1 Enumeration

The first half of the MATH 239 course, which is the one that accounts for most of the material in the midterm, is the one about enumeration problems. We will speak about sets of configurations, each of them with an associated non-negative weight. In our problems, we will want to know how many configurations are there of an specific weight.

1.1 Basic tools for enumeration

Definition 1. For any real number a and non-negative integer k, we define *a choose k*, denoted $\binom{a}{k}$, as

$$\frac{a(a-1)\dots(a-k+1)}{k(k-1)\dots 1}.$$

If a is equal to an integer n, then $\binom{n}{k}$ is the number of subsets of size k of a set of size n:

Proof. Let L be the set of all ordered lists of k distinct numbers from the set of size n. There are n ways to choose the first element, $n-1$ ways to choose the second, and so on, so $|L| = n(n-1)...(n-k+1)$. However, there are $k*(k-1)*(k-2)\dots*1$ ways to permute each list of k elements (by choosing the first element, then the second element, and so on). Hence, the number of ways to choose subsets of size k from n elements is $\frac{n(n-1)...(n-k+1)}{k(k-1)...1} = \binom{n}{k}$ $\square$

Definition 2 (Bijective function)**.** A function $f : X \to Y$ is *bijective* if for each $y \in Y$ there is exactly one $x \in X$ such that $f(x) = y$.

A way to show that two expressions are equivalent is to associate each of them with a set, prove that they are equal to the number of elements in the set, and give a bijection between the sets.

Example 1 (Problem Set 1.4 , 3.a)**.** We want to prove

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$$

Let A and B be disjoint sets of size m and n, respectively. Then, as we proved when we defined the choose function, $\binom{m+n}{k}$ is the number of subsets of size k of $A \cup B$.

Similarly, the right hand side is the number of pairs (X, Y) of sets $X \subseteq A$ and $Y \subseteq B$ such that $|X| + |Y| = k$.

There is a bijection between the subsets of size k of $A \cup B$, and the pairs of subsets of A and B with total size k. This bijection simply splits a subset S of $A \cup B$ into two sets, one with the elements of S belonging to A, and the other with the elements of S belonging to B. The fact that it is a bijection follows from the fact that every pair (X, Y) of subsets of A and B with total size k can be obtained in this way from an unique subset of $A \cup B$, $X \cup Y$.

Therefore,

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i}.$$

1.2 Generating functions

Generating functions are a way of writing down the answer to a counting problem. If there are 3 configurations of weight 0, 5 configurations of weight 1, 6 configurations of weight 2 and so on, the generating function for the problem is $3 + 5x + 6x^2 + \dots$ More formally:

Definition 3. Let S be a set of configurations with a non-negative weight function w. The *generating function* for S with respect to w is

$$\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)}.$$

Remark 1. By collecting terms with the same power of x, we obtain

$$\Phi_S(x) = \sum_{k \geq 0} a_k x^k,$$

where a_k is the number of configurations of weight k.

Remark 2. The coefficient for x^k in $\Phi_S(x)$ can also be denoted as $[x^k]\Phi_S(x)$.

In case there is a bound on the weights, the generating function will just be a polynomial. An example is the case in which the configurations are the faces of a die, and the corresponding weight is the number of dots in the face. The generating function is then $x + x^2 + x^3 + x^4 + x^5 + x^6$. However, in case there is no bound on the weights, the number of terms of this polynomial can be infinite. In that case, we call it a *formal power series*. Some important facts about power series are the following ones:

- If $A(x)$ and $B(x)$ are formal power series and $B(x)$ has no constant term, then $A(B(x))$ is a formal power series.
- $B(x)$ is the inverse of $A(x)$ iff $A(x)B(x) = 1$. If $B(x)$ is the inverse of $A(x)$, we write $B(x) = \frac{1}{A(x)} = A(x)^{-1}$.
- A formal power series has an inverse if and only if it has a non-zero constant term and this inverse is unique.
- The sum of two formal power series $\sum\limits_{i\geq 0} a_i x^i$ and $\sum\limits_{i\geq 0} b_i x^i$ is equal to $\sum\limits_{i\geq 0}(a_i + b_i)x^i$.
- The product of two formal power series $\sum\limits_{i\geq 0} a_i x^i$ and $\sum\limits_{i\geq 0} b_i x^i$ is equal to $\sum\limits_{i\geq 0}(\sum\limits_{j\geq 0}^{i} a_j b_{i-j})x^i$. Note that we can also write this as $\sum\limits_{i\geq 0}(\sum\limits_{j,k\geq 0, j+k=i} a_j b_k)x^i$.
- The derivative of a formal power series $\sum\limits_{i\geq 0} a_i x^i$ is equal to $\sum\limits_{i\geq 0}(i+1)a_{i+1}x^i$.

1.3 Properties of generating functions

- If we evaluate the generating function at $x = 1$, we obtain the total number of configurations.
- If we evaluate the derivative of the generating function at $x = 1$, we obtain the sum of the weights over all configurations.
- From the previous two properties, we obtain that the ratio between the value of the generating function at $x = 1$, and the value of the derivative of the generating function at $x = 1$, is equal to the average weight of a configuration. (Note that this only makes sense when the number of configurations is finite, as well as the sum of the weights).

1.3.1 The Sum Lemma

Lemma 1. *Let A and B be two disjoint sets of configurations. Then, if their generating functions are $\Phi_A(x)$ and $\Phi_B(x)$, the generating function $\Phi_{A\cup B}(x)$ for $A \cup B$ is $\Phi_A(x) + \Phi_B(x)$. More generally, for any two sets of configurations A and B, the generating function $\Phi_{A\cup B}(x)$ is equal to $\Phi_A(x) + \Phi_B(x) - \Phi_{A\cap B}(x)$.*

Remark 3. This can be extended to the union of more than two disjoint sets of configurations (see Exercise 2 in Problem Set 1.4).

1.3.2 The Product Lemma

Lemma 2. *Let A and B be two sets of configurations, with generating functions $\Phi_A(x)$ and $\Phi_B(x)$. If the weight of a pair $(a, b) \in A \times B$ is the sum of the weight of a and the weight of b, then the generating function $\Phi_{A\times B}$ for $A \times B$ is $\Phi_A(x)\Phi_B(x)$.*

Remark 4. This can be extended to the cartesian product of more than two sets of configurations (see Exercise 3 in Problem Set 1.4).

1.4 The Binomial Theorem

Theorem 1 (The Binomial Theorem)**.** *For any rational number a,*

$$(1+x)^a = \sum_{k\geq 0} \binom{a}{k} x^k.$$

You do not need to know the proof for general case. However, you should know the combinatorial proof for the case of non-negative integer n.

Example 2 (Problem Set 1.8, 8)**.** We want to prove

$$\binom{m-n}{t} = \sum_{r+s=t} (-1)^r \binom{n+r-1}{r} \binom{m}{s}.$$

By the Binomial Theorem, the coefficient for x^t in $(1+x)^{m-n}$ is $\binom{m-n}{t}$. However, we can also write $(1+x)^{m-n}$ as

$$(1+x)^m (1+x)^{-n} = \left(\sum_{s\geq 0} \binom{m}{s} x^s \right) \left(\sum_{r\geq 0} (-1)^r \binom{n+r-1}{n-1} x^r \right),$$

using the formula for $\binom{-n}{r}$ in the course notes.

The coefficient for x^t in this new product is, using our definition for the product of two formal power series, equal to

$$\sum_{r+s=t} \binom{m}{s}(-1)^r \binom{n+r-1}{n-1}.$$

We have then proved the given identity, computing the coefficient of x^t in $(1+x)^{m-n}$ in two possible ways.

1.5 Recurrences

Theorem 2. *Suppose we know that* $\sum_k a_k x^k = \Phi(x) = \frac{P(x)}{Q(x)}$*, for two polynomials* $P(x)$ *and* $Q(x)$*. This implies that* $Q(x)\Phi(x) = P(x)$*. Then, by making the coefficient of* x^i *in* $Q(x)\Phi(x)$ *be equal to the one in* $P(x)$*, we can find a linear recurrence for the* a_k*, as well as the initial conditions.*

Example 3. (To be) discussed at review session.

1.6 Solving problems using generating functions

We will solve now different problems concerning combinatorial objects, using generating functions. The general idea is always the same.

- We identify the underlying set of configurations.
- We assign them a weight function.
- Using the information obtained in the previous steps, we calculate the generating function.

1.6.1 Partitions of an integer

Theorem 3. *The number of solutions to* $t_1 + \ldots + t_k = n$*, with* n, k, t_i *all* ≥ 0*, is equal to* $\binom{n+k-1}{n}$*.*

Remark 5. The previous theorem is proved in the course notes using generating functions. However, there is a simple combinatorial solution. Consider a sequence of $n+k-1$ cells of length 1 in a line. Then, a choice of $k-1$ cells will delimit k possibly empty ranges for which the sum of their lengths is n. Every set of such ranges is uniquely determined by a choice of $k-1$ delimiters. Therefore, the number of solutions to $t_1 + \ldots + t_k = n$ is $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$.

Remark 6. The proof in the previous remark can be adapted to show that the number of ways to pick k elements from a larger set of n elements, allowing for repetition, and not caring about the order, is equal also to $\binom{n+k-1}{k-1}$.

Theorem 4. *The number of solutions to* $t_1 + \ldots + t_k = n$*, with* n, k, t_i *all* ≥ 1*, is equal to* $\binom{n-1}{k-1}$.

Definition 4. A *composition* of n with k parts is an ordered list of positive integers $c_1, \ldots, c_k$ such that $c_1 + \ldots + c_k = n$. If $n \geq 1$, note that $k \geq 1$. If $n = 0$, we allow a single composition of 0, with 0 parts.

Remark 7. Theorem 4 is equivalent to saying that the number of compositions of n with k parts is $\binom{n-1}{k-1}$.

Example 4 (Problem Set 2.2, Exercise 4)**.** A travel agency has pamphlets of 6 different kinds. We can take at most 7 pamphlets, and at most 2 pamphlets of each kind We want to know how many choices can we make.

Each configuration is a selection of pamphlets. If the weight of a sequence of pamphlets is the number of pamphlets, the generating function for all possible choices of pamphlets picking at most two of any one kind is equal to

$$(1 + x + x^2)^6$$

Using the Binomial Theorem, we expand this as

$$\sum_{i=0}^{6} \binom{6}{i} (x + x^2)^i = \sum_{i=0}^{i} \binom{6}{i} x^i (1 + x)^i.$$

Using the Binomial Theorem again, we obtain

$$\sum_{i=0}^{6} \binom{6}{i} x^i \sum_{j=0}^{i} \binom{i}{j} x^j.$$

Grouping together terms with the same sum of $i + j$, we obtain

$$\sum_{i+j=k, j \leq i \leq 6} \binom{6}{i} \binom{i}{j} x^k.$$

The number of choices in which we pick at most 7 pamphlets is then equal to the sum of the coefficients of x^k, for $0 \leq k \leq 7$.

This is equal then to

$$\binom{6}{0}\binom{0}{0}+\binom{6}{1}\binom{1}{0}+\binom{6}{2}\binom{2}{0}+\binom{6}{1}\binom{1}{1}+\binom{6}{3}\binom{3}{0}+\binom{6}{2}\binom{2}{1}+\binom{6}{4}\binom{4}{0}+\binom{6}{3}\binom{3}{1}+\binom{6}{2}\binom{2}{2}$$

$$+\binom{6}{5}\binom{5}{0}+\binom{6}{4}\binom{4}{1}+\binom{6}{3}\binom{3}{2}+\binom{6}{6}\binom{6}{0}+\binom{6}{5}\binom{5}{1}+\binom{6}{4}\binom{4}{2}+\binom{6}{3}\binom{3}{3}.$$

This is equal to
$1+6+15+6+20+30+15+60+15+6+60+60+1+30+90+20 = 435.$

Example 5 (Problem Set 2.2, Exercise 11)**.** (To be) discussed at review session.

1.6.2 Binary strings

Definition 5. We will first remember some notation:

- There is a single binary string ϵ of length 0, called the *empty string*.
- Let A and B be sets of binary strings. Then, $AB = \{ab$ s.t. a belongs to A, b belongs to $B\}$.
- Let A be a set of binary strings. Then, $A^* = \epsilon \cup A \cup AA \cup AAA \cup \ldots$

Theorem 5. *Let A and B be sets of binary strings. If the elements of AB are uniquely created, then $\Phi_{AB}(x) = \Phi_A(x)\Phi_B(x)$. If the elements of A^* are uniquely created, then $\Phi_{A^*}(x) = (1 - \Phi_A(x))^{-1}$.*

The following formulas generate all binary strings uniquely:

- $\{1\}^* \left(\{0\}\{1\}^*\right)^*$
- $\{0\}^* \left(\{1\}\{0\}^*\right)^*$
- $\{1\}^* \left(\{0\}\{0\}^*\{1\}\{1\}^*\right)^* \{0\}^*$
- $\{0\}^* \left(\{1\}\{1\}^*\{0\}\{0\}^*\right)^* \{1\}^*$

Example 6 (Includes parts (a), (c) and (f) of Problem Set 2.5, Question 3)**.** We can start from the previous formulas to create formulas that generate uniquely sets of binary strings:

- The binary strings that have no substrings of 0s with length 3:

  We modify the second of the four decompositions, and obtain $\{\epsilon, 0, 00\}\left(\{1\}\{\epsilon, 0, 00\}\right)^*$.

- The binary strings that have no blocks of 1s with length 3:

  We modify the first of the four decompositions, and obtain $\left(\{\epsilon, 1, 11\} \cup 1111\{1\}^*\right)\left(\{0\}\left(\{\epsilon, 1, 11\} \cup 1111\{1\}^*\right)\right)^*$.

- Part (a) of Problem Set 2.5, Question 3. The binary strings that have no substring of 0s with length 3, and no substring of 1s with length 2:

  We modify the fourth of the four decompositions, and obtain $\{\epsilon, 0, 00\}\left(\{1\}\{0, 00\}\right)^*\{\epsilon, 1\}$.

- Part (c) of Problem Set 2.5, Question 3. The binary strings in which 011 does not occur as a substring:

  We modify the first of the four decompositions, and obtain $\{1\}^*\left(\{0\}\{\epsilon, 1\}\right)^*$.

- Part (f) of Problem Set 2.5, Question 3. The binary strings in which each odd-length block of 0s is followed by a non-empty block of 1s, and each even-length block of 0s is followed by an odd-length block of 1s:

  We modify the third of the four decompositions, and obtain $\{1\}^*\left(\left(\{0\}\{00\}^*\{11\}\{11\}^*\right) \cup \left(\{00\}\{00\}^*\{1\}\{11\}^*\right)\right)^*$.

The previous decompositions all generate strings uniquely. It follows from them being restrictions of unique decompositions of the set of all binary strings.

1.6.3 Recursion

Sometimes, we characterize the set of configurations recursively. That gives us then a functional equation where our variable is the generating function.

Example 7 (Problem Set 2.5, Question 14a)**.** Let A be the set of strings in which every block of 0's is followed by a block of 1's of the same length. We can decompose A uniquely into $\{1\}^*B$, where B is the empty string, plus the set of strings in which the initial block is a block of 0's, and every block of 0's is followed by a block of 1's of the same length.

Therefore,

$$\Phi_A(x) = \frac{1}{1-x}\Phi_B(x).$$

Now, we can also uniquely decompose B as $\epsilon \cup \{01, 0011, 000111, \ldots\}B$. Therefore,

$$\Phi_B(x) = 1 + (x^2 + x^4 + x^6 + \ldots)\Phi_B(x) = 1 + \left(\frac{1}{1-x^2} - 1\right)\Phi_B(x).$$

We obtain then

$$\Phi_B(x)\left(2(1-x^2) - 1\right) = 1 - x^2 \iff \Phi_B(x) = \frac{1-x^2}{1-2x^2}.$$

Therefore,

$$\Phi_A(x) = \frac{1}{1-x}\frac{(1+x)(1-x)}{1-2x^2} = \frac{1+x}{1-2x^2}.$$

Remark 8. If in the previous solution we said "Now, we can also uniquely decompose B as $\{\epsilon, 01, 0011, 000111, \ldots\}B$.", we would be making a mistake. Why?

1.6.4 Binary trees

Definition 6. A *binary tree* is a tree with a fixed root node; each node has a left branch and a right branch (either of which might be empty). We allow for an empty binary tree, denoted by ϵ.

Theorem 6. *The number of binary trees with n edges is $\frac{1}{n+1}\binom{2n}{n}$, the n^{th} Catalan number.*

1.6.5 Bivariate generating functions

Theorem 7. *A bivariate weight function assigns two different weights $w_1(s)$ and $w_2(s)$ to each configuration s. The corresponding generating function is then $F(x,y) = \sum_s x^{w_1(s)}y^{w_2(s)}$. This can be extended to three weights, with a third variable z, and so on.*

Remark 9. $F(x,1)$ is the generating function for only the first weight. Similarly, $F(1,y)$ is the generating function for only the second weight.

Theorem 8. *The average value of the second weight over all configurations with first weight n is given by*

$$\frac{[x^n]\frac{\partial F}{\partial y}(x,1)}{[x^n]F(x,1)}.$$

Remark 10. Note that this is different from saying that the average value of the second weight over all configurations with first weight n is given by

$$[x^n]\frac{\frac{\partial F}{\partial y}(x,1)}{F(x,1)}.$$

This appears in your course notes in the particular case in which our configurations are binary strings, the first weight is the length, and the second weight is the number of occurrence of some property P:

Theorem 9. *Let R be a set of binary strings, and $c_{n,k}$ be the number of strings in R of length n with k occurrences of property P, $n,k \geq 0$. Let the corresponding bivariate generating function (with weights the length and the number of occurrences of P) be*

$$F(x,y) = \sum_{n\geq 0}\sum_{k\geq 0} c_{n,k}x^n y^k.$$

Then, the average number of occurrences of property P among the binary strings in R of length n is

$$\frac{[x^n]\frac{\partial F}{\partial y}(x,1)}{[x^n]F(x,1)}.$$

Example 8 (Problem Set 2.6, Exercise 2)**.** (To be) discussed at review session.

1.7 Homogeneous recurrence relations

Definition 7. The *characteristic polynomial* of the recurrence $c_n + q_1c_{n-1} + \ldots + q_kc_{n-k} = 0$ is the polynomial $x^k + q_1x^{k-1} + \ldots + q_{k-1}x + q_k$.

Theorem 10. *If the characteristic polynomial of this recurrence has roots $\beta_1, \ldots, \beta_j$ with multiplicities $m_1, \ldots, m_j$, for $i = 1, \ldots, j$, then the unique solution to the recurrence is given by*

$$c_n = P_1(n)\beta_1^n + \ldots + P_j(n)\beta_j^n,$$

where $P_i(n)$ is a polynomial in n with degree less than m_i, and the P_i are determined by the initial conditions.

Remark 11. Note that if all the roots are unique, all the P_i are just constants.

PART 2 – GRAPH THEORY

INTRODUCTION

Def^n: A graph G is a set of vertices $V(G)$ together with a set of edges $E(G)$ such that each edge is an unordered pair $\{u, v\}$ (or uv for short), where $u, v \in V(G)$.

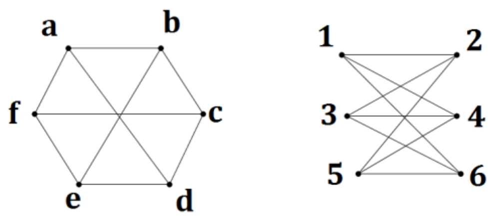
Def^n: If $uv \in E$, we say uv are adjacent, the edge $e{=}uv$ joins u and v, e is incident with u and v, and v is a neighbour of u and vice versa.

$N(v) = \{all\ neighbours\ of\ v\}$

Deg$(v) = |N(v)|$ is the degree of v.

Def^n: Two graphs G and H are isomorphic if there is a bijection $f: V(G) \to V(H)$ (called an isomorphism) such that $uv \in E(G)\ \ iff\ \ f(u)f(v) \in E(H)$. We denote this $G \cong H$.

I.e. $G \cong H$ if the vertices of G can be renamed so that G becomes H.



Theorem If G is a graph, then

$$\sum_{v \in v(G)} \deg(v) = 2q\ ,\ \ where\ q = |E(G)|$$

Corollary: "Handshaking Theorem"

In any graph, the number of vertices of odd degree is even.

Def^n: G is regular if all vertices have the same degree. It is k-regular if all vertices have degree k.

Theorem If G is k-regular with p vertices, then

$$|E(G)| = \frac{kp}{2}$$

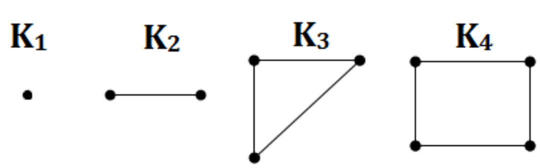
TYPES OF GRAPHS

Complete Graph K_p

A complete graph, K_p is a graph with p vertices, with an edge uv for every $u, v \in V$.

Note: $|E(K_p)| = \frac{p(p-1)}{2} = \binom{p}{2}$

Note: K_p is $(p$-1$)$-regular.

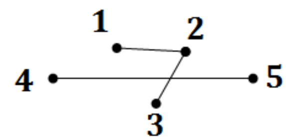


Bipartite Graph

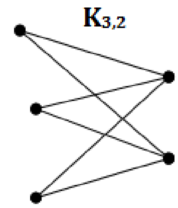
G is bipartite if its vertex set *V(G)* can be partitioned into subsets *A* and *B* such that all edges join a vertex in *A* to a vertex in *B*. *(A,B)* is called a bipartition of *G*.

Example For the bipartite graph on the left, there are 2 possible bipartitions:

- A = {1,3,5} and B = {2,4}
- A = {1,3,4} and B = {2,5}



Complete Bipartite Graph $K_{s,t}$



A complete bipartite graph is one with bipartition *(A,B)* such that
$|A| = s \;\; |B| = t$
Note: $|E(K_{s,t})| = st$

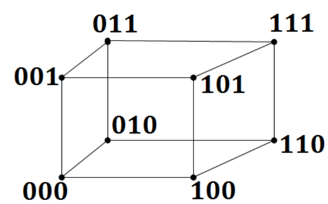
The n-cube Q_n

A graph is an *n*-cube if it satisfies the following:

- $V(Q_n) = \{all\ binary\ strings\ of\ length\ n\}$, $n \geq 1$
- Two vertices (strings) are adjacent if they differ in exactly one coordinate.

Note: $|V| = 2n$
Note: $|E(Q_n)| = \frac{n2^n}{2} = n2^{n-1}$
Note: Q_n is *n*-regular.

Q_3=



PATHS AND CYCLES

Def^n: A walk in a graph *G* is a sequence
$$v_o e_1 v_1 e_2 v_2 \ldots . e_n v_n \;\; where \;\; v_i \in V(G), e_i \in E(G)$$
and e_i is incident with v_{i-1} and v_i.
We call this a (v_0,v_n)-walk.

Def^n: A path is a walk with no repeated vertex, i.e. $v_0, v_1, \ldots, v_n$ are all distinct.

Def^n: A closed walk is one where the first and last vertex is the same.

Def^n: A cycle is a closed walk with no repeated vertices except the first/last one. Note that it must have length ≥ 3.

Def^n: A subgraph of a graph *G* is any graph *H* with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We denote this $H \subseteq G$.
Note: Paths and cycles may be referred to as subgraphs.

Theorem If there is a *(u,v)*-walk, then there is a *(u,v)*-path.

Theorem If there is a *(u,v)*-path and a *(v,w)*-path, then there is a *(u,w)*-path.

CONNECTIVITY

Def^n: A graph G is connected if for all $u, v \in V(G)$, there exists a *(u,v)*-path in *G*.

Theorem Let $v \in V(G)$. If there exists a *(u,v)*-path for all $u \in V(G), u \neq v$, then G is connected.

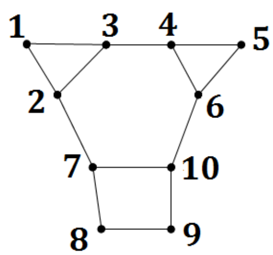
Def^n: A component of G is a maximal connected subgraph.

Theorem u and v are in the same component of G iff there is a *(u,v)*-path in *G*.

Corollary Q_n is connected for all $n \geq 1$.

Def^n: If $U \subseteq V(G)$, the cut induced by U is the set of edges which join a vertex $\in U$ to a vertex $\notin U$.

Example In the graph below, {27, 34} is a cut where U={1,2,3}. {27, 34, 610} is not a cut because there is no suitable set U.



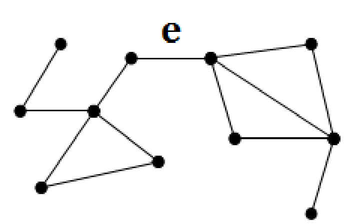
Theorem A graph is connected iff no set $U \subseteq V$ induces an empty cut, except $U=V$ and $U=\emptyset$.

Notation: If $e \in E(G)$, *G-e* denotes the graph with

- *V(G-e)=V(G)*
- *E(G-e)=E(G)\{e}*

Bridges

Def^n: A bridge of a connected graph G is an edge e such that *G-e* is disconnected. A bridge of a disconnected graph is any bridge of a connected component.



Theorem Let $e = uv \in E(G)$.
Then e is a bridge iff u and v are in different components of *G-e*.

Theorem Let $e \in E(G)$. Then e is a bridge iff *{e}* is a cut.

Theorem Let $e \in E(G)$. Then e is a bridge iff e is not on any cycle.

Corollary If there are two *(u,v)*-paths in G that are distinct, then G has a cycle.

TREES

Def[n]: A tree is a graph that is connected and has no cycles.

Theorem A graph G is a tree iff there is a single unique path between $u, v \in V(G)$.

Lemma Every edge in a tree is a bridge.

Theorem If T is a tree with at least 2 vertices, then it has at least one vertex v such that $\deg(v) = 1$

Theorem If T is a tree with p vertices where $p \geq 1$, then T has exactly p-1 edges.

Theorem If a tree T has a vertex of degree k, then it has at least k vertices of degree 1.

Def[n]: A spanning subgraph of G is a subgraph H with $V(H)=V(G)$. A spanning tree of G is a spanning subgraph of G that is a tree.

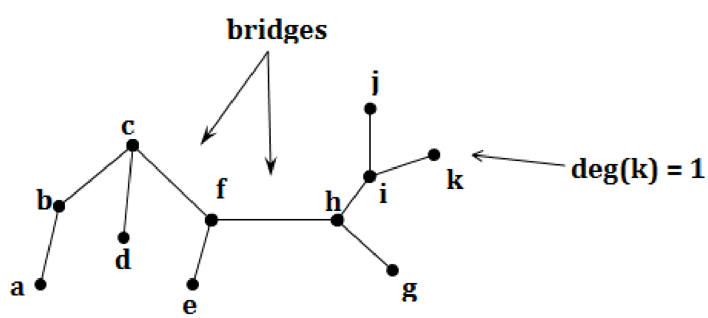
Theorem A graph G is connected iff it has a spanning tree.

Corollary If G is connected, and has p vertices and p-1 edges, then G is a tree.

Lemma Every tree is bipartite.

Theorem A graph G is bipartite iff G has no odd cycles.

Example In the tree on the right, one possible bipartition is $A=\{a,c,e,h,j,k\}$ and $B=\{b,d,f,g,i\}$. It has 11 vertices and therefore 10 edges.



SPANNING TREE ALGORITHM

Tree Search Algorithm

Given a graph G, and a vertex r of G, create $D \subseteq G$ as follows:

1. $V(D)=\{r\}$, $E(D)=\emptyset$
2. Let $u \in V(D), uv \in E(G)$ but $v \notin V(D)$. Add uv to $E(D)$, and v to $V(D)$.
3. Repeat until no such uv exists.

At the end, D is a spanning tree in the component of G containing r.
Notation: If uv is in the algorithm, u is the parent of v. We denote this $u=pr(v)$.

Corollary G is connected iff $V(D)=V(G)$ at the end of the algorithm.

<u>Breadth-First Search (BFS)</u>

BFS is a tree search algorithm in which the neighbours of a vertex v in D are added to D before any other vertices without neighbours in D. If all of v's neighbours are already in D, then v is "exhausted".

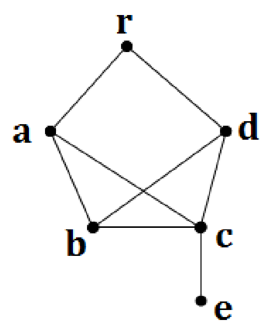
<u>Note</u>: Recall that there may be some edges $\in E(G)$ but $\notin E(D)$. We will call these "non-BFS-tree" edges.

<u>Note</u>: Each vertex in D is given a level, such that the root is at the highest level, 0.

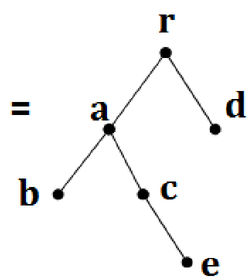
<u>Example</u> To create a BFS starting from r, you would add vertices in this order

- r
- a, d (neighbours of r)
- b, c (neighbours of a)
- $\emptyset$ (d is exhausted)
- $\emptyset$ (b is exhausted)
- e (neighbour of c)
- $\emptyset$ (e is exhausted)

G =



D =



<u>Theorem</u> In BFS, vertices are added to the tree in non-decreasing order of level.

<u>Theorem</u> In BFS, if $uv \in E(G)$ and $uv \notin E(D)$, then u and v's levels differ by at most 1.

<u>Applications of BFS</u>

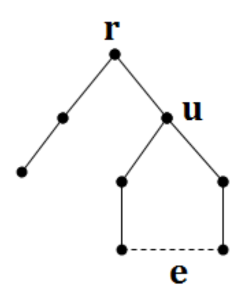
1. <u>Def^n</u>: The <u>distance</u> from u to v in G is the length of the shortest (u,v)-path.

<u>Theorem</u> If u and v are in the same component, then $dist(u,v)$ is equal to the level of v in any BFS tree with root u.

2. We want to determine whether or not G is bipartite.

<u>Lemma</u> Let G be a graph and D be its BFS tree. A non-BFS-tree edge joining 2 vertices at the same level in D gives an odd cycle in G.

<u>Example</u> In the BFS graph D to the right, e is an edge in G but not in D. Thus there is a cycle in G of length 5.



3. <u>Def^n</u>: The <u>girth</u> of a graph is the length of its shortest cycle.

<u>Lemma</u> Let G be a graph and D be its BFS tree. A non-BFS-tree edge placed highest in D gives the shortest cycle in G.

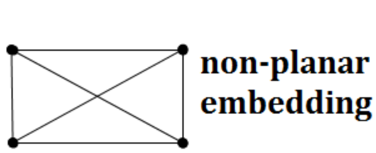
PLANAR GRAPHS

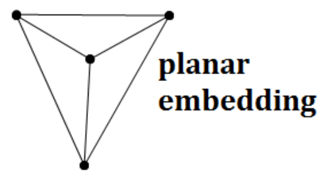
Def^n: A planar embedding of a graph G is a way of drawing G such that

- Vertices = distinct points in a plane
- Edges = lines (can be curved) in the plane joining the vertices as usual, but such that no edges meet except at the vertices

Def^n: A graph is planar if it has some planar embedding.

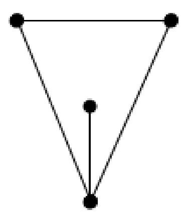
Example Below shows a planar embedding and a non-planar embedding of the planar graph K_4





Def^n: Suppose we cut (with scissors) along all edges of a planar embedding. Each piece that remains connected is a face of the planar embedding.

Example This face has degree 5:

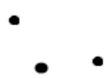


Theorem If $f_1,\ldots,f_n$ are the faces of a planar embedding, then

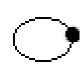
$$\sum_k \deg(f_k) = 2|E(G)|$$

Consider graphs that have a face of degree $d \leq 3$:

Degree 0: # edges = 0



Degree 1: no loops are allowed, so this is not possible



Degree 2: # edges = 1

Theorem: Euler's Formula If a graph has p vertices, q edges, c components, and a planar embedding with f faces, then

$$p - q + f = c + 1$$

Def^n: A graph is platonic if it is connected, k-regular with $k \geq 3$, and has a planar embedding with all faces of the same degree $d \geq 3$.

Theorem If G is platonic with p vertices of degree k, q edges, and f faces of degree d, then

$$(k, d) \in \{(3,3), (3,4), (4,3), (3,5), (5,3)\}$$

Non-planar graphs

How do we prove that a graph is non-planar? Use the theorems below either directly or contrapositively.

Lemma If a planar embedding of a graph (with p vertices and q edges) has faces all of degree $d \geq 3$, then

$$q(d-2) \leq d(p-2)$$

Theorem If G is a planar graph with $p \geq 3$ vertices and q edges, then

$$q \leq 3p-6$$

Note: Converse may not be true.

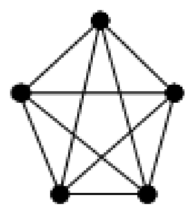
Example K_5

$p = 5$

$3p\text{-}6 = 9$

$q = 10 \not\leq 9$

So K_5 is not planar.



Def^n: The boundary of a face in a planar embedding is the subgraph consisting of all vertices and edges incident with the face.

Lemma If G has a cycle, then every face boundary in every planar embedding of G has a cycle.

Theorem If G is planar and has a finite girth $n \geq 3$, then

$$q \leq \frac{n(p-2)}{n-2}$$

Example $K_{3,3}$

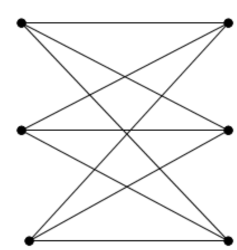
Because $K_{3,3}$ is bipartite, there are no odd cycles.

Thus there are no cycles of length 3.

So girth ≥ 4, and p=6

$\frac{4(6-2)}{4-2} = 8$

But q=9 $\not\leq$ 8. Therefore $K_{3,3}$ is not planar.



Theorem If G is planar, then it has a vertex of degree ≤ 5.

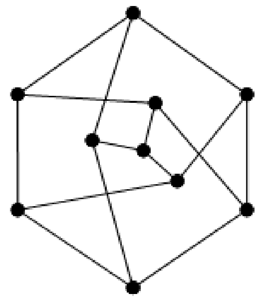
Def^n: Given a graph G, if we replace some edges of G by paths to obtain a graph H, we say H is an edge subdivision of G.

Note: G is considered an edge subdivision of itself.

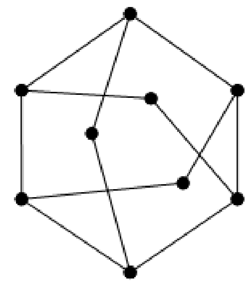
Lemma If G is non-planar, then every edge subdivision of G is non-planar.

Kuratovski's Theorem G is non-planar iff it has a subgraph that is an edge subdivision of K_5 or $K_{3,3}$.

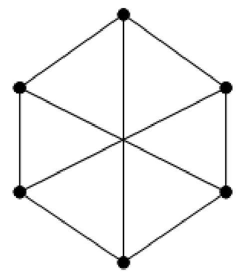
Example Petersen graph:



One Subgraph is:



Which is an edge subdivision of $K_{3,3}$:



COLOURINGS

Defn: A colouring of G is an assignment of colours to its vertices, such that u and v are different colours if $uv \in E(G)$.

A colouring is a function $f: V \rightarrow C$ where C is a set of colours.

Note: If $|C|=k$, it is called a k-colouring.

Note: If uv is an edge, then $f(u) \neq f(v)$.

Note: A k-colouring is also a $(k+n)$-colouring for n=1,2,…

Lemma G is 2-colourable iff it is bipartite.

Theorem K_p is p-colourable, but not $(p-1)$-colourable.

Notation: Let G be a graph with $v \in V(G)$. Then $G-v$ is the graph with v and all its incident edges deleted.

Theorem Every planar graph is 6-colourable. (Review proof)

Defn: If $e=uv$, we can contract e by deleting e and identifying u and v (i.e. make $u=v$).

We denote this G/e.

Note: G/e may have double edges. Thus G/e may not be a graph.

Note: If G is planar, then G/e is planar. However, the converse may not be true.

Theorem Every planar graph is 5-colourable. (Review proof)

Defn: Let G be a planar graph. Create a new graph H as follows:

- Every face in G is replaced with a vertex in H.
- An edge connects two vertices in H if the corresponding two faces in G were adjacent.
- Delete all loops and fuse all double edges.

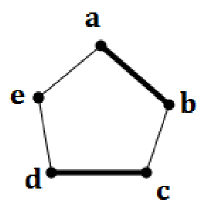
The new graph H is the dual of G.
Note: Colouring H gives a colouring of the faces of G.

Theorem Every planar graph is 4-colourable.

MATCHINGS

Def^{n}: A matching M is a set of edges such that no two of them are incident with the same vertex.

Example In the graph below, $M = \{ab, cd\}$ is a matching.



Def^{n}: A maximum matching M is a matching of largest possible cardinality in G.

Def^{n}: A matching M saturates a vertex $v \in V(G)$, if v is incident with an edge in M.

Theorem If $|M|=k$, then M saturates $2k$ vertices.

Corollary If $|V(G)| = 2|M|$ or $2|M| + 1$, then M is a maximum matching.
Note: The converse may not be true.

Def^{n}: A perfect matching saturates all vertices in the graph.

Def^{n}: Given a matching M of G, a path $v_0, v_1, \ldots v_n$ is an alternating path with respect to M if one of the following properties hold:

- $\{v_i, v_{i+1}\} \in M$ for even i, $\{v_i, v_{i+1}\} \notin M$ for odd i
- $\{v_i, v_{i+1}\} \in M$ for odd i, $\{v_i, v_{i+1}\} \notin M$ for even i

I.e. Consecutive edges in G alternate between matching and non-matching edges.

Def^{n}: An augmenting path P with respect to M is an alternating path between two unsaturated vertices.
Note: $|M|$ is odd.

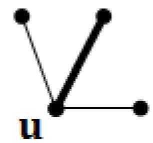
Method Given P and M, we can therefore choose a new matching M' such that every matching edge in M now becomes a non-matching edge in M', and vice versa.
I.e. We are "flipping" the path P.
Note: $|M'|=|M|+1$

Lemma If there is an augmenting path in M, then M is not a maximum matching.

Def[n]: Given a graph G, a cover C is a set of vertices such that every edge of G has at least one end in C.

Example For all graphs G, $V(G)$ is a cover.

Example In the graph to the right, $C = \{u\}$ is a cover.



Example For complete graph K_p, any cover should have $|C| \geq p - 1$.

Example For a bipartite graph with bipartition (A,B), A or B can be a cover.

Lemma $|M| \leq |C|$ for any matching M and any cover C of the same graph.

Theorem If $|M| = |C|$, then M is a maximum matching and C is a minimum cover.

König's Theorem Let M be a maximum matching and C be a minimum cover for a bipartite graph. Then $|M| = |C|$.

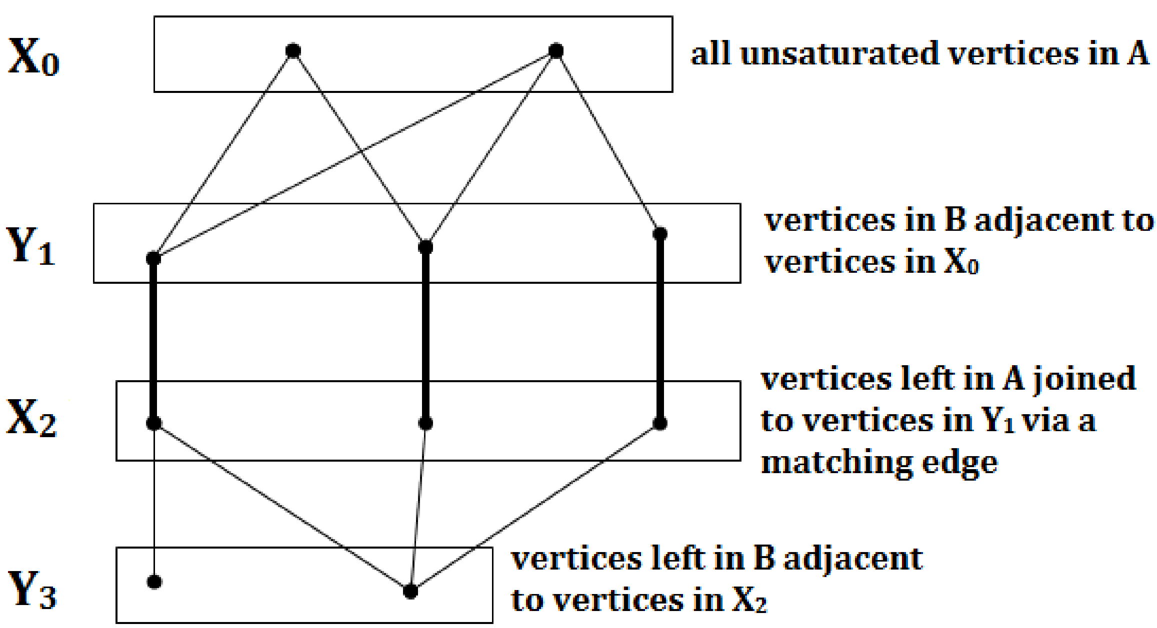
The XY-Construction

Here is an algorithm for finding the maximum matching M and its corresponding cover C for a bipartite graph G, with bipartition (A,B).

We start with X_0 = the set of unsaturated vertices in A.
Then we find the sets $Y_1, X_2, Y_3, X_4, \ldots$ as follows. Also let

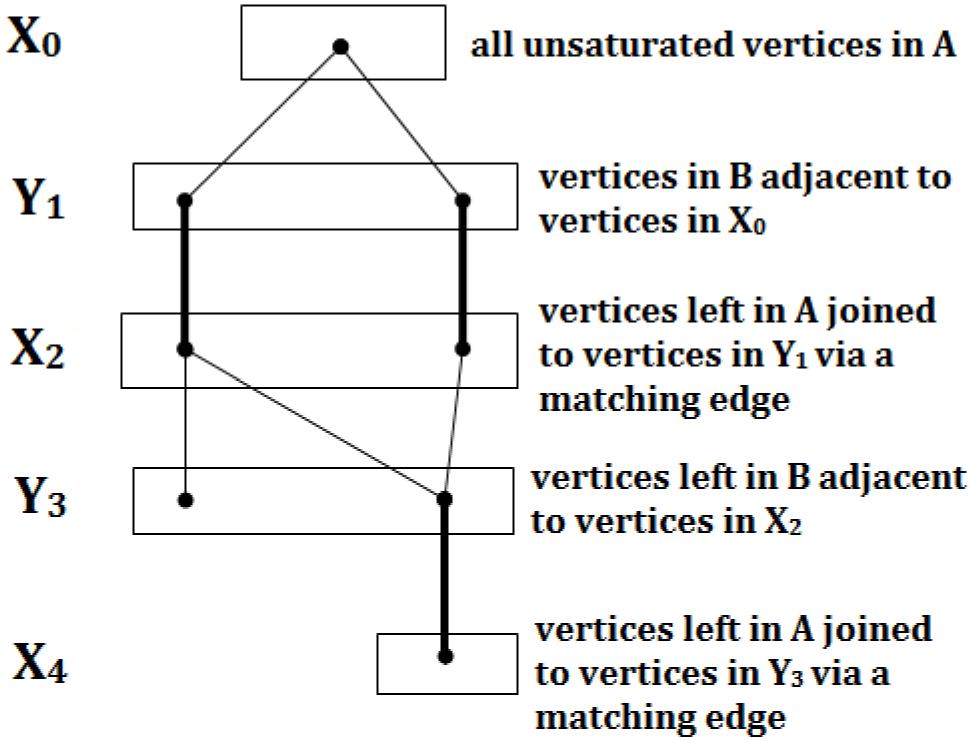
$$X = X_0 + X_2 + X_4 + \ldots$$
$$Y = Y_1 + Y_3 + Y_5 + \ldots$$

(Note that $X \in A$, and $Y \in B$)



At this point, if all vertices in Y_3 are saturated, continue the algorithm.

However, if any vertices in Y_3 are unsaturated, we have an augmenting path. Flip the augmenting path to create a new matching. Start the XY-construction all over again.

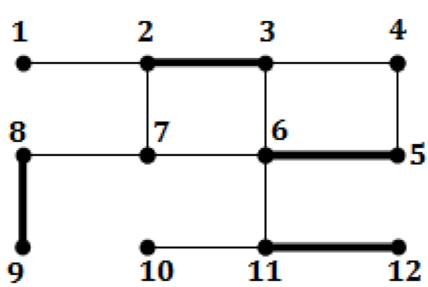


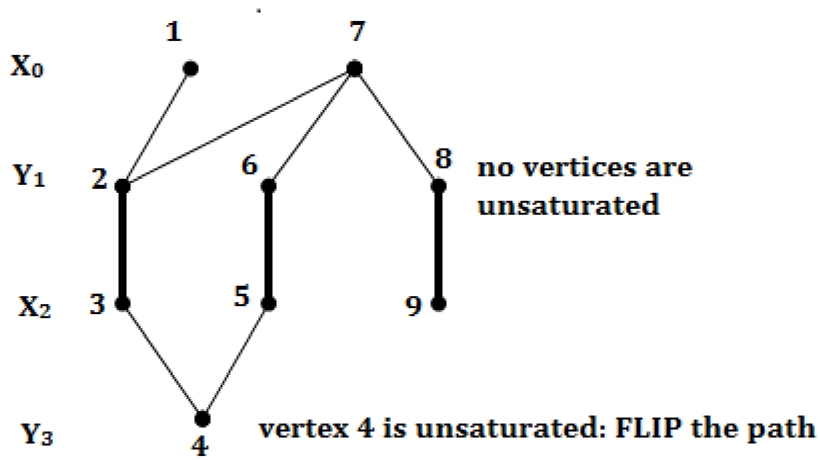
Continue the algorithm (start, flip, restart, flip, restart, etc.) until an empty Y_i or X_0 is created. In the example above, Y_5 is empty. At this point, we have found the maximum matching and its cover:

- M = the final set of matching edges (after flipping all the augmenting paths)
- C = $Y \cup A \backslash X$

<u>Note</u>: There is no matching edge from Y to A\X, and there is no edge from X to B\Y.

<u>Example</u> Given the matching below for G, let A={odd #s} and B={even #s}. Find its max matching.

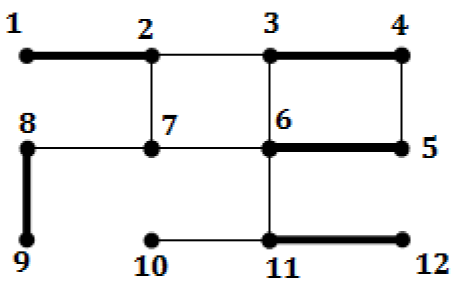
G= 

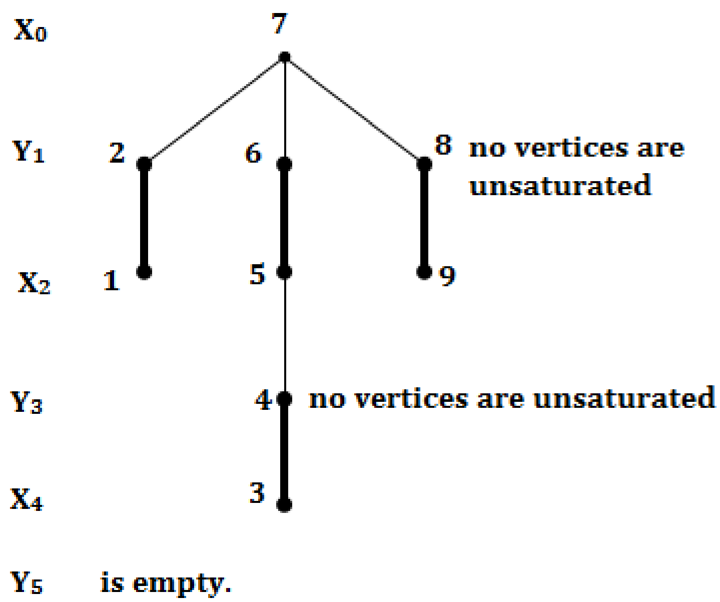


So  becomes 

Now we start the XY construction again with the new matching which has the flipped path.

New matching:

G = 



Therefore,

$M = \{12, 34, 56, 89, 1112\}$

$C = Y \cup A \backslash X$

$= \{2, 6, 8, 4\} \cup \{odd\} \backslash \{7,1,5,9,3\}$

$= \{2, 6, 8, 4, 11\}$

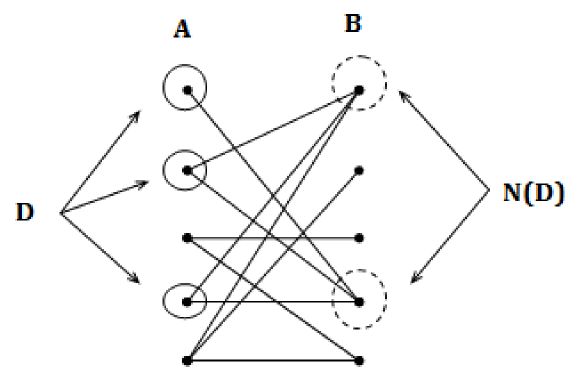
<u>Lemma</u> Let G be a graph with bipartition *(A,B)* and matching M. Then $|M| \leq |A|$.

<u>Hall's Theorem</u> Let G be a graph with bipartition *(A,B)*. Then there exists a matching that saturates A iff

$$|N(D)| \geq |D|, for\ all\ subsets\ D \subseteq A$$

<u>Example</u> Consider the graph on the right.

Since $|N(D)| \leq |D|$, there is no perfect matching.



<u>Corollary</u> If G has bipartition *(A,B)*, then G has a perfect matching iff

$$|A| = |B| \text{ and } |N(D)| \geq |D|, for\ all\ subsets\ D \subseteq A$$

<u>Corollary</u> Every k-regular bipartite graph with $k \geq 1$ has a perfect matching.