Math 239 - Lecture 2

Graham Cooper

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Bijections

Let A, B be finite sets.
Consider a function $f : \overset{domain}{A} \to \overset{codomain}{B}$

One to One

f is 1 to 1 if $x \neq\in A \implies f(x) \neq f(y) \in B$
$(f(x) = f(y) \implies x = y)$

If f is 1 to 1 then —B— $\geq$ —A—

Onto

f is onto if for all $y \in B$, there exists $x \in A$ such that $f(x) = y$ (Every element B is being mapped to by something in A).

Bijection (One to One and Onto)

f is a bijection if f is one to one and onto.
If f is a bijection, then —A— = —B—

Example: A = {1, 2, 3} B = {a, b, c} Define $f : A \to B$
by: $f(q) = a$ $f(2) = b$, $f(3) = c$ f is a bijection

Example: Let S be the set of all subsets of [n] of size k.
Let T be the set of all subsets of [n] of size n - k

$$n = 4$$

$$k = 1$$

$$S = \{\{1\}, \{2\}, \{3\}, \{4\}\}$$

$$T = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$$

$$\{1\} \to \{2, 3, 4\}$$

$$\{2\} \to \{1,3,4\}$$

$$\{3\} \to \{1,2,4\}$$

$$\{4\} \to \{1,2,3\}$$

The fractions below means that x is not in [n]

Define $f : s \to T$ by $f(x) = \frac{[n]}{x}$, for any $x \in S$
Check $f(x) \in T$. Since $x \leq [n]$ of size k, $\frac{[n]}{x}$ is also a subset of [n], now of size n-k. so $f(x) \in T$

Inverse:
The inverse of $f : A \to B$ is the function $f^{-1} : B \to A$ such that for all $x \in A$, $f^{-1}(f(x)) = x$, and for all $y \in B$, $f(f^{-1}(y)) = y$

Theorem: $f : A \to B$ is a bijection if and only if its inverse exists

Back to example, f has an inverse: $f^{-1} : T \to S$ where
$f^{-1}(y) = \frac{[n]}{y}$ for all $y \in T$
For any $x \in S$, $f^{-1}(f(x)) = f^{-1}(\frac{[n]}{x}) = \frac{[n]}{\frac{[n]}{x}}$

This establishes that —S— = —T— $|S| = \binom{n}{k}$ $|T| = \binom{n}{n-k}$
So $\binom{n}{k} = \binom{n}{n-k}$
The bijection serves as a combiinatorial proof of this equation.

Example:
Let S be the set of all subsets of [n]
Let T be the set of all binary strings of length n

$$n = 3$$

$$S = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

$$T = \{000, 001, 010, 011, 100, 101, 110, 111\}$$

$$111 \to 1,2,3$$

$$110 \to 1,2$$

$$100 \to 1$$
$$011 \to 2, 3$$
$$\downarrow$$

Each 1 in the binary string is "on" for one of the digits 1, 2, and 3.

Define $f : T \to S$ where $f(a_1, a_2, ...a_n)$ ={i — $i \in [n]$, $a_i = 1$}
(if a_i is 1, put element i in the subset)

The inverse is $f^{-1} : S \to T$ where for each $x \in S$
f(x) = $a_1 a_2 ... a_n$ hwere a_i = {1 if i ∈ x — 0 if i not ∈ x}