

# Math 239 - Lecture 2

Graham Cooper

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## Bijections

Let  $A, B$  be finite sets.

Consider a function  $f : \overset{\text{domain}}{A} \rightarrow \overset{\text{codomain}}{B}$

### One to One

$f$  is 1 to 1 if  $x \notin A \implies f(x) \neq f(y) \in B$   
 $(f(x) = f(y) \implies x = y)$

If  $f$  is 1 to 1 then  $|B| \geq |A|$

### Onto

$f$  is onto if for all  $y \in B$ , there exists  $x \in A$  such that  $f(x) = y$  (Every element  $B$  is being mapped to by something in  $A$ ).

### Bijection (One to One and Onto)

$f$  is a bijection if  $f$  is one to one and onto.

If  $f$  is a bijection, then  $|A| = |B|$

**Example:**  $A = \{1, 2, 3\}$   $B = \{a, b, c\}$  Define  $f : A \rightarrow B$   
by:  $f(1) = a$   $f(2) = b$ ,  $f(3) = c$   $f$  is a bijection

**Example:** Let  $S$  be the set of all subsets of  $[n]$  of size  $k$ .  
Let  $T$  be the set of all subsets of  $[n]$  of size  $n - k$

$$n = 4$$

$$k = 1$$

$$S = \{\{1\}, \{2\}, \{3\}, \{4\}\}$$

$$T = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$$

$$\{1\} \rightarrow \{2, 3, 4\}$$

$$\{2\} \rightarrow \{1, 3, 4\}$$

$$\{3\} \rightarrow \{1, 2, 4\}$$

$$\{4\} \rightarrow \{1, 2, 3\}$$

The fractions below means that  $x$  is not in  $[n]$

Define  $f : S \rightarrow T$  by  $f(x) = \frac{[n]}{x}$ , for any  $x \in S$

Check  $f(x) \in T$ . Since  $x \leq [n]$  of size  $k$ ,  $\frac{[n]}{x}$  is also a subset of  $[n]$ , now of size  $n-k$ . so  $f(x) \in T$

**Inverse:**

The inverse of  $f : A \rightarrow B$  is the function  $f^{-1} : B \rightarrow A$  such that for all  $x \in A$ ,  $f^{-1}(f(x)) = x$ , and for all  $y \in B$ ,  $f(f^{-1}(y)) = y$

*Theorem:*  $f : A \rightarrow B$  is a bijection if and only if its inverse exists

Back to example,  $f$  has an inverse:  $f^{-1} : T \rightarrow S$  where

$$f^{-1}(y) = \frac{[n]}{y} \text{ for all } y \in T$$

$$\text{For any } x \in S, f^{-1}(f(x)) = f^{-1}\left(\frac{[n]}{x}\right) = \frac{[n]}{\frac{[n]}{x}}$$

This establishes that  $|S| = |T|$

$$|S| = \binom{n}{k} \quad |T| = \binom{n}{n-k}$$

$$\text{So } \binom{n}{k} = \binom{n}{n-k}$$

The bijection serves as a combinatorial proof of this equation.

**Example:**

Let  $S$  be the set of all subsets of  $[n]$

Let  $T$  be the set of all binary strings of length  $n$

$$n = 3$$

$$S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

$$T = \{000, 001, 010, 011, 100, 101, 110, 111\}$$

$$111 \rightarrow 1, 2, 3$$

$$110 \rightarrow 1, 2$$

$$100 \rightarrow 1$$

$$011 \rightarrow 2, 3$$

$\downarrow$

Each 1 in the binary string is "on" for one of the digits 1, 2, and 3.

Define  $f : T \rightarrow S$  where  $f(a_1, a_2, \dots, a_n) = \{i \mid i \in [n], a_i = 1\}$   
 (if  $a_i$  is 1, put element  $i$  in the subset)

The inverse is  $f^{-1} : S \rightarrow T$  where for each  $x \in S$   
 $f^{-1}(x) = a_1 a_2 \dots a_n$  where  $a_i = \{1 \text{ if } i \in x \mid 0 \text{ if } i \notin x\}$