Math 239 - Lecture 3

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May 11th, 2015

Combinatoial Proofs

Recal $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$
$\binom{7}{4} = \binom{6}{3} + \binom{6}{4}$
$= \binom{6}{3} + \binom{5}{3} + \binom{5}{4}$
$= \binom{6}{3} + \binom{5}{3} + \binom{4}{3} + \binom{3}{3}$

Identity: $\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$

Combinatorial proof: Let S be the set of all subsets of [n+k] of size n. So $|S| = \binom{n+k}{n}$.

[IMAGE 1]

For i = 0, k, let S_i be all subsets of $[n+k]$ of size n whose largest element is n+i. Then each element of S_i consists of n+k-i together with:

[{ $\underbrace{\_, \_, ...}_{n-1 spots}$ $\underline{\text{n+i}}$} n spots] in [n +i - 1]
So $|S_i| = \binom{n+i-1}{n-1}$
Since $S = S_0 \cup S_1 \cup ... \cup S_k$ is a disjoint union
$\therefore |S| = \sum_{i=0}^{k}$ Identity holds.

Hockey stick identity with pascal's triangle.

Generating Series

Example: How many subsets of [3] have size k? Let S be all subsets of [3].

Give each element δ of S a weight w where $w(\delta) = |\delta|$ (Related to the counting problem)

Our problem becomes "How many elements of S have weight k?"

$\delta in S$	$W(\delta)$	$x^{W(\delta)}$
θ	0	1
{1}	1	x
{2}	1	x
{3}	1	x
{1,2}	2	x^2
{1,3}	2	x^2
{2,3}	2	x^2
{1,2,3}	3	x^3

For each element δ, contribute $x^{W(\delta)}$ to the "generating series" of S. Sum them all up. In this example, the generating series for S is $\Phi_s(x) = 1 + 3x + 3x^2 + x^3 = (1 + x)^3$
The coeff of x^k records the answer to our counting problem.

Definition: Given set S where each eleent $\delta \in S$ is given a non-negative integer weight $W(\delta)$, the generating series for S with respect to w is $\Phi_s(x) = \sum_{\delta \in S} x^{w(\delta)}$.

Let a_k be the number of elements in S of weight k.
Then $\Phi_s(x) = \sum_{k \geq 0} a_k x^k$

Example: How many subsets of [n] have size k? Let S be all subsets of [n]. For any $\delta \in S$, define $w(\delta) = |\delta|$

The number of elements in S of weight k is $\binom{n}{k}$:
The generating series for S is $\Phi_s(x) = \sum_{k=0} n \binom{n}{k} x^k = (1 + x)^n$
The answer is coeff of x^k in $(1 + x)^n$

Example: How many ways can we throw 2 6-sided dice to get a sum of k?

Let S = [6] $\times$ [6]
For each $(a, b) \in S$, define w(a,b) = a + b.

(2,5)	wt 7	x^7
(5,6)	wt 11	x^{11}

$\Phi_s(x) = \sum_{(a,b) \in S} x^{a+b}$ Coeff of x^k is the number of ways to get a sum of k.

$\Phi_s(x) = x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^10 + 2x^11 + x^12$
$= (x + x^2 + x^3 + x^4 + x^5 + x^6)^2$