A,x)\mid A\subseteq\{1,2,\ldots,n\}, x\in A\}$.)

   Solution. Let $S = \{(A,x)\mid A\subseteq[n], x\in A\}$. We partition S into $S_0,S_1,\ldots,S_n$ where S_k contains all elements $(A,x)\in S$ where $|A|=k$. To count the size of S_k, we see that A can be any subset of $[n]$ of size k, so there are $\binom{n}{k}$ choices for A. Then there are k ways to choose an element $x\in A$. So $|S_k| = \binom{n}{k}\cdot k$.

   There is another way to count S. There are n choices for x. After having chosen x, the set A can be any subset of $[n]$ that consists of x. The number of ways to choose A is 2^{n-1} as we are picking any subset of $[n]\setminus\{x\}$. So $|S| = n\cdot 2^{n-1}$.

   We see that $S = \bigcup_{k=0}^{n} S_k$ is a disjoint union, so $|S| = \sum_{k=0}^{n}|S_k|$. And the result follows.

5. Let C be the set of all compositions of any non-negative integer with at least 3 parts where each part is even.

   (a) {2 marks} Write down two different compositions of 12 that are in C.

   Solution. $(4,4,4)$ and $(2,2,4,4)$.

(b) {5 marks} For any integer $n \geq 0$, let c_n be the number of compositions of n in C. Determine c_n. (You may answer this by stating that c_n is equal to the coefficient of a certain rational expression. Do not find an explicit formula for c_n.)

Solution. Let $E = \{2, 4, 6, \ldots\}$ be the set of all positive even integers. Then we can represent C as

$$C = \bigcup_{k \geq 3} E^k.$$

Let the weight of a composition to be the sum of its parts. Then the generating series for E is

$$\Phi_E(x) = x^2 + x^4 + x^6 + \cdots = \frac{x^2}{1 - x^2}.$$

Using the sum and product lemmas, we get

$$\begin{aligned}
\Phi_C(x) &= \sum_{k \geq 3} \Phi_{E^k}(x) = \sum_{k \geq 3} (\Phi_E(x))^k = (\Phi_E(x))^3 \sum_{k \geq 0} (\Phi_E(x))^k \\
&= \left(\frac{x^2}{1 - x^2}\right)^3 \sum_{k \geq 0} \frac{x^2}{1 - x^2} = \frac{x^6}{(1 - x^2)^3} \cdot \frac{1}{1 - \frac{x^2}{1 - x^2}} \\
&= \frac{x^6(1 - x^2)}{(1 - x^2)^3(1 - 2x^2)} = \frac{x^6}{(1 - x^2)^2(1 - 2x^2)}.
\end{aligned}$$

So the number of compositions of n in C is

$$c_n = [x^n] \frac{x^6}{(1 - x^2)^2(1 - 2x^2)}.$$

6. {3 marks} You are given that the following is an unambiguous expression for a certain set of strings S. Determine the generating series for S with respect to the lengths of the strings, and express your answer as the ratio of two polynomials.

$$S = (\{1\}^*\{0\}\{1\}^*\{0\})^*\{1\}^*$$

Solution. The generating series is

$$\begin{aligned}
\Phi_S(x) &= \frac{1}{1 - \frac{1}{1 - x} \cdot x \cdot \frac{1}{1 - x} \cdot x} \cdot \frac{1}{1 - x} \\
&= \frac{(1 - x)^2}{(1 - x)^2 - x^2} \cdot \frac{1}{1 - x} \\
&= \frac{1 - x}{1 - 2x}.
\end{aligned}$$

7. {4 marks} Write an unambiguous expression for the set of all strings where every block of 0's must be followed by a block of 1's of length divisible by 3.

Solution. Consider the block decomposition $\{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*$. The last $\{0\}^*$ must be removed since it is not followed by any block of 1's. The block of 0's $\{0\}\{0\}^*$ must be followed by a block of 1's of length divisible by 3, which can be expressed as $\{111\}\{111\}^*$. So an unambiguous expression for our set of strings is

$$\{1\}^*(\{0\}\{0\}^*\{111\}\{111\}^*)^*.$$

8. {5 marks} Let $\{a_n\}$ be the sequence where $a_0 = 2, a_1 = 15$, and for $n \geq 2$,

$$a_n - 6a_{n-1} + 9a_{n-2} = 0$$

Determine an explicit formula for a_n.

Solution. The characteristic polynomial of this recurrence is $x^2 - 6x + 9 = (x - 3)^2$. So it has root 3 with multiplicity 2. The general solution to a_n is

$$a_n = (An + B) \cdot 3^n$$

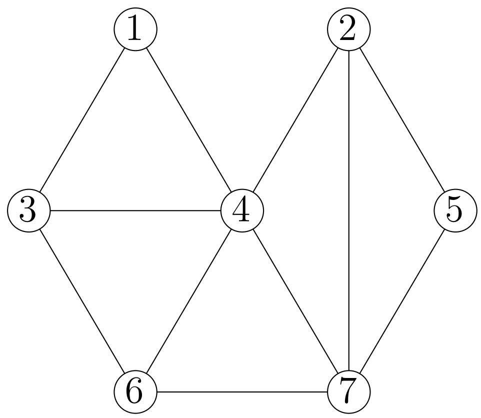
for some constants A, B. Using the initial conditions $a_0 = 2, a_1 = 15$, we get

$$2 = B, \quad 15 = 3A + 3B.$$

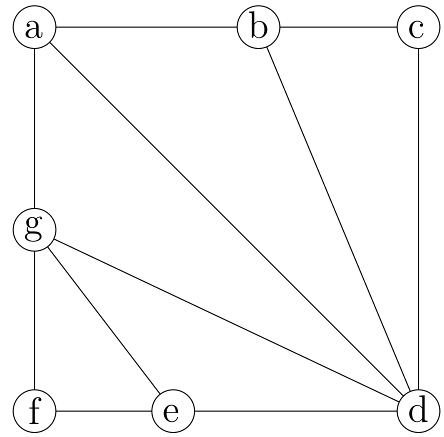
This gives us $A = 3, B = 2$. Hence the solution to the recurrence is

$$a_n = (3n + 2) \cdot 3^n.$$

9. {2 marks} The following two graphs are isomorphic. Write down a function that is an isomorphism from G to H.



Graph G



Graph H

Solution. One possible isomorphism: $f : V(G) \to V(H)$ where

v	1	2	3	4	5	6	7
$f(v)$	c	e	b	d	f	a	g

10. {3 marks} Suppose G is a graph with 10 vertices of degree 3, 20 vertices of degree 2, x vertices of degree 1, and no vertices of other degrees. In addition, G has 40 edges. Determine the value of x.

Solution. The sum of the degrees of all vertices in G is

$$10 \cdot 3 + 20 \cdot 2 + x \cdot 1 = 70 + x.$$

By the handshaking lemma, this is equal to twice the number of edges, which is 80. So we have $70 + x = 80$, this gives us $x = 10$.

11. {3 marks} Prove that any graph with 10 vertices and 26 edges cannot be bipartite.

Solution. Suppose by way of contradiction that there is a bipartite graph G with 10 vertices and 26 edges. Let (A, B) be a bipartition of the vertices of G. Since each edge joins a vertex in A with a vertex in B, the maximum number of edges in G is $|A| \cdot |B|$. Since $|A| + |B| = 10$, $|A| \cdot |B| = |A| \cdot (10 - |A|)$. This value is maximized when $|A| = 5$, and the maximum number of edges is $5 \cdot 5 = 25$, which is a contradiction. Hence any graph with 10 vertices and 26 edges cannot be bipartite.

12. {3 marks} A ternary tree is either
   (a) the empty tree ε; or
   (b) a tree with a fixed root vertex $\bullet$ such that each vertex has a left branch, a centre branch, and a right branch, each of which is a ternary tree.

Let t_n be the number of ternary trees with n vertices, and let $T(x) = \sum_{n \geq 0} t_n x^n$. Prove that

$$T(x) = 1 + xT(x)^3.$$

Solution. Let $\mathcal{T}$ be the set of all ternary trees. For each tree T in $\mathcal{T}$, define the weight function $w(T)$ to be the number of vertices in T. Define a function $f : \mathcal{T} \to (\{\varepsilon\} \cup \{\bullet\} \times \mathcal{T}^3$ as follows: $f(\varepsilon) = \varepsilon$, and for any nonempty tree $T \in \mathcal{T}$, $f(T) = (\bullet, T_1, T_2, T_3)$ where T_1, T_2, T_3 are the left, centre and right branches of T respectively. This is a bijection. For $\{\bullet\} \times \mathcal{T}^3$, if we define $w'(\bullet, T_1, T_2, T_3) = 1 + w(T_1) + w(T_2) + w(T_3)$, then $w(T) = w'(f(T))$ (the corresponding elements have the same weight). So the generating series for $\mathcal{T}$ with respect to w is the same as the generating series for ε plus the generating series for $\{\bullet\} \times \mathcal{T}^3$ with respect to w'. The generating series for $\{\bullet\}$ is x, and the generating series for each of the 3 copies of $\mathcal{T}$ is $T(x)$. So using the product lemma, we have

$$T(x) = 1 + xT(x)^3.$$