M239 - Lecture 6

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May 15th, 2015

Power Series

Multiplication

$$A(x)B(x) = \sum_{n\geq 0}\sum_{i=0} n([x^i]A(x)[x^{n-i}]B(x))x^n$$
$$[x^n]x^kA(x) = [x^{n-k}]A(x) \ when \ k \leq n$$
$$[x^n]x^kA(x) = 0 \ when \ k > n$$

Example:

$$A(x) = (1+2x)^2$$
$$B(x) = 1 + 2x + 4x^2 + ... = \sum_{i\geq 0} 2^i x^i$$
$$[x^n]A(x)B(x) = [x^n](1+4x+4x^2)B(x)$$
$$= [x^n]B(x) + [x^n]4xB(x) = [x^n]4x^2B(x)$$
$$= 2^n + 4[x^{n-1}]B(x) + 4[x^{n-2}]B(x)$$
$$= 2^n + 4\cdot 2^{n-1} + 4\cdot 2^{n-2} = 4\cdot 2^n = 2^{n+2}$$
$$= 2^n + 2\cdot 2^n + 2^n$$

This works for $n \geq 2$ for $n = 1$, we have $1 + 6x$ (do them seperately). So $A(x)B(x) = 1 + 6x + \sum_{n\geq 2} 2^{n+2}x^n$

Inverses

$$\frac{1}{A(x)} = B(x)$$

Definition: The <u>inverse</u> of A(x) is a power series B(x) such that $A(x)B(x) = 1$
Let B(d) be the inverse of 1-x. Let $B(x) = \sum_{i\geq 0} b_i x^i$
We Want $B(x)(1-x) = 1$

$$1 = B(x)(1-x)$$
$$= B(x) - xB(x)$$

$$= b_0 + b_1x + b_2x^2 + ... - b_0x - b_1x^2 - ...$$

$$= b_0 + (b_1 - b_0)x + (b_2 - b_1)x^2 + ...$$

This equals to 1. By comparing coefficients, we get $b_0 = 1, b_1 - b_0 = 0, b_2 - b_1 = 0$
$\implies b_1 = 1, b_2 = 1...$
So $B(x) = 1 + x + x^2 + x^3... = \frac{1}{1-x}$

Let C(x) be the inverse of x. $C(x) = \sum_{i \geq 0} c_i x^i$
Want C(x)x = 1, $1 = c_0x + c_1x^2 + c_2x^3 + ...$
So x does not have an inverse because there is no constant term on the right to balance out the constant term on the left.
Never do $\frac{1}{x}$

Theorem A(x) has an inverse if and only if the constant term of A(x) is not 0.

Common Series

1. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{i \geq 0} x^i$ Geometric Series
2. $1 + x + x^2 + ... + x^k = \frac{1-x^{k+1}}{1-x}$
3. $\frac{1}{(1-x)^k} = \sum_{n \geq 0} (\frac{n+k-1}{k-1}) x^n$

Compositions

Let $G(x) = \frac{1}{1-x} = 1 + x + x^2 + ...$
$G(3x^2) = 1 + 3x^2 + 9x^4 + 27x^6 + ... = \sum_{i \geq 0} (3x^2)^i$