Math 239 - Lecture 9

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Integer Compositions

n=5
(1,2,3), (4,1)

How many compositions of n have k parts?

$$N \times N \times ... \times N = N^k$$

$$w(a_1, a_2, ...a_k) = a_1 + a_2 + ... + a_k$$

$$\Phi_{N^k}(x) = (\Phi_N(x))^k = (\frac{x}{1-x})^k$$

Our answer is $[x^n]\Phi_{N^k}(x) = [x^n]\frac{x^j}{(1-x)^k}$
$= \binom{(n-k)+k-1}{k-1} = \binom{n-1}{k-1}$

Combinatorial Interpretation

00|00|....|0
n 0's
k-1 1's

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1's have to be in between 0's, no duplicates in the same spot.

There are n-1 spots to put k-1 1's $\implies \binom{n-1}{k-1}$

Example

How many compositions of n have 2k parts, where the first k parts are at least 5 and hte last k parts are multiples of 3?
Define A = {5,6,7,...}, B={3,6,9,12...}
No 0 because parts are ≥ 1

$A^k \times B^k$ enumerates all comps of our property (Cartesian product)
$A^k \times B^k = \{(a_1, ...a_k, b_1...b_k) | a_i \in A \ b_i \in B\}$

Using w(a) = a for A, B

$$\Phi_A(x) = x^5 + x^6 + x^7 + ... = \frac{x^5}{1-x}$$

$$\Phi_B(x) = x^3 + x^6 + x^9 + ... = \frac{x^3}{1-x^3}$$

By Product Lemma,

$$\Phi_{A^k \times B^k}(X) = (\Phi_A(x))^k (\Phi_B(x))^k$$

$$= \frac{x^{5k}}{(1-x)^k} \cdot \frac{x^{3k}}{(1-x^3)^k}$$

$$= \frac{x^{8k}}{(1-x)^k (1-x^3)^k}$$

Our answer is:

$$= [x^n] \frac{x^{8k}}{(1-x)^k (1-x^3)^k}$$

Explicit Formula:

$$= [x^{n-8k}] \frac{1}{(1-x)^k (1-x^3)^k}$$

$$= [x^{n-8k}] (\sum_{m \geq 0} \binom{m+k-1}{k-1} x^m)(\sum_{p \geq 0} \binom{p+k-1}{k-1} x^{3p})$$

$$= [x^{n-8k}] (\sum_{m \geq 0} \sum_{p \geq 0} \binom{m+k-1}{k-1} \binom{p+k-1}{k-1} x^{m+3p}$$

$$= \sum_{(m,p) \in N_0 \times N | m+3p = n-8k} \binom{m+k-1}{k-1} \binom{p+k-1}{k-1}$$

General Method

How many compositinos of n have certain properties?

1. Define a set S of all compositions which satisfy these properties (disregard n)
2. Find $\Phi_S(X)$ with weight function being the sum of all parts
3. Answer is $[x^n]\Phi_S(x)$

Example

How many compositions of n are there? (number of parts not fixed)
n = 2 (2), (1,1)
n = 3 (3), (1,2), (2,1), (1,1,1)
n = 4 (4), (1,3), (2,2), (3,1) (1,1,2), (1,2,1), (2,1,1), (1,1,1,1)

Partition the set of all compositions according to the number of parts (elements)

The set of all compositions $S = N^0 \cup N^1 \cup N^2 \cup N^3 ... = \bigcup_{k \geq 0} N^k$
This is a disjoint union, so we can use the sum lemma. The weight function of a composition is the sum of al parts.

We have $\Phi_{N^k})x_{=}(\frac{x}{1-x})^k$
By Sum Lemma:

$$\Phi_s(x) = \sum_{k \geq 0} \Phi_{N^k}(x) = \sum_{k \geq 0} (\frac{x}{1-x})^k$$

$$= \frac{1}{(1 - \frac{x}{1-x})}$$

We can do this since the constant term of $\frac{x}{1-x}$ is 0.