

Numerical Solution of the Advection-Diffusion Equation using the Discontinuous Enrichment Method (DEM)

Abstract

This paper presents a discontinuous enrichment method for the solution of the 2D advection-diffusion equation, the usual model for the Navier-Stokes equations, the central equations in fluid dynamics. The enrichment basis consists of the free-space solutions to the governing homogeneous PDE. Continuity of the solution across element boundaries is enforced using Lagrange multipliers. Preliminary results reveal that whereas the Galerkin solution exhibits spurious oscillations unless a very fine mesh is used, the DEM solution is excellent on a significantly coarser grid.

1 Introduction

In the standard Galerkin finite element method (FEM), the solution is approximated by continuous, piecewise polynomial basis functions. The FEM is considered quasi-optimal for elliptic boundary value problems (BVPs), meaning the approximate solutions generated by this method reproduce the properties of the “best approximations” in the underlying finite element space. Although this property assures good performance at any mesh resolution for the Laplace operator, the FEM can be prohibitively expensive for many other BVPs, particularly those whose solutions exhibit sharp gradients or rapid oscillations, e.g., problems involving boundary layers.

In the discontinuous enrichment method (DEM) [1], the standard finite element polynomial field is “enriched” by the free-space solutions of the homogeneous partial differential equation (PDE) governing the BVP. Since the enrichment field is related to the underlying equation, it is more effective in resolving sharp gradients and rapid oscillations than piecewise polynomial basis functions. As continuity across element boundaries is no longer automatic, it must be enforced weakly using appropriate Lagrange multipliers (LMs).

The DEM has shown much promise when applied to the 2-dimensional (2D) and 3-dimensional (3D) Helmholtz equation by Farhat *et al.* [1]. The Helmholtz equation describes acoustic vibrations in a fluid and may lose ellipticity with increasing wave number k . This causes a pollution effect in the Galerkin solution, which leads to spurious dispersion in the computation and makes the Galerkin FEM intractable for medium to high frequency simulations due to overwhelming cost. 3-dimensional (3D) hexahedral DEM elements have been developed, tested and shown to be superior to polynomial approximations for a scattering problem by a capped cylinder. For elements with roughly the same convergence rates, 4 – 8 times fewer degrees of freedom (dofs) are

required to achieve the same level of accuracy. This translates to a measured decrease in computation time up to a factor of 60.

Given the stellar performance of the DEM in acoustic scattering problems, it is natural to ask whether the method will perform equally well for other BVPs in which standard finite elements run into difficulties. One such problem is the advection-diffusion equation in fluid mechanics, governed by the asymmetric operator $\mathcal{L}u = -\Delta u + 2\mathbf{k} \cdot \nabla u$ and the usual model for the more challenging Navier-Stokes equations. When the Peclet number is high (the equation is advection-dominated), spurious oscillations pollute the Galerkin solution unless a very fine mesh is used in the region of the sharp gradient present in the solution. Preliminary results for the 2D problem show that the DEM solution, on the other hand, is virtually indistinguishable from the exact solution in the entire domain even when very few elements are used.

This paper is organized as follows. In §2, a model 2D advection-diffusion BVP is presented in its strong and its hybrid variational form, and the derivation of the enrichment basis and LMs is discussed. The so-called $R - 8 - 2$ element is constructed in §3. Its numerical performance is then evaluated and compared to the Galerkin Q_1 and Q_2 elements. Conclusions are offered in §4.

2 2D advection-diffusion boundary value problem

Consider the following BVP for the advection-diffusion equation:

$$\begin{aligned} 2\mathbf{k} \cdot \nabla u - \Delta u &= 0 \quad \text{on } \Omega \\ u|_{g_1} &= 1, u|_{g_2} = 0 \\ \nabla u \cdot \mathbf{n}|_{h_1} &= \nabla u \cdot \mathbf{n}|_{h_2} = 0 \end{aligned} \quad (*)$$

Fig. 1. Domain Ω for BVP (*)

where $\mathbf{k} = \frac{1}{2}(a \ b)^T$, Ω is the unit square $[0, 1] \times [0, 1]$ rotated clockwise by the angle $\alpha = \tan^{-1}(b/a)$ and g_i , h_i for $i = 1, 2$ are its boundaries (see Fig. 1). Ω is partitioned into n_{el} non-overlapping, square elements Ω^e with boundaries Γ^e . We denote the intersection of two element boundaries Γ^e and $\Gamma^{e'}$ by $\Gamma^{e,e'} = \Gamma^e \cup \Gamma^{e'}$. a and b are constant x and y advection coefficients, respectively; the diffusion coefficient is taken to be 1. The exact solution to (*), derived using separation of variables, is a function having a sharp gradient whose steepness depends on the values of the advection coefficients a and b :

$$u_{ex} = \frac{1}{e^{-\sqrt{a^2+b^2}} - 1} \left[e^{ax+by-\sqrt{a^2+b^2}} - 1 \right] \quad (1)$$

The fact that the domain Ω is the rotated unit square motivates one to define

the following transformation from $\Omega \rightarrow \Omega' \triangleq [0, 1] \times [0, 1]$: $x' = \frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y$, $y' = \frac{b}{\sqrt{a^2+b^2}}x + \frac{a}{\sqrt{a^2+b^2}}y$. In the transformed variables, (1) does not depend on y' . This observation suggests that an equivalent problem to (*) can be formulated and solved on Ω' with a zero y advection coefficient, and then rotated and scaled to represent the solution of (*) on Ω with $b \neq 0$.

The advection-diffusion equation is governed by the following asymmetric bilinear operator

$$a(u, v) = \int_{\Omega} (2v\mathbf{k} + \nabla v) \cdot \nabla u d\Omega \quad (2)$$

Let $\mathcal{V} = \{v \in L^2(\tilde{\Omega}) | v|_{\Omega^e} \in H^1(\Omega^e)\}$ be the space of test functions and denote the space of LMs by $\mathcal{W} = \{\Pi_e \Pi_{e' < e} H^{-1/2}(\Gamma^{e,e'})\}$. $\tilde{\mathcal{V}} \subset \mathcal{V}$ and $\tilde{\mathcal{W}} \subset \mathcal{W}$ are the corresponding finite-dimensional approximation spaces. The basic idea of the DEM is to seek an approximate solution $(u^h, \lambda^h) \in \tilde{\mathcal{V}} \times \tilde{\mathcal{W}}$ such that

$$a(u^h, v^h) + b(\lambda^h, v^h) = 0, \quad b(\mu^h, u^h) = -r(\mu^h), \quad \forall (v^h, \mu^h) \in \tilde{\mathcal{V}} \times \tilde{\mathcal{W}} \quad (3)$$

where, for the BVP (*),

$$b(\lambda, v) = \sum_e \sum_{e' < e} \int_{\Gamma^{e,e'}} \lambda (v_{e'} - v_e) dS, \quad r(\lambda) = \int_{g_1} \lambda dy \quad (4)$$

The variational equation (3) leads to the following element stiffness matrices and load vectors

$$\mathbf{k}^e = \begin{pmatrix} \mathbf{k}^{EE} & \mathbf{k}^{EC} \\ \mathbf{k}^{EC^T} & 0 \end{pmatrix}, \quad \mathbf{r}^e = \begin{pmatrix} 0 \\ -\mathbf{r}^C \end{pmatrix} \quad (5)$$

The solution vector is $(\mathbf{u} \ \lambda)^T$, with $u^h = u^E \in \tilde{\mathcal{V}}$ being the primal unknown¹. The most efficient way to solve (5) is by eliminating the first of the equations at the element level using static condensation, taking the Schur complement $\mathbf{f}^{CC} = -\mathbf{k}^{EC^T}(\mathbf{k}^{EE})^{-1}\mathbf{k}^{EC}$. The global interface problem $\mathbf{f}\lambda = \mathbf{r}$ is then built and assembled. Remark that the cost of computing a statically-condensed DEM solution is dependent on n^λ , the number of LM dofs, and *not* on n^E , the number of enrichment functions in the DEM basis.

An infinite set of solutions to the free-space version of (*) is derived using separation of variables. This set contains 8 linearly independent functions ($n^E = 8$), which are used to build up the enrichment field:

$$\begin{aligned} u^h = e^{\frac{a}{2}(x-x_r) + \frac{b}{2}(y-y_r)} & \left[e^{||\mathbf{k}|| (x-x_r)} v_1 + e^{||\mathbf{k}|| (y-y_r)} v_2 + e^{-||\mathbf{k}|| (x-x_r)} v_3 + e^{-||\mathbf{k}|| (y-y_r)} v_4 \right. \\ & + \cosh(\sqrt{2}||\mathbf{k}|| (x-x_r)) \{ \cos(||\mathbf{k}|| (y-y_r)) v_5 + \sin(||\mathbf{k}|| (y-y_r)) v_6 \} \\ & \left. + \cosh(\sqrt{2}||\mathbf{k}|| (y-y_r)) \{ \cos(||\mathbf{k}|| (x-x_r)) v_7 + \sin(||\mathbf{k}|| (x-x_r)) v_8 \} \right] \end{aligned} \quad (6)$$

The v_i are the enrichment dofs to be solved for and (x_r, y_r) is an arbitrary reference point, added to counteract the ill-conditioning of the element matrices.

It can be shown from the variational form (3) that the LMs should be

¹ For the homogeneous equation, we drop the Galerkin polynomial field u^P .

taken as boundary normal derivatives of the enrichment field: $\lambda^h = \frac{\partial u_e^h}{\partial \nu} = -\frac{\partial u_e^h}{\partial \nu}$ on $\Gamma^{e,e'}$. In order for the system (5) to have a solution, the number of LMs should be limited to $\leq n_{\text{eq}}$, the number of equations stemming from the enrichment basis. Asymptotically, this translates to a global limit of $n^\lambda \leq \frac{n^E}{2}$ on the number of LMs. Unfortunately, the number of linearly independent LMs computed as normal derivatives of (6) does not satisfy this global bound. There are 6 LM dofs/edge on the transformed domain Ω'^2 :

$$\begin{aligned} \lambda_1^{tb} &= e^{ax'}, \lambda_2^{tb} = e^{\frac{a}{2}x'}, \lambda_3^{tb} = 1, \lambda_4^{tb} = e^{\frac{a}{2}x'} \cosh\left(\frac{a\sqrt{2}}{2}x'\right), \\ \lambda_5^{tb} &= e^{\frac{a}{2}x'} \cos\left(\frac{a\sqrt{2}}{2}x'\right), \lambda_6^{tb} = e^{\frac{a}{2}x'} \sin\left(\frac{a\sqrt{2}}{2}x'\right) \\ \lambda_1^{lr} &= 1, \lambda_2^{lr} = e^{\frac{a}{2}y'}, \lambda_3^{lr} = e^{-\frac{a}{2}y'}, \lambda_4^{lr} = \cos\left(\frac{a}{2}y'\right), \lambda_5^{lr} = \sin\left(\frac{a}{2}y'\right), \lambda_6^{lr} = \cosh\left(\frac{a\sqrt{2}}{2}y'\right) \end{aligned} \quad (7)$$

Choosing which LMs to include in the LM basis is not so obvious a priori. Sending $h \rightarrow 0$, observe that

$$\lambda_6^{tb} \rightarrow \lambda_3^{tb}, \quad \lambda_4^{tb}, \lambda_5^{tb} \rightarrow \lambda_2^{tb}, \quad \lambda_4^{lr}, \lambda_5^{lr}, \lambda_6^{lr} \rightarrow \lambda_1^{lr} \quad (8)$$

These limits suggest keeping λ_2^{tb} and λ_3^{tb} on the top/bottom edges and λ_1^{lr} on the left/right edges and motivates the construction of the so-called $R - 8 - 2$ element, a rectangular element with 8 enrichment basis functions and 2 LMs/edge. Implementation confirms that this theoretically “best” choice of LMs does indeed produce the smallest error in the DEM solution. Although the bound $n^\lambda \leq \frac{n^E}{2}$ would allow as many as 4 LMs/edge, in practice, using more than 2 results in severe ill-conditioning of the global matrices, which takes away any benefit the additional LMs would have in theory. If static condensation is implemented, the asymptotic number of dofs is $4n^2$, making the element comparable to the Galerkin Q_2 element in computational cost.

3 Numerical results

Since the DEM uses free-space solutions of the governing PDE, it is expected to deliver a very accurate solution using few elements. Preliminary numerical results show just that. The plots below compare the standard Galerkin FEM solution using the Q_1 element and the DEM solution at $n_{el} = 100$ and $a = b = 25$. One can see clearly that the Galerkin FEM solution exhibits spurious oscillations and overshoots the exact solution (plotted in black) in the region of the sharp gradient. Indeed, at $n_{el} = 100$, the L^2 relative error for the $R - 8 - 2$ element is of order 10^{-15} , compared to 8% for the Q_1 and 3% for the Q_2 . Because the true solution to the BVP is represented in the DEM basis, it turns out that a single element having only 12 LM dofs will yield the exact solution, continuous to machine precision, with a relative error of order

² λ^{tb} are the LMs on the top/bottom edges and λ^{lr} are those on the left/right edges. The LMs in (7) are computed for $b = 0$, since the transformation discussed at the beginning of §2 shows that (*) is equivalent to a problem on $\Omega' = [0, 1] \times [0, 1]$ with zero y advection coefficient.

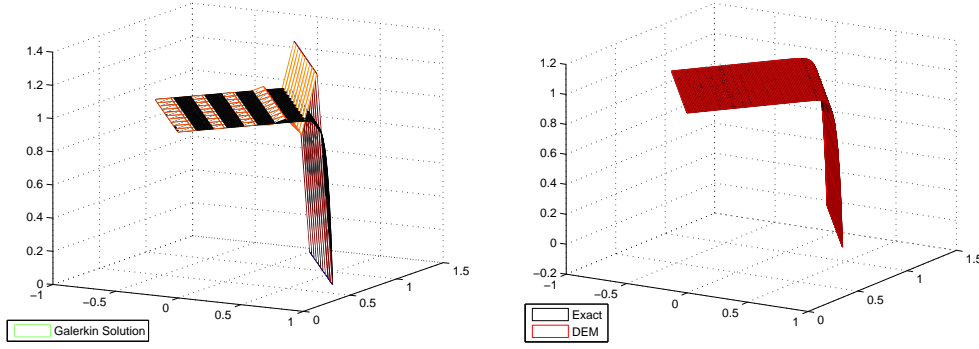


Fig. 2. Q_1 solution vs. u_{ex} (left), $R - 8 - 2$ solution vs. u_{ex} (right)

10^{-9} . The Q_1 element, on the other hand, requires almost 800 dofs and a Q_2 around 625 dofs to achieve only a 1% relative error.

4 Conclusions

In this paper, the discontinuous enrichment method [1] is extended to the advection-diffusion equation. The enrichment basis, spanned by free space solutions to the governing homogeneous PDE, is derived. Continuity across element boundaries is enforced weakly using LMs, taken as normal traces of the enrichment field. Following a discussion of the choice of the LM field, the $R - 8 - 2$ element is constructed and its numerical performance is evaluated.

As in the 2D Helmholtz equation considered in [1], preliminary testing on the 2D Neumann-Dirichlet BVP shows that the DEM solution is continuous and exact to machine precision whereas spurious oscillations pollute the Galerkin solution for larger Peclet numbers. The numerical results and plots presented above demonstrate the potential of the method, suggesting that it is a cost effective choice for cases in which the standard finite element method runs into difficulties. The chosen BVP is a simple model problem and the best case scenario for DEM; the full version of this paper will report on similar results for more challenging problems. Since the ultimate goal is to extend the DEM to the more challenging Navier-Stokes equations, arguably the most important equations in fluid flow, research in this domain has a significant potential for improving finite element computations in the field of fluid mechanics.

References

- [1] C. Farhat, I. Harari, L. P. Franca. The discontinuous enrichment method. *Comput. Methods Appl. Mech. Engrg.*, **190**:455-6479, 2001.