Math239 Tutorial 6

Graham Cooper

May 4th, 2015

1)

Find an explicit formula for a_n where $\{a_n\})n = 0^\infty$ defined by:
$a_n - a_{n-1} - 8a_{n-2} + 12a_{n-3} = 0$
$a_0 = 1$
$a_1 = 13$
$a_2 = 23$

Solution:

The characteristic polynomial is:

$$p(x) = 12 - 8x - x^2 + x^3$$

Observe that p(2) = 0
Hence:

$$(x-2)|p(x)$$

Once cna show
p(x) = (x-2)(x-2)(x+3)

$\therefore$ the roots of p(x) are x = 2 with multiplicity 2, x-3 with multiplicity 1

The general for of the solution is:

$$a_n = (\alpha + \beta n)2^n + 8(-3)^n$$

Substituting the given values of a_0, a_1 and a_2 we get

$$a_0 = 1 = \alpha + \gamma$$

$$a_1 = 13 = 2\alpha + 2\beta - 3\gamma$$

$$a_2 = 23 = 4\alpha + 8\beta + 9\gamma$$

$\alpha = 2$
$\beta = 3$
$\gamma = -1$

The explicit formula for a_n is

$$a_n = (2+3n)2^n - (-3)^n$$

2)

Let $\{b_n\}_{n=0}^{\infty}$
Define $b_n = \sqrt{5}(\frac{3+\sqrt{5}}{2})^n - \sqrt{5}(\frac{3-\sqrt{5}}{2})^n$
And let B(x) $= \sum_{n=0}^{\infty} b_n x^n$ find a rational expression for B(x)

Solution:

$$\begin{aligned} B(x) &= \sum_{n=0}^{\infty} b_n x^n \\ &= \sum_{n=0}^{\infty}(\sqrt{5}(\frac{3+\sqrt{5}}{2})^n - \sqrt{5}(\frac{3-\sqrt{5}}{2})^n)x^n \\ &= \sqrt{5}\sum_{n=0}^{\infty}(\sqrt{5}(\frac{3+\sqrt{5}}{2})^n x^n - \sqrt{5}\sum_{n=0}^{\infty}(\frac{3-\sqrt{5}}{2})^n)x^n \\ &= \sqrt{5}\cdot\frac{1}{1-(\frac{3+\sqrt{5}}{2})x} - \sqrt{5}\cdot\frac{1}{1-(\frac{3-\sqrt{5}}{2})x} \\ &= \sqrt{5}(\frac{\sqrt{5}x}{1-3x+x^2}) \\ &= \frac{5x}{1-3x+x^2} \end{aligned}$$

3

Let A_n be the nxn matrix which is tridiagonal with parameters 1,-2, 3
ie $A_1 = [1]$
$A_2 = \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}$ $A_3 = \begin{vmatrix} 1 & -2 & 0 \\ 3 & 1 & -2 \\ 0 & 3 & 1 \end{vmatrix}$ Define $a_n = det(A_n)$. Find a recursive formula for a_n, along wiht enough initial conditions to full determine $\{a_n\}_{n=1}^{\infty}$

then find an explicit formula for a_n

Solution
$a_n = det(A_n)$

$$A_n = \left|\begin{array}{c|cccc} 1 & \text{-2} & 0 & ... & 0 \\ \hline 3 & & & & \\ 0 & & & & \\ ... & & A_{n-1} & & \\ 0 & & & & \end{array}\right|$$

$$\begin{aligned} a_n &= det(A_n) \\ &= 1 \cdot det(A_{n-1}) - 3det(A_{n-2}) \\ &= 1 \cdot det(A_{n-1}) - 3 \cdot (-2)(det(A_{n-2})) \\ &= a_{n-1} + 6a_{n-2} \end{aligned}$$

The something I can't read are:
$a_1 = det[1] = 1$
$a_2 = det \begin{vmatrix} 1 & \text{-2} \\ 3 & 1 \end{vmatrix} = 1^2 + 6 = 7$

The characteristic polynomial is:
$p(x) = x^2 - x - 6$
$= (x - 3)(x + 2)$
The roots of p(x) are
x = 3, mult 1
x = -2, mult 1

The general form of the stuff??? is
$a_n = \alpha 3^n + \beta(-2)^n$
$a_1 = 1 = 3\alpha - 2\beta$
$a_2 = 7 = 9\alpha + 4\beta$

The solution is
$a_n = \frac{3}{5} \cdot 3^n + \frac{2}{5}(-2)^n$

4)

For k $\in N$, a k-ary tree is a root tree in which any vertex has up to k outgoing edges.

Let t_n be the number of k-ary trees on n vertices. Let T(x) = $\sum_{n=0}^{\infty} t_n x^n$, and T be the set of all k-ary trees

a) Show that T(x) = 1 + xT(x)^k

Solution:
Note that any k-ary tree is either empty, or has a root.

If such tree τ on n vertices has a root r, consider the following construction

Let $\gamma_1, ...\gamma_k$ be the children of r. Remove r from τ and defin $r_1, ..r_k$ to be the roots of the connected components of the resulting graph. You are left with k k-ary trees with roots $r_1, ...r_k$

The same procedure works in reverse)

If we define ϕ in the following way:
ϕ :
$\epsilon \rightarrow \epsilon$
$\tau \rightarrow$as defined earlier
ϕ is a bijection map between
T and $\{\epsilon\} \cup \{r\} \times$ T^k

Define w(τ) = n

This tells us that the generating function for T(x) = 1 + xT(x)^k