Math 239 Lecture 26

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Euler's Formula

Euler's formula for a planar embedding of a connected planar graph G with n vertices, m edges and S faces,

$$n - m + S = 2$$

Proof: Fix the number of vertices n, do induction on the number of edges m.

Base Case: m = n-1 (a tree, smallest connected planar graph)
In a tree S = 1 So n - m + s = n - (n-1) + 1 = 2

Induction Hypothesis: Assume any connected planar embedding with n vertices and m-1 edges satisfy Euler's formula

Induction Step: Consider a connected planar emb. if a graph G with n vertices, m edges, s faces. Since G is not a tree, it contains a cycle. Let e be an edge in a cycle. Then G-e is connected (sicne e is not a bridge), planar, with m-1 eges.
By Induction hypothesis, Euler's formula holds for G-e. The edge e has 2 different faces on both sides. In G-e, these two faces merge into 1 so G-e has S-1 faces. Using Euler's formular for G-e, n-(m-1)+(s-1) = 2. So n-m+s = 2 and Euler's formula hodls for G.

Platonic Solids

Any planar embedding can bedrawn on the sphere, cut off the faces to obtain a polyhedron.
Definition A connected graph is platonic if it has an embedding where every verte has the same degree (≥ 3)

Suppose a platonic graph has n verices, m edges, s faces, $d_v \geq 3$ verte deg, $d_f \geq 3$ face deg

1. Handshaking lemma: 2m = n$\cdot d_v \implies$ n = $\frac{2m}{d_v}$

2. Handshaking lemma for faces: 2m = S $\cdot d_f \implies$ S = $\frac{2m}{d_f}$

3. Euler's Formula: n - m + s = 2

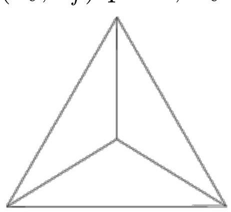
$$\implies \frac{2m}{d_v} - m + \frac{2m}{d_f} = 2$$

multiply by $d_v d_f$

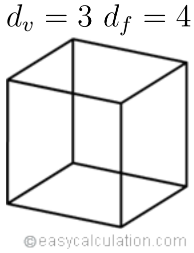
$$\implies 2md_f - md_vd_f + 2md_v = 2d_vd_f$$
$$\implies m(2d_f - d_vd_f + 2d_v) = 2d_vd_f$$
$$\implies 2d_f - d_vd_f + 2d_v > 0$$
$$\implies 2d_f - d_vd_f + 2d_v - 4 + 4$$
$$= -(d_v - 2)(d_f - 2) + 4 > 0$$
$$\implies (d_v - 2)(d_f - 2) < 4$$

d_v	d_f
3	3,4,5
4	3
5	3

Only possible
(d_v, d_f) pairs, $d_v = 3$ $d_f = 3$



$d_v = 3$ $d_f = 4$



$d_v = 4$ $d_f = 3$