

Module 2: Priority Queues

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Spring 2015

Abstract Data Types

Abstract Data Type (ADT): A description of *information* and a collection of *operations* on that information.

The information is accessed *only* through the operations.

We can have various *realizations* of an ADT, which specify:

- How the information is stored (*data structure*)
- How the operations are performed (*algorithms*)

Dynamic Arrays

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Dynamic arrays offer a compromise:

$O(1)$ element access, and $O(1)$ insertion/deletion *at the end*.

Two realizations of dynamic arrays:

- Allocate one HUGE array, and only use the first part of it.
- Allocate a small array initially, and double its size as needed.
(*Amortized analysis* is required to justify the $O(1)$ cost for insertion/deletion at the end — take CS 341/466!)

Stack ADT

Stack: an ADT consisting of a collection of items with operations:

- *push*: inserting an item
- *pop*: removing the most recently inserted item

Items are removed in LIFO (*last-in first-out*) order.

We can have extra operations: *size*, *isEmpty*, and *top*

Applications: Addresses of recently visited sites in a Web browser,
procedure calls

Realizations of Stack ADT

- using arrays
- using linked lists

Queue ADT

Queue: an ADT consisting of a collection of items with operations:

- *enqueue*: inserting an item
- *dequeue*: removing the least recently inserted item

Items are removed in FIFO (*first-in first-out*) order.

Items enter the queue at the *rear* and are removed from the *front*.

We can have extra operations: *size*, *isEmpty*, and *front*

Realizations of Queue ADT

- using (circular) arrays
- using linked lists

Priority Queue ADT

Priority Queue: An ADT consisting of a collection of items (each having a *priority*) with operations

- *insert*: inserting an item tagged with a priority
- *deleteMax*: removing the item of *highest priority*

deleteMax is also called *extractMax*.

Applications: typical “todo” list, simulation systems

The above definition is for a *maximum-oriented* priority queue. A *minimum-oriented* priority queue is defined in the natural way, by replacing the operation *deleteMax* by *deleteMin*.

Using a Priority Queue to Sort

PQ – *Sort*(*A*)

1. initialize *PQ* to an empty priority queue
2. **for** $i \leftarrow 0$ **to** $n - 1$ **do**
3. *PQ.insert*(*A*[*i*], *A*[*i*])
4. **for** $i \leftarrow 0$ **to** $n - 1$ **do**
5. $A[n - 1 - i] \leftarrow PQ.deleteMax()$

Realizations of Priority Queues

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- insert: $O(n)$
- deleteMax: $O(1)$

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This realization used for sorting yields *insertion sort*.

Third Realization: Heaps

A *heap* is a certain type of binary tree.

Recall binary trees:

A binary tree is either

- empty, or
- consists of three parts: a node and two binary trees (left subtree and right subtree).

Terminology: root, leaf, parent, child, level, sibling, ancestor, descendant, etc. .

Heaps

A *max-heap* is a binary tree with the following two properties:

- 1 **Structural Property:** All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are *left-justified*.
- 2 **Heap-order Property:** For any node i , *key* (priority) of parent of i is larger than or equal to key of i .

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A *min-heap* is the same, but with opposite order property.

Lemma: Height of a heap with n nodes is $\Theta(\log n)$.

Storing Heaps in Arrays

Let H be a heap (binary tree) of n items and let A be an array of size n . Store root in $A[0]$ and continue with elements *level-by-level* from top to bottom, in each level left-to-right.

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It is easy to find parents and children using this array representation:

- the *left child* of $A[i]$ (if it exists) is $A[2i + 1]$,
- the *right child* of $A[i]$ (if it exists) is $A[2i + 2]$,
- the *parent* of $A[i]$ ($i \neq 0$) is $A[\lfloor \frac{i-1}{2} \rfloor]$ ($A[0]$ is the root node).

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bubble-up(v)

v : a node of the heap

1. **while** $\text{parent}(v)$ exists **and** $\text{key}(\text{parent}(v)) < \text{key}(v)$ **do**
2. swap v and $\text{parent}(v)$
3. $v \leftarrow \text{parent}(v)$

The new item bubbles up until it reaches its correct place in the heap.

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Time: $O(\text{height of heap}) = O(\log n)$.

deleteMax in Heaps

- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
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bubble-down(v)

v : a node of the heap

1. **while** v is not a leaf **do**
2. $u \leftarrow$ child of v with largest key
3. **if** $\text{key}(u) > \text{key}(v)$ **then**
4. swap v and u
5. $v \leftarrow u$
6. **else**
7. **break**

Time: $O(\text{height of heap}) = O(\log n)$.

Priority Queue Realization Using Heaps

heapInsert(A, x)

A : an array-based heap, x : a new item

1. $\text{size}(A) \leftarrow \text{size}(A) + 1$
2. $A[\text{size}(A) - 1] \leftarrow x$
3. *bubble-up*($A, \text{size}(A) - 1$)

heapDeleteMax(A)

A : an array-based heap

1. $\text{max} \leftarrow A[0]$
2. *swap*($A[0], A[\text{size}(A) - 1]$)
3. $\text{size}(A) \leftarrow \text{size}(A) - 1$
4. *bubble-down*($A, 0$)
5. **return** max

Insert and deleteMax: $O(\log n)$

Building Heaps

Problem statement: Given n items (in $A[0 \dots n - 1]$) build a heap containing all of them.

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Solution 1: Start with an empty heap and insert items one at a time:

heapify1(A)

A : an array

1. initialize H as an empty heap
2. **for** $i \leftarrow 0$ **to** *size*(A) $- 1$ **do**
3. *heapInsert*($H, A[i]$)

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This corresponds to going from $0 \dots n - 1$ in A and doing *bubble-ups*
Worst-case running time: $\Theta(n \log n)$.

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Solution 2: Using *bubble-downs* instead:

heapify(A)

A : an array

1. $n \leftarrow \text{size}(A) - 1$
2. **for** $i \leftarrow \lfloor n/2 \rfloor$ **downto** 0 **do**
3. *bubble-down*(A, i)

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```
heapify( $A$ )  
 $A$ : an array  
1.  $n \leftarrow \text{size}(A) - 1$   
2. for  $i \leftarrow \lfloor n/2 \rfloor$  downto 0 do  
3.   bubble-down( $A, i$ )
```

A careful analysis yields a worst-case complexity of $\Theta(n)$.
A heap can be built in linear time.

HeapSort

HeapSort(A)

1. initialize H to an empty heap
2. **for** $i \leftarrow 0$ **to** $n - 1$ **do**
3. $\text{heapInsert}(H, A[i])$
4. **for** $i \leftarrow 0$ **to** $n - 1$ **do**
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Running time of HeapSort: $O(n \log n)$

Selection

Problem Statement: The k th-max problem asks to find the *k th largest item* in an array A of n numbers.

Solution 1: Make k passes through the array, deleting the maximum number each time.

Complexity: $\Theta(kn)$.

Solution 2: First sort the numbers. Then return the k th largest number.

Complexity: $\Theta(n \log n)$.

Solution 3: Scan the array and maintain the k largest numbers seen so far in a min-heap

Complexity: $\Theta(n \log k)$.

Solution 4: Make a max-heap by calling *heapify*(A). Call *deleteMax*(A) k times.

Complexity: $\Theta(n + k \log n)$.