CS240 Tutorial 2

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Topics:

- small o proof
- loop analysis
- asymptotic proofs

Recall: We say that $f(n) \in o(g(n)) \iff \forall e > 0, \exists n_0 > 0$ such that f(n) $\leq$ cg(n).

Q1)

$2^{\sqrt{\log n}} \in o(n)$: Show from first principals.
Goal: $2^{\sqrt{\log n}} < cn$

Observe that $2^{\sqrt{\log n}} \underset{1}{<} 2^{log_4 n} \underset{2}{<} cn$
What values of n does 1 hold?

$$\sqrt{\log n} < log_4 n$$

$$\sqrt{\log n} < 1/2log(n)$$

$$2 < \sqrt{logn}$$

$$4 < log(n)$$

$$16 < n$$

$$2^{log_4 n} < cn$$

$$\sqrt{n} < cn$$

$$1/c < \sqrt{n}$$

$$1/c^2 < n$$

Pick $n_0 = max(16, 1/c^2)$

Q2)

Analyze the following:

```
for i = 1 to n {
for j = 1 to i {
k = j
while k > 1 {
            k ⇐ k / 2
}
}
}
```

$$T(n) = \sum_{i=1}^{n} \sum_{j=1}^{i} \log j$$

$$= \sum_{i=1}^{n} \log i!$$

Upper bound: $\leq \sum_{i=1}^{n} \log n!$
$\leq \sum_{i=1}^{n} cn \log n$

Recall that
$\log n! \in O(n \log n) \implies \sum_{i=1}^{n} cn \log n = cn^2 \log n \in O(n^2 \log n)$
Lower Bound:

$$\sum_{i=1}^{n} \log i!$$

$$\geq \sum_{i=n/2+1}^{n} \log i!$$

$$\geq \sum_{i=n/2+1}^{\log n/2!}$$

$$n/2(c(n/2) \log n/2) = n/2 \log n/2!$$

$$\geq c(n/4)(\log n/2)$$

$$\in \Omega(n^2 \log(n))$$

Q3)

for i = 1 to n
j = i
while j $\leq$ n
j=2j

Aside:
After t iterations(of the inner loop), j = $i2^t$
$i2^{t-1} < n < i2^t$
$t = \log n/i$
Answer:

$$T(n) = \sum_{i=1}^{n} \log n/i$$

$$= \sum_{i=1} n \log n - \log i$$

$$= n \log n - \sum_{i=1}^{n} \log i$$

$$= n \log n - \log n!$$

$$\approx n \log n - log([\frac{n}{e}]^n)$$

$$= n \log n - n \log(\frac{n}{e})$$

$$= n \log n - (n \log n - n \log e)$$

$$= n \log e \in \Theta(n)$$

ASIDE: Stirling's Approximation:

$$n! \approx (\frac{n}{e})^n \sqrt{2\pi n}$$

or

$$n! \approx (\frac{n}{e})^n$$

Q4

Prove or disprove:
If $f(n) \in \Theta(g(n))$ is $f(n) - g(n) \in \Theta(1)$
This is false, as shown above

$$f(n) = n^2, g(n) = n^2 + n$$
$$f(n) \in \Theta g(n)$$
$$f(n) - g(n) = n \notin \Theta(1)$$

Q5

If $f(n) \in O(g(n))$, and $g(n) \in O(h(n))$, is $f(n) \in O(h(n))$
True $\rightarrow f(n) \leq cg(n), \forall n \geq n_0$
$g(n) \leq c'h(n), \forall n \geq n_0'$
$f(n) \leq cg(n) \leq cc'h(n)$
Pick $c' = cc'$
$f(n) \leq c''h(n), \forall n \geq max(n_0, n_0')$

Q6

If $f(n) \in O(g(n))$, and $g(n) \in \Omega(h(n))$, is $f(n) \in \Omega(h(n))$

Consider $f(n) = n$ and $g(n) = n^3$ and $h(n) = n^2$
check: $f(n) \in O(g(n))$ - True
$g(n) \in \Omega(h(n))$ - True
$f(n) \in \Omega(h(n))$ - False