M239 Tutorial 1

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$$\binom{n}{k} = \frac{n}{k(n-k)}$$

Number of ways we can choose k elements from a set of size n.
n! is the number of ways to arrange n objects
2^n is the number of binary strings length n
2^n is the number of subsets of {1, 2, ... n}

Problem 1

Given $0 \leq r \leq k \leq n$ how many subsets of $[n] = \{1, ...n\}$ have exactly r elements in common with {1,..k}
{1,2,3,4, ..., k, k+1,k+2,...n} Every considered set is of the form $R \cup S$ where $R \subseteq \{1, ...k\} |R| = r$ and $S \subseteq \{k+1, ...n\}$
Number of ways to construct R = $\binom{k}{r}$
Number of ways to constrct S = 2^{n-k}
Answer = $\binom{k}{r} \times 2^{n-k}$

Problem 3

For integers $0 \leq r \leq k \leq n$. Give a combinatorial proof of the following identity $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$
$S = \{(X, Y); X \leq Y \leq [n]; |Y| = k, |X| = r\}$ Lets count S into two different ways

1. Find all possible Y's and then consturct $X \leq Y$ Number of possible y's = $\binom{n}{k}$, once y is fixed, how many $X \leq Y$ have $|X| = r$, $\binom{k}{4}$. Total = $\binom{n}{k}\binom{k}{r}$
2. Find all possible $X \leq [n]$ with $|X| = r$, $\binom{n}{r}$. Then find all Y's $Y \leq [n]$ $X \leq$ and $|Y| = k$ Total: $\binom{n}{r}\binom{n-r}{k-r}$

Problem 2

Define
E_n = subsets of [n] with even cardinality
O_n = subsets of [n] with odd cardinality

(a) find a bijection between E_n and O_n
$f : E_n \to O_n$ where $f(s) = S \cup \{n\} if n \notin S$ and $S/\{n\} if n \in S$
f(S) is in O_n because $|f(S)| = |S| + / - 1$
$f^{-1} : O_n \to E_n$ $f^{-1}(s) = S \cup \{n\} if n \notin S$ or $S\ \{n\} if n \in S$
$f^{-1}(f(S)) = S$

(b) to determine $|E_n|, |O_n|$
$|E_n| = |O_n|$
$E_n \cup O_n = \{$subsets of [n]$\}$
$|E_n| = 1/2$ $\{$subsets of [n]$\}$
$= 1/2 \times 2^n = 2^{n-1}$

(c) Using (a) show $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

$$\sum_{k is even, 0 \leq k \leq n} \binom{n}{k} + \sum_{k odd, 0 \leq k \leq n} \binom{n}{k}$$

$$|E_n| = |O_n|$$