Math 239 Lecture 15

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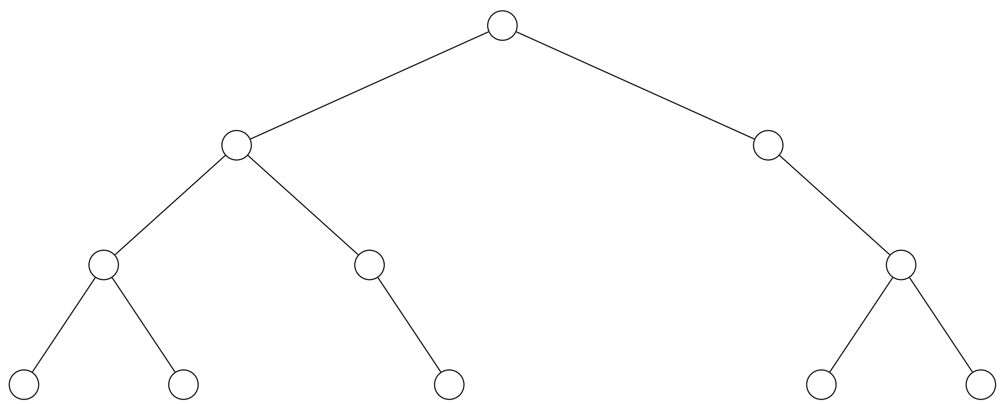
Counting Binary Trees

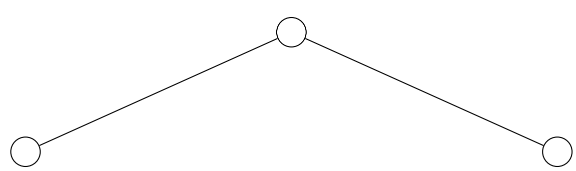


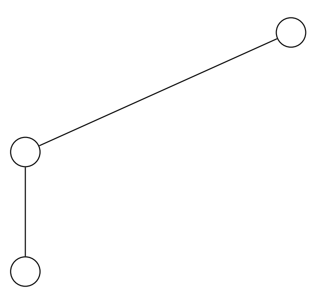
<u>**Definition:**</u> A binary tre is either:

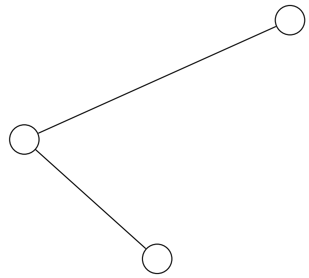
1. an emtpy tree ϵ
2. a root vertex with a left branch anda right branch ech of which is a binary tree

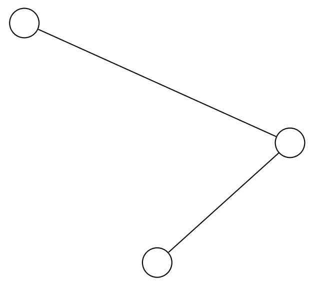
how many binary trees have n nodes?

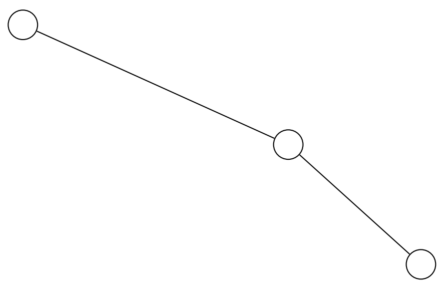
for 3 nodes:









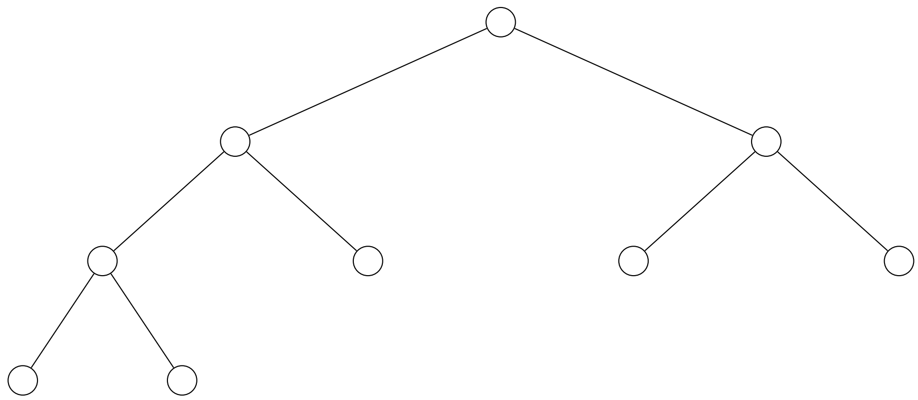


Let T be the set of all binary trees of any size. For each t $\in$ T, define weight functions w(t) to be the number of nodes in T. Let T(x) be the generating series fo T with respect to w. The answer to our question is $[x^n]t(x)$

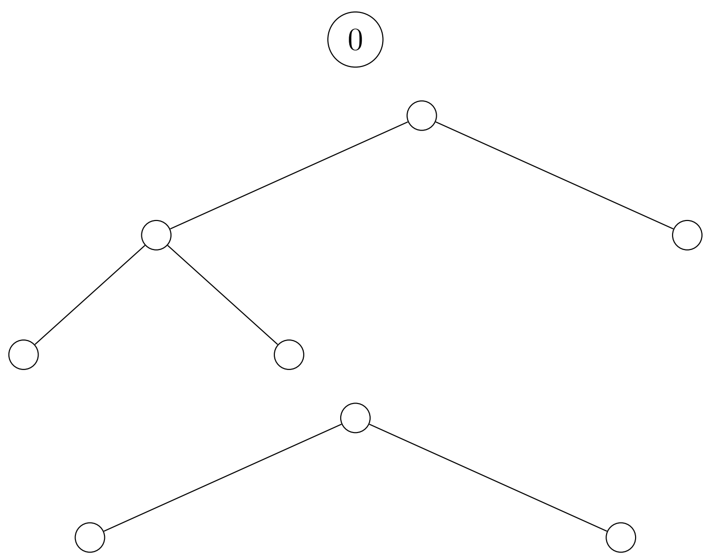
For any non-empty tree t, we can decompose it into 3 parts, root node r, the left tree t_1, the right tree t_2. t_1, $t_2 \in T$. This belongs to the cartesian product $r \times T \times T$

Let S = $\{\epsilon\} \cup \{r\} \times T \times T$

Then each tree t $\in$ T corresponds to one element of S in a bijection.



becomes



$$T \rightleftarrows (r, t_1, t_2)$$

$$w(r, t_1, t_2) = 1 + w(t_1) + w(t_2) = w(t)$$

We can appy product lemma

$$\Phi_S(x) = 1 + x \cdot t(x) \cdot t(x)$$

Due to the bijection t(x) = $\Phi_S(x)$

$$\implies t(x) = 1 + xt(x)^2$$

$$
\begin{aligned}
0 &= xt(X)^2 - t(x) + 1 \\
&= 4x(xt(x)^2 - t(x) + 1) \\
&= 4x^2t(x)^2 - 4xt(x) + 4x \\
&= (4x^2t(x)^2 - 4xt(x) + 1) - 1 + 4x \\
&= (2xt(x) - 1)^2 - 1 + 4x
\end{aligned}
$$

so:

$$(2xt(x) - 1)^2 = 1 - 4x = ((1 - 4x)^{1/2})^2$$

from A2:

$$
\begin{aligned}
2xt(x) - 1 &= +/-(1 - 4x)^{1/2} \\
&= +/-(1 - 2\sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n}x^{n+1})
\end{aligned}
$$

Constant term is -1 so we pick - over +, which means:

$$2xt(x) - 1 = -1 + 2\sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n}x^{n+1}$$

$$\implies t(x) = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n}x^{n}$$

So the number of binary trees iwth n nodes is $\frac{1}{n+!}\binom{2n}{n}$
(n = 3, $\frac{1}{4}\binom{6}{3} = \frac{1}{4}\frac{6*5*4}{3*2} = 5$)

Catalan number: $\frac{1}{n+1}\binom{2n}{n}$

Graph Theory

Definition: A graph G is a pair of sets (V, E) (or (V(G), E(G)) where V is a set of objects called vertices and E is a set of unordered pairs of V called edges.

Example: Define G = (V,E) where
V = {a,b,c,d}, E = {{a,b}, {b,c}, {c,d}, {a,d} }

Graphical Representation of G.
See notes.

In graph theory, we mainly care about the "structure" of the graphs, e.g. what are the vertices, which pairs for edges

Terminologies:

- Two vertices u,v are adjacent if {u,v} is an edge Example above, a is adjacent to b and d but not c
- If u is adjacent to v, then u is a neighbour of v, the set of all neighbours of u is the neighbourhood, denoted by N(v). Example: N(a) = {b,d}
- An edge, e = {u,v} is incident with u and v, e joins u and v