Because of the $r^{-(n+1)}$ dependence of ψ the importance of the higher-order terms decreases rapidly with distance from the Earth. Hence, much of trapped radiation theory is developed based on the dominant $n = 1$ or dipole term. However, when comparing radiation belt measurements taken at various points, it is necessary to use a magnetic field representation which is accurate enough to specify the geomagnetic field. Computer programs exist for calculating **B** at any point in space and for tracing geomagnetic field lines. The most recent programs contain additional functions to represent the various current systems in space as well as those in the interior of the Earth. These programs usually specify current distributions or potential functions for the various current systems and add the magnetic fields produced by these systems to give an overall magnetospheric field.

The dipole field

The lowest order, but dominant, term in (3.11) is the dipole term with $n = 1$, $m = 0$. Because many of the important features of the radiation belts can be illustrated with a dipole field, some useful relations for this field will be derived. The dipole potential from (3.11) is

$$\psi = \mathrm{R_E}\left(\frac{\mathrm{R_E}}{r}\right)^2 g_1^0 \cos\theta \tag{3.12}$$

where the distance r is measured from the center of the dipole and θ is the polar angle or colatitude (see Figure 3.1). The magnetic field **B** is equal to $-\nabla\psi$. In spherical polar coordinates the components of **B** are

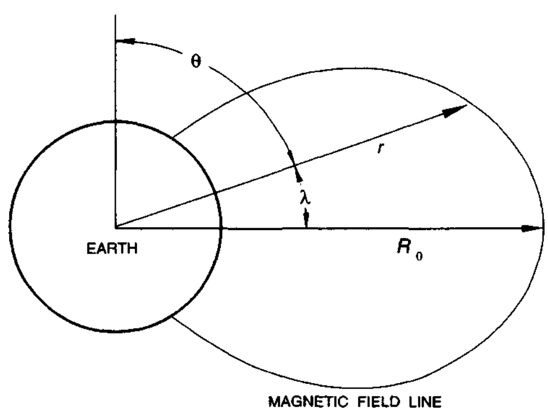

Figure 3.1. Dipole field coordinate system.

$$B_r = -\frac{\partial\psi}{\partial r} = 2\left(\frac{R_E}{r}\right)^3 g_1^0 \cos\theta = -2B_0\left(\frac{R_E}{r}\right)^3 \cos\theta \tag{3.13}$$

$$B_\theta = -\frac{1}{r}\frac{\partial\psi}{\partial\theta} = \left(\frac{R_E}{r}\right)^3 g_1^0 \sin\theta = -B_0\left(\frac{R_E}{r}\right)^3 \sin\theta \tag{3.14}$$

where B_0 is the mean value of the field on the equator at the Earth's surface. The components are negative since the direction of the field is in the minus $\hat{\mathbf{e}}_\theta$ direction and in the northern hemisphere it is in the minus $\hat{\mathbf{e}}_r$ direction. When considering only the magnitude of the field the minus signs can be omitted. For the Earth $B_0 = 3.12 \times 10^{-5}$ T. The dipole field is symmetric about its axis so that $B_\phi = 0$ everywhere.

The strength of a dipole can be characterized by the magnetic moment $\mathcal{M}$ whose units are ampere meters². In terms of the dipole moment the radial field component is

$$B_r = -\left(\frac{4\pi}{\mu_0}\mathcal{M}\right)\frac{2\cos\theta}{r^3}$$

Some authors designate the quantity $(4\pi/\mu_0)\mathcal{M}$ as the dipole moment so some confusion exists in published reports. The use of the equatorial surface field B_0 in (3.13) and subsequent equations avoids this ambiguity.

The intensity of the dipole field at any point in space is

$$B = \sqrt{(B_r^2 + B_\theta^2)} = B_0\left(\frac{R_E}{r}\right)^3 \sqrt{(1 + 3\cos^2\theta)} \tag{3.15}$$

The field intensity falls as r^{-3} with distance above the Earth and at constant r the intensity increases as one moves towards the poles. For a given value of r the field strength is twice as high over the poles as it is over the equator. Note that in equation (3.15) the r and θ dependences are separable. Hence, along any constant latitude line the field decreases as r^{-3}.

The equation for a geomagnetic field line in spherical coordinates is obtained by noting that the ratio of the lengths of the $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ components of the field line is

$$\frac{dr}{r d\theta} = \frac{B_r}{B_\theta} = \frac{2\cos\theta}{\sin\theta} \tag{3.16}$$

This equation can be integrated to give

$$r = R_0 \sin^2\theta \tag{3.17}$$

where R_0 is the value of r when $\theta = 90°$, namely the distance from the dipole to the point where the field line crosses the equatorial plane.

Expressed in terms of latitude λ,

$$r = R_0 \cos^2 \lambda \tag{3.18}$$

The distance along a field line for a dipole can be obtained analytically by integrating the equation for a distance element ds

$$ds = \sqrt{((dr)^2 + (r\, d\theta)^2)}$$

where dr and $r\,$dθ are constrained to be on a field line. Expressing dr in terms of dθ by differentiating the field line equation (3.17):

$$\begin{aligned} dr &= 2R_0 \sin\theta \cos\theta \, d\theta \\ ds &= \sqrt{(4\, R_0{}^2 \sin^2\theta \cos^2\theta + R_0{}^2 \sin^4\theta)}\, d\theta \\ &= R_0 \sqrt{(1 + 3\cos^2\theta)} \sin\theta \, d\theta \end{aligned} \tag{3.19}$$

By changing the variable to $\chi = \cos\theta$ equation (3.19) can be integrated from $\chi = 0$ (equatorial plane) to some off-equator value χ giving

$$s = \int_0^\chi ds = \frac{R_0}{2}\left[\chi\sqrt{(1 + 3\chi^2)} + \frac{1}{\sqrt{3}} \ln(\sqrt{(1 + 3\chi^2)} + \sqrt{3}\,\chi)\right] \tag{3.20}$$

The intesity of the magnetic field along a field line passing through R_0 is obtained as a function of colatitude by inserting r from equation (3.17) into (3.15), giving

$$B(\theta) = B_0\left(\frac{\mathrm{R_E}}{R_0}\right)^3 \frac{\sqrt{(1 + 3\cos^2\theta)}}{\sin^6\theta} \tag{3.21}$$

$$\begin{aligned} &= B_{\mathrm{eq}} \frac{\sqrt{(1 + 3\cos^2\theta)}}{\sin^6\theta} \\ &= B_{\mathrm{eq}} \frac{\sqrt{(1 + 3\sin^2\lambda)}}{\cos^6\lambda} \end{aligned} \tag{3.22}$$

B_{eq} is the value of B in the equatorial plane at distance R_0 from the dipole. From these equations it is apparent that the magnetic field along a field line increases monotonically with latitude as one moves from the equator to the poles.

If the Earth's field were a pure dipole located at the center of the Earth, contours of constant B on the Earth's surface would be lines of constant latitude. However, asymmetries in the interior current system introduce higher-order terms and the actual isointensity lines are as shown in Figure 3.2. Much of the distortion is caused by the fact that the magnetic axis is not aligned with the spin axis of the Earth and the center of the magnetic dipole is not at the center of the Earth. The poles are over northern Canada and southern Australia on the Mercator projection. Note, particularly, the large region of reduced field on the east coast of South America.