Math 239 - Lecture 3

Graham Cooper

May 8th, 2015

More on Bijections

S = subsets of [n]
T = bin str of length n
$f : T \to S$ bijection
f(11010) = {1,2,4}

f implies that $|S| = |T|$. $|T| = 2^n$ (n bits, 2 choices)
So $|S| = 2^n$ each element of [n] is either in or out of the subset

Proving Bijections

For this course you need:

- clear definitions of $f : A \to B$
- show that $f(x) \in B$ for any $x \in A$
- Define the inverse $f^{-1} : B \to A$

Combinatorial Proofs

Binomial Theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Prove by counting.

$$(1+x)^n = (1+x)(1+x)...(1+x)$$

Each term in the expansion is a product of n things, one from each bracket. $(1+x)^n$ is the sum of all such terms. Each term as the form $a_1 \cdot a_2 ... a_n$ where each a_i is either 1 or x. This gives x^k when k of the a_i's are x's, nk of the a_i's are 1's. There are $\binom{n}{k}$ ways to do so. So the coefficient of x^k in $(1+x)^n$

is $\binom{n}{k}$. This proves the binomial theorem.

Example: Plug in x = 1 into the binomial theorem, we get:

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

$$2^3 = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$$

Combinatorial Proof: Let S be all binary strings of length n.

So $|S| = 2^n$ Let S_k be the set of binary strings of length n with k 1's. Then $S = S_0 \cup S_1 \cup ... \cup S_n$ is a disjoint union. (each string has 0,1 r n 1's, and $S_i \cap S_j = \theta$ for $i \neq j$). We know $|S_k| = \binom{n}{k}$ (n bits, choose k to be 1s). So $|S| = |S_0| + |S_1| + ... + |S_n|$ and $2^n = \binom{n}{0} + \binom{n}{1} + ... + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k}$.

General: Given S, count $|S|$ in 2 different ways. Since $|S|$ is fixed, the two ways are equal.
(IMAGE OF PASCAL'S TRIANGLE)

Identity: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ where $1 \leq k \leq n-1$

Combinatorial Proof: Let S b the set of all subsets of $[n]$ of size k. Then $|S| = \binom{n}{k}$. Partition S into 2 sets S_1, S_2 where
S_1 are subsets of $[n]$ of size k that include element n.
S_1 are subsets of $[n]$ of size k that do not have element n.

$$n = 5$$

$$k = 3$$

$$S = subsets of \{1,2,3,4,5\} of size 3$$

$$S_1 = \{\{1,2,5\},\{1,3,5\},\{1,4,5\},\{2,3,5\},\{2,4,5\},\{3,4,5\}\}$$

$$S_2 = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$$

Then $S = S_1 \cup S_2$ is a disjoint union, and $|S| = |S_1| + |S_2|$
Each element of S_1 consists of n together with a subset of $[n-1]$ of size k -1
So $|S_1| = \binom{n-1}{k-1}$

Each element of $S_2 = \binom{n-1}{k}$
$\implies \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$