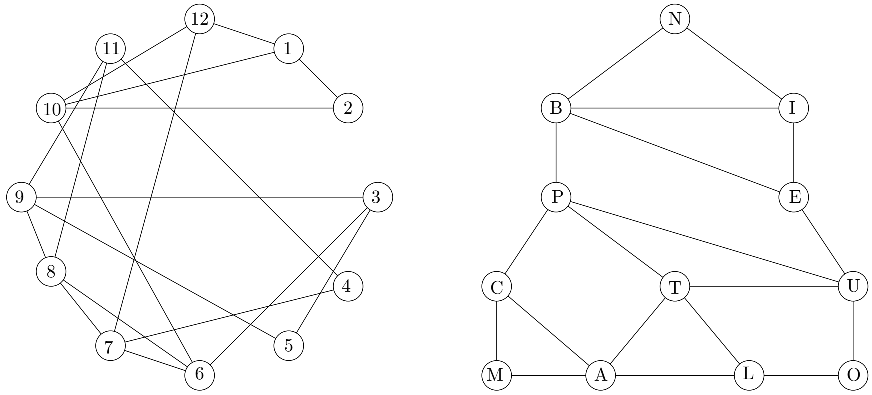
Math 239 Fall 2014 Assignment 6 Solutions

1. {3 marks} The following two graphs G and H are isomorphic. Find an isomorphism. (You do not need to prove that your mapping is an isomorphism.)



Graph G Graph H

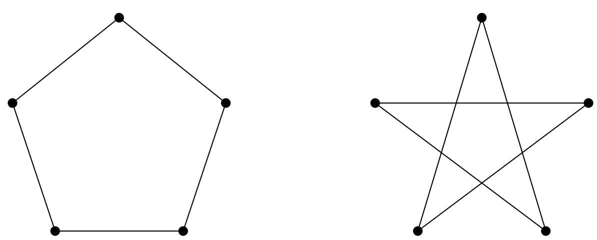
Solution. There is only one possible isomorphism $f : V(G) \to V(H)$, where

v	1	2	3	4	5	6	7	8	9	10	11	12
$f(v)$	I	N	C	O	M	P	U	T	A	B	L	E

2. Let G be a graph. The complement of G, denoted $\overline{G}$, is the graph where $V(\overline{G}) = V(G)$, and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

   (a) {2 marks} Draw a graph G on 5 vertices such that G is isomorphic to $\overline{G}$.

   Solution.



G $\overline{G}$

   (b) {3 marks} Suppose a graph G with n vertices is isomorphic to $\overline{G}$. Prove that $n \equiv 0, 1 \pmod 4$.

   Solution. There are $\binom{n}{2} = \frac{n(n-1)}{2}$ edges in K_n, and each edge belongs to either G or $\overline{G}$ (but not both). Since G and $\overline{G}$ have the same number of edges, G must have exactly $\frac{n(n-1)}{4}$ edges. This means that $n(n-1) \equiv 0 \pmod 4$. By checking $n \equiv 0, 1, 2, 3 \pmod 4$, we see that only 0 and 1 satisfy the equation. Hence $n \equiv 0, 1 \pmod 4$.

3. Let G be a bipartite graph with bipartition (A, B).

   (a) {2 marks} Prove that

$$\sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(v).$$

Solution. Since G is bipartite, each edge uv has one end in A (say it is u) and one end in B (say it is v). Then uv contributes one to each side of the equation, 1 for $\deg(u)$ for the sum on the left hand side, and 1 for $\deg(v)$ for the sum on the right hand side. Hence the two sums are equal.

(b) {2 marks} Let a, b be the number of odd-degree vertices in A, B respectively. Prove that $a \equiv b \pmod 2$.

Solution. From the corollary of the handshaking lemma, we know that the number of odd-degree vertices is even. The number of odd-degree vertices in G is $a + b$, so $a + b \equiv 0 \pmod 2$. Hence $a \equiv -b \equiv b \pmod 2$ (in other words, they are both even or both odd).

(c) {2 marks} Let $k \geq 1$ be an integer. Prove that if G is k-regular, then $|A| = |B|$.

Solution. Since every vertex has degree k, the equation from part (a) gives us

$$\sum_{v \in A} k = \sum_{v \in B} k.$$

This implies that

$$|A|k = |B|k.$$

Since $k \geq 1$, we can divide both sides to get $|A| = |B|$.

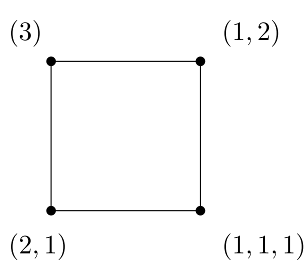
4. {4 marks} There are n participants at a certain meeting with no purpose whatsoever ($n \geq 2$). During this gathering, certain participants exchanged phone numbers with each other. Prove that at least two participants received the same number of phone numbers.

   Solution. In graph theory terms, this translates into "every graph with at least 2 vertices has 2 vertices of the same degree." Suppose by way of contradiction that there exists a graph G with n vertices where no two vertices have the same degree. The smallest and largest possible degrees are 0 and $n - 1$ respectively, so the possibilities of vertex degrees in G are $0, 1, 2, \ldots, n-1$. There are n possibilities, but there are n vertices, so there must exist one vertex of each degree. In particular, there is one vertex of degree 0 and one vertex of degree $n - 1$. This is not possible since the vertex of degree $n - 1$ is adjacent to all the other vertices, including the vertex of degree 0. Therefore, there must exist two vertices of the same degree.

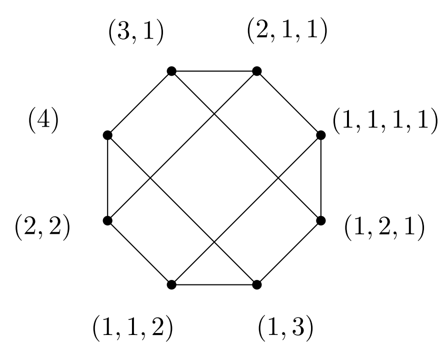
5. For $n \in \mathbb{N}$, let G_n be the graph whose vertices are all the compositions of n and there is an edge between two compositions if we can add two consecutive terms in the longer composition to get the shorter composition. For example, in G_{11}, $(1, 4, 3, 2, 1)$ is adjacent to $(1, 7, 2, 1)$ by combining the second and third part (4 and 3) to get 7. Furthermore, $(1, 7, 2, 1)$ is adjacent to $(8, 2, 1)$.

(a) {2 marks} Draw G_3 and G_4.

Solution.



G_3



G_4

(b) {2 marks} Prove that G_n is bipartite.

Solution. Let A be the set of compositions of n with even number of parts, and let B be the set of compositions of n with odd number of parts. If C, D are two compositions that are adjacent, then the number of parts in C is one more or one less than the number of parts in D. Therefore, the number of parts in C and D have different parities, hence one is in A and the other is in B. So G_n is bipartite.

(c) {2 marks} Prove that G_n is $(n-1)$-regular.

Solution. Let $(a_1, \ldots, a_k)$ be a composition of n with k parts. There are $k-1$ ways to combine two parts to get an adjacent composition (combine a_1, a_2, or a_2, a_3, etc.). This composition is also adjacent to those that can be obtained by splitting one part. Notice that for a part a_i, there are $a_i - 1$ ways to split it (for example, 4 can be split into $(1,3), (2,2), (3,1)$, and 1 cannot be split). So the number of ways we can split the composition is $(a_1-1)+(a_2-1)+\cdots+(a_k-1) = a_1+\cdots+a_k-k = n-k$. Together, this composition has degree $(k-1)+(n-k) = n-1$, hence the graph is $(n-1)$-regular.

(d) {2 marks} Determine the number of edges in G_n.

Solution. There are 2^{n-1} vertices, and every vertex has degree $n-1$. So the sum of the degrees is $(n-1)2^{n-1}$. By the handshaking lemma, the number of edges is half of this, which is $(n-1)2^{n-2}$.

(e) {Extra credit: 2 marks} Describe how you can construct G_n based on the graph G_{n-1}.

Solution. We make two copies of G_{n-1}: In one copy, we add the part 1 to the end of each composition; in the second copy, we add 1 to the last part of each composition. This covers all of the compositions of n by our bijections in assignment 5. In addition, we add the edges of the "corresponding" vertices between the two copies, i.e. we make $(a_1, \ldots, a_k, 1)$ adjacent to $(a_1, \ldots, a_k + 1)$. Notice that we can combine the last two parts of the first composition to get the second one. So this gives us G_n. (Note: G_n is isomorphic to the $(n-1)$-cube.)