MATH 239 Final Practice Problems

Note: These are some of the problems that appeared in previous finals. This is good for practice purposes, however it is not an indication of the content nor the difficulty of the final you will be writing. Abbreviated solutions will be provided later, and you are free to discuss them with TAs and instructors.

1. Determine the coefficient of x^{23} in the power series
$$\frac{1+x^2}{(1-2x^3)^7}.$$

2. Let $0 \le k \le n$ where k, n are integers. Consider the following set:
$$S = \{(A, B) \mid A, B \subseteq [n], |A| = |B| = k\}.$$
By counting S in two different ways, prove that
$$\binom{n}{k}^2 = \sum_{i=0}^{k} \binom{n}{i}\binom{n-i}{k-i}\binom{n-k}{k-i}.$$
(Hint: Consider the possible values of $|A \cap B|$.)

3. Let a_n be the number of compositions of n where each part is congruent to 1 (mod 3) and is at least 4. (There is no limitation on the number of parts.)
   (a) Prove that
   $$\sum_{i \ge 0} a_i x^i = \frac{1-x^3}{1-x^3-x^4}.$$
   (b) Determine a homogeneous linear recurrence relation that the sequence $\{a_n : n \ge 0\}$ satisfies, with sufficient initial conditions to uniquely specify $\{a_n\}$.

4. Let S be the set of binary strings that have the unambiguous decomposition
$$\{00\}^*(\{1\}\{11\}^*\{0\}\{0\}^*)^*(\{1\}\{11\}^*).$$
Determine the generating series for S (with the weight of a string being its length).

5. Find an unambiguous expression for the set of all strings that do not contain 110000 as a substring.

6. Let $\{a_n\}$ be the sequence where $a_0 = 2$, and for $n \ge 1$,
$$a_n - 3a_{n-1} = 0.$$
Determine an explicit formula for a_n.

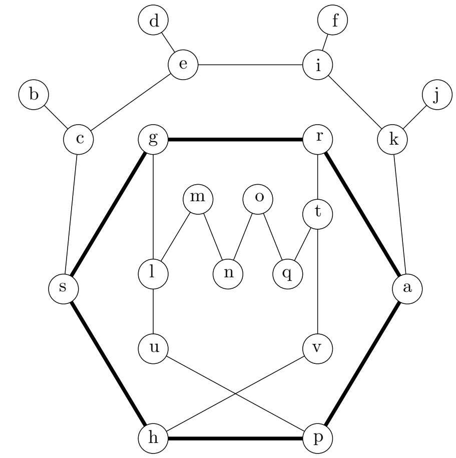
7. Prove that if every vertex of a graph G has even degree, then the edges of G can be partitioned into cycles.

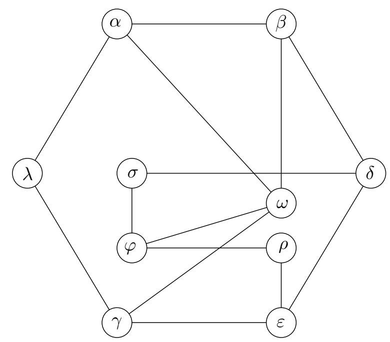
8. Let $a, b \in \mathbb{N}$. For each of the following conditions, determine all possible pairs of values (a, b) for which the complete bipartite graph $K_{a,b}$ satisfies the condition.
   (a) $K_{a,b}$ has a Hamilton cycle, that is, $K_{a,b}$ has a cycle containing all the vertices of $K_{a,b}$.
   (b) $K_{a,b}$ has an Euler tour.
   (c) $K_{a,b}$ is planar.

9. (a) Define what is meant by a tree in graph theory.
   (b) What is the largest t for which there is a tree having t vertices of degree 3, and at most 42 vertices of degree 1? Prove your claim.

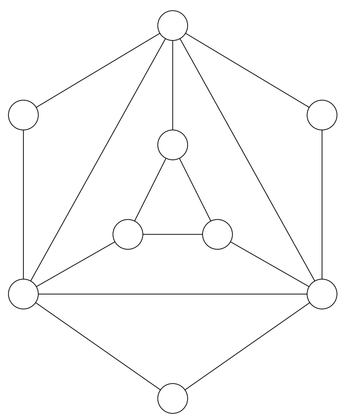
10. Let G be a planar graph on $n \ge 5$ vertices. Suppose G has an embedding where every face boundary is a cycle of length exactly 5. Determine the number of edges and the number of faces of G in terms of n.

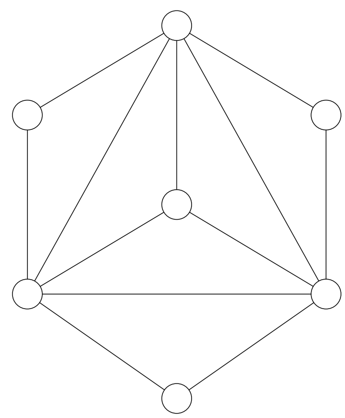
11. Determine whether or not the following graphs are planar. Prove your assertions.



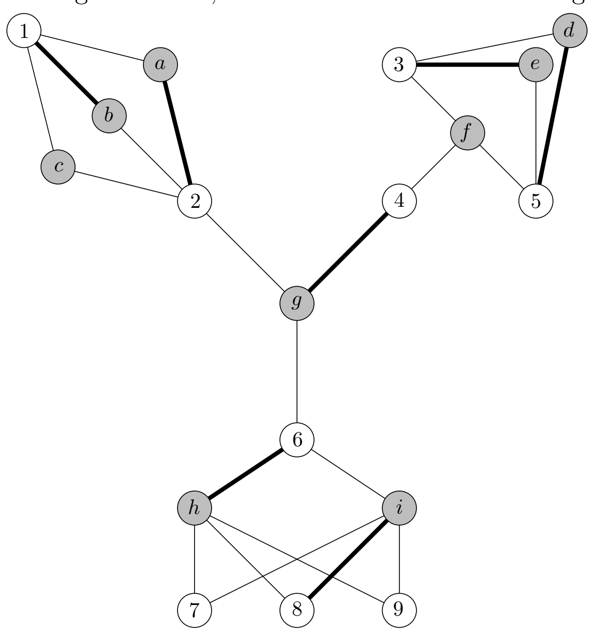


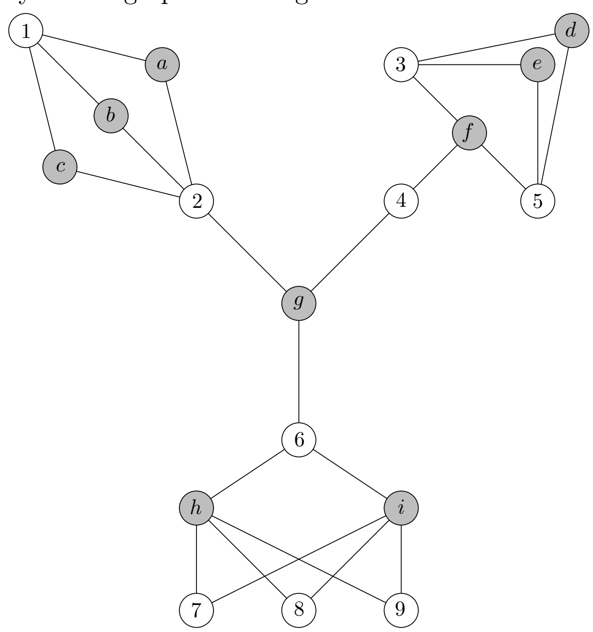
12. Determine whether or not the following graphs are 3-colourable. Prove your assertions.





13. (a) Consider the bipartite graph on the left below (the shaded vertices form one part of the bipartition, the unshaded ones form the other part). You are given a matching M represented by the bolded edges. Find an augmenting path with respect to M, and indicate the path clearly on the graph to the left. Use this path to find a matching larger than M, and indicate the new matching clearly on the graph to the right.





(b) Prove that the matching you have produced in part (a) is maximum by providing a vertex cover. Explain why your cover proves the maximality of your matching.

14. For each of the following statements, determine whether it is true or false. Give an appropriate proof or counterexample.

    (a) Every edge in a 3-regular planar graph is in some cycle.

    (b) Any bipartite graph is not 3-colourable.

    (c) In any non-bipartite graph, the size of a maximum matching is always *strictly* less than the size of a minimum vertex cover.