Math 239 Lecture 27

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Platonic Solids

Platonic graph, planar, vertices have the same degree, faces have the same degree
d_v = vertex degree
d_f = face degree

Non-Planar Graphs

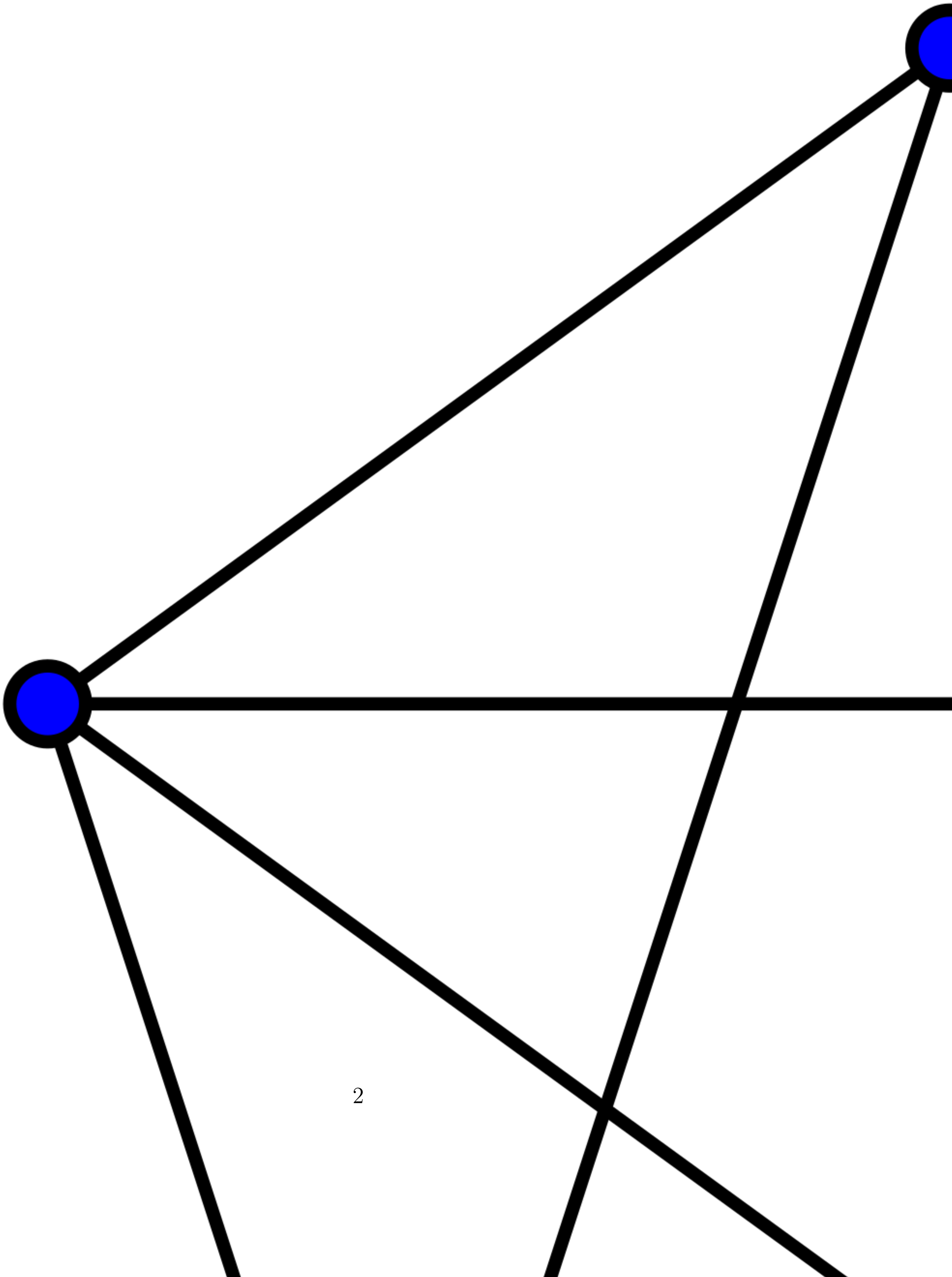
To prove that a graph is planar, we give a planar embedding

Theorem: When $n \geq 3$ a connected planar graph with n vertices has at most 3n-6 edges

Proof: Let G be a planar graph with a planar embedding with $n \geq 3$ in edges, S faces. We claim that every face has degree at least 3. If G is not a tree, then G has at least one cycle. So every face contains a cycle, which has at least 3 edges. So each face has degree ≥ 3. If G is a tree, then it has only once face. Its degree must be twice the number of edges, 2(n-1) = 2n-2 (by handshaking lemma for faces). Since $n \geq 3, 2n-2 \geq 3$. This proves the claim.

Using Handshaking Lemma for faces,
2m = $\sum_{f \in F} deg(f) \geq \sum_{f \in F} 3 = 3S$
$m \geq \frac{3}{2}S$
By Euler's formula: (n-m+s = 2)
$= \frac{3}{2}(2 - n + m)$
$= 3 - \frac{3}{2}n + \frac{3}{2}m$
$\implies \frac{3}{2}n - 3 \geq \frac{1}{2}m$
$\implies 3n - 6 \geq m$

Corollary: K_5 is not planar



Proof: K_5 has 5 vertices and 10 edges. Any planar graph wiht 5 vertices has at most $3 \cdot 5$- 6 = 9 edges. So K_5 is not planar

The converse of the theorem is false

Theorem: when n $\geq$ 3, a connected bipartite planar graph with n vertice has at most 2n-4 edges

Proof: (Similar to the proof of hte previous thoerem)
IF G has a cycle, then every face is bounded by a cycle of length $\geq$ 4 (since no triangle exists in a bipartite graph)
If G is a tree, its only face has deg 2n - 2 $\geq$ 4. Since n $\geq$ 3.
Using Hand shaking lemma for faces,
2m $\geq$ 4s = 4(2-n+m)
= 8 - 4n+4m
$\Longrightarrow$ 4n-8 $\geq$ 2m
$\Longrightarrow$ 2n-4 $\geq$ m

Corollary: $k_{3,3}$ has 6 vertices and 9 edges. Any planar bipartite graph with 6 vertices has at most $2 \cdot 6 - 4 = 8$ edes. So $k_{3,3}$ is not planar