Module 8: Tries and String Matching

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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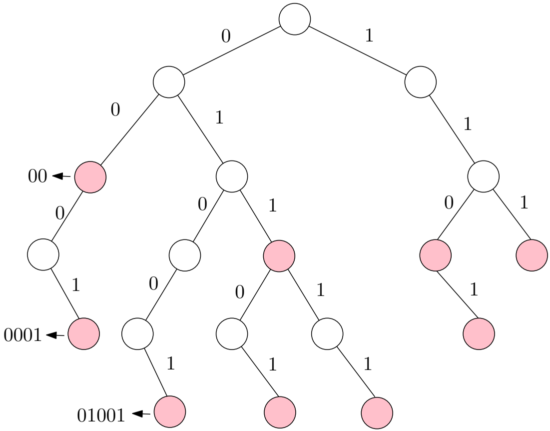
Tries

- Trie (Radix Tree): A dictionary for binary strings
  - Comes from retrieval, but pronounced "try"
  - A binary tree based on bitwise comparisons
  - Similar to radix sort: use individual bits, not the whole key
- Structure of trie:
  - A left child corresponds to a 0 bit
  - A right child corresponds to a 1 bit
- Keys can have different number of bits
- Keys are not stored in the trie: a node x is flagged if the path from root to x is a binary string present in the dictionary

Tries

- Example: A trie for
  $S = \{00, 0001, 01001, 011, 01101, 01111, 110, 1101, 111\}$



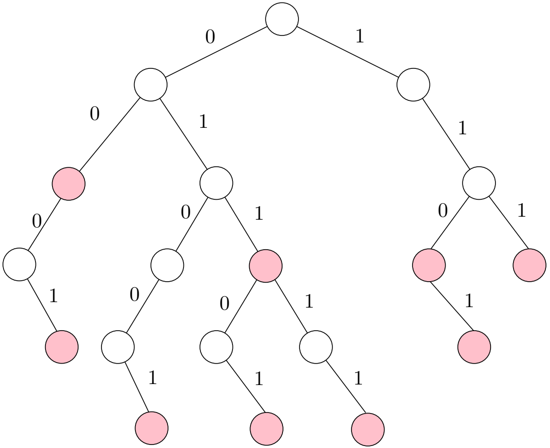
Tries: Search

Search(x):

- start from the root
- take the left link if the current bit in x is 0 and take the right link if it is 1 (return failure if the link is missing)
- if there are no extra bits in x left and the current node is flagged then - success (x is found)
- else, if the current node is a leaf, then - failure (x is missing)
- recurse

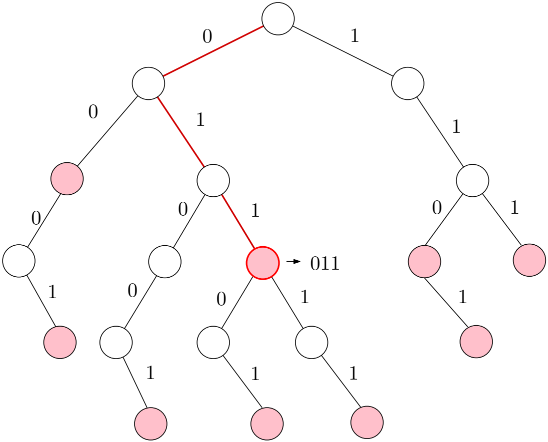
Tries: Search

Example: Search(011)



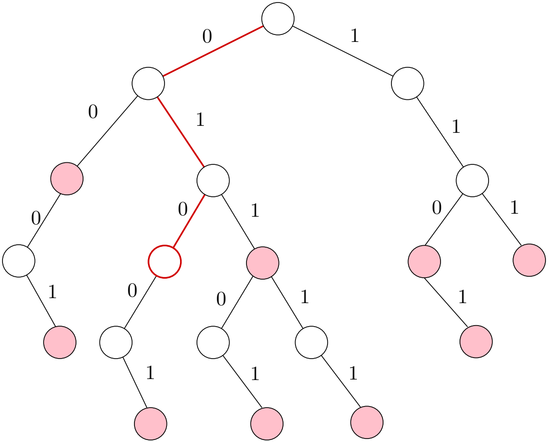
Tries: Search

Example: Search(011) successful



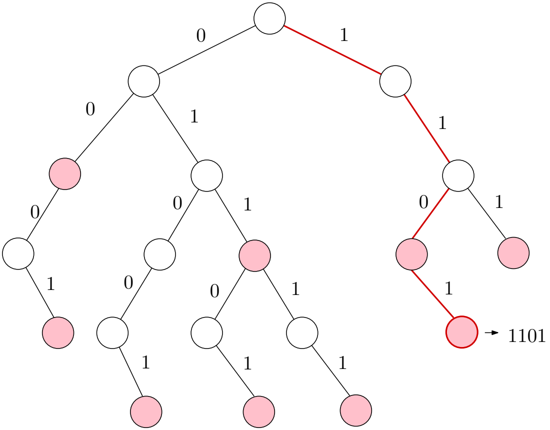
Tries: Search

Example: Search(0101) unsuccessful



Tries: Search

Example: Search(1101) successful

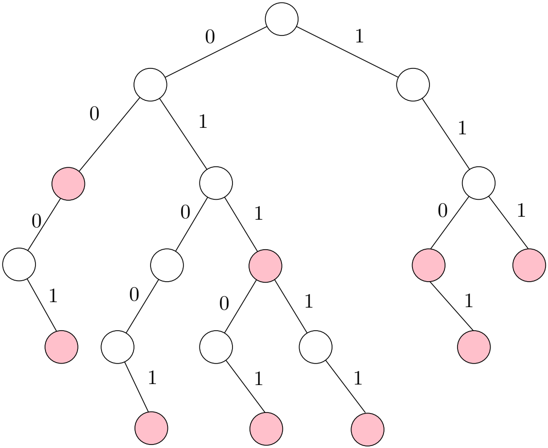


Tries: Insert

- Insert(x)
  - First search for x
  - If we finish at a leaf with key x, then x is already in trie: do nothing
  - If we finish at a leaf v and x has extra bits then flag v and expand the trie from the node v by adding necessary nodes that correspond to extra bits.
  - If we finish at an internal node and there are no extra bits: the node is then flagged
  - If we finish at an internal node and there are extra bits: expand trie by adding necessary nodes that correspond to extra bits

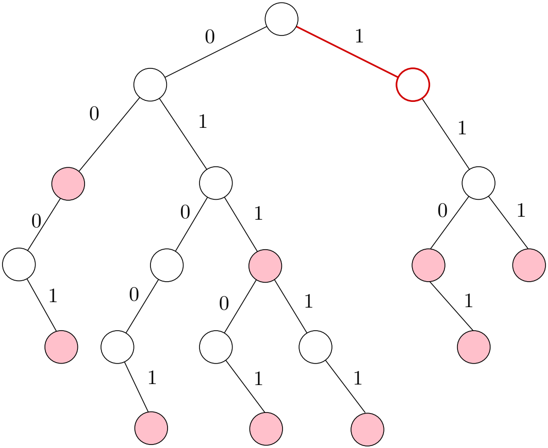
Tries: Insert

Example: Insert(101)



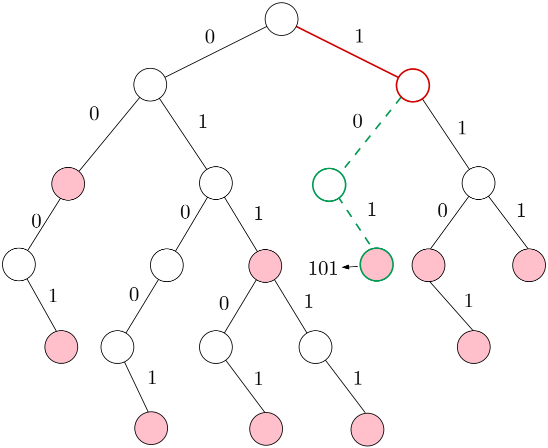
Tries: Insert

Example: Insert(101)



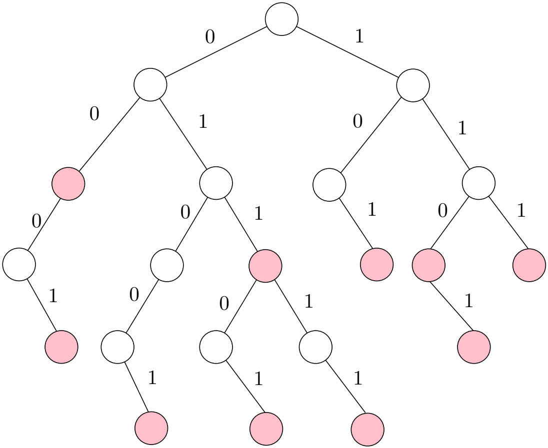
Tries: Insert

Example: Insert(101)



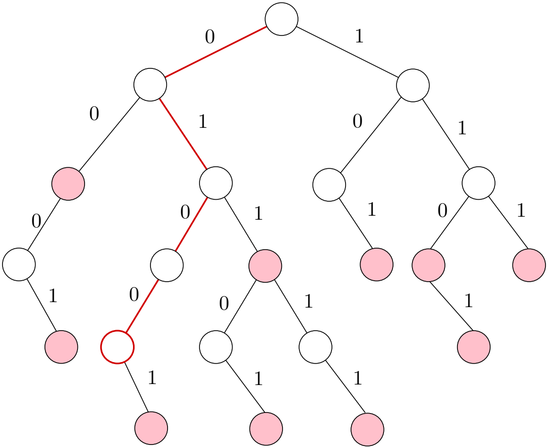
Tries: Insert

Example: Insert(0100)



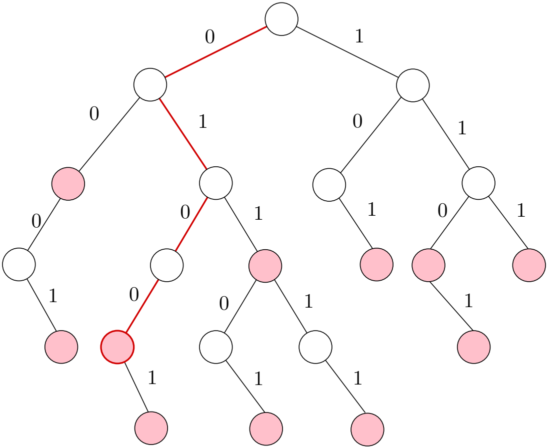
Tries: Insert

Example: Insert(0100)



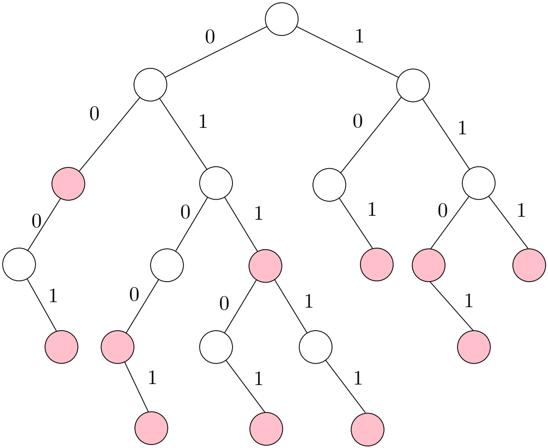
Tries: Insert

Example: Insert(0100)



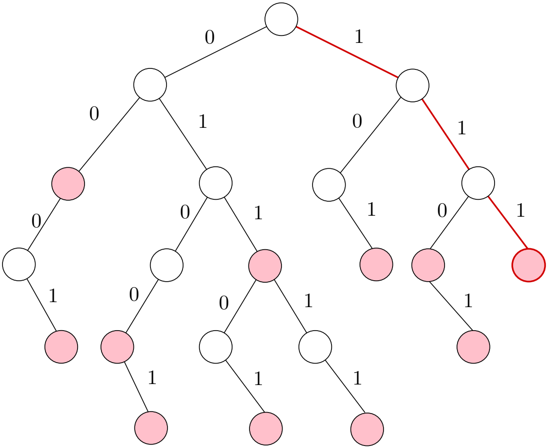
Tries: Insert

Example: Insert(11101)



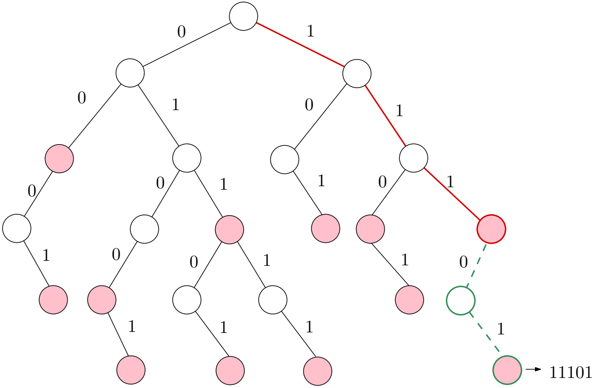
Tries: Insert

Example: Insert(11101)



Tries: Insert

Example: Insert(11101)

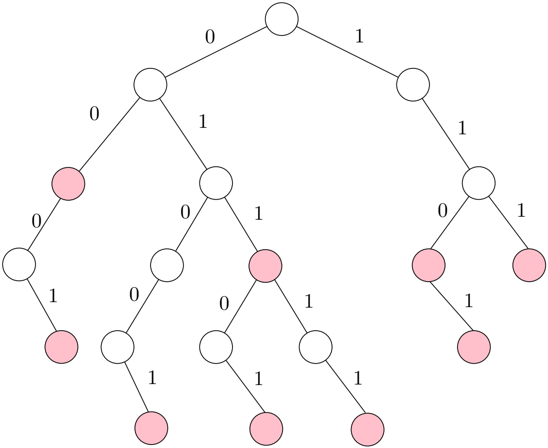


Tries: Delete

- Delete(x)
  - Search for x
  - if x found at an internal flagged node, then unflag the node
  - if x found at a leaf v_x, delete the leaf and all ancestors of v_x until
    - we reach an ancestor that has two children or
    - we reach a flagged node

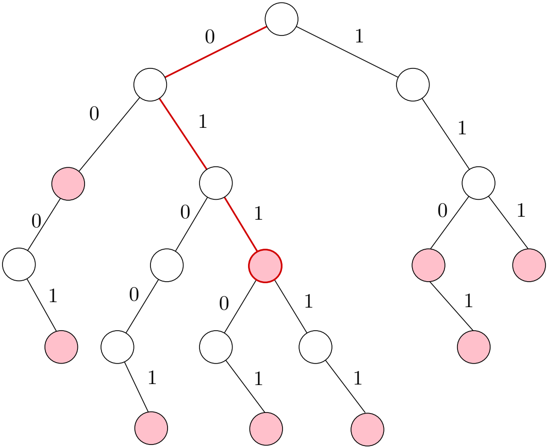
Tries: Delete

Example: Delete(011)



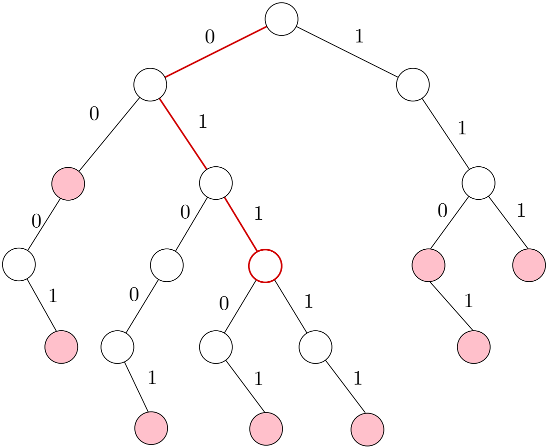
Tries: Delete

Example: Delete(011)



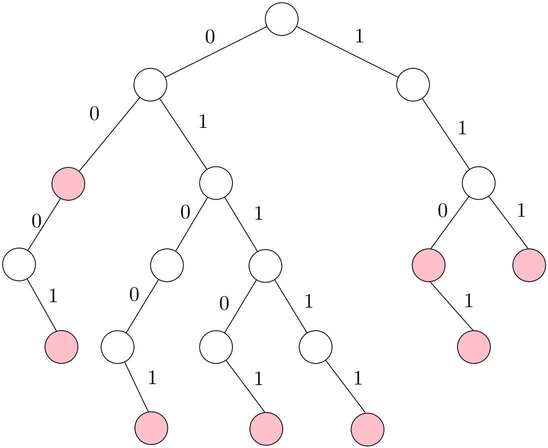
Tries: Delete

Example: Delete(011)



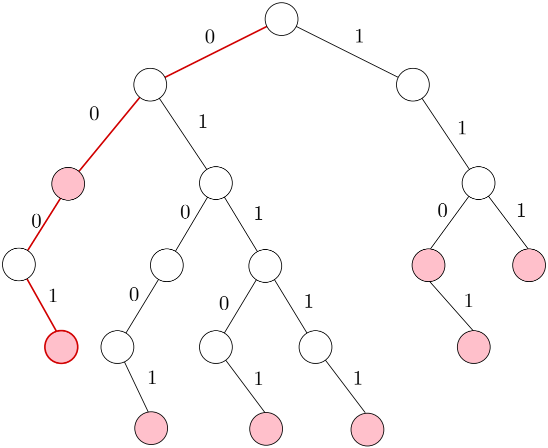
Tries: Delete

Example: Delete(0001)



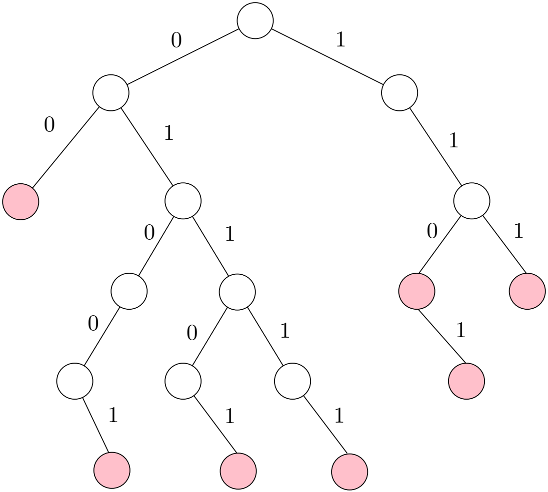
Tries: Delete

Example: Delete(0001)



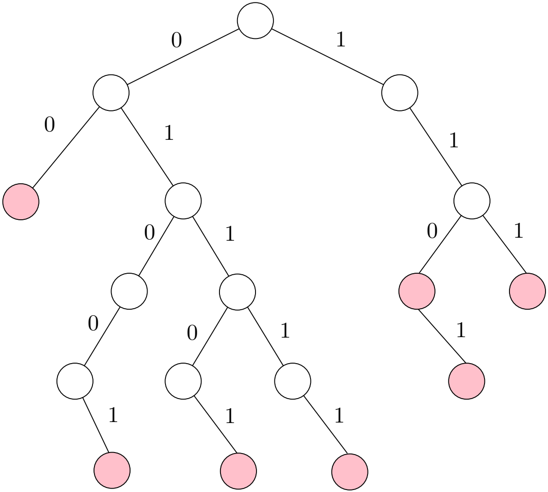
Tries: Delete

Example: Delete(0001)



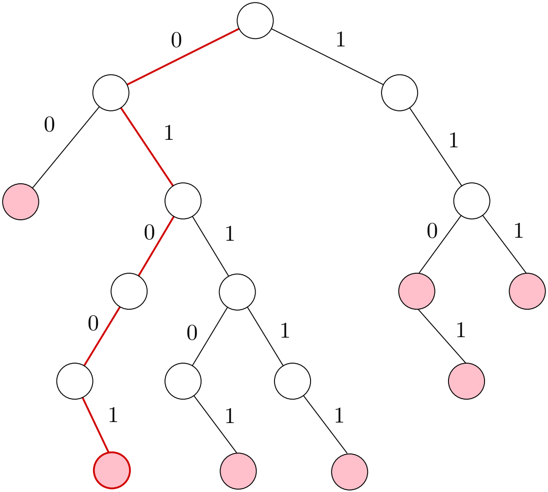
Tries: Delete

Example: Delete(01001)



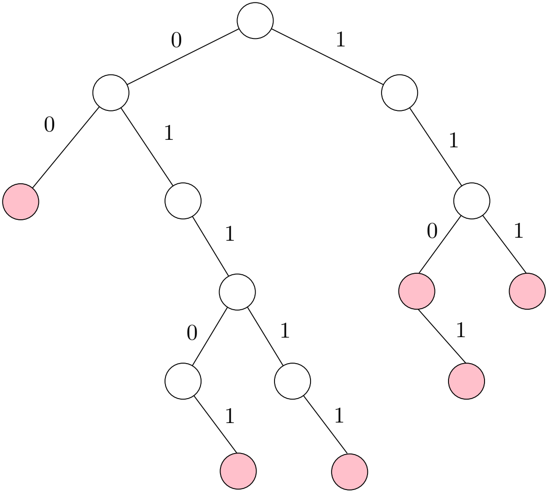
Tries: Delete

Example: Delete(01001)



Tries: Delete

Example: Delete(01001)



Tries: Operations

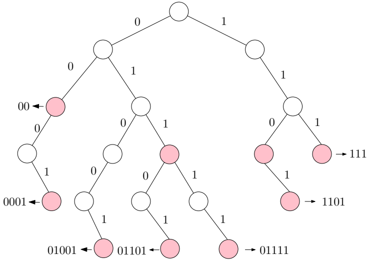
- Search(x)
- Insert(x)
- Delete(x)
- Time Complexity of all operations: $\Theta(|x|)$
  $|x|$: length of binary string x, i.e., the number of bits in x

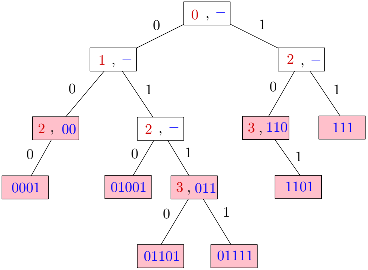
Compressed Tries (Patricia Tries)

- Patricia: Practical Algorithm To Retrieve Information Coded in Alphanumeric
- Introduced by Morrison (1968)
- Reduces storage requirement: eliminate unflagged nodes with only one child
- Every path of one-child unflagged nodes is compressed to a single edge
- Each node stores an index indicating the next bit to be tested during a search (index$= 0$ for the first bit, index$= 2$ for the second bit, etc)
- A compressed trie storing n keys always has at most $n - 1$ internal (non-leaf) nodes

Compressed Tries (Patricia Tries)

- Each node stores an index indicating the next bit to be tested during a search
- Example: A trie and the equivalent compressed trie



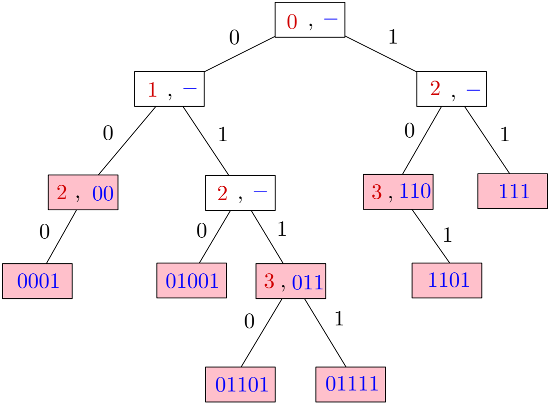


Compressed Tries: Operations

- Search(x):
  - Follow the proper path from the root down in the tree to a leaf
  - If search ends in an internal flagged node, it is successful
  - If search ends in an internal unflagged node, it is unsuccessful
  - If search ends in a leaf, we need to check again if the key stored at the leaf is indeed x

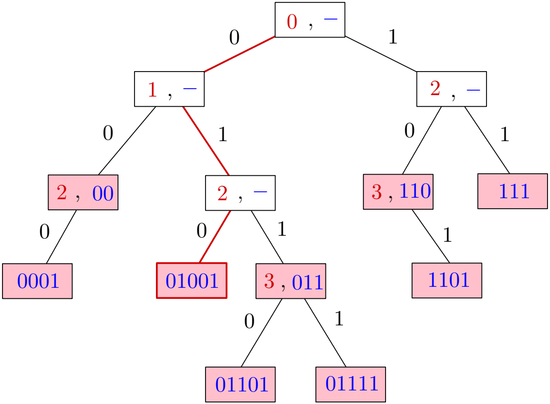
Compressed Tries: Operations

Example: Search(01001)



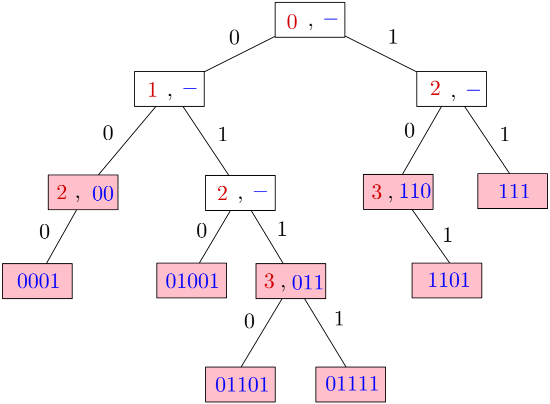
Compressed Tries: Operations

Example: Search(01001) - successful



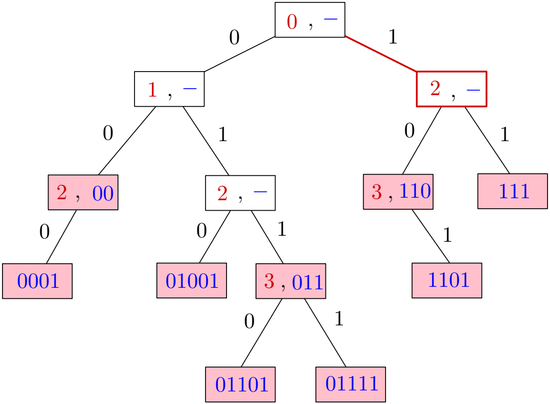
Compressed Tries: Operations

Example: Search(11)



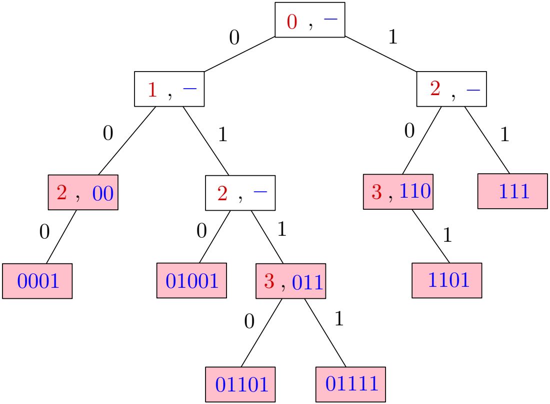
Compressed Tries: Operations

Example: Search(11) - unsuccessful



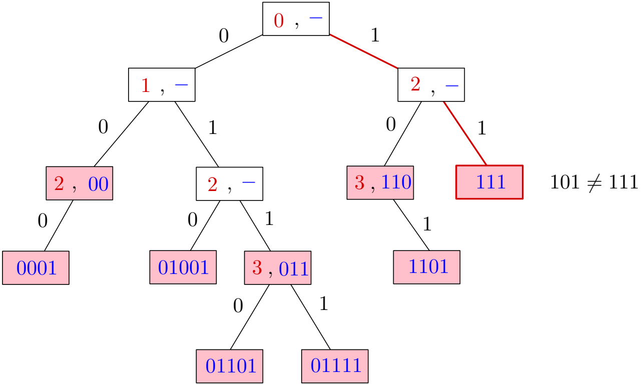
Compressed Tries: Operations

Example: Search(101)



Compressed Tries: Operations

Example: Search(101) - unsuccessful

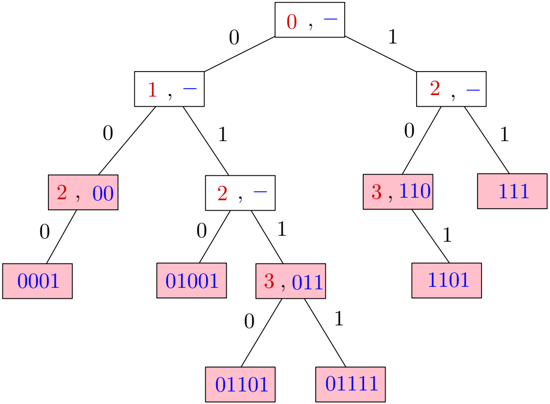


Compressed Tries: Operations

- Delete(x):
  - Perform Search(x)
  - if search ends in an internal node, then
    - if the node has two children, then unflag the node and delete the key
    - else delete the node and make his only child, the child of its parent
  - if search ends in a leaf, then delete the leaf and
  - if its parent is unflagged, then delete the parent

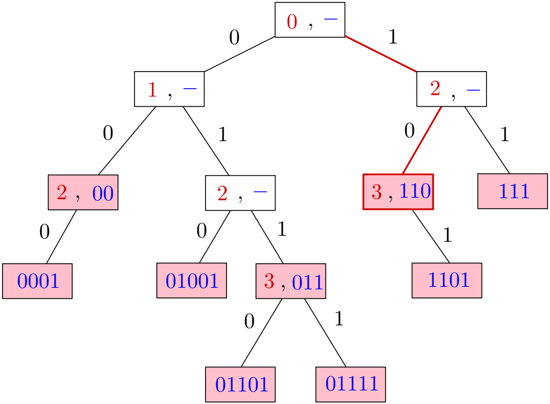
Compressed Tries: Operations

Example: Delete(110)



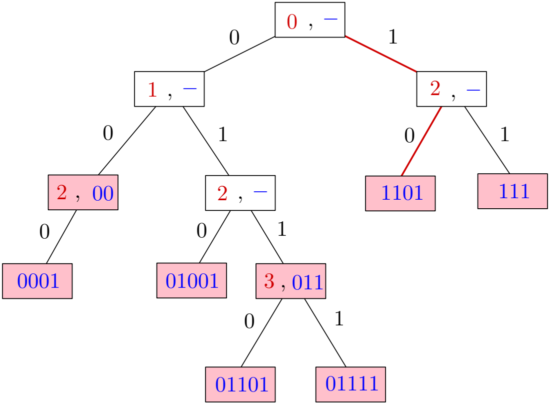
Compressed Tries: Operations

Example: Delete(110)



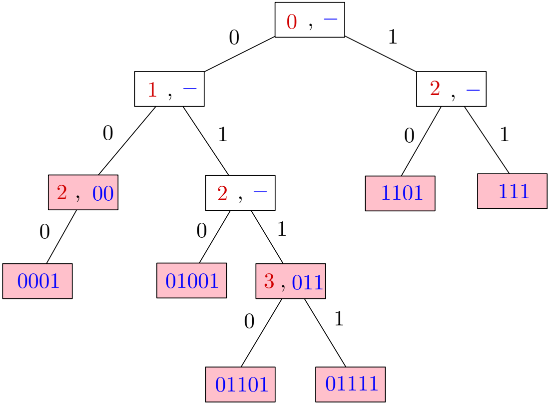
Compressed Tries: Operations

Example: Delete(110)



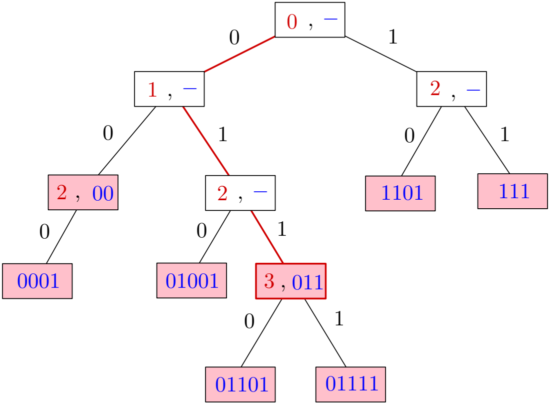
Compressed Tries: Operations

Example: Delete(011)



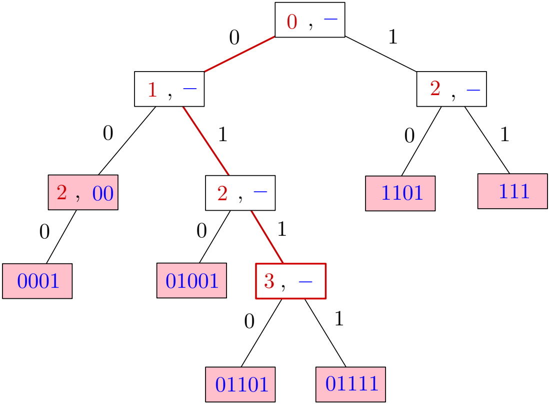
Compressed Tries: Operations

Example: Delete(011)



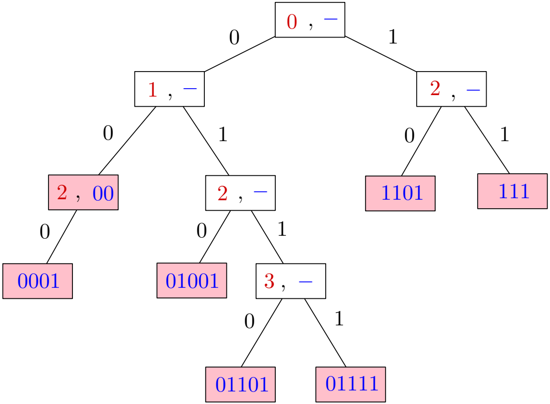
Compressed Tries: Operations

Example: Delete(011)



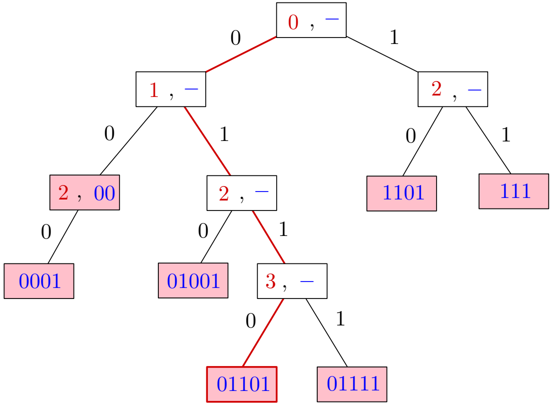
Compressed Tries: Operations

Example: Delete(01101)



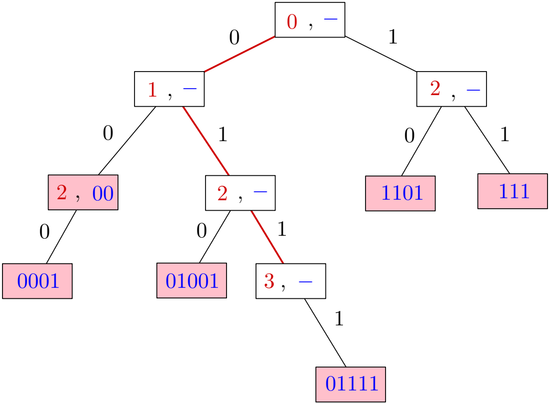
Compressed Tries: Operations

Example: Delete(01101)



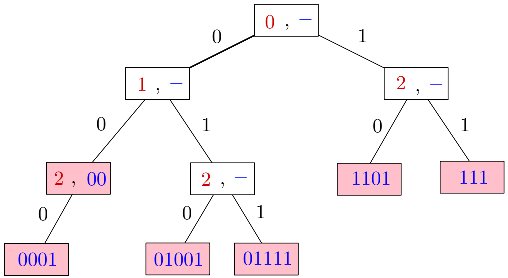
Compressed Tries: Operations

Example: Delete(01101)



Compressed Tries: Operations

Example: Delete(01101)

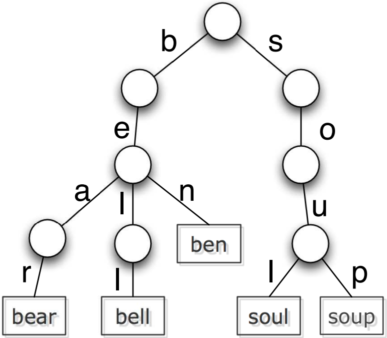


Compressed Tries: Operations

- Insert(x):
  - Perform Search(x)
  - If the search ends at a leaf L with key y, compare x against y to determine the first index i where they disagree.
    Create a new node N with index i.
    Insert N along the path from the root to L so that the parent of N has index $< i$ and one child of N is either L or an existing node on the path from the root to L that has index $> i$.
    The other child of N will be a new leaf node containing x.
  - If the search ends at an internal node, we find the key corresponding to that internal node and proceed in a similar way to the previous case.

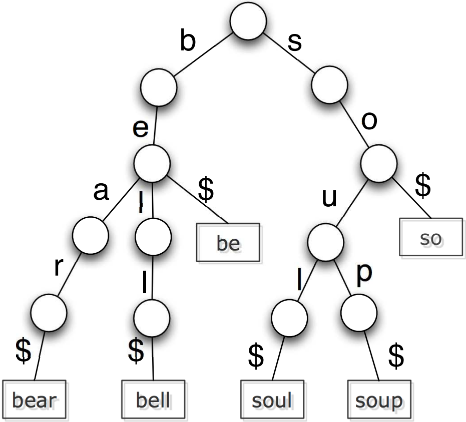
Multiway Tries

- To represent Strings over any fixed alphabet Σ
- Any node will have at most $|\Sigma|$ children
- Example: A trie holding strings $\{$bear, bell, ben, soul, soup$\}$



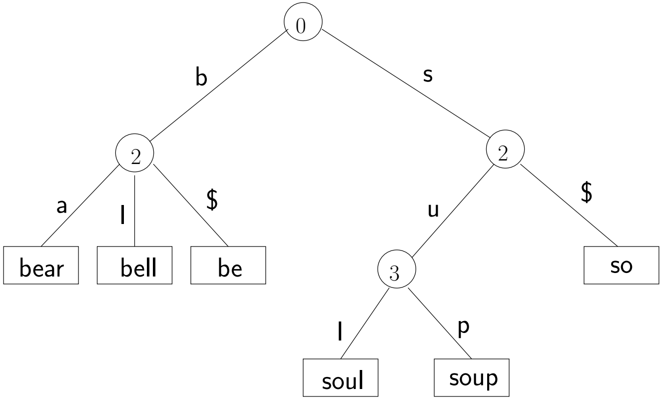
Multiway Tries

- Append a special end-of-word character, say \$, to all keys
- Example: A trie holding strings {bear, bell, be, so, soul, soup}



Multiway Tries

- Compressed multi-way tries
- Example: A compressed trie holding strings {bear, bell, be, so, soul, soup}



Pattern Matching

- Search for a string (pattern) in a large body of text
- $T[0..n-1]$ – The text (or haystack) being searched within
- $P[0..m-1]$ – The pattern (or needle) being searched for
- Strings over alphabet Σ
- Return the first i such that

$$P[j] = T[i+j] \quad \text{for} \quad 0 \leq j \leq m-1$$

- This is the first occurrence of P in T
- If P does not occur in T, return `FAIL`
- Applications:
  - Information Retrieval (text editors, search engines)
  - Bioinformatics
  - Data Mining

Pattern Matching

Example:

- $T =$ "Where is he?"
- $P_1 =$ "he"
- $P_2 =$ "who"

Definitions:

- Substring $T[i..j]$ $0 \le i \le j < n$: a string of length $j - i + 1$ which consists of characters $T[i], \ldots T[j]$ in order
- A prefix of T:
  a substring $T[0..i]$ of T for some $0 \le i < n$
- A suffix of T:
  a substring $T[i..n-1]$ of T for some $0 \le i \le n-1$

General Idea of Algorithms

Pattern matching algorithms consist of guesses and checks:

- A **guess** is a position i such that P might start at $T[i]$. Valid guesses (initially) are $0 \leq i \leq n - m$.
- A **check** of a guess is a single position j with $0 \leq j < m$ where we compare $T[i + j]$ to $P[j]$. We must perform m checks of a single correct guess, but may make (many) fewer checks of an incorrect guess.

We will diagram a single run of any pattern matching algorithm by a matrix of checks, where each row represents a single guess.

Brute-force Algorithm

Idea: Check every possible guess.

BruteforcePM$(T[0..n-1], P[0..m-1])$
T: String of length n (text), P: String of length m (pattern)

for $i \leftarrow 0$ **to** $n-m$ **do**
    match $\leftarrow$ *true*
    $j \leftarrow 0$
    while $j < m$ **and** *match* **do**
        if $T[i+j] = P[j]$ **then**
            $j \leftarrow j+1$
        else
            match $\leftarrow$ *false*
    if *match* **then**
        return i
return `FAIL`

Example

- Example: $T =$ abbbababbab, $P =$ abba

a	b	b	b	a	b	a	b	b	a	b
a	b	b	a							
	a									
		a								
			a							
				a	b	b				
					a					
						a	b	b	a	

- What is the worst possible input?
  $P = a^{m-1}b, \; T = a^n$
- Worst case performance $\Theta((n-m+1)m)$
- $m \leq n/2 \Rightarrow \Theta(mn)$

Pattern Matching

More sophisticated algorithms

- **KMP** and **Boyer-Moore**
- Do extra preprocessing on the pattern P
- We eliminate guesses based on completed matches and mismatches.

KMP Algorithm

- Knuth-Morris-Pratt algorithm (1977)
- Compares the pattern to the text in left-to-right
- Shifts the pattern more intelligently than the brute-force algorithm
- When a mismatch occurs, what is the most we can shift the pattern (reusing knowledge from previous matches)?

$T =$	a	b	c	d	c	a	b	c	?	?	?
	a	b	c	d	c	a	b	a			
						a	b	c	d	c	a

- KMP Answer: the largest prefix of $P[0..j]$ that is a suffix of $P[1..j]$

KMP Failure Array

T:	a	b	b	c	a	b	c	d	...
P:	a	b	b	c	a	b	a	a	

KMP Failure Array

T:	a	b	b	c	a	b	c	d	...
P:	a	b	b	c	a	b	a	a	

what next slide would match with the text?

KMP Failure Array

T:	a	b	b	c	a	b	c	d	…
P:	a	b	b	c	a	b	a	a	
		a	b	b	c	a	b	a	a

KMP Failure Array

T:	a	b	b	c	a	b	c	d ...		
P:	a	b	b	c	a	b	a	a		
		a	b	b	c	a	b	a	a	
			a	b	b	c	a	b	a	a

KMP Failure Array

T:	a	b	b	c	a	b	c	d	...		
P:	a	b	b	c	a	b	a	a			
		a	b	b	c	a	b	a	a		
			a	b	b	c	a	b	a	a	
				a	b	b	c	a	b	a	a

KMP Failure Array

T:	a	b	b	c	a	b	c	d	…				
P:	a	b	b	c	a	b	a	a					
✗		a	b	b	c	a	b	a	a				
✗			a	b	b	c	a	b	a	a			
✗				a	b	b	c	a	b	a	a		
✓					a	b	b	c	a	b	a	a	

KMP Failure Array

- Define $F[j]$ as the value of the first sliding position past the current one that matches the text T, up to position $T[i-1]$
- This can be computed by trying all sliding positions until finding the first one matching the text (as in previous example).

KMP Failure Array

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- This can be computed by trying all sliding positions until finding the first one matching the text (as in previous example).
- Observation 1: $T[i-j..i-1] = P[0..j-1]$
- Observation 2: $T[i-F[j-1]..i-1] = P[0..F[j-1]-1]$

KMP Failure Array

- Define $F[j]$ as the value of the first sliding position past the current one that matches the text T, up to position $T[i-1] = P[j-1]$
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- Observation 1: $T[i-j..i-1] = P[0..j-1]$
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- Observation 2: $T[i-F[j-1]..i-1] = P[0..F[j-1]-1]$
- Observation 3: $F[j]$ is the length of the largest prefix of $P[0..j]$ that is also a suffix of $P[1..j]$
- so we can preprocess the pattern to find matches of prefixes of the pattern with the pattern itself

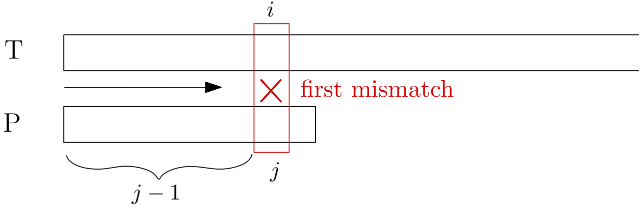
KMP Failure Array

- $F[0] = 0$
- $F[j]$, for $j > 0$, is the length of the largest prefix of $P[0..j]$ that is also a suffix of $P[1..j]$
- Consider $P =$ abacaba

j	$P[1..j]$	P	$F[j]$
0	—	abacaba	0
1	b	abacaba	0
2	ba	abacaba	1
3	bac	abacaba	0
4	baca	abacaba	1
5	bacab	abacaba	2
6	bacaba	abacaba	3

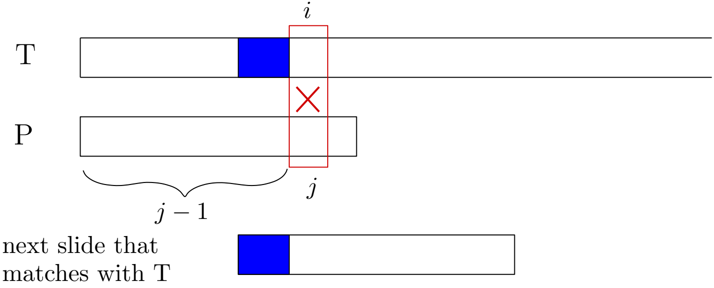
KMP Failure Array

Schematically:



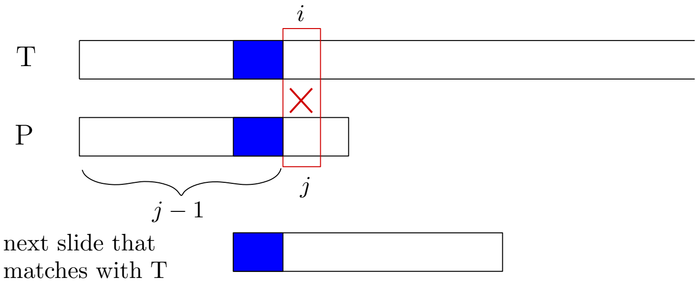
KMP Failure Array

Schematically:



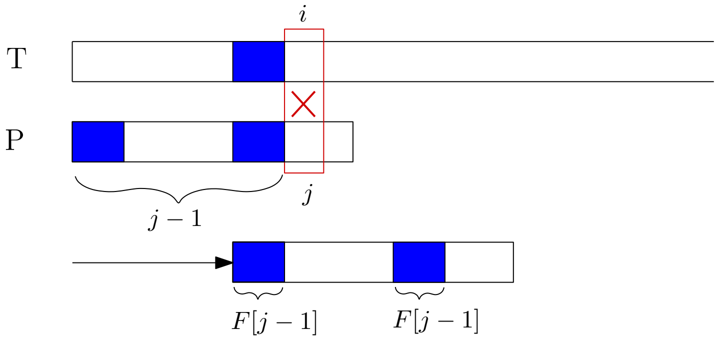
KMP Failure Array

Schematically:



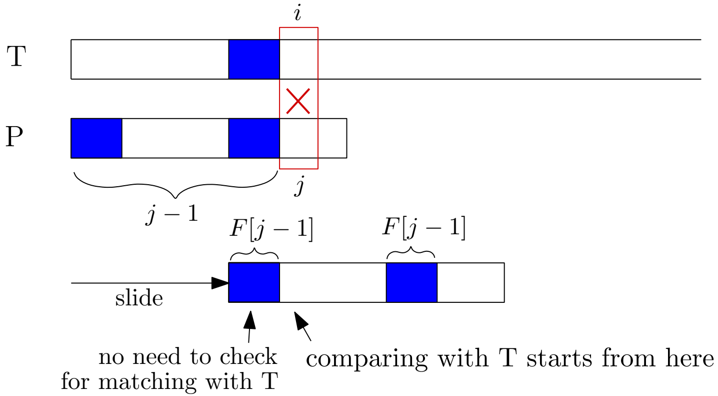
KMP Failure Array

Schematically:



KMP Failure Array

Schematically:



Computing the Failure Array

```
failureArray(P)
P: String of length m (pattern)
    F[0] ← 0
    i ← 1
    j ← 0
    while i < m do
        if P[i] = P[j] then
            F[i] ← j + 1
            i ← i + 1
            j ← j + 1
        else if j > 0 then
            j ← F[j − 1]
        else
            F[i] ← 0
            i ← i + 1
```

KMP Algorithm

```
KMP(T, P)
T: String of length n (text), P: String of length m (pattern)
    F ← failureArray(P)
    i ← 0
    j ← 0
    while i < n do
        if T[i] = P[j] then
            if j = m − 1 then
                return i − j //match
            else
                i ← i + 1
                j ← j + 1
        else
            if j > 0 then
                j ← F[j − 1]
            else
                i ← i + 1
    return −1 // no match
```

KMP: Example

$P = \texttt{abacaba}$

$T = \underline{\texttt{abaxyabacabb}}\texttt{aababacaba}$

j	0	1	2	3	4	5	6
$F[j]$	0	0	1	0	1	2	3

0	1	2	3	4	5	6	7	8	9	10	11
a	b	a	x	y	a	b	a	c	a	b	b
a	b	a	c								
		(a)	b								
			a				✓				
				a							
					a	b	a	c	a	b	a
									(a)	(b)	a

Exercise: continue with $T = \texttt{abaxyabacabb}\underline{\texttt{aababacaba}}$

KMP: Analysis

failureArray

- At each iteration of the while loop, either
  1. i increases by one, or
  2. the guess index $i - j$ increases by at least one $(F[j-1] < j)$
- There are no more than $2m$ iterations of the while loop
- Running time: $\Theta(m)$

KMP: Analysis

failureArray

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  1. i increases by one, or
  2. the guess index $i - j$ increases by at least one $(F[j-1] < j)$
- There are no more than $2m$ iterations of the while loop
- Running time: $\Theta(m)$

KMP

- failureArray can be computed in $\Theta(m)$ time
- At each iteration of the while loop, either
  1. i increases by one, or
  2. the guess index $i - j$ increases by at least one $(F[j-1] < j)$
- There are no more than $2n$ iterations of the while loop
- Running time: $\Theta(n)$

KMP: Another Example

- $T =$abacaabaccabacabaabb
- $P =$abacab

Boyer-Moore Algorithm

Based on three key ideas:

- Reverse-order searching: Compare P with a subsequence of T moving backwards
- Bad character jumps: When a mismatch occurs at $T[i] = c$
  - If P contains c, we can shift P to align the last occurrence of c in P with $T[i]$
  - Otherwise, we can shift P to align $P[0]$ with $T[i+1]$
- Good suffix jumps: If we have already matched a suffix of P, then get a mismatch, we can shift P forward to align with the previous occurence of that suffix (with a mismatch from the actual suffix). Similar to failure array in KMP.
- Can skip large parts of T

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

			o								

P = m o o r e

T = b o y e r m o o r e

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

			o								
							o				

P = m o o r e

T = b o y e r m o o r e

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

			o								
							o				
											o

P = m o o r e

T = b o y e r m o o r e

Bad character examples

P = a l d o
T = w h e r e i s w a l d o

w	h	e	r	e	i	s	w	a	l	d	o
			o								
							o				
										d	o

P = m o o r e
T = b o y e r m o o r e

b	o	y	e	r	m	o	o	r	e

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

			o								
							o				
									l	d	o

P = m o o r e

T = b o y e r m o o r e

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

w	h	e	r	e	i	s	w	a	l	d	o
			o								
							o				
								a	l	d	o

P = m o o r e

T = b o y e r m o o r e

b	o	y	e	r	m	o	o	r	e

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

w	h	e	r	e	i	s	w	a	l	d	o
			o								
							o				
								a	l	d	o

6 comparisons (checks)

P = m o o r e

T = b o y e r m o o r e

b	o	y	e	r	m	o	o	r	e

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

w	h	e	r	e	i	s	w	a	l	d	o
			o								
							o				
								a	l	d	o

6 comparisons (checks)

P = m o o r e

T = b o y e r m o o r e

b	o	y	e	r	m	o	o	r	e
				e					

Bad character examples

P = a l d o
T = w h e r e i s w a l d o

w	h	e	r	e	i	s	w	a	l	d	o
			o								
							o				
								a	l	d	o

6 comparisons (checks)

P = m o o r e
T = b o y e r m o o r e

b	o	y	e	r	m	o	o	r	e
				e					
				(r)	e				

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

			o								
							o				
								a	l	d	o

6 comparisons (checks)

P = m o o r e

T = b o y e r m o o r e

				e					
				(r)	e				
					(m)				e

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

w	h	e	r	e	i	s	w	a	l	d	o
			o								
							o				
								a	l	d	o

6 comparisons (checks)

P = m o o r e

T = b o y e r m o o r e

b	o	y	e	r	m	o	o	r	e
				e					
				(r)	e				
					(m)			r	e

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

			o								
							o				
								a	l	d	o

6 comparisons (checks)

P = m o o r e

T = b o y e r m o o r e

				e					
				(r)	e				
					(m)	o	o	r	e

7 comparisons (checks)

Good suffix examples

P = sells␣shells

s	h	e	i	l	a	␣	s	e	l	l	s	␣	s	h	e	l	l	s

P = odetofood

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o

Good suffix examples

$P =$ sells␣shells

s	h	e	i	l	a	␣	s	e	l	l	s	␣	s	h	e	l	l	s
							h	e	l	l	s							

$P =$ odetofood

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o

Good suffix examples

$P =$ sells␣shells

s	h	e	i	l	a	␣	s	e	l	l	s	␣	s	h	e	l	l	s
							h	e	l	l	s							
								(e)	(l)	(l)	(s)							s

$P =$ odetofood

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o

Good suffix examples

$P =$ sells␣shells

s	h	e	i	l	a	␣	s	e	l	l	s	␣	s	h	e	l	l	s
							h	e	l	l	s							
							s	(e)	(l)	(l)	(s)	␣	s	h	e	l	l	s

$P =$ odetofood

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o

Good suffix examples

$P =$ sells␣shells

s	h	e	i	l	a	␣	s	e	l	l	s	␣	s	h	e	l	l	s
							h	e	l	l	s							
							s	(e)	(l)	(l)	(s)	␣	s	h	e	l	l	s

$P =$ odetofood

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o
				o	f	o	o	d										

Good suffix examples

P = sells␣shells

s	h	e	i	l	a	␣	s	e	l	l	s	␣	s	h	e	l	l	s
							h	e	l	l	s							
							s	(e)	(l)	(l)	(s)	␣	s	h	e	l	l	s

P = odetofood

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o
				o	f	o	o	d										
				(e)						d								

Good suffix examples

$P =$ sells␣shells

s	h	e	i	l	a	␣	s	e	l	l	s	␣	s	h	e	l	l	s
							h	e	l	l	s							
							s	(e)	(l)	(l)	(s)	␣	s	h	e	l	l	s

$P =$ odetofood

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o
				o	f	o	o	d										
							(o)	(d)							d			

- Good suffix moves further than bad character for 2nd guess.

Good suffix examples

$P =$ sells␣shells

s	h	e	i	l	a	␣	s	e	l	l	s	␣	s	h	e	l	l	s
							h	e	l	l	s							
							s	(e)	(l)	(l)	(s)	␣	s	h	e	l	l	s

$P =$ odetofood

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o
				o	f	o	o	d										
							(o)	(d)							d			

- Good suffix moves further than bad character for 2nd guess.
- Bad character moves further than good suffix for 3rd guess.
- This is out of range, so pattern not found.

Last-Occurrence Function

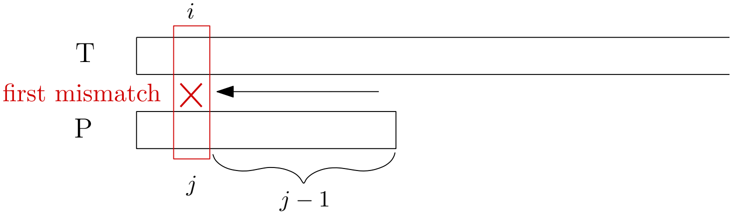
- Preprocess the pattern P and the alphabet Σ
- Build the last-occurrence function L mapping Σ to integers
- $L(c)$ is defined as
  - the largest index i such that $P[i] = c$ or
  - -1 if no such index exists
- Example: $\Sigma = \{a, b, c, d\}, P = abacab$

c	a	b	c	d
$L(c)$	4	5	3	-1

- The last-occurrence function can be computed in time $O(m + |\Sigma|)$
- In practice, L is stored in a size-$|\Sigma|$ array.

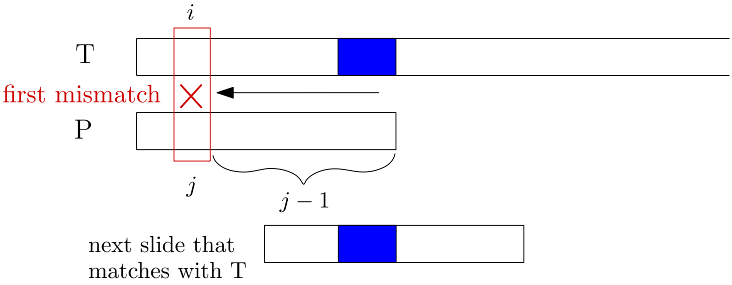
Good Suffix array

Schematically:



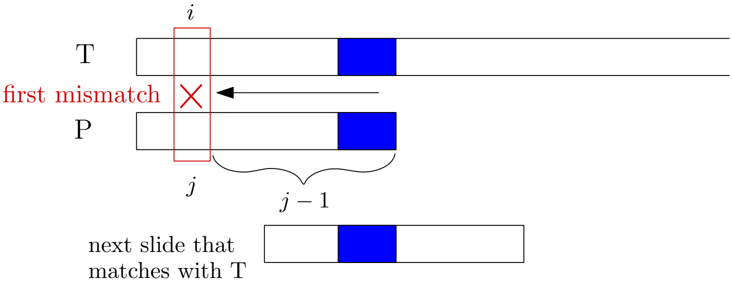
Good Suffix array

Schematically:



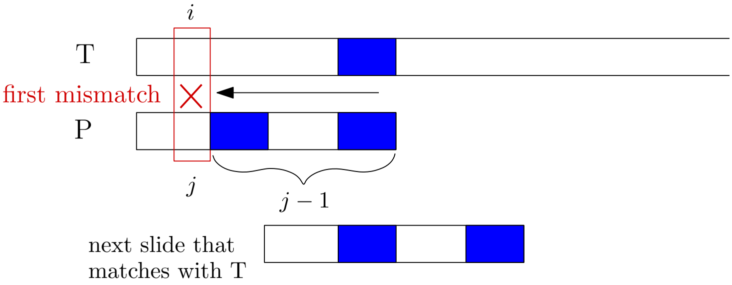
Good Suffix array

Schematically:



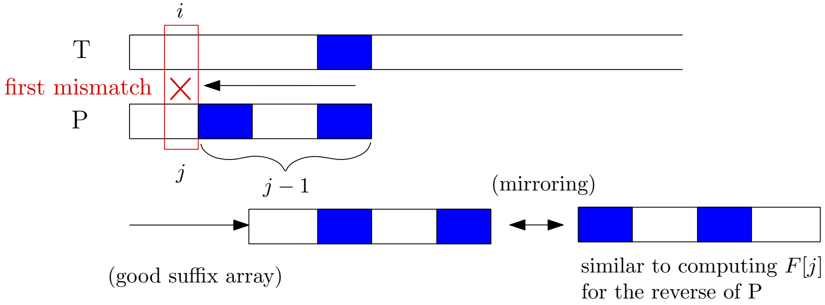
Good Suffix array

Schematically:



Good Suffix array

Schematically:



Boyer-Moore Algorithm

boyer-moore(T,P)

1. $L \leftarrow$ last occurrance array computed from P
2. $S \leftarrow$ good suffix array computed from P
3. $i \leftarrow m-1, \qquad j \leftarrow m-1$
4. **while** $i < n$ **and** $j \geq 0$ **do**
5.   **if** $T[i] = P[j]$ **then**
6. $i \leftarrow i-1$
7. $j \leftarrow j-1$
8.   **else**
9. $i \leftarrow i + m - 1 - \min(L[T[i]], S[j])$
10. $j \leftarrow m-1$
11. **if** $j = -1$ **return** $i+1$
12. **else return** `FAIL`

Exercise: Prove that $i - j$ always increases on lines 9–10.

Boyer-Moore algorithm conclusion

- Worst-case running time $\in O(n + |\Sigma|)$
- This complexity is difficult to prove.
- What is the worst case?
- On typical English text the algorithm probes approximately 25% of the characters in T
- Faster than KMP in practice on English text.

Suffix Tries and Suffix Trees

- What if we want to search for many patterns P within the same fixed text T?
- Idea: Preprocess the text T rather than the pattern P
- Observation: P is a substring of T if and only if P is a prefix of some suffix of T.

We will call a **suffix trie**, a trie that stores all suffixes of a text T, and a **suffix tree** the compressed suffix trie of T.

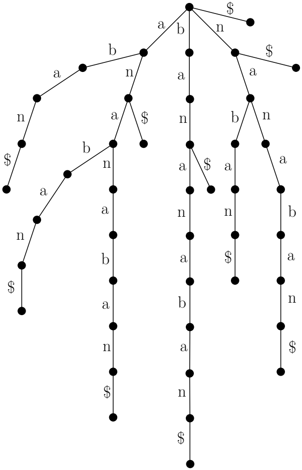
Suffix Trees

- Build the suffix trie, i.e. the trie containing all the suffixes of the text
- Build the suffix tree by compressing the trie above (like in Patricia trees)
- Store two indexes l, r on each node v (both internal nodes and leaves) where node v corresponds to substring $T[l..r]$

Suffix Trie: Example

T =bananaban

{bananaban, ananaban, nanaban, anaban, naban, aban, ban, an, n}

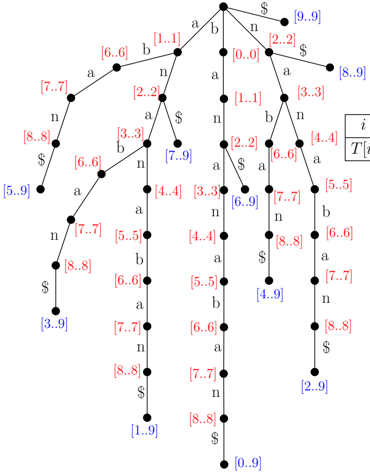


i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	$\$$

Suffix Trie: Example

T =bananaban

{bananaban, ananaban, nanaban, anaban, naban, aban, ban, an, n}

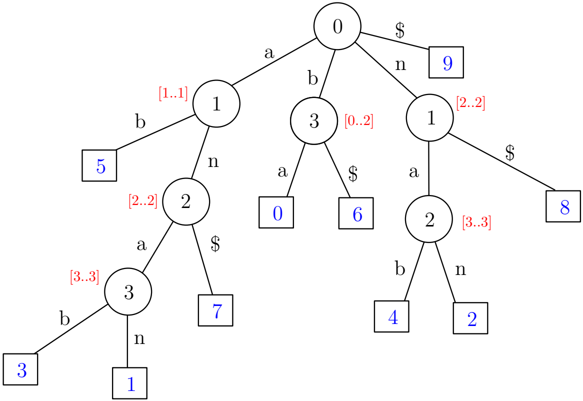


i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	\$

Suffix Tree (compressed suffix trie): Example

T =bananaban

{bananaban, ananaban, nanaban, anaban, naban, aban, ban, an, n}



i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	$

Suffix Trees: Pattern Matching

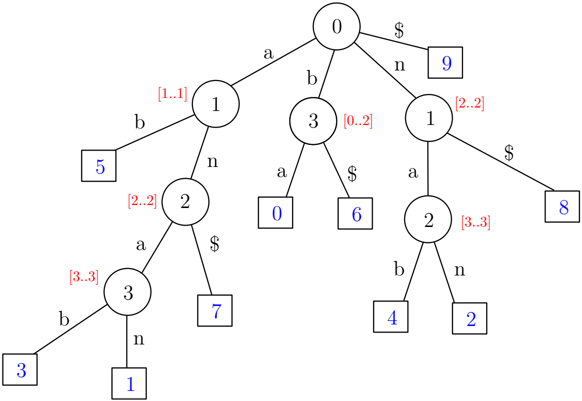
To search for pattern P of length m:

- Similar to Search in compressed trie with the difference that we are looking for a prefix match rather than a complete match
- If we reach a leaf with a corresponding string length less than m, then search is unsuccessful
- Otherwise, we reach a node v (leaf or internal) with a corresponding string length of at least m
- It only suffices, to check the first m characters against the substring of the text between indices of the node, to see if there indeed is a match

Suffix Tree: Example

T = `bananaban`

P = `ana`

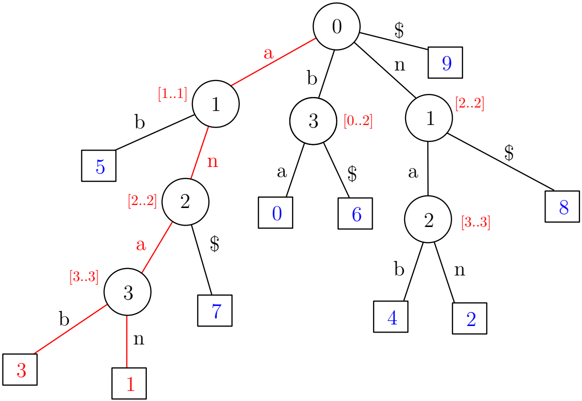


i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	\$

Suffix Tree: Example

T = `bananaban`

P = `ana`

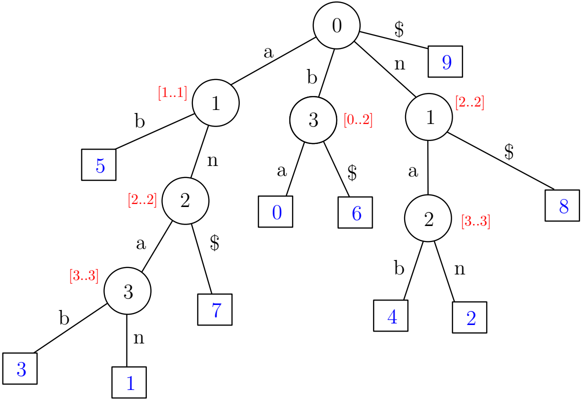


i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	\$

Suffix Tree: Example

$T =$ `bananaban`

$P =$ `ban`

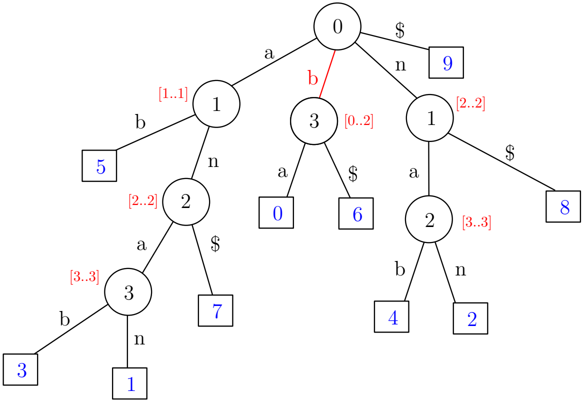


i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	$

Suffix Tree: Example

T = `bananaban`

P = `ban`

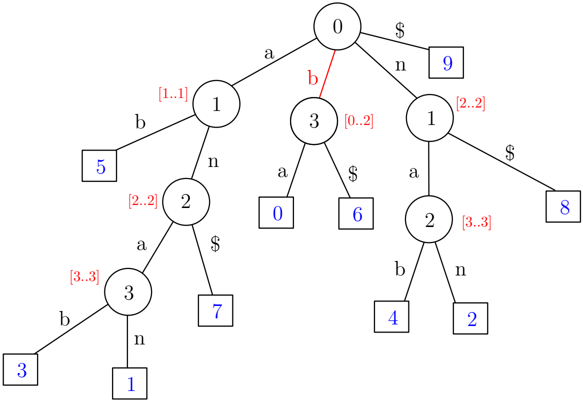


i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	\$

Suffix Tree: Example

T = bananaban

P = ban

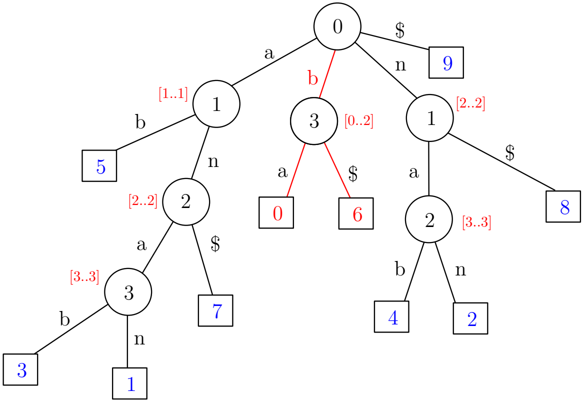


i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	\$

Suffix Tree: Example

T = `bananaban`

P = `ban`

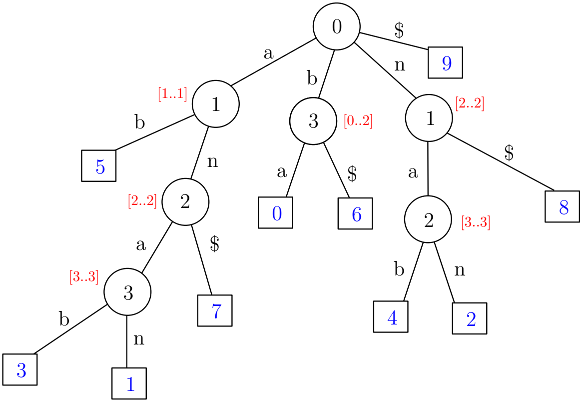


i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	\$

Suffix Tree: Example

T = `bananaban`

P = `nana`

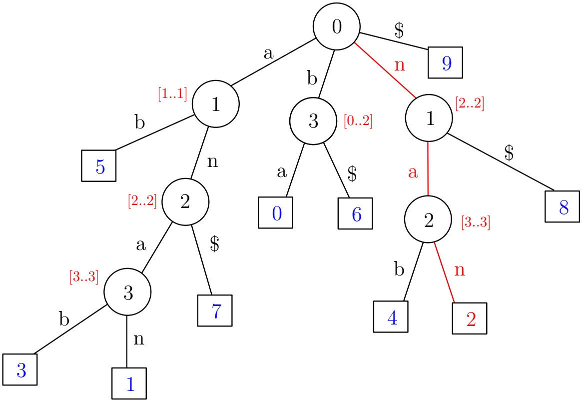


i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	$\$$

Suffix Tree: Example

T = `bananaban`

P = `nana`

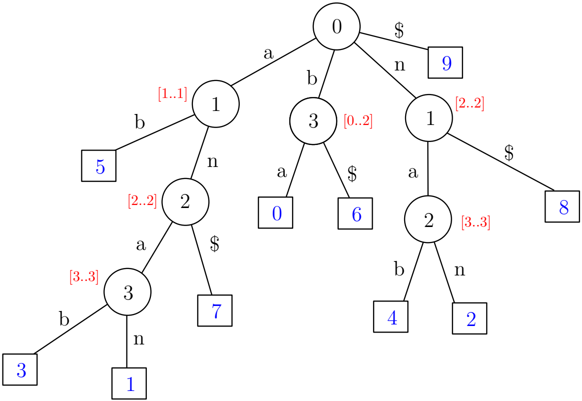


i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	\$

Suffix Tree: Example

T = `bananaban`

P = `bbn`

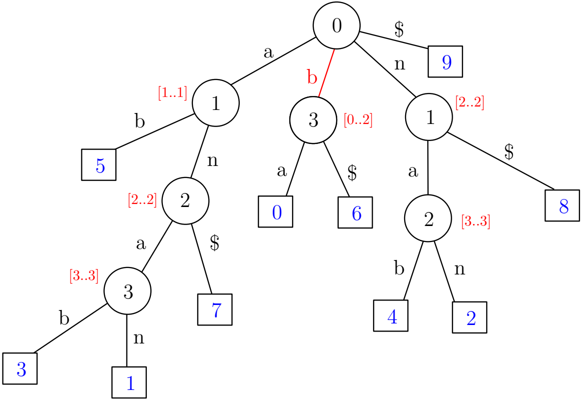


i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	\$

Suffix Tree: Example

$T =$ `bananaban`

$P =$ `bbn`

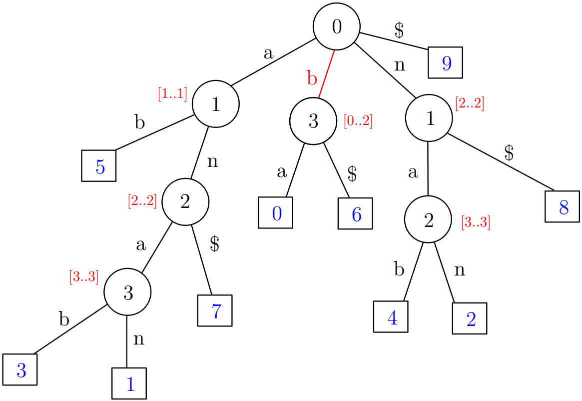


i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	$\$$

Suffix Tree: Example

T = bananaban

P = bbn



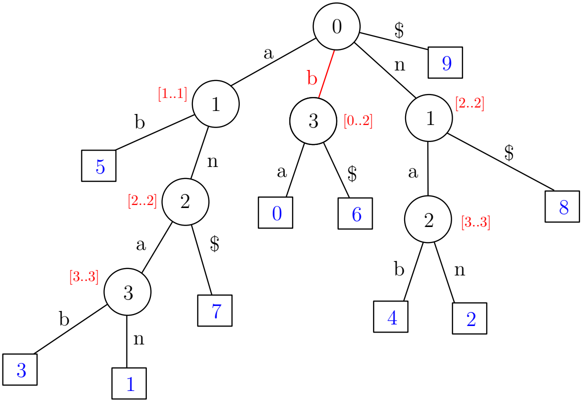
i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	\$

b b n

Suffix Tree: Example

T = `bananaban`

P = `bbn` not found



i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	\$
	b	b	n							

Pattern Matching Conclusion

	Brute-Force	KMP	BM	Suffix trees
Preproc.:	–	$O(m)$	$O(m + \|\Sigma\|)$	$O(n^2)$ $(\rightarrow O(n))$
Search time:	$O(nm)$	$O(n)$	$O(n)$ (often better)	$O(m)$
Extra space:	–	$O(m)$	$O(m + \|\Sigma\|)$	$O(n)$