Problem Set 2

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1 Problem 1

Use Ito's Lemma to prove that

$$\int_0^t W_s^2 dWs = \frac{1}{3}W_t^3 - \int_0^t W_s ds$$

Answer:

Using Ito's Lemma for a function $f(t, x)$:

$$df(t, W_t) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right) dt + \frac{\partial f}{\partial x} dW_t$$

For the function $f(t, x) = \frac{1}{3}x^3$:

$$\frac{\partial f}{\partial t} = 0$$
$$\frac{\partial f}{\partial x} = x^2$$
$$\frac{\partial^2 f}{\partial x^2} = 2x$$

Plugging these into Ito's Lemma:

$$df(t, W_t) = W_t^2 dW_t + W_t dt$$

Integrating both sides from 0 to t:

$$\int_0^t df(s, W_s) = \int_0^t W_s dt + \int_0^t W_s^2 dW_s$$

On the left side:

$$\int_0^t df(s, W_s) = \frac{1}{3}W_t^3 - \frac{1}{3}W_0^3$$

Given $W_0 = 0$, we have:

$$\int_0^t df(s, W_s) = \frac{1}{3}W_t^3$$

Rearranging terms:

$$\int_0^t W_s^2 dW_s = \frac{1}{3}W_t^3 - \int_0^t W_s dt$$

2 Problem 2

Check whether the process $X(t) = W_1(t) \times W_2(t)$ is a martingale, where $W_1(t)$, $W_2(t)$are (independent) brownian motions.

Answer:

We want to compute the differential of the product $X(t) = W_1(t) \times W_2(t)$ using Ito's product rule:

$$
\begin{aligned}
d(X(t)) &= d(W_1(t) \times W_2(t)) \\
dX(t) &= W_1(t)dW_2(t) + W_2(t)dW_1(t) + dW_1(t)dW_2(t).
\end{aligned}
$$

Given that the differential $dW_1(t)dW_2(t) = 0$ (because the two Brownian motions are independent), the last term drops out. Thus, we have:

$$dX(t) = W_1(t)dW_2(t) + W_2(t)dW_1(t).$$

This differential does not have a deterministic component (a "drift" term). Thus, the process $X(t)$ has zero drift, which makes it a martingale, provided the process is integrable.

3 Problem 3

Let $g(s)$ be a bounded, deterministic function of time. Show that

$$X_t = \int_0^t g(s)\, dW_s$$

is normally distributed with mean zero and standard deviation $\sqrt{\int_0^t g^2(s)\, ds}$. Hint:

1. Use Itô's Lemma to show that

$$Z_t = e^{-\frac{\eta^2}{2}\int_0^t g^2(s)\, ds + \eta \int_0^t g(s)\, dW_s}$$

is a martingale.

2. Use this observation to conclude that

$$E(e^{\eta X_t}) = e^{\frac{\eta^2}{2}\int_0^t g^2(s)\, ds}$$

which is the moment generating function of a normally distributed variable with zero mean and standard deviation $\sqrt{\int_0^t g^2(s)\, ds}$.

Answer:
For a function $f(t,x)$, Ito's Lemma states:

$$df(t, X_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2$$

Considering $f(t,x) = e^{\eta x}$ and applying Ito's Lemma:

$$df(t, X_t) = \eta e^{\eta X_t} dX_t + \frac{1}{2}\eta^2 e^{\eta X_t}(dX_t)^2$$

Substitute $X_t = \int_0^t g(s)\, dW_s$ and calculate its differential.

$$
\begin{aligned}
dX_t &= g(t)\, dW_t \\
df(t, X_t) &= \eta e^{\eta X_t} g(t)\, dW_t + \frac{1}{2}\eta^2 e^{\eta X_t} g^2(t)\, dt \\
dZ_t &= e^{-\frac{\eta^2}{2}\int_0^t g^2(s)\, ds}\left(\eta e^{\eta X_t} g(t)\, dW_t + \frac{1}{2}\eta^2 e^{\eta X_t} g^2(t)\, dt\right) \\
dZ_t &= \eta e^{-\frac{\eta^2}{2}\int_0^t g^2(s)\, ds + \eta X_t} g(t)\, dW_t.
\end{aligned}
$$

From the above, we see that dZ_t has no dt term, so Z_t is a martingale. Then,

$$\mathbb{E}[Z_t] = \mathbb{E}[Z_0]$$
$$\mathbb{E}\left[e^{\eta X_t - \frac{\eta^2}{2}\int_0^t g^2(s)\,ds}\right] = 1$$
$$\mathbb{E}\left[e^{\eta X_t}\right] = e^{\frac{\eta^2}{2}\int_0^t g^2(s)\,ds}.$$

Given this is the moment generating function of a normally distributed variable with zero mean and variance $\int_0^t g^2(s)ds$, we can conclude that:

$$X_t = \int_0^t g(s)dW_s$$

is normally distributed with mean zero and standard deviation.

4 Problem 4

Let

$$\beta_k(t) = \mathbb{E}\left[W_t^k\right] \text{ for } k = 0, 1, 2, \ldots$$

Use Ito's formula to prove that

$$\beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)\,ds$$

Answer: Given a standard Brownian motion W_t, we have the following properties:

1. $W_0 = 0$
2. $W_t - W_s$ is normally distributed with mean 0 and variance $t - s$ for $0 \leq s < t$.
3. W_t has independent increments.

Using Ito's formula for $f(t, W_t) = W_t^k$. Its derivatives are:

$$\frac{\partial f}{\partial t} = 0$$
$$\frac{\partial f}{\partial x} = kW_t^{k-1}$$
$$\frac{\partial^2 f}{\partial x^2} = k(k-1)W_t^{k-2}$$

Then we get:

$$d(W_t^k) = \frac{1}{2}k(k-1)W_t^{k-2}dt + kW_t^{k-1}dW_t$$

Taking the expectation on both sides:

$$d\beta_k(t) = \mathbb{E}\left[d(W_t^k)\right]$$

$$= \mathbb{E}\left[\frac{1}{2}k(k-1)W_t^{k-2}dt + kW_t^{k-1}dW_t\right]$$

Since the expectation of dW_t is zero, we have:

$$d\beta_k(t) = \frac{1}{2}k(k-1)\mathbb{E}\left[W_t^{k-2}\right]dt$$

$$d\beta_k(t) = \frac{1}{2}k(k-1)\beta_{k-2}(t)dt$$

Integrating both sides from 0 to t:

$$\beta_k(t) - \beta_k(0) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)ds$$

Given $W_0 = 0$, we have:

$$\beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)ds$$

Thus, the result is proven.