Math 239 Fall 2014 Midterm Solutions

1. {8 marks total, 2 marks part} Short answers. For this question only, no justication required.

   (a) How many subsets of [314] have size 159?

   Solution. $\binom{314}{159}$.

   (b) Write an example of a composition of 7 with exactly 4 parts.

   Solution. $(4, 1, 1, 1)$.

   (c) Write an example of a binary string with exactly 3 blocks.

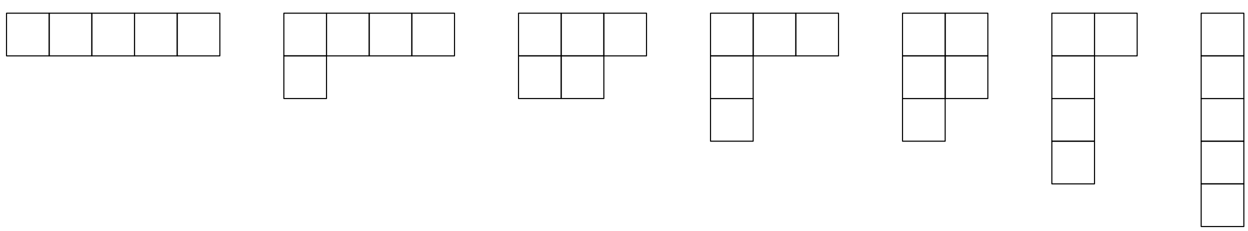
   Solution. 101.

   (d) Let $k \in \mathbb{N}$. Define what it means for a graph G to be k-regular.

   Solution. Every vertex has degree k.

2. Let $n \in \mathbb{N}$. A *partition* of n is a composition $(\lambda_1, \ldots, \lambda_k)$ where $\lambda_1 + \cdots + \lambda_k = n$, and the numbers are arranged in non-increasing order, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. Let P_n be the set of all partitions of n. For example, there are 7 partitions in P_5, illustrated below where each row of boxes represent one part of the partition, from top to bottom.

(5) $(4, 1)$ $(3, 2)$ $(3, 1, 1)$ $(2, 2, 1)$ $(2, 1, 1, 1)$ $(1, 1, 1, 1, 1)$



   (a) {3 marks} For each partition $\sigma = (\lambda_1, \ldots, \lambda_k)$, define the weight function $w(\sigma) = \lambda_1$ (i.e. the largest part). Determine the generating series $\Phi_{P_5}(x)$ of P_5 with respect to w.

   Solution. The weights of the individual partitions (from left to right) are $5, 4, 3, 3, 2, 2, 1$, so the generating series is
$$\Phi_{P_5}(x) = x^5 + x^4 + 2x^3 + 2x^2 + x.$$

   (b) {2 marks} For each partition $\sigma = (\lambda_1, \ldots, \lambda_k)$, define the weight function $w^*(\sigma) = k$ (i.e. the number of parts). Let $\Phi^*_{P_5}(x)$ be the generating series for P_5 with respect to w^*. Without calculating $\Phi^*_{P_5}(x)$, explain why $\Phi_{P_5}(1) = \Phi^*_{P_5}(1)$. (You may not use part (c).)

   Solution. For any weight function, $\Phi_{P_5}(1) = \sum_{\sigma \in P_5} 1^{w(\sigma)} = \sum_{\sigma \in P_5} 1 = |P_5|$. This value is independent of the weight function. So $\Phi_{P_5}(1) = \Phi^*_{P_5}(1)$.

   (c) {Extra credit: 2 marks} Prove that for all $n \in \mathbb{N}$, $\Phi_{P_n}(x) = \Phi^*_{P_n}(x)$.

   Solution. We can find a bijection $f : P_n \to P_n$ by transposing the diagram corresponding to the input partition (i.e. flip top-right with bottom-left), and output the corresponding partition. Notice that in a partition, parts are non-increasing, and this property is preserved when we transpose the digram. The inverse is the same function. Now the largest part of of a partition is the first horizontal line in the diagram, and this corresponds to the first vertical line in the mapped partition, which corresponds to the number of parts. So $w(\sigma) = w^*(f(\sigma))$. Therefore,
$$\Phi_{P_n}(x) = \sum_{\sigma \in P_n} x^{w(\sigma)} = \sum_{\sigma \in P_n} x^{f(\sigma)} = \sum_{\sigma' \in P_n} x^{\sigma'} = \Phi^*_{P_n}(x).$$

3. {3 marks} Determine the coefficient of x^{15} in the following power series. You do not need to evaluate large powers and binomial coefficients.

$$A(x) = \frac{x^2 + x^3}{(1 - 5x^2)^{10}}.$$

Solution. We see that

$$[x^{15}]\frac{x^2 + x^3}{(1 - 5x^2)^{10}} = [x^{13}]\frac{1}{(1 - 5x^2)^{10}} + [x^{12}]\frac{1}{(1 - 5x^2)^{10}}.$$

Notice that

$$\frac{1}{(1 - 5x^2)^{10}} = \sum_{n\geq 0}\binom{n+10-1}{10-1}(5x^2)^n = \sum_{n\geq 0}\binom{n+9}{9}5^n x^{2n}.$$

The coefficient is 0 for all odd powers of x, so the coefficient of x^{13} is 0. The coefficient of x^{12} can be obtained by substituting $n = 6$, and we get $\binom{15}{9}5^6$.

4. {5 marks} For any integer $n \geq 0$, let a_n be the number of compositions of n with even number of parts where no part is divisible by 3. For example, $a_4 = 2$ with the compositions $(2, 2), (1, 1, 1, 1)$. Determine a_n, and express it as the coefficient of a rational expression. (Do not find an explicit formula for a_n.)

Solution. Let $T = \{1, 2, 4, 5, 7, 8, 10, 11, \ldots\}$ be the set of all positive integers not divisible by 3. Then the set of all compositions we need can be enumerated as

$$S = T^0 \cup T^2 \cup T^4 \cup \cdots = \bigcup_{k\geq 0} T^{2k}.$$

Define the sum of the parts as the weight of a composition. Then

$$\Phi_T(x) = \sum_{i\geq 0} x^{1+3i} + \sum_{i\geq 0} x^{2+3i} = \frac{x}{1-x^3} + \frac{x^2}{1-x^3} = \frac{x+x^2}{1-x^3}.$$

Using the sum and product, we get

$$\begin{aligned}
\Phi_S(x) &= \sum_{k\geq 0}\Phi_{T^{2k}}(x) \\
&= \sum_{k\geq 0}(\Phi_T(x))^{2k} \\
&= \frac{1}{1-\Phi_T(x)^2} \\
&= \frac{1}{1-\frac{(x+x^2)^2}{(1-x^3)^2}} \\
&= \frac{(1-x^3)^2}{(1-x^3)^2-(x+x^2)^2}
\end{aligned}$$

The answer is then the coefficient of x^n in this power series.

5. {3 marks} The following is an unambiguous expression for the set S of all binary strings with even number of 0's. Using the length of a string as its weight, determine the generating series $\Phi_S(x)$ and express it as a simplified rational expression.

$$S = (\{1\}^*\{0\}\{1\}^*\{0\})^*\{1\}^*.$$

Solution.

$$\Phi_S(x) = \frac{1}{1-\frac{1}{1-x}x\frac{1}{1-x}x}\frac{1}{1-x} = \frac{1}{1-\frac{x^2}{(1-x)^2}}\frac{1}{1-x} = \frac{(1-x)^2}{(1-x)^2-x^2}\frac{1}{1-x} = \frac{1-x}{1-2x}.$$

6. {3 marks} Write an unambiguous expression for the set of all binary strings where every block of 0's of even length cannot be followed by a block of 1's of length at least 3.

   Solution. Starting with the block decomposition $\{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*$, we need to modify the center portion by removing those where an even block of 0's is followed by a block of 1's of length at least 3. So a decomposition is

   $$\{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^* \setminus \{00\}\{00\}^*\{111\}\{1\}^*)^*\{0\}^*.$$

   Alternatively, we can modify the center so that even block of 0's is followed by a block of 1's of length 1 or 2, and an odd block of 0's can be followed by any block of 1's.

   $$\{1\}^*(\{00\}\{00\}^*\{1, 11\} \cup \{0\}\{00\}^*\{1\}\{1\}^*)^*\{0\}^*$$

7. {7 marks} Let $\{a_n\}$ be the sequence which satisfies

   $$a_n - 6a_{n-1} + 5a_{n-2} = 8$$

   for $n \geq 2$ with initial conditions $a_0 = 3, a_1 = 5$. Determine an explicit formula for a_n.

   Solution. The characteristic polynomial is $x^2 - 6x + 5 = (x - 5)(x - 1)$, so the roots are 5 and 1. Notice that on the right hand side of the recurrence, there is a hidden 1^n, so in guessing a specific solution b_n, we cannot use $b_n = \alpha$. Instead, we will guess $b_n = \alpha n$. Then

   $$b_n - 6b_{n-1} + 5b_{n-2} = \alpha n - 6\alpha(n - 1) + 5\alpha(n - 2) = -4\alpha.$$

   This equals 8, so $\alpha = -2$. A specific solution is then $b_n = -2n$.

   Then the explicit formula for a_n has the form

   $$a_n = A \cdot 5^n + B \cdot 1^n - 2n.$$

   Substituting the initial conditions to get

   $$\begin{aligned} a_0 &= 3 = A + B \\ a_1 &= 5 = 5A + B - 2 \end{aligned}$$

   Solving gives $A = 1, B = 2$. So an explicit formula for a_n is

   $$a_n = 5^n + 2 - 2n.$$

8. {4 marks} Let $n \in \mathbb{N}$. Consider the following set:

   $$S = \{(A, B) \mid A, B \subseteq [n], |A \cap B| = 1\}.$$

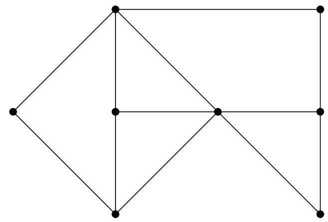
   By counting S in two different ways, prove that

   $$\sum_{k=1}^{n} \binom{n}{k} \cdot k \cdot 2^{n-k} = n \cdot 3^{n-1}.$$

   Solution. We partition S into n different sets $S_1, S_2, \ldots, S_n$ where each S_k consists of those pairs $(A, B) \in S$ where $|A| = k$. Then there are $\binom{n}{k}$ ways to pick A. We have k ways to choose the element that is in $A \cap B$. There are $n - k$ elements remaining from $[n]$, and each element could be in B or not in B, so there are 2^{n-k} ways to fill in B. Therefore, $|S_k| = \binom{n}{k}k2^{n-k}$. Since $S = \bigcup_{k=1}^{n} S_k$ is a disjoint union, we see that $|S| = \sum_{k=1}^{n} \binom{n}{k}k2^{n-k}$.

   On the other hand, we can count S as follows: Pick the element that is in the intersection $A \cap B$. There are n ways to do so. For the remaining $n - 1$ elements, each one could be in A only, in B only, or in neither A nor B. So there are 3 choices for each. Therefore, $|S| = n \cdot 3^{n-1}$. This proves the equation.

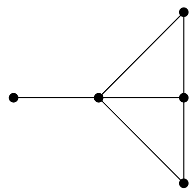
9. The *degree sequence* of a graph is the list of vertex degrees $d_1, d_2, \ldots, d_n$, usually written in non-decreasing order, i.e. $d_1 \leq \cdots \leq d_n$. For example, in the graph below, its degree sequence is $2, 2, 2, 3, 3, 3, 4, 5$.



Prove or disprove each of the following statements using an appropriate proof, example, or counterexample.

(a) {2 marks} There exists a graph with degree sequence $1, 2, 2, 3, 4$.

Solution. True.

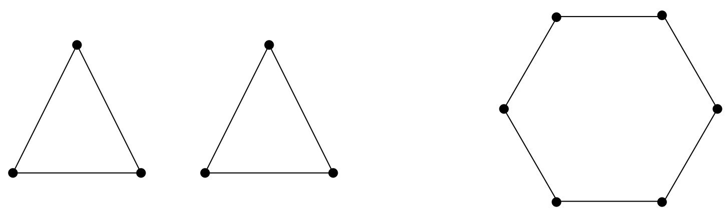


(b) {2 marks} There exists a graph with degree sequence $1, 1, 2, 2, 2, 3$.

Solution. False. There are 3 vertices of odd degrees, but every graph has even number of odd-degree vertices.

(c) {2 marks} All graphs with degree sequence $2, 2, 2, 2, 2, 2$ are isomorphic.

Solution. False. The graphs G and H below have this degree sequence but are not isomorphic. (One is connected, the other is not.)



G $\qquad\qquad$ H

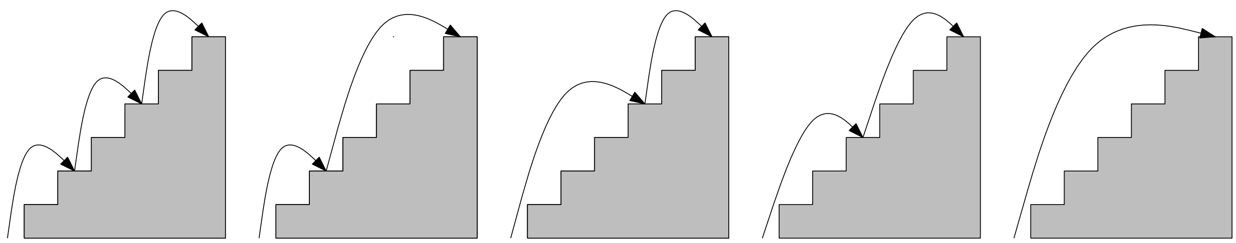
10. {3 marks} Let G be a graph on n vertices such that $\overline{G}$ is bipartite. Prove that G contains the complete graph $K_{\lceil \frac{n}{2} \rceil}$.

(Recall: The complement of a graph G, denoted $\overline{G}$, is the graph where $V(\overline{G}) = V(G)$, and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.)

Solution. Consider $\overline{G}$, which is bipartite. Let (A, B) be its bipartition. Since there are n vertices, one of A, B has size at least $\lceil \frac{n}{2} \rceil$. Without loss of generality, say it is A. Notice that in $\overline{G}$, no two vertices in A are adjacent. So in particular, in G, every pair of vertices in A are adjacent, which forms a $K_{|A|}$. Since $|A| \geq \lceil \frac{n}{2} \rceil$, there exists $K_{\lceil \frac{n}{2} \rceil}$ in G.

11. {3 marks} Mr. Fibonacci has really long legs. Whenever he climbs the stairs, he always takes at least 2 steps in one stride. Let $n \in \mathbb{N}$, and let a_n be the number of different ways that Mr. Fibonacci can climb n steps. (For example, $a_6 = 5$ as illustrated in the following diagram.) Prove that for all $n \geq 1$, $a_n = f_{n-1}$ where $\{f_n\}$ is the Fibonacci sequence.

(Recall: The Fibonacci sequence $\{f_n\}$ satisfies $f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.)



Solution. We may consider each way of climbing as a composition where each part is at least 2. Let $T = \{2, 3, 4, \ldots\}$. Then the set of all such compositions is

$$S = \bigcup_{k \geq 0} T^k.$$

Using the usual weight function for compositions, we see that

$$\Phi_T(x) = \frac{x^2}{1-x},$$

and

$$\Phi_S(x) = \sum_{k \geq 0} \Phi_T(x)^k = \frac{1}{1 - \frac{x^2}{1-x}} = \frac{1-x}{1-x-x^2}.$$

Then $a_n = [x^n]\frac{1-x}{1-x-x^2}$. So if $A(x) = \sum_{n \geq 0} a_n x^n = \frac{1-x}{1-x-x^2}$, we can multiply both sides to get

$$1 - x = (1 - x - x^2)A(x) = a_0 + (a_1 - a_0)x + \sum_{n \geq 2}(a_n - a_{n-1} - a_{n-2})x^n.$$

By comparing coefficients, this gives us $a_0 = 1$ and $a_1 - a_0 = -1$, so $a_1 = 0$. The next term $a_2 = a_1 + a_0 = 1$. So we see that $a_1 = f_0$ and $a_2 = f_1$. By induction, for $n \geq 3$, $a_n = a_{n-1} + a_{n-2} = f_{n-2} + f_{n-3} = f_{n-1}$.

Alternate solution. Let S_n be the set of all compositions of n where every part is at least 2. Then $S_1 = \emptyset, S_2 = \{(2)\}$, so $a_1 = 0 = f_0, a_2 = 1 = f_1$. For $n \geq 2$, there is a bijection $f : S_n \to S_{n-1} \cup S_{n-2}$ as follows:

$$f(a_1, \ldots, a_k) = \begin{cases} (a_1, \ldots, a_{k-1}) & a_k = 2 \\ (a_1, \ldots, a_{k-1}, a_k - 1) & a_k > 2 \end{cases}$$

In the first case $a_k = 2$, the composition is mapped to S_{n-2}. In the second case $a_k > 2$, notice that by subtracting 1 from a_k, it is still at least 2, so this is mapped to S_{n-1}. The inverse is

$$f^{-1}(b_1, \ldots, b_l) = \begin{cases} (b_1, \ldots, b_l, 2) & b_1 + \cdots + b_l = n - 2 \\ (b_1, \ldots, b_{l-1}, b_l + 1) & b_1 + \cdots + b_l = n - 1 \end{cases}$$

This establishes that $|S_n| = |S_{n-1}| + |S_{n-2}|$, or $a_n = a_{n-1} + a_{n-2}$. Using induction, we see that $a_n = a_{n-1} + a_{n-2} = f_{n-2} + f_{n-3} = f_{n-1}$.