Math 239 Theorems and Definitions

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1 Combinatorial Analysis

1.3 Binomial Coefficients

1.3.1 Theorem: For non-negative integers n and k, the number of k-element subsets of an n-element set is:

$$\frac{n(n-1)...(n-k+1)}{k!} = \binom{n}{k} = \binom{n}{n-k}$$

1.3.2 Theorem: For any non-negative integer n,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

1.3.3 Problem: For any non-negative integers n and k:

$$\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$$

1.4 Generating Series

1.4.2 Definition: Let S be a set of configurations with a weight function w. The generating series for S with respect to w is defined by:

$$\begin{aligned}\Phi_S(x) &= \sum_{\sigma \in S} x^{w(\sigma)} \\ &= \sum_{k \geq 0} a_k x^k\end{aligned}$$

1.4.3 Theorem: Let $\Phi_S(x)$ be the generating series for a finit set S with respect to a weight function w. Then,

- $\Phi_S(1) = |S|$
- the sum of the weights of the elements in S is $\Phi'_S(1)$, and
- the average weight of an element in S is $\Phi'_S(1)/\Phi_S(1)$

1.5 Formal Power Series

1.5.0 Definition: For a sequence of $(a_0, a_1, a_2...)$ which are rational numbers, then A(x) = $a_0 + a_1x + a_2x^2 + ...$ is called the formal power series. We say that a_n is the coefficient of x^n and we write $a_n = [x^n]A(x)$.
Also:

$$A(x) + B(x) = \sum_{n\geq 0}(a_n + b_n)x^n$$

$$A(x)B(x) = \sum_{n\geq 0}(\sum_{k=0}^{n} a_k b_{n-k})x^n$$

1.5.2 Theorem: Let $A(x) = a_0 + a_1x + a_2x^2 + ...$, $P(x) = p_0 + p_1x + p_2x^2 + ...$ and $Q(x) = 1 - q_1x - q_2x^2 - ...$ be formal power series. Then:

$$Q(x)A(x) = P(x)$$

if and only if for each n ≥ 0

$$a_n = p_n + q_1a_{n-1} + q_2a_{n-2} + ... + q_na_0$$

1.5.3 Corollary: Let P(x) and Q(x) be formal power series. If the constant term of Q(x) is non-zero, then there is a formal power series A(x) satisfying:

$$Q(x)A(x) = P(x)$$

Moreover, the solution A(X) is unique

1.5.4 Definition: We say that B(x) is the inverse of A(X) if

$$A(x)B(x) = 1$$

we denote this by B(x) = A(x)$^{-1}$ or by B(x) = $\frac{1}{A(x)}$

1.5.7 Theorem: A formal power series has an inverse if and only if it has a non-zero costant term. Moreover, if the constant term is non-zero, then the inverse is unique

1.5.8 Definition: The composition of formal power series A(x) = $a_0 + a_1 x + a_2 x^2 + ...$ B(x) is defined by:

$$A(B(x)) = a_0 a_1 B(x) + a_2 (B(x))^2 + ...$$

However unlink for polynomials, this composition operation is not always well defined. Consider, for example, the case that A(X) = 1 + x + x^2 + ... and B(x) = (1+x). Then

$$A(B(x)) = 1 + (1 + x) + (1 + x)^2 + ...$$

The constant term of the right-hand side has non-zero contributions from an infinite number of terms, so A(B(x)) is not a formal power series. The following result shows that A(B(x)) is well-defined so long as B(x) has its constant term equal to zero (that is B(0) = 0).

1.6 The Sum and Product Lemma

1.6.1 (Sum Lemma) Theorem: Let (A,B) be a partition of a set S. (That is, A and B are disjoint sets whose union is S.) Then,

$$\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$$

1.6.2 (Product Lemma) Theorem: Let A and B be sets of configurations with weight functions α and β respectively. If $w(\sigma) = \alpha(a) + \beta(b)$ for each $\sigma = (a, b) \in A \times B$, then

$$\Phi_{A\times B}(x) = \Phi_A(x) \cdot \Phi_B(x)$$

1.6.5 Theorem: For any positive integer k and non-negative integer n,

$$(1 - x)^{-k} = \sum_{n\geq 0} \binom{n + k - 1}{k - 1} x^n$$

2 Copositions and Strings

2.1 Compositions of an Integer

2.1.1 Definition For non-negative integers n and k, a composition of n with k parts is an ordered list $(c_1, ...c_k)$ of positive integers $c_1, ...c_k$ such that

$c_1 + ... + c_k = n$. The positive integers $c_1...c_k$ are called the parts of the composition. There is one composition of 0, the empty composition, which is a composition with 0 parts.

2.3 Binary Strings

2.3.1 Definition: A binary string or {0,1}-string is a string of 0's and 1's, its length is the number of occurences of 0 and 1 in the string. We use ϵ to denote the empty string of length 0

2.4 Unambiguous Expressions

Definition: We say that the expression AB is ambiguous if there exists distinct pairs (a_1, b_1) and (a_2, b_2) in $A \times B$ with $a_1 b_1 = a_2 b_2$ - otherwise we say that AB is an unambigious expression.

If A and B are finite sets, then AB is unambiguous if and only if $|AB| = |A \times B|$

2.5 Decomposition Rules

Basic string decompositions, all are unambiguous

- Decompose a string after each 0 or 1

$$S = \{0,1\}^*$$

- Decompose a string after each occurence of 0. Each piece in the decomposition will be from {0,10,110,...} = $\{1\}^*\{0\}$ except possibly for the last piece which may consist only of 1s. This gives rise to the expression:

$$S = (\{1\}^*\{0\})^*\{1\}^*$$

- Decompose a string after each block of 0s. Each piece in the decomposition, except possibly the first and last pieces, will constist of a block of 1s followed by a block of 0s. The first piece may consist only of a block of 0s and the last piece may consist only of a block of 1s. This gives rise to the expression:

$$S = \{0\}^*(\{1\}\{1\}^*\{0\}\{0\}^*)^*\{1\}^*$$

2.6 Sum and Product Rules for Strings

<u>**2.6.1 Theorem:**</u> Let A,B be a set of {0,1}-strings.

1. If $A \cap B = \emptyset$ then

$$Phi_{A \cup B}(x) = \Phi_A(x) + \Phi_B(x)$$

2. If the expression AB is unambiguous then:

$$\Phi_{AB}(X) = \Phi_A(x)\Phi_B(x)$$

3. If the expression A^* is unambiguous then:

$$\Phi_{A^*}(x) = (1 - \Phi_A(X))^{-1}$$

2.8 Recursive Decompositions of Binary Strings

All strings can be written as:

- The empty string is in S
- Any other element of S consists of a symbol (either 0 or 1) followed by an element of S

Recursive definition leads directly into the recusrive decomposition:

$$S = \{\epsilon\} \cup \{0,1\}S$$

3 Recurrences, Binary Trees and Sorting

3.1 Coefficients of Rational Functions

<u>3.1.1 Lemma:</u> If f(x) is a polynomial of degree less than r, then there is a polynomial P(x) with degree less than r such that

$$[x^n]\frac{f(x)}{(1-\theta x)^r} = P(n)\theta^n$$

<u>3.1.2 Lemma:</u> Suppose f and g are polynomials with deg(f) < deg(g). If g(x) = $g_1(x)g_2(x)$ where g_1, g_2 are coprime, there are polynomials f_1, f_2 such that $deg(f_i) < deg(g_i)$ for i = 1,2 and

$$\frac{f(x)}{g(x)} = \frac{f_1(x)}{g_1(x)} + \frac{f_2(x)}{g_2(x)}$$

<u>3.1.3 Theorem:</u> Suppose f and g are polynomials such that deg(f) < deg(g). If for i = 1...k there are complex numbers θ_i and positive integers m_i such that

$$g(x) = \prod_i (1-\theta_i x)^{mi}$$

then there are polynomials P_i such that deg(P_i) < m_i and

$$[x^n]\frac{f(x)}{g(x)} = \sum_{i=1}^{k} P_i(n)\theta_i^n$$

3.2 Solutions to Recurrence Equations

<u>3.2.1 Theorem:</u> Let $C(x) = \sum_{n\geq 0} c_n x^n$ where the coefficients c_n satisfy the recurence:

$$c_n + q_1 c_{n-1} + ... + q_k c_{n-k} = 0$$

IF

$$g(x) := 1 + q_1 x + ... + q_k x^k$$

there is a polynomial f(x) with degree less than k such that:

$$C(x) = \frac{f(x)}{g(x)}$$

<u>3.2.2 theorem</u> Suppose $(c_n)_{n\geq 0}$ satisfies the recurrence equation (3.2.1) if the characteristic polynomial of this recurrence has root β_i whith multiplicity m_i, for i = 1...j, then the general solution to (3.2.1) is:

$$c_n = P_1(n)\beta_1^n + ... + P_j(n)\beta_j^n$$

where $P_i(n)$ is a polynomial in n with degree less than m_i and these polynomials are determined by the $c_0 ... c_{k-1}$.

3.3 Nonhomogeneous Recurrence Equations

<u>3.3.1 Theorem:</u> Suppose that $a_0, a_1, ...$ is a solution to (3.3.1)(any solution without checking the initial conditions). Then the general solution to (3.3.1) is given by:

$$b_n = a_n + c_n$$

where c_n is given by theorem 3.2.2 and the k constants $b_{11} ... b_{j,m_j}$ in c_n can be chosent to fit the initial conditions for b_n

3.6 Binary Trees

<u>Definition</u> A binary tree is either:

- the empty tree
- a tree with a fixed root vertex such that each verte has a left branch and a right branch (either of which may be empty)

<u>3.6.2 Theorem:</u> The number of binary trees with $n \geq 0$ vertices is:

$$\frac{1}{n+1}\binom{2n}{n}$$

3.7 The Binomial Series

<u>3.7.1 Theorem the Binomial Series</u>: For any rational number a:

$$(1+x)^a = \sum_{k\geq 0}\binom{a}{k}a^k$$

$$(1+x)^{-n} = \sum_{k\geq 0} \binom{-n}{n} (-x)^k = \sum_{k\geq 0} \binom{n+k-1}{n-1} x^k$$

<u>3.7.2 Lemma:</u>

$$(1-4x)^{\frac{1}{2}} = 1 - 2\sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n$$

4 Introduction to Graph Theory

4.1 Definitions

<u>4.1.1 Definition</u>: A graph G is a finite non-empty set, V(G) of objects, called vertices, together with a set E(G), of unordered pairs of distinct vertices. The elements of E(G) are called edges.
More definitions!

- Adjacent if an edge connects two vertices
- Incident is the edge between two adjacent vertices (or the edge joins)
- Degree of vertex u is the number of vertices adjacent with u

4.2 Isomorphism

<u>4.2.1 Definition</u>: Two graphs G_1 and G_2 are isomorphic if there exists a bijection f : V(G_1) $\rightarrow$ V(G$_2$) such that vertices f(u) and f(v) are adjacent in G_2 if and only if u and v are adjacent in G_1.

4.3 Degree

<u>4.3.1 Theorem:</u> For any graph G we have:

$$\sum_{v\in V(G)} deg(v) = 2|E(G)|$$

<u>4.3.2 Corollary:</u> The number of vertices of odd degree is even

4.3.3 Corollary: The average degree of a vertex in the graph G is:

$$\frac{2|E(G)|}{|V(G)|}$$

4.3.4 Definition: A complete graph is one in which all pairs of distinct vertices are adjacent. (Thus each vertex is joined to every other vertex). The complete graph with p vertices is denoted by K_p $p \geq 1$

4.4 Bipartite Graphs

Definition: A grpah in which the vertices can be partitioned into two sets A and B, so that all edges join a vertex in A to a vertex in B is called a bipartite graph.
4.4.1 Definition: For $n \geq 0$ the n-cube is the graph whose vertices are the {0,1}-strings of length n, and two strings are adjacent if and only if they differe in exactly one position.

4.5.1 Defintion Adjacency Matrix (I DONT HTINK WE DID THIS)

4.5.2 Definition Incidence Matrix (I DONT THINK WE DID THIS??)

4.6 Paths and Cycles

4.6.1 Definition: A subgraph of a graph G is a graph whose vertex set is a subset U of V(G) and whose edge set is a subset of those edges in G that have both vertices in U

Definitions!

- Spanning graph of G if a subgraph has all of the vertexes in G
- A proper-subgraph of G if the subgraph is not equal to G
- A walk in a graph G from v_0 to v_n $n \geq 0$ is an alternating sequence of vertices and edges in G. The length of a walk is the number of edges in it.

- A closed walk is when the walk starts and ends at the same point
- A path is a walk in which all vertices are distinct

4.6.2 Theorem: If there is a walk from vertex x to vertex y in Gm then there is a path from x to y in G.

4.6.3 Corollary: Let x,y,z be vertices of G. If there is a path from x to y in G and a path from y to z in G then there is a path from x to z in G.

Hamilton Cycle: A spanning cycle in graph.

4.8 Connectedness

4.8.1 Definition: A graph G is connected if for each two vertices x and y there is a path from x to y.

4.8.2 Theorem Let G be a graph and let v be a vertex in G. If for each vertex w in G there is a path from v to w in G, then G is connected

4.8.4 Definition A component of G is a subgraph C of G such that

- C is connected
- No subgraph of G that properly contains C is connected

4.8.5 Theorem A graph G is not connected if and only if there exists a proper nonempty subset X of V(G) such that the cut induced by X is empty.

4.9 Bridges

4.9.1 Definition: An edge e of G is a bridge if G-e has more components than G.

4.9.2 Lemma: If e = {x,y} is a bridge of a connected graph G, then G-e has precisely two components; furthermore, x and y are in different components.

4.9.3 Theorem: An edge e is a bridge of a graph G if and only if it is not contained in any cycle of G.

4.9.4 Corollary: If there are two distinct paths from vertex u to vertex v in G then G contains a cycle

4.10 (NOT IN BOOK) Euler Tours

Proofs to the below theorems are provided due to them not being covered in the book. I got these proofs from the following source:
https://www.youtube.com/watch?v=1V_6nUUNoms

4.10.1 Definition: An Euler Path (or trail) is a path that visits every edge in a graph exactly once
4.10.2 Definition: An Euler Tour is an Euler Path that is closed.
4.10.3 Definition: A graph is Eulerian if it contains an Euler tour

4.10.4 Theorem: A connected graph G is Eularian if and only if every vertex of G has even degree
4.10.4 Proof:
$\implies$ Suppose G is a connected Eularian Graph
Let w : u $\xrightarrow{*}$ u be an Euler tour.
Let u $\neq$ v of G that occurs k times in w
Since we need to enter and exit v every time we pass it degree(v) = 2k
we also know that every time we pass through u we need to enter and exit, and we also need a start and end point at u, so this has even degree as well

$\impliedby$ Let G be a non-trivial connected graph whose vertices all have even degree.

Let w be a longest trail in the graph.
w: $v_0 e_1 v_1 e_2 ... e_{i-1} v_{i-1} e_i v_i$
$v_0 \xrightarrow{i} v_i$

Since all edges with v_i are used in w_i and v_i has an even degree, $v_i = v_0$
Otherwise, w could be extended to a longer trail.

If $v_i \neq v_0$ and v_i occurs k times in w, then deg(v_i) = 2(k-1) + 1 which is odd (contradiction)
Thus, w is a closed path.
Suppose w is not an Euler Tour. Since G is not connected, there is an edge f = $\{v_q, u\} \in E(G)$ and $f \notin E(W)$
But if we add f onto the graph, then f must be on the walk, which is a contradiction.

<u>4.10.5 Theorem:</u> A connected graph has an Euler trail if and only if it has a tmost 2 vertices of odd degree.

$\Longrightarrow$ If G has an Euler Trail u $\xrightarrow{*}$ v then as in the proof of the pverious theorem, all $w \notin \{u, v\}$ has an even degree

$\Longleftarrow$

- If there are no vertices which have odd degree then there is an Euler Tour
- If there are two vertices of odd degree:
  Let u,v have odd degree in G, if we add some vertex w, and new edges {u,w} and {v,w} the new graph H has vertices which all have even degree, so H has an Euler tour. If we take the section of the euler tour which does not include w, we have an Euler path that starts with u and ends with v

5 Trees

5.1 Trees

<u>5.1.1 Definition:</u> A tree is a connected graph with no cycles.
<u>Definition:</u> A forest is a graph with no cycles

<u>5.1.2 Lemma:</u> there is a unique path between every pair of vertices u and v in a tree T.

5.1.3 Lemma: Every edge of a tree T is a bridge.

5.1.4 Theorem: A tree with at least two vertices has at least two vertices of degree one.

5.1.5 Theorem: If T is a tree, then $|E(T)| = |V(T)| - 1$

Definition: A leaf in a tree is a vertex of degree 1
Theorem: Every tree with at least two vertices has at least two leaves

Theorem: A tree is bipartite

5.2 Spanning Trees

Definition: T is a spanning tree of G if T is a subgraph of G that is a tree and uses every vertex in G

5.2.1 Theorem: A graph G is connected if and only if it has a spanning tree

5.2.2 Corollary: If G is connected, with p vertices and q = p - 1 edges, then G is a tree.

Theorem: Let G be a graph with n vertices. If any of the two following points are true then G is a tree:

- G is connected
- G has no cycles
- G has n-1 edges

Theorem: If T is a spanning tree of G and e is an edge in E(G)/E(T) then T + e contains exactly one cycle C. Moreover, if e' is any edge in C then T + e - e' is also a spanning tree of G.

5.3 Characterizing Bipartite Graphs

5.3.1 Lemma: An odd cycle is not bipartite

5.3.2 Theorem: A graph is bipartite if and only if it has no odd cycles

5.3.3 Lemma: Let T be a spanning tree in a connected graph G. If G is not bipartite, then there is an odd cycle in G that uses exactly one edge not from T.

7 Planar Graphs

7.1 Planarity

7.1.1 Definition: A graph G is planar if it has a drawing in the plan so that its edges intersect only at their ends, and so that no two vertices coincide. The actual drawing is called a planar embedding of G or a planar map

Defnition: A face of a planar embedding is an area surrounded by edges. The degree of a face is the number of edges that surround it.

7.1.2 Theorem: If we have a planar embedding of a connected graph G with faces $f_1, ...f_s$ then

$$\sum_{i=1}^{s} deg(f_i) = 2|E(G)|$$

(Handshaking Lemma for Faces
)

7.1.3 Corollary: if the connected graph G has a planar embedding with f faces, the average degree of a face in the embeddings is $\frac{2|E(G)|}{f}$

Jordan Curve Theorem: Every simple closed curve on the plane, seperates the plane into two parts

7.2 Euler's Formula

<u>7.2.1 Theorem:</u> (Euler's Formula) Let G be a connected graph with p vertices and q edges. If G has a planar embedding with f faces, then:

$$p - q + f = 2$$

7.4 Platonic Solids

<u>Definition:</u> A connected graph is platonic if it has an embedding where every vertex has the same degree(> 3)

<u>7.4.1 Theorem:</u> There are exactly five platonic graphs

<u>7.4.2 Lemma:</u> Let G be a planar embedding with p vertices and q edges and s faces, in which each vertex has degree ≥ 3 and each face has degree $d^* \geq 3$. Then (d,d^*) is one of the five pairs:
{(3,3),(3,4),(4,3),(3,5),(5,3)}

<u>7.4.3 Lemma:</u> If G is a platonic graph with p vertices and q edges and f faces, where each vetex has degree d and each face degree d^* then:

$$q = \frac{2dd^*}{2d + 2d^* - dd^*}$$

and $p = 2q/d$ and $f = 2q/d^*$

7.5 Nonplanar Graphs

<u>7.5.1 Lemma:</u> If G is connected and not a tree, then in a planar embedding of G, the boundary of each face contains a cycle

<u>7.5.2 Lemma:</u> Let G be a planar embedding with p vertices and q edges. If each face of G has degree at least d^* then $(d^* - 2)q \leq d^*(p - 2)$.

<u>7.5.3 Theorem:</u> In a connected planar graph with $p \geq 3$ vertices and q edges we have:

$$q \leq 3p - 6$$

7.5.4 Corollary: K_5 is not planar

7.5.5 Corollary: A planar graph has a vertex of degree at most five.

7.5.6 Lemma: $K_{3,3}$ is not planar

7.6 Kuratowski's Theorem:

Definition: An edge subdivision of a graph G is obtained by applying the following operation, independentyl, to each edge of G: replace the edge by a path of legnth 1 or more; if the path has length m > 1, then there are m-1 new vertices and m-1 new edges created; if the path has length m = 1, then the edge is unchanged.

7.6.1 Theorem: A graph is not planar if and only if it has a subgraph that is an edge subdivision of K_5 or $K_{3,3}$

Colouring and Planar Graphs

7.7.1 Definition: A k-colouring of a graph G is a function from V(G) to a set of size k (whose elements are called colours), so that adjacent vertices always have different colours. A graph with k-colouring is called a k-colourable graph.

Theorem: If a graph
7.7.2 Theorem: A graph is 2-colourable if and only if it is bipartite

7.7.3 Theorem: K_n is n-colourable and not k-colourable for any $K < n$.

7.7.4 Theorem: Every planar graph is 6-colourable

7.7.5 Definition: Let G be a graph and let e = {x,y} be an edge of G. The graph G/e obtained from G by contracting the edge e is the graph with the vertex set V(G)/{x,y} ∪ {z} where z is a new vertex and the edge set:

$\{\{u,v\} \in E(G) : \{u,v\} \cap \{x,y\} = \emptyset\} : u \notin \{x,y\}, \{u,w\} \in E(G) \exists w \in \{x,y\}\}$

7.7.6 Theorem: Every planar graph is 5-colourable

7.7.7 Theorem: Every planar graph is 4-colourable

7.8 Dual Planar Maps

Definition: Let G be a planar graph with an embedding. The dual G* of the embedding has one vertex v_f corresponding to each face f of G, and for each edge in G whose two sides are f_1, f_2, G* has a corresponding edge v_{f_1}, v_{f_2}

Properties of Duality:

- If G is planar then G* is also planar
- (G*)* = G
- - # of faces in G = # of vertices in G*
  # of vertices in G = # of faces in G*
  # of edges in G = # of edges in G*
- The degree of a vertex in G is the degree of the corresponding face in G*
- The dual of a platonic graph is platonic

Theorem: The dual of a Eularian planar graph is bipartite

8 Matchings

8.1 Matching

Definition: A matching of a graph G is a set of M edges of G such that no two edges in M have a common end.

Definitions: We say that a vertex v of G is saturated by M, or that M saturates v if v is incident with an edge in M.
Definition: Maximum Matching is the largest amount of matchings in a graph G. A perfect matching is when the number of matchings is with the number of vertices p, # of matchings = p/2

Definition: If we have a matching M of G, we say that a path $v_0v_1v_2...v_n$ is an alternating path with respect to M if one of the following is true:

- $\{v_i, v_{i+1}\}\in$ M if i is even and $\{v_i, v_{i+1}\} \notin M$ if i is odd
- $\{v_i, v_{i+1}\}\notin$ M if i is even and $\{v_i, v_{i+1}\} \in M$ if i is odd

Definition: An augmenting path with respect to M is an alternating path joining two distinct vertices neither of which is saturated by M.
8.1.1 Lemma If M has an augmenting path, it is not a maximum matching.

8.1.1 Lemma(part2) If there is no augmenting path with respect to M then M is a maximum matching.

8.2 Covers

Defintion: A vertex cover C of a graph G is a set of vertices such that each edge in G has at least one end in C

8.2.1 Lemma If M is a matching of G and C is a cover of G then $|M| \leq |C|$

8.2.2 Lemma IF M is a matching and C is a cover and $|M| = |C|$ then M is a maximum matching and C is a minimum cover

8.3 Konig's Theorem

8.3.1 Theorem(Konig's Theorem) In a bipartite graph the maxmimum size of a matching is the minimum size of a cover

XY-Construction Find all possible alternating paths starting at an unsaturated vertex in A. Any Augumenting path that starts in A must end in B

8.3.2 Lemma Let M be a matching of a bipartite graph G with bipartition A,B and let X and Y are sets of an XY construction.

1. There is no edge of G from X to B/Y
2. C = $Y \cup (A/X)$ is a cover of G
3. There is no edge of M from Y to A/X
4. $|M| = |C| - |U|$ where U is the set of unsaturated vertices in Y
5. There is an augumenting paht to each vertex in U

8.4 Applications of Konig's Theorem

8.4.1 Theorem(Hall's Theorem) A bipartite graph G with bipartition A, B has a matching saturating every vertex in A, if and only if every subset D of A satisfies:

$$|N(D)| \geq |D|$$

where N(D) are the neightbour set of D

Corollary If G is a K-regular bipartite graph with k$\geq$ 1 then G has a perfect matching