Math 239 Lecture 13

Graham Cooper

June 5th, 2015

String Recursion

S = strings with no 1010
T = Strings with one 1010 at the right

$\{\epsilon\} \cup S\{0,1\} = S \cup T$

S{1010} = T

$\subseteq$ IF we add 1010 to a string in S, it has at least 1 copy of 1010 if it is the only copy, then it is in T. For those strings in S that ends with 10, there are 2 copies of 1010: _____101010. In this case it is in T{10}.
$\subseteq$ (Other way). Any string in T or T{10} ends with 1010. By removing the last 4 bits we destroy all copies of 1010 in the string so it is in S{1010}

Generating Series: <u>**1**</u>: $1 + \Phi_S(x) \cdot 2x = \Phi_S(x) + \Phi_T(x)$
<u>**2**</u>: $\Phi_S(x) \cdot x^4 = \Phi_T(x) + \Phi_T(x) \cdot x^2$
<u>**2 $\implies$**</u> $\Phi_T(x) = \frac{x^4}{1+x^2}\Phi_S(x)$

<u>**SUb into 1**</u>: $1 + \Phi_S(x) \cdot 2x = \Phi_S(x) + \frac{x^4}{1+x^2}\Phi_S(x)$
$\Phi_S(x) = \frac{1}{1+\frac{x^4}{1+x^2}-2x} = \frac{1+x^2}{1-2x+x^2-2x^3+x^4}$

Coefficients of Rational Expressions

<u>**Goal:**</u> Find explicit formula for $[x^n]f(x)/g(x)$

<u>**Example**</u>

$$A(x) = \sum_{n\geq 0} a_n x^n$$

where

$$A(X) = \frac{4-11x}{1-7x+10x^2}$$

$$1 - 7x + 10x^2 = (1-2x)(1-5x)$$

By partial fractions there exists constants C_1, C_2 such that:

$$A(x) = \frac{C_1}{1-2x} + \frac{C_2}{1-5x}$$

Solving ives $C_1 = 1, c_2 = 3$
So:

$$\begin{aligned} A(x) &= \frac{1}{1-2x} + \frac{3}{1-5x} \\ &= 2^n + 3 \cdot 5^n \end{aligned}$$

So

$$[x^n]A(x) = [x^n]\frac{1}{1-2x} + [x^n]\frac{3}{1-5x}$$

Example:

$$A(x) = \frac{-1+8x-4x^2}{(1-2x)^3} = \frac{c_1}{1-2x} + \frac{C_2}{(1-2x)^2} + \frac{C_3}{(1-2x)^3}$$

$$C_1 = -1, C_2 = -2, C_3 = 2$$

$$A(x) = \frac{-1}{1-2x} + \frac{-2}{(1-2x)^2}\frac{2}{(1-2x)^3}$$

SIDE NOTE:

$$(\frac{1}{(1-x)^k} = \sum_{n \geq 0} \binom{n+k-1}{k-1} x^n)$$

Continuing

$$\begin{aligned} [x^n]A(x) &= (-1)2^n - 2\binom{n+2-1}{2-1} \cdot 2^n + 2\binom{n+3-1}{3-1}2^n \\ &= (-1)2^n - 2 \cdot \binom{n+1}{1}2^n + 2\binom{n+2}{2}2^n \\ &= (-1)2^n - 2(n+1)2^n + 2\frac{(n+2)(n+1)}{2}2^n \\ &= (-1)2^n - 2(n+1)2^n + (n^2+3n+2)2^n \\ &= (n^2+n-1)2^n \end{aligned}$$

Expand

$$\binom{n+k-1}{k-1} = \frac{(n+k-1)!}{(k-1)!n!} = \frac{(n+k-1)(n+k-2)...(n+1)}{(k-1)!}$$

This numerator is a polynomial in n of deg k-1.
In general:

$$[x^n]\frac{C}{1-rx)^k} = P(n) \cdot r^n$$

Where P(n) has degree k-1

Generalize more:

$$A(x) = \frac{f(x)}{g(x)}$$

deg(f(x)) < deg(g(x)) and g(x) = (1-$r_1 x)^{e_1}(1-r_2x)^{e_2}...(1-r_kx)^{e_k}$
Then:

$$A(x) = \frac{C_{1,1}}{1-r_1x} + ... + \frac{C_{1,e_1}}{(1-r_1x)^{e_1}} + ... + \frac{C_{k,1}}{1-r_kx} + \frac{C_{k,e_k}}{(1-r_kx)^{e_k}}$$

Then,

$$[x^n]A(x) = P_1(n)r_i^n + ... + P_k(n)r_k^n$$

Where $P_i(n)$ is a polynomial in n of degree $e_i - 1$

example:

$$A(x) = \frac{1-2x}{(1+3x)^2(1-5x)^3}$$

$$[x^n]A(x) = (An+B)(-3^n) + (Cn^2+Dn+E)5^n$$

For constants A,B,C,D and E

Characteristic Polynomial

$$g(x) = (1-r_1x)^{e_1}...(1-r_kx)^{e_k}$$

Define:

$$g*(x) = (x-r_1)^{e_1}...(x-r_k)^{e_k}$$

Switch 1-rx to x-r
Then $r_1, ...r_k$ are roots of g*(x) with multiplicatives $e_1, ...e_k$ g*(x) is the char polynomial