Math 239 Graph Theory Strategies

Isomorphic Graphs:

How to show 2 graphs are isomorphic?

1.) Exhibit an isomorphism... can you find one right away?

2.) If not, then find a substructure in one of the graphs that do not exist in another!

ie.) length of cycles, degree of vertexes, length of paths etc (a6p3)

Bipartite Graphs:

How to show a graph G is bipartite?

Try to partition the graph G into two different sets A and B. Show that every edge is incident to an element in A and an element in B if they're connected. (a7p2b)

Connected Graphs:

How do you tell if a graph is connected?

A: To show a graph is connected: Use the definition or find a vertex v and show that for every $x \in V(G)$there exists a path to v.

To show that a graph is not connected: find a proper subset x that induces an empty set (a7p5)

Properties and Lemmas:

Lemma: Let G be a graph. Suppose there is a walk in G from x to y. Then there is a path in G from x to y.

Lemma: Let G be a graph with p≥1 vertices. Then G is connected iff there exists a vertex v such that for all $x \in V(G)$ there is a path in G from x to v.

Lemma: Let G be a graph. Then G is connected iff every proper subset $x \subseteq V(G)$ the cut induced by x is non-empty.

Lemma: Let G be a connected graph with a bridge e=xy. Then G-e has exactly 2 components, $C_x\ containing\ x\ and\ C_y\ containing\ y$

Lemma: An edge e is a bridge of the connected graph G iff e is not in a cycle in G.

Trees:

How do you prove that a graph is a tree?

A: Show that the graph is connected and also show that G has no cycles. (a8p2)

To show that it's not a Tree, disprove the above or show that there's an edge that is not a bridge or a path of length greater than 1.

Properties and Lemmas:

Lemma: Every tree with p vertices has p-1 edges:

Lemma: Let T be a tree with $p \geq 2\ vertices. For\ 1 \leq i \leq p$ let n decide the number of vertices of T with degree i. Then: $n = 2 + n_3 + \ 2n_4 + \cdots (p-2)n_p$

Spanning Trees:

How do you show a graph has a spanning tree?

A: Show that the graph is connected.

Properties and Lemmas:

Lemma: Let G be a graph. Then G is a spanning tree iff G is connected.

Lemma: If G is a connected graph with p vertices and p-1 edges, then it is a tree.

Lemma: A graph G is bipartite iff it does not contain an odd cycle.

Breadth-First Search Trees:

How do you create a BFST?

A: Follow the algorithm! (a8p4)

Properties and Lemmas:

Lemma: Let T be a BFST in G. Then every edge of G either joins 2 vertices on the same level or two vertices of consecutive levels.

Lemma: Let T be a BFST in a connected graph G. Then G is bipartite iff there is no edge of G joining two vertices in the same level of T.

Lemma: Let G be a connected graph. The length of the shortest path from x to y in G is the level k of y in a BFST in G rooted at x.

Planar Graphs:

How do you tell if a graph is planar?

A: You must exhibit a planar drawing. TIP: Look for a long cycle. Draw it in the plane then the rest of the vertices and edges must either go inside the cycle or outside!

ASK: is $q \leq 3p - 6 \;\; or \; q(k-2) \leq k(p-2)$ where q is the number of edges, p is the number of vertices, and k is the girth(shortest cycle) of the graph. If any of these simple tests fail, then the graph is not planar. If you can find a subgraph that is a subdivision of $K_5 or\ K_{3,3}$ then G is not planar. (a8p5 and a9p4)

Properties and Lemmas:

Face Shake Lemma: Let G be a connected planar graph. Let F(G) be the set of faces of G then

$$\sum_{f \varepsilon F(G)} deg(f) \;=\; 2|E(G)|$$

Euler's Formula: Let G be a connected planar graph.

Let p = |V(G)| and q = |E(G)| s = |F(G)| then p - q + s = 2

Lemma: Let G be a planar drawing of a connected graph. Let f be a face of G. If the boundary B(f) of f does not contain a cycle, then G is a tree.

Lemma: in any planar drawing of a connected planar graph that contains a cycle, every face has a cycle in its boundary.

Lemma: Let G be a planar graph with p vertices and q edges, where $p \geq 3\ then\ q \leq 3p - 6$

Lemma: If G is a planar graph of girth k then $q(k-2) \leq k(p-2)$

Lemma: Every planar graph has a vertex of degree less than 5

Kuratowski's Theorem: A graph G is planar iff it does not contain a subdivision of $K_5 or\ K_{3,3}$

Colouring:

Given G, how do you show that G is k-colourable?

A: Proof by induction on the number of vertices.

Base Case: Any G with $p \leq k$ vertices is k-colourable.

Induction Hypothesis: Assume that $p > k\ and\ any\ p' < \ p\ \ is\ k - colourable$

Induction Step: Obtain a p' < p from removing an arbitrary vertex that would still maintain the properties and restrictions of G. (a9p5b)

Properties and Lemmas:

Four-Colour Theorem: Every planar graph is 4-colourable

Lemma: A graph is 2-colourable iff G is bipartite

Lemma: K_p is p-colourable but not (p-1)-colourable

Matching:

How do you find a maximum matching and a minimum cover in a bipartite graph?

A: Use the Bipartite Matching Algorithm! (a10p1)

How do you prove a graph has a k-perfect matching?

Proof by induction on k.

Base Case: k = 1

Induction Hypothesis: Assume the claim holds for any $k' < k$

Induction Step: Show that k' holds the claim. (a10p3)

How do you prove a graph has a perfect matching in general?

Proof by contradiction, contradict the definition of a perfect matching. (a10p5)

Use Hall's Theorem! (a10p7)

Properties and Lemmas:

A perfect matching in G has size |V(G)|/2 every vertex of G is incident to exactly one edge in the matching.

Lemma: Let m be a matching in G if there exists an m-augmenting path two, m is not a maximum matching.

Lemma: Let G be a graph. Let M be a matching in G, and C be a cover in G. Then

$$|M| \leq |C|$$

Lemma: Suppose M is a matching in G and C is a cover in G, and $|M| \leq |C|$ then M is a maximum matching and C is a minimum cover

Konig's Theorem: Let G be a bipartite graph. Let M be a maximum matching in G then there exists a cover C in G with |C|=|M|.

Bipartite Matching Algorithm:

aim: to find a maximum matching in any bipartite graph

-start with a bipartite graph G and a matching M in G

-construct the sets X and Y

-if we find $y \in Y$ that is M-exposed we set an augmenting path P and a bigger matching $M = \Delta E(p)$

-if we complete the construction of x and y without finding an augmenting path, then the matching M is maximum and we find a cover with |C|=|M|

Input: A bipartite graph G with vertex classes A and B

Output: A maximum matching M_0 in G.

Let M be any arbitrary matching in G.

1.) Let $\hat{x} = \{v \in A: v\ is\ M - exposed\}\hat{y} = 0$

2.) if there exists $v \in B - \hat{y}\ such\ that\ v \in E(G) for\ some\ u\ \in \hat{x}\ then\ add\ v\ to\ \hat{y}\ and\ set\ pr(u) = v$

3.) if no such v exists, then M is a maximum matching and set $M_0 = M\ and\ C = \hat{y} \cup (A \backslash \hat{x})$

4.) if v is m-exposed, then the pr function given an M-augmenting path p from v to an M-exposed vertex on $\hat{x}\ set\ M \coloneqq M \Delta E(p)$ Replace M by M and go back to 0.

5.) If there exists $w \in A \backslash \hat{x}\ and\ an\ edge\ wz \in M\ with\ z = \hat{y}, add\ w\ to\ \hat{x}.$

$$Set\ pr(w) = z\ go\ to\ 2$$

Halls Theorem: Let G be a bipartite graph with vertex classes A and B. Then G has a matching of size |A| iff $|N(S)| \geq |S| forall\ S \subseteq A$

Lemma: Let G be a bipartite graph with vertex classes A and B. Then G has a perfect matching iff $|N(S)| \geq |S| forall\ S \subseteq A\ and\ |A| = |B|$

Lemma: Given a bipartite graph G with at least one edge, there is a matching in G saturating all the vertices of maximum degree.

Lemma: Let G be a bipartite graph. Then G has a $\Delta(G) - edge - colouring$