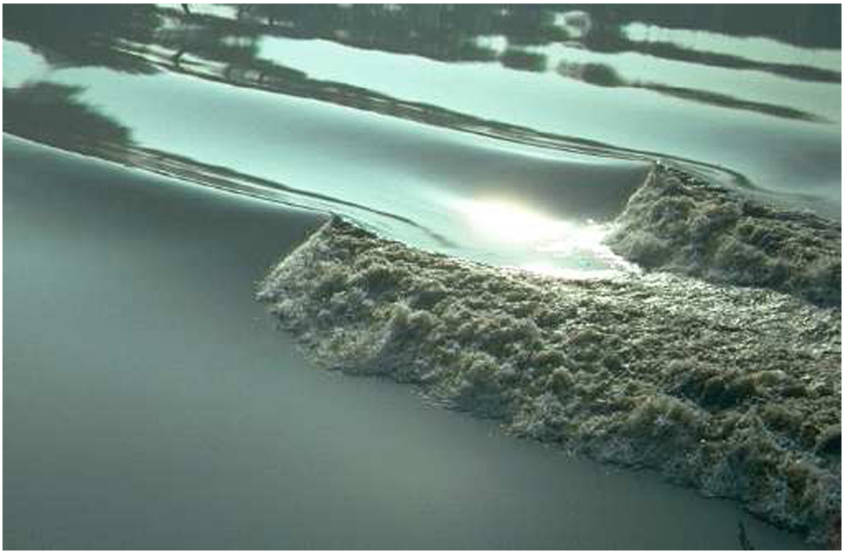

Tidal bore in Severn River (Peregrine, 1981). Reproduced with permission by Cambridge University Press

Chapter 4

Shallow-Water Approximation

We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearances. To this purpose the philosophers say that Nature does nothing in vain, and more is in vain when less will serve; for Nature is pleased with simplicity, and affects not the pomp of superfluous causes.

Isaac Newton, Principia, Vol. 2, 1686

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4.1 INTRODUCTION

The free-surface flow equations developed in Chapter 2 are nonlinear partial differential equations. Even under the assumptions of irrotationality and zero viscosity, which result in the linear Laplace equation, the free-surface boundary conditions introduce a nonlinear behavior in the description of the problem that does not allow an analytical solution. Alternatively, we my seek an approximation based on examination of the relative magnitude of the physical dimensions of the flow field. By choosing relevant scales of the flow problem, we can recast the governing equations in a form that reveals the true influence of the factors controlling the flow. If there exists a small parameter in the problem, it is logical to expect that an expansion in a power series of this parameter will converge rapidly, and therefore the expansion should lead to an asymptotic approximation. Then, if more terms of this expansion are included, the approximation will yield results almost identical to the exact solution.

In most cases, it is difficult to assess the relative effects of the various terms in the equations of continuity, momentum, and energy unless the dependent and independent variables are normalized with respect to some physically meaningful quantities. The resulting dimensionless equations contain partial derivatives that are usually of order one. If certain terms are then accompanied by dimensionless parameters that are very small compared to unity, these terms may be neglected altogether, leading to the so-called lowest order of approximation.

Selecting characteristic quantities for rendering the governing equations dimensionless is not an easy task, especially if one does not already know a great deal about the physics of the problem in question. In other words, it is unlikely that some earth-shaking perturbation parameters will emerge out of a random scaling of the equations for fluid flow, unless we have some idea about the nature of the parameters. In establishing this necessary link between the mathematics and the physics of the problem, one must rely on linearized equations for which analytical solutions can be found and studied. In the next section we proceed to construct the simplest possible analytical solution for a linear free-surface flow problem, and attempt to utilize the properties of this solution in our search for a rational approximation of the complete nonlinear equations.

4.2 SHALLOW-WATER EQUATIONS

We proceed with the derivation of the shallow-water approximation for waves of finite amplitude by means of dimensional analysis. This is facilitated by rewriting the hydrodynamic equations in dimensionless form, in which the magnitude of individual terms can be explicitly evaluated.

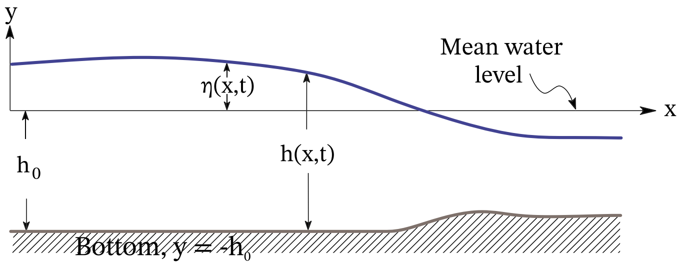

FIGURE 4.1 Definition sketch for shallow-water flow

Consider the flow a shallow stream, as shown in Fig. 4.1. The x axis is along the horizontal, streamwise direction, and the y axis is along the vertical. For simplicity, the bottom is shown to be horizontal, with a constant depth, h_0. However, an uneven bottom does not affect the analysis, as it will be shown later. The free-surface elevation is given by $\eta(x, z)$ although only its streamwise variation is shown in the figure. The transverse horizontal direction is normal to the figure, and usually does not present any conceptual difficulties.

The choice of proper normalizing quantities is essential to the success of the small perturbation analysis, and must account for the dynamic changes of the flow variables. The geometric difference between the horizontal and vertical scales in shallow-water flow suggests that the spatial coordinates should be normalized as follows

$$x_* = \frac{x}{L} \qquad y_* = \frac{y}{h_0} \qquad z_* = \frac{z}{L} \tag{4.1}$$

where L is a characteristic length associated with the streamwise extent of the channel. The wave amplitude is allowed to be of the same order as the depth of flow, thus we can introduce its dimensionless counterpart as follows

$$\eta_* = \frac{\eta}{h_0} \qquad h_* = \eta_* + 1 \tag{4.2}$$

where h_* is the dimensionless thickness of the flow layer, as defined by Eq. (3.1). The selection of a characteristic velocity is not as trivial, but an obvious choice is the asymptotic value that the wave speed of small amplitude approaches in shallow water. Then, the dimensionless horizontal velocity components read

$$u_* = \frac{u}{\sqrt{gh_0}} \qquad w_* = \frac{w}{\sqrt{gh_0}} \tag{4.3}$$

Recognition of the fact that in shallow water the ratio of vertical to horizontal velocity is of order h_0/L, allows us to introduce the following expression for the vertical velocity

$$v_* = \frac{vL}{\sqrt{gh_0^3}} \tag{4.4}$$

Finally, the corresponding definitions for dimensionless time and pressure are conveniently adopted as follows

$$t_* = \frac{t\sqrt{gh_0}}{L} \qquad\qquad p_* = \frac{p}{\rho g h_0} \tag{4.5}$$

When the above dimensionless variables are substituted in the incompressible continuity equation, i.e. Eq. (I-5.12), and the momentum equations on a rotating frame, as given by Eq. (I-10.138), the following result is obtained

$$\frac{\partial u_*}{\partial x_*} + \frac{\partial v_*}{\partial y_*} + \frac{\partial w_*}{\partial z_*} = 0 \tag{4.6}$$

$$\begin{aligned}
\frac{\partial u_*}{\partial t_*} + u_*\frac{\partial u_*}{\partial x_*} + v_*\frac{\partial u_*}{\partial y_*} + w_*\frac{\partial u_*}{\partial z_*} &= -\frac{\partial p_*}{\partial x_*} + f w_* \\
\left(\frac{h_0}{L}\right)^2 \left(\frac{\partial v_*}{\partial t_*} + u_*\frac{\partial v_*}{\partial x_*} + v_*\frac{\partial v_*}{\partial y_*} + w_*\frac{\partial v_*}{\partial z_*}\right) &= -1 - \frac{\partial p_*}{\partial y_*} \\
\frac{\partial w_*}{\partial t_*} + u_*\frac{\partial w_*}{\partial x_*} + v_*\frac{\partial w_*}{\partial y_*} + w_*\frac{\partial w_*}{\partial z_*} &= -\frac{\partial p_*}{\partial z_*} - f u_*
\end{aligned} \tag{4.7}$$

In Eqs. (4.7), we have omitted the stress terms without justification in order to simplify the presentation. The viscous and turbulent stresses will be included in a later section without a major conceptual revision of our analysis, therefore presently we can concentrate on the behavior of the inertial, gravitational, Coriolis, and pressure forces under shallow water conditions.

It is clear from Eqs. (4.7) that as the ratio h_0/L becomes small, the governing equations of flow can be approximated by neglecting all terms of order h_0^2/L^2. This results in the shallow-water equations of lowest order, in which the vertical acceleration is negligible. This reduces the vertical momentum equation to the hydrostatic pressure condition, and allows for the consideration of vertically-averaged horizontal velocities.

Notice that the remaining dimensionless terms are not necessarily of order one in all cases. However, because the ratio h_0/L is typically very small, minor discrepancies in the magnitude of other terms play no role in this lowest-order approximation. As we will see later, however, this may not be true for higher-order approximations, in which case the magnitude of all dimensionless terms must be assessed before any simplifications can be made.

4.2.1 Depth-Averaged Equations

Since acceleration in the vertical direction is negligible, the vertical velocity may be assumed to vary slowly. Notice that in the three-dimensional problem, v is not zero for otherwise the free surface could not rise or fall. On the other hand, u and w may be integrated over the depth to yield depth-averaged quantities without loss of accuracy.

4.2.1.1 Equation of Continuity

We first integrate the continuity equation in the vertical direction, and eliminate the vertical component of velocity altogether. Returning to dimensional variables, integration of Eq. (4.6) from the channel bottom to the free surface yields

$$\int_{-h_0}^{\eta} \frac{\partial u}{\partial x}\, dy + \int_{-h_0}^{\eta} \frac{\partial w}{\partial z}\, dy = -\int_{-h_0}^{\eta} \frac{\partial v}{\partial y}\, dy \tag{4.8}$$

The integral on the right hand side can be evaluated by means of the fundamental theorem of calculus to give

$$-\int_{-h_0}^{\eta} \frac{\partial v}{\partial y}\, dy = -v\big|_{y=\eta} + v\big|_{y=-h_0} \tag{4.9}$$

The integration of the left hand side of Eq. (4.8) can be performed by recalling the Leibniz rule, described in section I-2.9.4. Thus, the horizontal components of the continuity equation can be written as follows

$$\begin{aligned}\int_{-h_0}^{\eta} \frac{\partial u}{\partial x}\, dy + \int_{-h_0}^{\eta} \frac{\partial w}{\partial z}\, dy &= \frac{\partial}{\partial x}\int_{-h_0}^{\eta} u\, dy + \frac{\partial}{\partial z}\int_{-h_0}^{\eta} w\, dy \\ &\quad - u\big|_{y=\eta}\frac{\partial \eta}{\partial x} - w\big|_{y=\eta}\frac{\partial \eta}{\partial z} - u\big|_{y=-h_0}\frac{\partial h}{\partial x} - w\big|_{y=-h_0}\frac{\partial h_0}{\partial z}\end{aligned} \tag{4.10}$$

Then, equating the right hand sides of Eqs. (4.9) and (4.10) yields

$$\begin{aligned}\frac{\partial}{\partial x}\int_{-h_0}^{\eta} u\, dy + \frac{\partial}{\partial z}\int_{-h_0}^{\eta} w\, dy &= \left(-v\big|_{y=\eta} + u\frac{\partial \eta}{\partial x} + w\frac{\partial \eta}{\partial z}\right) \\ &\quad + \left(v\big|_{y=-h_0} + u\frac{\partial h_0}{\partial x} + w\frac{\partial h_0}{\partial z}\right)\end{aligned} \tag{4.11}$$

The first group of terms on the right hand side is equal to $-\frac{\partial \eta}{\partial t}$ because of the kinematic free-surface condition, i.e. Eq. (3.6). A similar kinematic condition can be invoked for the bottom, which makes the second group of terms on the right vanish. A final simplification of Eq. (4.11) is possible by making the following definitions. Let

$$q_x = \int_{-h_0}^{\eta} u\, dy \qquad\qquad q_z = \int_{-h_0}^{\eta} w\, dy \tag{4.12}$$

be the discharge per unit width in the x and z directions, respectively. Then the depth-averaged continuity equation under the assumption of shallow-water flow takes the following form

$$\frac{\partial \eta}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_z}{\partial z} = 0 \tag{4.13}$$

Alternatively, the continuity equation can be written in terms of the total depth, h, and the depth-averaged velocities, $\bar{u}$ and $\bar{w}$, as follows

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(\bar{u}h) + \frac{\partial}{\partial z}(\bar{w}h) = 0 \tag{4.14}$$

where the depth-averaged horizontal velocities are defined as follows

$$\bar{u} = \frac{q_x}{h} \qquad\qquad \bar{w} = \frac{q_z}{h} \tag{4.15}$$

4.2.1.2 Equation of Streamwise Momentum

In similar fashion, the horizontal momentum equations, i.e. the second and third of Eqs. (4.7), can be integrated over the vertical, as follows. Since terms of order h_0^2/L^2 have been dropped, the expression for the pressure profile is obtained by integration of the hydrostatic equation over the total depth, i.e.

$$\int_p^{p_0} dp = -\rho g \int_y^{\eta} dy \tag{4.16}$$

where the pressure at the free surface was given the value of p_0 by means of the dynamic surface condition. Then

$$p = p_0 + \rho g(\eta - y) \tag{4.17}$$

The hydrostatic pressure thrust over the vertical is found by integrating once more over the depth, as follows

$$\int_{-h_0}^{\eta} p\,dy = \frac{1}{2}\rho g(\eta + h_0)^2 = \frac{1}{2}\rho g h^2 \tag{4.18}$$

Finally, under the present assumption of negligible viscous effects, and a horizontal channel bed, the Bernoulli equation, Eq. (I-6.77) can be written as follows

$$\frac{\bar{V}^2}{2g} + \frac{p}{\rho} + \zeta = \frac{\bar{V}^2}{2g} + h = \text{const} \tag{4.19}$$

Following these preliminary calculations, the horizontal momentum equations are next integrated from the bottom to the free surface. For the x-component, we obtain

$$\int_{-h_0}^{\eta}\left[\frac{\partial u}{\partial t}+\frac{\partial(u^2)}{\partial x}+\frac{\partial(uv)}{\partial y}+\frac{\partial(uw)}{\partial z}\right]dy=-\frac{1}{\rho}\int_{-h_0}^{\eta}\frac{\partial p}{\partial x}\,dy+\int_{-h_0}^{\eta}f w\,dy \tag{4.20}$$

where Eq. (4.6) was used to express the convective terms in divergence form. The integration once again employs the Leibniz rule, i.e. Eq. (I-2.111), and the kinematic free-surface and bottom conditions, as it was earlier shown in Eq. (4.10). Then, setting $p_0 = 0$ for convenience, the result reads

$$\frac{\partial}{\partial t}(\bar{u}h)+\frac{\partial}{\partial x}\left(\bar{u}^2 h\right)+\frac{\partial}{\partial z}(\bar{u}\bar{w}h)=-\frac{1}{2}g\frac{\partial h^2}{\partial x}+f q_z \tag{4.21}$$

where the depth-averaged momenta per unit mass are given by

$$\bar{u}^2=\frac{1}{h}\int_{-h_0}^{\eta}u^2\,dy \qquad \bar{u}\bar{w}=\frac{1}{h}\int_{-h_0}^{\eta}uw\,dy \tag{4.22}$$

It is convenient to also write the momentum equation in terms of the primitive variables. Then, by expanding the products in the derivatives, and using the continuity equation, i.e. Eq. (4.14), we obtain

$$\frac{\partial\bar{u}}{\partial t}+\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{w}\frac{\partial\bar{u}}{\partial z}=-g\frac{\partial h}{\partial x}+f\bar{u} \tag{4.23}$$

4.2.1.3 Equation of Transverse Momentum

Following a similar process, we can derive the z-component of the momentum equation, as follows

$$\frac{\partial}{\partial t}(\bar{w}h)+\frac{\partial}{\partial x}(\bar{u}\bar{w}h)+\frac{\partial}{\partial z}\left(\bar{w}^2 h\right)=-\frac{1}{2}g\frac{\partial h^2}{\partial z}-f q_x=0 \tag{4.24}$$

where

$$\bar{w}^2=\frac{1}{h}\int_{-h_0}^{\eta}w^2\,dy \tag{4.25}$$

Finally, the corresponding equation with primitive variables reads

$$\frac{\partial\bar{w}}{\partial t}+\bar{u}\frac{\partial\bar{w}}{\partial x}+\bar{w}\frac{\partial\bar{w}}{\partial z}=-g\frac{\partial h}{\partial z}+f\bar{w} \tag{4.26}$$

4.2.1.4 Vector Form of Shallow-Water Equations

It is often useful to express the continuity and momentum equations for shallow-water flow in vector form. To this end, let $\bar{\mathbf{V}}=\left(\bar{u}\quad\bar{w}\right)^T$ be the depth-averaged, horizontal velocity vector. Then, the material derivative in shallow water can be written as follows

$$\frac{D_s}{Dt} = \frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} + \bar{w}\frac{\partial}{\partial z} \tag{4.27}$$

Therefore, the continuity equation can be written, as follows

$$\frac{D_s h}{Dt} = -h\left(\nabla \cdot \bar{\mathbf{V}}\right) \tag{4.28}$$

Similarly, the horizontal momentum equations in vector form read

$$\frac{D_s \bar{\mathbf{V}}}{Dt} = -\mathbf{f} \times \bar{\mathbf{V}} - g\nabla h \tag{4.29}$$

where $\mathbf{f} = f\mathbf{k}$ is the Coriolis parameter vector. In vector form, the shallow-water equations are clearly evolution equations for the depth and horizontal velocity. The dependent variables, h and $\bar{\mathbf{V}}$ appear in both equations, which are now fully coupled.

We conclude that in shallow water, the vertical acceleration is negligible, and that the pressure distribution remains hydrostatic even under waves of finite amplitude, as indicated by the presence of the nonlinear convective terms in Eqs. (4.21) and (4.24). As a result, the pressure has been replaced by the piezometric head or depth of flow as the dependent variable. Furthermore, the depth-averaged horizontal velocity is independent of the vertical coordinate, and the free-surface and bottom boundary conditions are implicitly satisfied.

Furthermore, recall that Eq. (3.34) indicates that the parameter h_0/L is a measure of wave dispersion, i.e. determines whether dispersion is strong or weak. However, the shallow-water approximation prevents any possibility of wave dispersion, regardless of the size of the ratio h_0/L. In fact, all waves under the shallow-water approximation travel with the same speed regardless of their frequency.

Finally, a significant change has been made in the mathematical content of the problem. The continuity equation is now a prognostic equation for the depth of flow, a feature that makes Eq. (4.13) fundamentally different from Eq. (I-5.12), which is only a diagnostic equation for the pressure. In this regard, the shallow-water equations are analogous to the compressible equations in gas dynamics than the incompressible Navier-Stokes equations. Equivalently, the elliptic equations of incompressible viscous flow with a free-surface have been transformed to hyperbolic equations with real characteristics. The mathematical implications are enormous. Although there are obvious simplifications in the reduction of the vertical dimension, there are also new challenges in the description of shallow-water waves.

The shallow-water equations were originally derived by Jean-Claude Barré de Saint-Venant (Barré de Saint-Venant, 1871), and they represent the cornerstone of what we commonly call *open-channel-flow*, a subject that will be pursued in detail later.

St.-Venant

Jean-Claude Barré de Saint-Venant (1797–1886) was a French engineer and mathematician, who developed the theory of unsteady flow in open channels. The corresponding governing equations are known today as "the Saint-Venant equations". They are the one-dimensional version of the shallow-water equations or, equivalently, an extension of the Euler equations that includes bed slope and resistance, thus creating a model for unsteady flow in a real channel. Barré de Saint-Venant entered the École Polytechnique when he was 16, and graduated at 19. For 27 years he worked as a civil engineer at the Corps des Ponts et Chaussées.

In 1850, Barré de Saint-Venant was appointed professor of Agricultural Engineering at the Agronomic Institute of Versailles. Although de Saint-Venant is known today for his equations of open-channel flow, his most brilliant work concerned the derivation of what we call "the Navier-Stokes equations". In 1843, Saint-Venant re-derived the equations of viscous flow, correcting the viscous stress terms, and for the first time identifying the viscosity coefficient. He was also the first to recognize that the product of viscosity and the velocity gradient acts as an internal stress in fluid flow. Stokes also derived the "Navier-Stokes equations," but he published the results two years after de Saint-Venant. Unfortunately, the English literature carried out an unexplainable injustice by adding the name of Stokes to what we should call the "Navier-de Saint Venant equations."

4.2.2 The Gas Dynamics Analogy

In the previous sections, the shallow-water equations were derived by averaging the incompressible Navier-Stokes equations over the depth of flow. This resulted in a simpler description of long waves in open channels without any significant loss of accuracy. However, the nature of the governing equations has changed dramatically. In fact, the shallow-water equations accept solutions of elevation and depression waves that are analogous to compression and rarefaction waves in gas dynamics. This analogy has led to benefits to both branches of fluid mechanics. As a result, analytical developments in gas dynamics find direct application in open-channel flow, and experiments in shallow-water flow are used to model the behavior of complex gas flows (Crossley, 1949).

It is important to quantify the analogy between the compressible gas and shallow-water regimes. To this end, let us consider gas flow with negligible viscous effects and heat transfer through the boundaries. Therefore, the isentropic relation for a perfect gas, Eq. (I-2.171), allows the density to be expressed as a function of pressure alone. Then, the following analogies become possible for two-dimensional flow.

Continuity Equation

The compressible gas continuity equation, Eq. (I-5.8), may be compared to the shallow water continuity equation, Eq. (4.14), as follows

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{V}) = 0 \qquad \Longleftrightarrow \qquad \frac{D_s h}{Dt} + h\left(\nabla \cdot \bar{\mathbf{V}}\right) = 0 \tag{4.30}$$

Thus, if $\rho \Longleftrightarrow h$, the two equations become analogous.

Momentum Equation

Similarly, if we neglect shear stresses, the momentum equation for a compressible gas, Eq. (I-5.51), can be written as follows

$$\frac{D}{Dt}(\rho \mathbf{V}) = -\nabla p \tag{4.31}$$

Notice that gravitational effects were also neglected since they are not significant in gas dynamics. Use of the continuity equation allows us to further write the momentum equation as follows

$$\frac{D\mathbf{V}}{Dt} = -\frac{1}{\rho}\nabla p = -\frac{1}{\rho}\frac{dp}{d\rho}\nabla \rho \tag{4.32}$$

Next, enforcing the isentropic relation for an ideal gas, i.e. Eq. (I-2.171), we can rewrite the density-pressure dependence as follows

$$\frac{dp}{d\rho} = c\gamma_h \rho^{\gamma_h - 1} \tag{4.33}$$

where c is some arbitrary constant. Then, the momentum equation for isentropic flow of a compressible gas can be contrasted to the momentum balance in shallow water, Eq. (4.29), with the Coriolis effect neglected, as follows

$$\frac{D\mathbf{V}}{Dt} = -c\gamma_h \rho^{\gamma_h - 2}\nabla \rho \qquad \Longleftrightarrow \qquad \frac{D_s \bar{\mathbf{V}}}{Dt} = -g\nabla h \tag{4.34}$$

Therefore, if water is interpreted as an ideal gas with $\gamma_h = 2$, and if we set $c\gamma_h = g$, the compressible momentum equation becomes analogous to the corresponding shallow water equation with $\rho \Longleftrightarrow h$.

Bernoulli Equation

For an ideal gas, the Bernoulli equation, Eq. (I-6.76), can be written as follows

$$e + \frac{p}{\rho} + \frac{1}{2}V^2 + g\zeta = \text{const} \tag{4.35}$$

where e is the internal energy of the gas, given by Eq. (I-2.133). Under typical temperatures and small elevation differences, the potential energy may be neglected when compared to the other terms in the Bernoulli equation. Furthermore, for an ideal gas, Eq. (I-2.143) allows us to contrast Eq. (4.35) to Eq. (4.19) as follows

$$C_pT + \frac{V^2}{2} = \text{const} \qquad \Longleftrightarrow \qquad gh + \frac{V^2}{2} \tag{4.36}$$

Therefore, the analogy is complete if $gh \Longleftrightarrow C_pT$.

Vorticity Equation

For an ideal gas, the vorticity transport equation, Eq. (I-7.49), can be simplified for frictionless flow such that it can be directly contrasted to the potential vorticity equation in shallow water, i.e. Eq. (4.42). Therefore

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = 0 \qquad \Longleftrightarrow \qquad \frac{D_s}{Dt}\left(\frac{\xi}{h}\right) = 0 \tag{4.37}$$

where it is implied that the gas vorticity is limited to the horizontal plane. The analogy between gas dynamics and shallow-water flow is once again established if $\rho \Longleftrightarrow h$.

Pressure Forces

The hydrostatic pressure thrust over the depth of flow, Eq. (4.18), should be analogous to the average pressure force, p, over a two-dimensional layer of a compressible gas, if the analogy between isentropic gas flow and shallow water is to be complete. Therefore, the following relation must hold true

$$p \qquad \Longleftrightarrow \qquad \frac{1}{2}\rho g h^2 \tag{4.38}$$

Equivalently, $h^2 \Longleftrightarrow 2p/\rho g$, ensures that when the depth averaged momentum equation is written over a finite control volume, the forces acting on the control surface faces are analogous. It should be noted that this analogy is true for continuous flows and flows with weak shocks, but becomes inaccurate for strong shocks. This will be further discussed in Chapter 12.

4.2.3 Vorticity Transport in Shallow Water

In the foregoing analysis and derivation of the shallow-water equations, it has been assumed for simplicity that viscous effects and fluid vorticity are negligible. Neither of these assumptions affect the shallow-water approximation, and therefore their contribution can be added to the equations at a later time. However, it is important to mention that in shallow water, fluid rotation is limited to the horizontal plane, therefore the depth-averaged vorticity can be defined as follows

$$\bar{\boldsymbol{\omega}} = \left(\frac{\partial \bar{w}}{\partial x} - \frac{\partial \bar{u}}{\partial z}\right)\mathbf{k} = \xi \mathbf{k} \tag{4.39}$$

The vorticity transport equation can then be derived following the procedure outlined in section I-7.6. Let us ignore viscous effects, and assume that the Coriolis parameter is constant, i.e. $f = f_0$. Then, taking the curl of the horizontal momentum equation, i.e. (4.29), we obtain

$$\frac{\partial \bar{\boldsymbol{\omega}}}{\partial t} + \left(\bar{\mathbf{V}} \cdot \nabla\right)\bar{\boldsymbol{\omega}} + \bar{\boldsymbol{\omega}}\left(\nabla \cdot \bar{\mathbf{V}}\right) = 0 \tag{4.40}$$

The last term can be eliminated by means of the continuity equation, (4.28). Then, use of the definition, (4.39), results in the following expression

$$\frac{\partial \xi}{\partial t} + \left(\bar{\mathbf{V}} \cdot \nabla\right)(\xi + f_0) - \frac{\xi + f_0}{h}\frac{D_s h}{Dt} = 0 \tag{4.41}$$

Therefore, recognizing that the first two terms comprise a material derivative in shallow water, inviscid vorticity transport is described by

$$\frac{D_s}{Dt}\left(\frac{\xi + f_0}{h}\right) = 0 \tag{4.42}$$

The ratio $(\xi + f_0)/h$ is called the *potential vorticity*, and represents a quantity that is conserved in shallow-water flows when viscous effects are negligible.

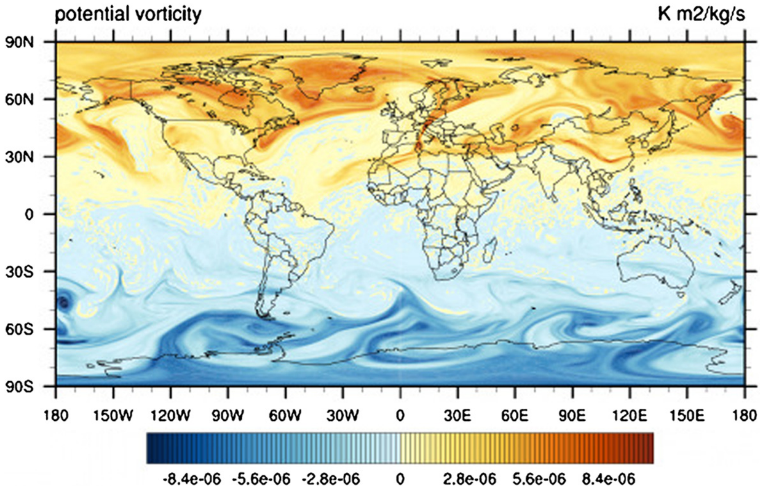

FIGURE 4.2 Snapshot of potential vorticity contours in the upper troposphere (200 *mb* level) from an Accelerated Climate Modeling for Energy (ACME) project simulation. Courtesy of Oak Ridge National Laboratory, U.S. Dept. of Energy

Fig. 4.2 shows the computed potential vorticity in the Earth's atmosphere. The potential vorticity shows a wide variation with latitude. Specifically, the potential vorticity is zero at the Equator while its magnitude reaches a maximum at the poles.

4.3 WAVES IN SHALLOW WATER

The shallow-water equations accept solutions that represent wave motion under the influence of various forces. By definition, these are all long waves since any significant curvature in the free surface has been disallowed.

4.3.1 Gravity Waves

As first example of the type of waves that can be expected in shallow water, we focus on small amplitude waves in the absence of rotation. Let us consider a horizontal channel, initially at rest with uniform depth h_0, and $u_0 = 0$. Then, a small perturbation is introduced, such that $h = h_0 + h'$, and $u = u_0 + u'$. The resulting wave motion is then counteracted by gravity that tries to restore the initial state, thus we speak of *gravity waves*. Introducing the foregoing perturbations in the continuity and momentum equations, allows a linearization that can be written as follows

$$\begin{aligned} \frac{\partial h'}{\partial t} + h_0 \frac{\partial u}{\partial x} = 0 \\ \frac{\partial u'}{\partial t} + g \frac{\partial h'}{\partial x} = 0 \end{aligned} \tag{4.43}$$

Cross differentiation of these equations, and elimination of the velocity derivatives leads to the linearized wave equation, i.e.

$$\frac{\partial^2 h'}{\partial t^2} = g h_0 \frac{\partial^2 h'}{\partial x^2} \tag{4.44}$$

This equation admits solutions of the form

$$h'(x,t) = h_0 e^{\mathfrak{J}(kx - \omega t)} \tag{4.45}$$

where k is the wave number, and ω the radian frequency. Hence, substitution in Eq. (4.44) leads to the following dispersion relation

$$\omega = \pm \sqrt{g h_0}\, k \tag{4.46}$$

Furthermore, the phase speed of the wave is given by

$$c_0 = \frac{\omega}{k} = \sqrt{g h_0} \tag{4.47}$$

Therefore, the perturbations generate bi-directional, non-dispersive waves of small amplitude in shallow water.

4.3.2 Gravity Waves on a Rotating Earth

If the rotation is significant, the shallow-water equations can be linearized by assuming that the Coriolis parameter is constant, i.e. limiting the rotational effects to a fixed latitude. Then, the linearized wave equation reads

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial t^2}+f_0^2-c_0^2\frac{\partial^2}{\partial x^2}\right)h'=0 \tag{4.48}$$

and the associated dispersion relation can be written, as follows

$$-\jmath\omega\left(-\omega^2+f_0^2+c_0^2k^2\right)=0 \tag{4.49}$$

The solution for the radian frequency includes the trivial case of $\omega=0$, which corresponds to the standard geostrophic balance discussed in section I-10.7.2. The remaining solution describes waves traveling with a phase speed

$$c=\pm\sqrt{\left(\frac{f_0}{k}\right)^2+c_0^2} \tag{4.50}$$

These are linear dispersive waves that are often called *Poincaré waves*, after the French mathematician Henri Poincaré (1854–1912), who made many significant contributions to pure and applied mathematics. The wave speed is now dependent on the wave length, with longer waves traveling faster while the group velocity is given by

$$C_g=\frac{kc_0^2}{\sqrt{f_0^2+k^2c_0^2}} \tag{4.51}$$

Notice that for small wave lengths, the effects of rotation become negligible, and we recover the speed of elementary gravity waves. At the other end of the spectrum, gravitational effects diminish for very long waves, thus the motion becomes an inertial oscillation, as described in section I-10.7.1.

4.3.3 Gravity Waves Along the Coast

Poincaré waves are inherently two-dimensional because the free-surface slope in the x direction induces a velocity component, w', in the transverse direction due to the Coriolis acceleration. This presents no problems in the foregoing analysis, except when plane gravity waves travel next to a solid boundary, such as a coastline parallel to the x axis, which enforces the boundary condition $w'=0$. Therefore, next to the boundary, the free surface should have a transverse slope given by the z component of the momentum equation that now reads

$$f_0u'+g\frac{\partial h'}{\partial z}=0 \tag{4.52}$$

A gravity wave parallel to a wall will thus have a solution described by a disturbance traveling with speed $\pm c_0$, but with its amplitude decaying exponentially away from the wall, as follows

$$h'(x,z,t) = e^{-\frac{zf_0}{c_0}} e^{\dot{\Im}(x-c_0t)} \tag{4.53}$$

These are called *Kelvin waves*, after William Thomson or Lord Kelvin (1824–1907), who was an Irish physicist. Notice that only the progressive gravity wave yields a physical solution, i.e. a wave that decays away from the wall. Thus, in the northern hemisphere the Kelvin wave always propagates with the coastline on the right hand side, and decays away from it, as shown in Fig. 4.3. The opposite is true in the southern hemisphere.

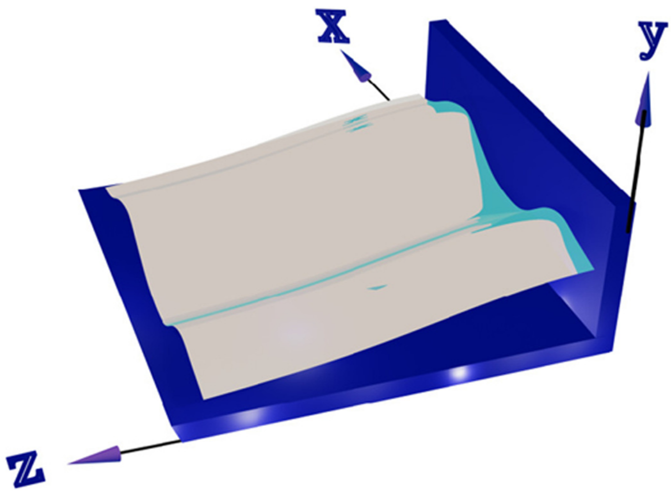

FIGURE 4.3 Kelvin wave against a solid wall

4.3.4 Barotropic Vorticity Waves

Let us consider shallow-water waves over a horizontal bottom. We will assume that the free-surface perturbations are negligible, i.e. $h' = 0$, and the rigid-lid approximation is valid. Therefore, the continuity equation, Eq. (4.28), requires that the divergence of the horizontal velocity must vanish. Thus, we can express the velocity components in terms of the stream function, as follows

$$\bar{u} = -\frac{\partial \psi}{\partial z} \qquad \bar{w} = \frac{\partial \psi}{\partial x} \tag{4.54}$$

Under these conditions, waves may be generated as a result of the variation of the Coriolis acceleration with latitude. For simplicity, we will assume that the streamwise direction, x, is at constant latitude, and the Coriolis parameter has the constant value, f_0, at $z = 0$. Then, expanding in a Taylor series, we obtain

$$f(z) = f(0) + \left.\frac{df}{dz}\right|_0 z + \cdots = f_0 + \beta z + \mathcal{O}(z^2) \tag{4.55}$$

where $f(0) = 2\Omega \sin\theta_0$, and $\beta = 2\Omega \cos\theta_0 / r_0$, is known as the *Rossby parameter*. Here θ_0 is the latitude at $z = 0$, and r_0 is the radius of the Earth, as shown in Fig. 4.4. Under these conditions, the potential vorticity equation, i.e. Eq. (4.42), can be written as follows

$$\frac{D_s \xi}{Dt} + \beta w = 0 \tag{4.56}$$

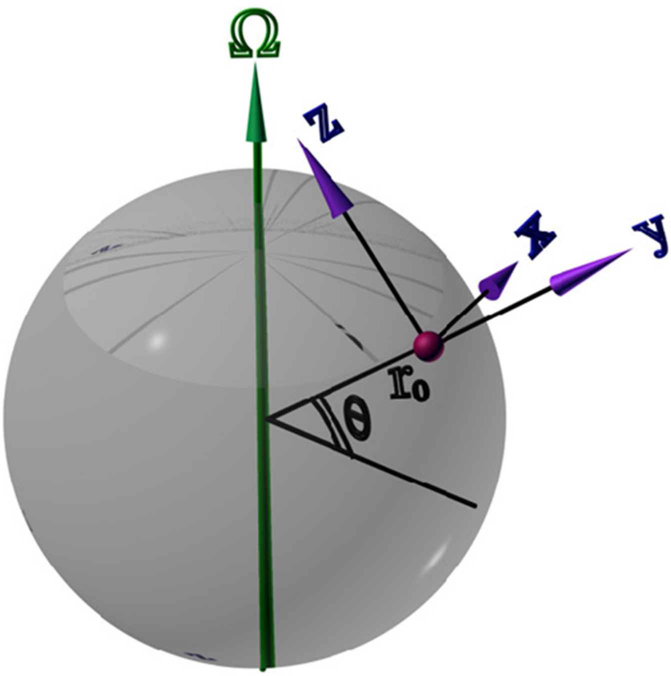

FIGURE 4.4 Rossby wave coordinate system

The vorticity equation can also be written in terms of the stream function, as follows

$$\frac{\partial \xi}{\partial t} - \frac{\partial \psi}{\partial z}\frac{\partial \xi}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \xi}{\partial z} + \beta \frac{\partial \psi}{\partial x} = 0$$

For small amplitude disturbances, the vorticity equation may be linearized. If there is a uniform base flow limited along the x direction, then $\psi_0 = -u_0 z$. Thus, a small perturbation in the stream function, i.e. $\psi = \psi_0 + \psi'$, results in a small change in vorticity, i.e. $\xi' = \nabla^2\psi'$. Therefore, the vorticity equation, (4.56), can be written in terms of the stream function, as follows

$$\frac{\partial}{\partial t}\left(\nabla^2\psi'\right) + u_0\frac{\partial}{\partial x}t\left(\nabla^2\psi'\right) + \beta\frac{\partial\psi'}{\partial x} = 0 \tag{4.57}$$

This is a generalized wave equation that admits solutions of the form (Whitham, 1974)

$$\psi'(x, z, t) = \psi_0 e^{\dot{\Im}(k_x x + k_z z - \omega t)} \tag{4.58}$$

where k_x, k_z are the wave numbers in the x and z directions, respectively. Thus, substitution in Eq. (4.57) leads to the following dispersion relation

$$\omega = u_0 k - \frac{\beta k}{k_x^2 + k_z^2} \tag{4.59}$$

The wave speed in the x direction, and the group velocity are then found, as follows

$$c = u_0 - \frac{\beta}{k_x^2 + k_z^2} \qquad C_g = u_0 - \frac{\beta(k_z^2 - k_x^2)}{(k_x^2 + k_z^2)^2} \tag{4.60}$$

The waves generated by the foregoing process are known as *Rossby waves*, after Carl-Gustaf Arvid Rossby (1898–1957), who was a Swedish meteorologist that made significant contributions to the study of large-scale flows in the atmosphere. They are considered *barotropic waves* since the motion does not change with elevation.

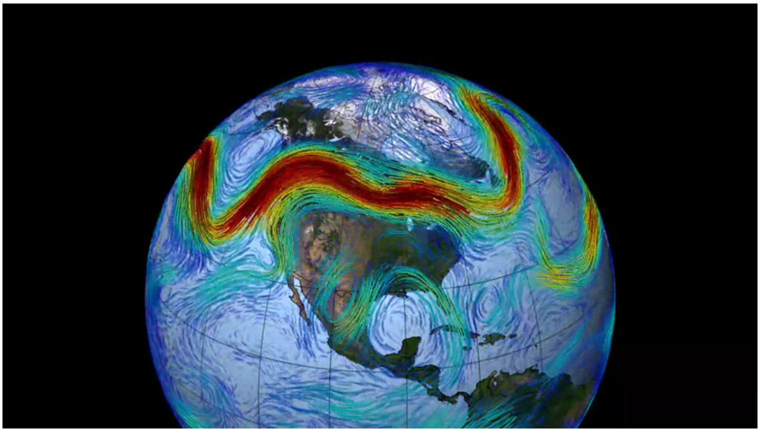

FIGURE 4.5 Rossby waves recorded by NASA's Goddard Space Flight Center along the jet stream. The colors represent the speed of the wind ranging from slowest (light blue colors) to fastest (dark red). Reproduced with permission by NASA

The primary mechanism behind the generation of Rossby waves is conservation of potential vorticity. The planetary vorticity gradient, β, is positive since the Coriolis parameter increases northward. As a fluid parcel moves to a lower latitude, where f_0 is smaller, ξ must increase since the potential vorticity needs to be conserved. As a result, cyclonic flow is induced in the northern hemisphere, an anti-cyclonic in the southern hemisphere. Therefore, the Rossby wave will move in the negative x direction, as shown in Fig. 4.5. The same is true in the southern hemisphere, once the signs of vorticity and β are taken into account.

In summary, Rossby waves are dispersive waves with a negative phase speed, and thus appear to always move westward in a motionless basic state on the surface of the Earth. These waves play a significant role in the determination of the Earth's climate, and weather patterns.

4.4 DISPERSION RELATIONS FOR NONLINEAR WAVES

As already mentioned, the shallow-water equations serve as a cut-off filter for dispersion, which under most conditions of channel flow is a reasonable approximation. There are, however, certain flow conditions that exacerbate the complete inability to include some effects of dispersion. One of the most unrealistic predictions by shallow-water models is that all waves eventually break, even on a horizontal bottom. This is not true, as first observed by the Scottish engineer named John Scott Russell (1808–1882). While conducting experiments to improve the design of navigation channels, Russell observed what we commonly call a *solitary wave*. He wrote in his report (Russell, 1845)

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped–not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".

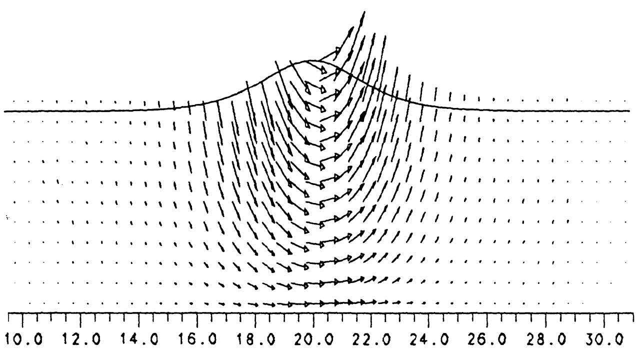

FIGURE 4.6 Solitary wave simulation by a Navier-Stokes model. Adapted from Bradford and Katopodes (1998)

The solitary wave is a wave which propagates without any change in shape or size. This is apparently accomplished by a perfect balance between nonlinearity and dispersion, which is possible when both of these wave properties are of moderate magnitude. Fig. 4.6 shows the free surface and two-dimensional velocity field of a solitary wave computed by a model based on the Navier-Stokes equations. The solution verifies that the complete hydrodynamic equations are indeed capable of preserving the shape and amplitude of the solitary wave. On the other hand, when the same wave is modeled by the shallow-water equations, the wave gradually steepens, and its permanent shape is lost.

Solitary waves represent more than a rare phenomenon in long canals. The properties of this wave are a manifestation of the fundamental properties of water waves. For example, computations show that the solitary wave retains its permanent structure, even after interacting with another solitary wave. Specifically, when two waves propagate in opposite directions, they can effectively pass through each other without breaking (Katopodes and Wu, 1986). Unfortunately, the shallow-water equations are incapable of capturing this behavior even approximately. Thus, the abrupt removal of dispersion effects from these equations leads to complete failure although the solitary is a long wave in shallow water. It is therefore important to seek equations for nonlinear dispersive waves.

The dispersion relation given by Eq. (3.34) is valid for general oscillatory gravity waves. In shallow water, the dispersion relation can be simplified, allowing the identification of dispersive wave equations that are specifically valid in shallow-water flow. This is important because dispersion can only be precisely defined for linear systems. Nonlinear dispersive systems are classified as dispersive if their linearized form admits dispersive waves as a solution. Thus, the derivation of dispersive wave equations proceeds more naturally by starting from a dispersion relation. We first render Eq. (3.34) dimensionless by introducing a reference frequency and a reference wave number, as follows

$$\Omega = \frac{\sqrt{gh_0}}{L} \qquad\qquad K = \frac{1}{L} \tag{4.61}$$

These definitions ensure that the reference wave speed

$$U = \frac{\Omega}{K} = \sqrt{gh_0} \tag{4.62}$$

is consistent with our previous definition for the speed of an elementary gravity wave in shallow water. Eq. (3.34) can now be written as follows

$$\omega_* = k_* \left[\frac{\tanh (k_* \beta)}{k_* \beta} \right]^{1/2} \tag{4.63}$$

where $\beta = h_0/L$, and the starred variables have been scaled by the reference values given by Eqs. (4.61). Under shallow water conditions, β becomes small, thus an approximate expression for the dispersion relation can be found by expanding the hyperbolic tangent in a power series, as follows

$$\omega_* = k_* \left(1 - \frac{1}{3}\beta^2 k_*^2 + \frac{2}{15}\beta^4 k_*^4 - \cdots \right)^{1/2} \tag{4.64}$$

The square root can further be expanded in a power series, yielding the following simplified expression

$$\omega_* = k_* \left(1 - \frac{1}{6}\beta^2 k_*^2 + \frac{19}{360}\beta^4 k_*^4 - \cdots\right) \tag{4.65}$$

Retaining only terms of $\mathcal{O}(\beta^2)$, yields the dispersion relation of gravity waves in shallow water

$$\omega_* \approx k_* - \frac{1}{6}\beta^2 k_*^3 \tag{4.66}$$

This is an important relationship because it establishes the fundamental relation between wave number and frequency in a long, shallow channel without any reference to the underlying conservation laws. We therefore ask ourselves the question: what model wave equation corresponds to this dispersion relation? Using the dispersion analysis techniques developed in Chapter 3, the answer can be written as follows

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} + \frac{1}{6}\beta^2 \frac{\partial^3 \eta}{\partial x^3} = 0 \tag{4.67}$$

This is a linear wave equation that corresponds precisely to the dispersion relation of Eq. (4.66). The first two terms in Eq. (4.67) represent a simple translation while the last term introduces the effects of dispersion. Similarly, if nonlinear effects are included, by introducing a second parameter, $\alpha = \eta/h_0$, the model equation reads

$$\frac{\partial \eta}{\partial t} + (1 + \alpha\eta)\frac{\partial \eta}{\partial x} + \frac{1}{6}\beta^2 \frac{\partial^3 \eta}{\partial x^3} = 0 \tag{4.68}$$

Finally, if dissipation effects are included, we obtain

$$\frac{\partial \eta}{\partial t} + (1 + \alpha\eta)\frac{\partial \eta}{\partial x} + \frac{1}{6}\beta^2 \frac{\partial^3 \eta}{\partial x^3} = \nu\frac{\partial^2 \eta}{\partial x^2} \tag{4.69}$$

The last two equations represent important mathematical models for wave propagation, and will be further discussed later. For present purposes, however, these equations show that a higher-order approximation may be important in the derivation of long-wave equations.

4.5 HIGHER-ORDER LONG-WAVE APPROXIMATION

It is clear from the foregoing analysis that the applicability of the shallow-water approximation depends on the smallness of the dimensionless parameter

$$\beta = \frac{h_0}{L} \tag{4.70}$$

Eqs. (4.13), (4.21), and (4.24) correspond to a zero-order approximation with respect to β although no formal expansion was attempted. The structure of the vertical momentum equation, and our scaling choice have led to an approximation that is expected to yield satisfactory results provided that β is less than about 0.05, according to the findings of the linearized gravity-wave analysis.

Unfortunately, the derivation of Eqs. (4.13), (4.21), and (4.24) does not give any information regarding the convergence of the approximate solution or whether or not a higher-order expansion is necessary. Since in many practical cases we may encounter problems that are near the limits of applicability of the zero-order approximation, it is important to derive the shallow-water equations by means of a more systematic approach.

A second point of departure in the pursuit of a higher-order approximation concerns the multiple scales of the shallow-water wave problem. Specifically, the shallow-water theory offers no measure for the relative magnitude of the wave amplitude and the depth of flow. Depending on the point of view, this has advantages and disadvantages. For, there is no limit on the size of the wave amplitude, thus it can be as high as the depth itself since the equations are fully nonlinear in this regard. On the other hand, the vertical acceleration is cut-off abruptly since there is no option to retain any powers of β. It is clear that a higher-order approximation must be cognizant of the multiple scales of the problem. To this end, consider an asymptotic expansion of the flow variables in terms of a second small parameter, α, as follows

$$\begin{aligned}
u_* &= u_*^{(0)} + \alpha u_*^{(1)} + \alpha^2 u_*^{(2)} + \ldots \\
v_* &= v_*^{(0)} + \alpha v_*^{(1)} + \alpha^2 v_*^{(2)} + \ldots \\
\eta_* &= \eta_*^{(0)} + \alpha \eta_*^{(1)} + \alpha^2 \eta_*^{(2)} + \ldots \\
p_* &= p_*^{(0)} + \alpha p_*^{(1)} + \alpha^2 p_*^{(2)} + \ldots
\end{aligned} \tag{4.71}$$

in which

$$\alpha = \frac{a}{h_0} \tag{4.72}$$

is the ratio of a typical wave amplitude to the undisturbed water depth. Unfortunately, the determination of the proper magnitude of α on the basis of physical arguments similar to those presented for β is not easy. First, even the smallest finite value of α renders the problem nonlinear, thus prohibiting the possibility of an analytical approach. It is obvious at the same time, that for any approximation theory to exist and contain products and powers of the two parameters, α and β must be of the same order of magnitude. For otherwise it would be impossible to arrive at a fixed-order approximation, since combinations of these parameters may assume arbitrary values.

Recall that for linear gravity waves, Eq. (3.34) indicates that β is a measure of wave dispersion while α is a measure of nonlinearity, therefore the two parameters have distinct roles in the approximation of the governing equations of flow. In reality, dispersion and nonlinearity are completely separate phenomena, although our imperfect theory cannot incorporate both unless they are of the same order of magnitude. Notice that there are no restrictions on the magnitude of either α or β if they appear separately. We can therefore derive approximations for linear non-dispersive waves, linear dispersive, and nonlinear non-dispersive waves without restrictions on the relative size of these parameters, as long as they both remain less than unity. For nonlinear dispersive waves, however, we have to ensure that the dispersion and nonlinearity effects remain of similar order of magnitude. This is succinctly expressed in terms of the *Ursell number*, i.e.

$$\mathbf{U} = \frac{\alpha}{\beta^2} = \frac{aL^2}{h_0^3} = \mathcal{O}(1) \tag{4.73}$$

The foregoing discussion justifies the introduction of α on the basis of mathematical arguments. A physical meaning can also be associated with the asymptotic expansion in Eq. (4.71), if we assume that the perturbation is superimposed on some typical flow conditions. The simplest unperturbed flow situation represents conditions of still water. We can then associate the zero-order term of the expansion with a static situation where no wave is present. The third of Eqs. (4.71) together with the definition of α can be used to confirm this hypothesis. When the free surface is perturbed about its mean value at which η_* is zero, and if the amplitude of this perturbation is small as assumed, then contribution of higher-order effects diminish rapidly.

Substitution of Eqs. (4.71) in the governing equations of flow leads to long-wave approximations of variable order. The pertinent equations are obtained by equating the coefficients with like powers of α, thus establishing an order relation between the flow variables. For simplicity, we are going to consider a one-dimensional problem. Thus, the dependence of the wave motion on z is ignored in the following. This results in no loss of generality since it is obvious that whatever conclusions are reached for the x-direction will also be applicable to the z-direction as well.

Based on these assumptions, the governing equations and boundary conditions can then be written as follows (Dingemans, 1973)

Continuity in the vertical plane

$$\frac{\partial u_*}{\partial x_*} + \frac{\partial v_*}{\partial y_*} = 0 \tag{4.74}$$

Horizontal momentum balance

$$\frac{\partial u_*}{\partial t_*} + u_* \frac{\partial u_*}{\partial x_*} + u_* \frac{\partial u_*}{\partial y_*} = -\frac{\partial p_*}{\partial x_*} \tag{4.75}$$

Vertical momentum balance

$$\beta^2 \left(\frac{\partial v_*}{\partial t_*} + u_* \frac{\partial v_*}{\partial x_*} + v_* \frac{\partial v_*}{\partial y_*} \right) = -1 - \frac{\partial p_*}{\partial y_*} \tag{4.76}$$

Irrotationality

$$\frac{\partial u_*}{\partial y_*} = \beta^2 \frac{\partial v_*}{\partial x_*} \tag{4.77}$$

Dynamic free-surface condition

$$p_*|_{y_* = \eta_*} = 0 \tag{4.78}$$

Kinematic free surface condition

$$v_*|_{y_* = \eta_*} = \frac{\partial \eta_*}{\partial t_*} + u_* \frac{\partial \eta_*}{\partial x} \tag{4.79}$$

Kinematic condition at the bottom

$$v_*|_{y_* = -h_0} = -u_* \frac{\partial h_0}{\partial x_*} \tag{4.80}$$

To avoid confusion between the one and two parameter approximations, we will call the former the *shallow-water approximation*, and the latter, the *moderate-amplitude long-wave approximation*. As it will be shown next, in linearized form these two approximations agree at the lowest order, but for non-linear or dispersive waves the results are different.

4.5.1 Zero-Order Approximation

The asymptotic expansions given by Eqs. (4.71) can now be used to establish the coefficients of the various powers of α by substitution in Eqs. (4.74) through (4.80). The zero-order approximation does not contain α at all, which means that no wave is present, and thus $\eta_*^{(0)} = 0$. This approximation can be considered as corresponding to conditions of still water, and in fact it can be seen that the only non-trivial coefficient is found on the right hand side of Eq. (4.76). Hence

$$\begin{aligned} \eta^{(0)} &= 0 \\ u^{(0)} &= 0 \\ v^{(0)} &= 0 \\ p^{(0)} &= -y \end{aligned} \tag{4.81}$$

In Eqs. (4.81) the asterisks have been omitted for simplicity although all variables are still dimensionless. This will also be done in the remainder of the analysis for clarity of the presentation. It can be seen from the fourth of Eqs. (4.81) that, as expected, the pressure distribution in the vertical is hydrostatic in the absence of wave motion.

4.5.2 First-Order Approximation

Proceeding with the coefficients of terms of order α, we observe that the vertical momentum equation has only one such term, from which it follows that

$$\frac{\partial p^{(1)}}{\partial y} = 0 \tag{4.82}$$

This can be integrated to yield

$$p^{(1)} = c_1(x,t) \tag{4.83}$$

where c_1 is an integration constant independent of y. This first-oder relation can be further explored by invoking the dynamic free-surface boundary condition, i.e.

$$\left[p^{(0)} + \alpha p^{(1)}\right]_{y=\eta} = 0 \tag{4.84}$$

which, because of Eq. (4.81) can be written as follows

$$-\alpha\eta^{(1)} + \alpha c_1(x,t) = 0 \tag{4.85}$$

Here we made use of the fact that $\eta = \eta^{(0)} + \alpha\eta^{(1)}$ in the first-order approximation. It then follows that $c_1 = \eta^{(1)}$ and therefore

$$p^{(1)} = \eta^{(1)} \tag{4.86}$$

The horizontal velocity component of order α is found from the condition of irrotationality, i.e.

$$\alpha\frac{\partial u^{(1)}}{\partial y} = 0 \tag{4.87}$$

This can be integrated to give

$$u^{(1)} = U_1(x,t) \tag{4.88}$$

in which U_1 is an integration constant. Clearly, the horizontal velocity of first order is constant in the vertical direction, thus we can integrate Eq. (4.88) with respect to y to obtain the discharge per unit width of first order, i.e.

$$q_x^{(1)} = \int_{-h}^{0} u^{(1)}\,dy + \int_{0}^{\alpha\eta^{(1)}} u^{(1)}\,dy = U_1 h \tag{4.89}$$

since the second integral can be neglected, as it is clearly of higher order. The corresponding expression for the vertical velocity component can be found by integration of the continuity equation over the depth of flow. Since the horizontal component of order one is not a function of y, the integration yields

$$\int_{-h}^{y} \frac{\partial v^{(1)}}{\partial y}\,dy = -\int_{-h}^{y} \frac{\partial u^{(1)}}{\partial x}\,dy \tag{4.90}$$

Then, after evaluation of the integrals, we obtain

$$v^{(1)} = v^{(1)}|_{y=-h} - (y+h)\frac{\partial U_1}{\partial x} \tag{4.91}$$

The first term on the right can be replaced by means of Eq. (4.80) to give

$$v^{(1)}(x,y,t) = -\frac{\partial (hU_1)}{\partial x} - y\frac{\partial U_1}{\partial x} \tag{4.92}$$

This shows that the vertical velocity to first-order of approximation varies linearly in the y direction. In order to obtain long-wave equations of first order with respect to wave amplitude, we make use of the kinematic free-surface condition once again. The latter now reads

$$v^{(1)} = \frac{\partial \eta^{(1)}}{\partial t} + u^{(1)}\frac{\partial \eta^{(1)}}{\partial x} \tag{4.93}$$

In view of Eq. (4.92), and after dropping all terms of order two and higher, we can write the following continuity equation

$$\frac{\partial \eta^{(1)}}{\partial t} + \frac{\partial (u^{(1)}h)}{\partial x} = 0 \tag{4.94}$$

Next, from the horizontal momentum equation, we obtain

$$\frac{\partial u^{(1)}}{\partial t} + \frac{\partial \eta^{(1)}}{\partial x} = 0 \tag{4.95}$$

Therefore, Eqs. (4.94) and (4.95) represent the long-wave equations of first order. They are linear partial differential equations for the wave amplitude and horizontal velocity. Notice that for the case of a horizontal bed, i.e. $h = 1$, these equations reduce to the well-known linear wave equation of classical mechanics. This can be verified by differentiating Eq. (4.94) with respect to time, Eq. (4.95) with respect to distance and eliminating the velocity. The result reads

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{\partial^2 \eta}{\partial x^2} \tag{4.96}$$

Eq. (4.96) is customarily associated with waves traveling at a constant speed in isotropic media. These waves are dispersion free, non-dissipative, and propagate in both the positive and negative x direction. Their propagation behavior, and several other properties of linear waves will be subject of discussion in later chapters.

4.5.3 Second-Order Approximation

We now proceed to develop equations corresponding to the coefficients of second-order terms. Since we assumed that α and β are of the same order, their product constitutes a second-order term. Thus, from Eq. (4.76) we obtain

$$\alpha\beta^2 \frac{\partial v^{(1)}}{\partial t} = -\alpha^2 \frac{\partial p^{(2)}}{\partial y} \tag{4.97}$$

Substitution of the expression for $v^{(1)}$ from Eq. (4.92) and integration yields

$$p^{(2)} = c_2(x,t) + \frac{y}{\mathbf{U}} \frac{\partial^2 (hU_1)}{\partial x \partial t} + \frac{y^2}{2\mathbf{U}} \frac{\partial^2 U_1}{\partial x \partial t} \tag{4.98}$$

Like c_1, c_2 is an integration constant independent of the vertical coordinate, and can thus be identified by means of the dynamic surface condition, as follows

$$\left[p^{(0)} + \alpha p^{(1)} + \alpha^2 p^{(2)}\right]_{y=\eta} = 0 \tag{4.99}$$

To second-order accuracy, the free surface is approximated by

$$\eta = \alpha\eta^{(1)} + \alpha^2\eta^{(2)} \tag{4.100}$$

Then, on the basis of Eqs. (4.81) and (4.86), we can rewrite the last equation, as follows

$$-[\alpha\eta^{(1)} + \alpha^2\eta^{(2)}] + [\alpha\eta^{(1)}] + [\alpha^2 c_2] = 0 \tag{4.101}$$

It then follows that $c_2 = \eta^{(2)}$, and therefore

$$p^{(2)} = \eta^{(2)} + \frac{y}{\mathbf{U}} \frac{\partial^2 (hU_1)}{\partial x \partial t} + \frac{y^2}{2\mathbf{U}} \frac{\partial^2 U_1}{\partial x \partial t} \tag{4.102}$$

The horizontal velocity component of order α^2 is found again from the condition of irrotationality, i.e.

$$\alpha^2 \frac{\partial u^{(2)}}{\partial y} = \alpha\beta^2 \frac{\partial v^{(1)}}{\partial x} \tag{4.103}$$

In view of Eq. (4.92), this expression can be integrated to give

$$u^{(2)} = U_2(x,t) - \frac{y}{\mathbf{U}} \frac{\partial^2 (hU_1)}{\partial x^2} - \frac{y^2}{2\mathbf{U}} \frac{\partial^2 U_1}{\partial x^2} \tag{4.104}$$

in which U_2 is an integration constant independent of the vertical coordinate. We can now write the second-order continuity equation by integrating Eq. (4.74) over the vertical. The discharge is first sought in a manner similar to that of Eq. (4.89). Then

$$\alpha^2 q_x^{(2)} = \int_0^{\alpha\eta^{(1)}} \alpha u^{(1)}\, dy + \int_{-h}^{0} \alpha^2 u^{(2)}\, dy + \int_0^{\alpha\eta^{(1)}} \alpha^2 u^{(2)}\, dy \tag{4.105}$$

The first integral on the right hand side corresponds to the second-order term not included in Eq. (4.89) since it was of first order. Similarly, the third integral is of third order, and can therefore be neglected from the present expression resulting in

$$q_x^{(2)} = U_1\eta^{(1)} + hU_2 + \frac{h^2}{2\mathbf{U}} \frac{\partial^2 (hU_1)}{\partial x^2} - \frac{h^3}{6\mathbf{U}} \frac{\partial^2 U_1}{\partial x^2} \tag{4.106}$$

In terms of these newly derived quantities, the continuity equation reads

$$\frac{\partial \eta^{(2)}}{\partial t} + \frac{\partial q_x^{(2)}}{\partial x} = 0 \tag{4.107}$$

Similarly, the horizontal momentum equation can be written as follows

$$\frac{\partial u^{(2)}}{\partial t} + U_1 \frac{\partial U_1}{\partial x} + \frac{\partial p^{(2)}}{\partial x} = 0 \tag{4.108}$$

Finally, substituting the expressions for $u^{(2)}$ and $p^{(2)}$ from Eqs. (4.102) and (4.104), we obtain

$$\frac{\partial U_2}{\partial t} + U_1 \frac{\partial U_1}{\partial x} + \frac{\partial \eta^{(2)}}{\partial x} = 0 \tag{4.109}$$

In summary, Eqs. (4.107) and (4.109) represent a system of partial differential equations for the second-order terms of the long-wave approximation. The solution depends on the corresponding solution of the first-order equations, suggesting that a simple way of obtaining the second-order solution is to first solve the first-order equations, and then substitute the result in the second-order ones. Thus, after substituting the expressions for pressure and discharge per unit width, the final system of second-order equations may be written as follows

$$\begin{aligned} \frac{\partial \eta^{(2)}}{\partial t} + \frac{\partial}{\partial x}\left(hu^{(2)}\right) &= -\frac{\partial}{\partial x}\left(\eta^{(1)}u^{(1)}\right) \\ &\quad - \frac{1}{\mathbf{U}}\frac{\partial}{\partial x}\left[\frac{1}{2}h^2\frac{\partial^2}{\partial x^2}\left(hu^{(1)}\right) - \frac{1}{6}h^3\frac{\partial^2 u^{(1)}}{\partial x^2}\right] \\ \frac{\partial u^{(2)}}{\partial t} + \frac{\partial \eta^{(2)}}{\partial x} &= -u^{(1)}\frac{\partial u^{(1)}}{\partial x} \end{aligned} \tag{4.110}$$

Finally, the second-order system of equations can be recast in the general form of Eq. (4.96) by elimination of the velocity as a dependent variable. The corresponding second-order wave equation is given by

$$\frac{\partial^2 \eta^{(2)}}{\partial t^2} - \frac{\partial^2 \eta^{(2)}}{\partial x^2} = -\frac{\partial^2(\eta^{(1)}u^{(1)})}{\partial x \partial t} - \frac{1}{3\mathbf{U}}\frac{\partial^4 u^{(1)}}{\partial x^3 \partial t} + \frac{\partial}{\partial x}\left(u^{(1)}\frac{\partial u^{(1)}}{\partial x}\right) \tag{4.111}$$

Notice that the right hand side represents a known source term in the linear wave equation, therefore, given an initial condition for the wave amplitude, we need to solve the wave equation twice.

4.5.4 Second-Order Oscillatory Wave

Consider the case of an oscillatory wave propagating in a channel of constant depth. To this end, let us assume a simple excitation of the free surface at $x = 0$, in the following familiar form

$$\eta(0, t) = a \cos \omega t \tag{4.112}$$

Then, it is not difficult to verify that the first-order solution is given by

$$\eta^{(1)} = a \cos(kx - \omega t) \qquad\qquad u^{(1)} = \frac{a\omega}{k}\cos(kx - \omega t) \tag{4.113}$$

The second-order wave equation can then be written as follows

$$\frac{\partial^2 \eta^{(2)}}{\partial t^2} - \frac{\partial^2 \eta^{(2)}}{\partial x^2} = -3\omega^2 a^2 \cos 2(kx - \omega t) + \frac{1}{3\mathbf{U}}k^2\omega^2 a \cos(kx - \omega t) \tag{4.114}$$

The solution for $\eta^{(2)}$ must satisfy the boundary condition at $x = 0$, which implies that $\eta^{(2)}(0, t) = 0$. Then, it can be shown that (Dingemans, 1973)

$$\alpha\eta^{(2)}(x, t) = -\frac{1}{6}\beta^2 kx\omega^2 a \sin\theta + \frac{3}{2}\frac{\alpha}{k}xa^2\omega^2 \sin 2\theta \tag{4.115}$$

where $\theta = kx - \omega t$. Thus, to second order, the free-surface elevation is given by

$$\eta(x,t) = \eta^{(1)} + \alpha\eta^{(2)} + \mathcal{O}(\alpha^3) \tag{4.116}$$

Notice that the solution grows linearly with x, thus the predicted value for $\eta(x,t)$ may not be valid for large x. Furthermore, the asymptotic analysis shows the generation of a second harmonic wave component that was not present in the boundary conditions. More importantly for the present analysis, however, is the presence of first-order effects in $\eta^{(2)}$. It seems that a sequential solution to a second order approximation may not be the best approach after all. In fact, it seems more appropriate to combine the first and second order approximations from the beginning in order to capture these first-order effects. This leads to the classical *Boussinesq Equations* (Boussinesq, 1872).

4.6 THE BOUSSINESQ EQUATIONS

To account for both first and second-order effects by means of a mixed-order equation, let us introduce a new dependent variable for the wave amplitude that contains both first and second-order components, as follows

$$\tilde{\eta} = \alpha\eta^{(1)} + \alpha^2\eta^{(2)} \tag{4.117}$$

The choice for the associated velocity variable is more complicated since both the first and second-order components are functions of x and t only, and independent of the vertical coordinate. Nwogu (1993) has shown that the linear dispersion characteristics depend on the choice of the velocity variable, thus the resulting equations are not equivalent. For example, in analogy to the shallow-water equations, we may choose the depth-averaged velocity, i.e.

$$\tilde{u} = \frac{q_x}{h+\tilde{\eta}} = \frac{\alpha q_x^{(1)} + \alpha^2 q_x^{(2)}}{h + \alpha\eta^{(1)} + \alpha^2\eta^{(2)}} \tag{4.118}$$

Expansion into a power series and truncation of higher-order terms leads to

$$\begin{aligned}\tilde{u} &= \frac{\alpha}{h}\left(q_x^{(1)} + \alpha q_x^{(2)}\right)\left(1 - \frac{\alpha}{h}\eta^{(1)}\right) + \mathcal{O}(\alpha^3)\\ &= \alpha q_x^{(1)} - \alpha^2\eta^{(1)}u^{(1)} + \alpha^2 q_x^{(2)} + \mathcal{O}(\alpha^3)\end{aligned} \tag{4.119}$$

which, upon substitution of the component variables, results in the expression for the depth-averaged velocity of mixed order, as follows

$$\tilde{u} = \alpha u^{(1)} + \alpha^2 u^{(2)} + \frac{\alpha\beta^2}{2}h\frac{\partial^2(hu^{(1)})}{\partial x^2} - \frac{\alpha\beta^2}{6}h^2\frac{\partial^2 u^{(1)}}{\partial x^2} \tag{4.120}$$

Equations for continuity and horizontal momentum in terms of the new flow variables are obtained by multiplying Eqs. (4.107)–(4.109) by α and adding them to Eqs. (4.94)–(4.95), respectively. The result reads

$$\begin{aligned}\frac{\partial}{\partial t}\left(\eta^{(1)} + \alpha\eta^{(2)}\right) + \frac{\partial}{\partial x}\left(hu^{(1)} + \alpha hu^{(2)}\right) + \alpha\frac{\partial(u^{(1)}\eta^{(1)})}{\partial x}&\\ = -\frac{\beta^2}{2}h^2\frac{\partial^2(hu^{(1)})}{\partial x^2} + \frac{\beta^2}{6}h^3\frac{\partial^2 u^{(1)}}{\partial x^2}&\end{aligned} \tag{4.121}$$

$$\frac{\partial}{\partial t}\left(u^{(1)} + \alpha u^{(2)}\right) + \alpha u^{(1)}\frac{\partial u^{(1)}}{\partial x} + \frac{\partial}{\partial x}\left(\eta^{(1)} + \alpha\eta^{(2)}\right) = 0 \tag{4.122}$$

We can now obtain first-order equations by truncating terms of order α^2 and $\alpha\beta^2$, and rewriting the above equations in terms of the newly defined variables $\tilde{u}$ and $\tilde{\eta}$. The resulting equations for continuity and momentum are of first order, but can account for second-order effects at moderate distances and times. These equations were first derived by Boussinesq, and describe nonlinear dispersive waves provided that the dispersion and nonlinearity effects are of the same order of magnitude. One of many possible forms of the Boussinesq equations can be written as follows

$$\frac{\partial \tilde{\eta}}{\partial t} + \frac{\partial}{\partial x}[\tilde{u}(\tilde{\eta} + h)] = \mathcal{O}(\alpha^2, \alpha\beta^2) \tag{4.123}$$

$$\frac{\partial \tilde{u}}{\partial t} + \bar{u}\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{\eta}}{\partial x} - \frac{\beta^2}{2}h\frac{\partial^3(\tilde{u}h)}{\partial x^2 \partial t} + \frac{\beta^2}{6}h^2\frac{\partial^3 \tilde{u}}{\partial x^2 \partial t} = \mathcal{O}(\alpha^2, \alpha\beta^2) \tag{4.124}$$

As it can be seen from Eqs. (4.123)–(4.124), α does not appear explicitly in these equations. This is a result of the definitions (4.117)–(4.120) for the flow variables, which are obviously of order α instead of order unity. It is therefore convenient in many cases to introduce yet another set of variables that are of order one. For instance, let

$$\hat{u} = \alpha^{-1}\tilde{u} \tag{4.125}$$

and

$$\hat{\eta} = \alpha^{-1}\tilde{\eta} \tag{4.126}$$

These variables would perhaps have been a more suitable choice in the first place, since our original scaling of the dimensional variables has evidently led to dimensionless variables for amplitude and velocity that are of order α instead of order one. The selection of a characteristic velocity can now be re-evaluated based on the findings of the oscillatory gravity wave analysis. From Eq. (3.31), we can see that in shallow water, the horizontal velocity can be approximated as follows

$$u \approx \frac{a}{h_0}\sqrt{gh_0} \tag{4.127}$$

Therefore, we can normalize the horizontal velocity component as follows

$$\hat{u}_* = \frac{u}{a\sqrt{g/h_0}} = \frac{\tilde{u}_*}{\alpha\sqrt{h_0}} \tag{4.128}$$

Thus, the dimensionless amplitude is scaled by its maximum possible value, which ensures that the associated terms in the equation remain of order unity at all times. With these new definitions, we can rewrite Eqs. (4.123)–(4.124) as follows

$$\frac{\partial \hat{\eta}}{\partial t} + \frac{\partial}{\partial x}[\hat{u}(\alpha\hat{\eta} + h)] = \mathcal{O}(\alpha^2, \alpha\beta^2) \tag{4.129}$$

$$\frac{\partial \hat{u}}{\partial t} + \alpha\hat{u}\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{\eta}}{\partial x} - \frac{\beta^2}{2}h\frac{\partial^3(\hat{u}h)}{\partial x^2 \partial t} + \frac{\beta^2}{6}h^2\frac{\partial^3 \hat{u}}{\partial x^2 \partial t} = \mathcal{O}(\alpha^2, \alpha\beta^2) \tag{4.130}$$

These equations show explicitly that the nonlinear terms are of order α while the dispersion terms are of order β^2. It is also clear from the magnitude of the neglected higher-order terms that for Eqs. (4.129)–(4.130) to make sense, α and β^2 must remain of the same order of magnitude, as they both approach zero. If α, for example, is much greater than β^2, the neglected terms of order α^2 may be larger than the retained terms of order β^2, which is completely inconsistent with our asymptotic approximation theory.

The inclusion of higher-order terms in the long-wave approximation is possible, but not very practical, as higher order partial derivatives appear in the governing equations of flow. This in turn complicates significantly the numerical solution of these equations, and the attractive features of depth averaging the equations quickly diminish. The Boussinesq equations are therefore the highest-order equations for moderately nonlinear dispersive waves that are used in practice. The emphasis here is on "moderately" nonlinear. It appears at first that the Boussinesq equations are more powerful than the shallow-water equations, but this is not true. For if the Ursell number becomes larger than order one, the increased nonlinearity breaks down the balance with the dispersion effects, and the approximation becomes invalid. For $\mathbf{U} = \mathcal{O}(1)$, we can rewrite Eqs. (4.129)–(4.130) in dimensional form, as follows

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[u(\eta + h)] = 0 \tag{4.131}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} = \frac{1}{2}h\frac{\partial^3 (uh)}{\partial x^2 \partial t} - \frac{1}{6}h^2\frac{\partial^3 u}{\partial x^2 \partial t} \tag{4.132}$$

The Boussinesq equations are an elegant model for predicting the behavior of the *solitary wave* where dispersion and nonlinearity are perfectly balanced. The Boussinesq equations can also provide an accurate solution for *cnoidal waves*, and the *undular bore*, for moderate times and distances. Furthermore, the structure of the Boussinesq equations allows us to consider them as a unified model for shallow-water waves.

Other model equations can be obtained from Eqs. (4.129)–(4.130) by observing the relative magnitude of the amplitude and dispersion terms. This is straightforward to accomplish in terms of the Ursell number. Thus, Eqs. (4.129)–(4.130) are valid when $\mathbf{U} = \mathcal{O}(1)$. As the Ursell number deviates from order one, certain terms in the Boussinesq equations may be eliminated, to recover simpler forms of shallow-water waves. When the Ursell number approaches zero, the nonlinear terms become negligible, thus the flow equations in dimensionless reduce to

$$\frac{\partial \hat{\eta}}{\partial t} + \frac{\partial \hat{u}}{\partial x} = 0 \tag{4.133}$$

$$\frac{\partial \hat{u}}{\partial t} + \frac{\partial \hat{\eta}}{\partial x} = \frac{1}{3}\beta^2\frac{\partial^3 \hat{u}}{\partial x^2 \partial t} \tag{4.134}$$

where we have assumed for simplicity that the bottom is horizontal, i.e. $h = 1$. These are equations for linear dispersive waves in shallow water, thus they are valid for very small wave amplitudes. At the other end of the spectrum, as the Ursell number becomes much larger than unity, the dispersion effects become negligible, thus the higher derivatives may be dropped from the momentum equation, leading to

$$\frac{\partial \hat{\eta}}{\partial t} + \alpha \frac{\partial}{\partial x}\left(\hat{u}\hat{\eta}\right) + \frac{\partial \hat{u}}{\partial x} = 0 \tag{4.135}$$

$$\frac{\partial \hat{u}}{\partial t} + \alpha \hat{u} \frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{\eta}}{\partial x} = 0 \tag{4.136}$$

Eqs. (4.135) and (4.136) describe nonlinear, non-dispersive waves of moderate amplitude. Notice, that since α is the only parameter, these equations are valid for values of α of order one, and thus they become identical in form to the shallow-water equations of lowest order derived earlier. It is important to recall, however, that the nonlinear terms in the Boussinesq equations may not become arbitrarily large without any consequences on the consistency of the approximation. Thus, Eqs. (4.135)–(4.136) can describe nonlinear waves of larger amplitude than can Eqs. (4.131)–(4.132), but they are not identical to the shallow-water equations. This leaves a void in the spectrum of long-wave models that does not permit the unified solution of nonlinear dispersive waves.

4.7 LONG WAVES IN TRAPEZOIDAL CHANNELS

Although theories and models are widely available for channels of rectangular cross section, little is known about the propagation of waves in trapezoidal channels. Observations in the field and laboratory indicate that a typical disturbance leading to a one-dimensional wave in a rectangular channel, results in a two-dimensional wave in a trapezoidal channel. The wave amplitude increases from the center to the banks of the channel with breaking at the banks occurring at even moderate amplitudes.

FIGURE 4.7 Undular Bore in Rhine-Danube Canal. Reproduced from Katopodes et al. (1987)

Fig. 4.7 shows a long wave propagating in the navigation channel connecting the rivers Rhine and Danube in Germany. The continental divide is overcome by shaft locks up to 25 m high. Lock operation creates undular waves that are not horizontal across the trapezoidal channel. Reflection at the lock entrances leads to even higher undulations, which reduce the free-board, and stress the lining of the channel and lock gates, and the ease of navigation.

Experiments on long waves in a trapezoidal channel were carried out at the Obernach Hydraulics Laboratory of the University of Munich. The test channel was 131.5 m long, with a bottom width of 1.24 m. The bank slope was 3:1, resulting in a top width of 2.56 m. Fig. 4.8 shows the depth hydrographs resulting from an undular wave front on an initially uniform depth equal to 0.1 m. Notice that the wave amplitude at gauge No. 5 is almost twice that of gauge No. 3, which verifies the two-dimensional character of this nonlinear dispersive wave.

4.7.1 Boussinesq Equations for Trapezoidal Channel

The theory of nonlinear dispersive waves that are fairly long and fairly low was extended to channels of arbitrary cross-sectional shape by Peregrine (1968), and specifically to trapezoidal channels by Peregrine (1969).

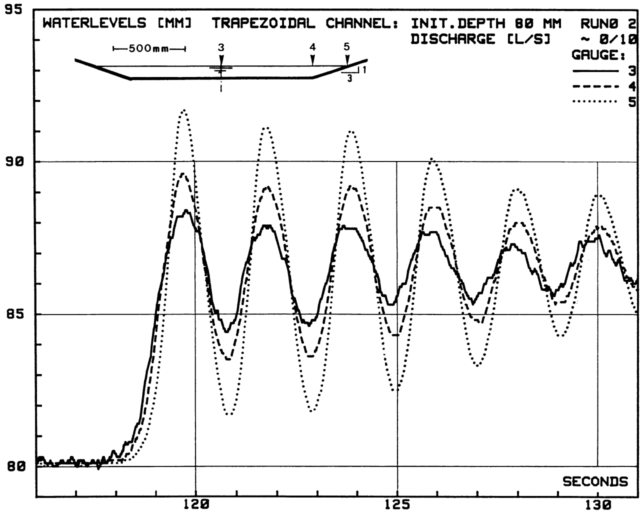

FIGURE 4.8 Recorded depth of undular bore in trapezoidal channel. Reproduced from Katopodes et al. (1987) with permission by Rhein-Main-Donau AG and Obernach Hydraulics Laboratory

The basis of the approximation theory consists of a careful examination of the order of magnitude of the flow quantities involved, and eventual omission of terms that are not as large as others. It is therefore advantageous to scale the original problem in terms of certain characteristic quantities. In Peregrine' model, it is assumed that $\alpha = \beta^2$, thus all dependent variables are scaled similar to the standard Boussinesq model.

4.7.1.1 First Approximation

The first-order approximation of the Euler equations is obtained following the procedure of section 4.5.2. Thus, from continuity and the irrotationality condition, it follows that the dimensionsless pressure identifies with the wave amplitude, and u_1 is a function of x and t only, i.e.

$$p_1 = \eta_1(x,t) \qquad\qquad u_1 = U_1(x,t) \tag{4.137}$$

where we have used subscripts to denote the approximation order. Then, the governing equations can be written as follows

$$\begin{aligned} \frac{\partial A_1}{\partial t} + \frac{\partial Q_1}{\partial x} &= 0 \\ \frac{\partial u_1}{\partial t} + \frac{\partial \eta_1}{\partial x} &= 0 \end{aligned} \tag{4.138}$$

where η_1, A_1, Q_1 are the first-order approximations of the free-surface elevation, cross-sectional area, and discharge, respectively. If the channel width is given by $b(y)$, the cross-sectional area can be approximated, as follows

$$A = A_0 + \int_0^{\eta} b(y)\,dy = A_0 + \alpha\eta_1 b(0) + \mathcal{O}(\alpha^2) \tag{4.139}$$

Therefore, $A_1 = b_0\eta_1$, with $b_0 = b(0)$. Then $Q_1 = A_0 u_1$, and the continuity equation can be combined with the equation of motion to yield the classical wave equation, as follows

$$\frac{\partial^2 \eta_1}{\partial t_1^2} = \frac{A_0}{b_0}\frac{\partial^2 \eta_1}{\partial x_1^2} \tag{4.140}$$

where the wave celerity is represented by the hydraulic depth A_0/b_0. Next, to represent the velocity field in a given cross section, we notice that to first order of approximation

$$\frac{\partial v_1}{\partial z} = \frac{\partial w_1}{\partial y} \tag{4.141}$$

Therefore, we may introduce the transverse velocity potential, ϕ_1, such that

$$v_1 = \frac{\partial \phi_1}{\partial y} \qquad\qquad w_1 = \frac{\partial \phi_1}{\partial z} \tag{4.142}$$

The continuity equation and kinematic boundary condition may now be used to determine the velocity potential ϕ_1. Since u_1 is not a function of either y or z, we may assume that

$$\phi(x, y, z, t) = -\frac{\partial u_1}{\partial x}\psi(y, z) \tag{4.143}$$

where ψ satisfies the Poisson equation over the channel cross-section, i.e.

$$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 1 \tag{4.144}$$

with the boundary conditions $\frac{\partial \psi}{\partial n} = 0$ on all solid boundaries, and $\frac{\partial \psi}{\partial y} = c_0^2$ at $y = 0$. Therefore, a unique solution for $\psi(y, z)$ can be determined, albeit within an arbitrary constant. Notice, however, that to first order, there are no variations of u and η in the transverse direction, thus a second-order model is necessary.

4.7.1.2 Second Approximation

A second-order approximation in a rectangular channel introduces the effects of vertical acceleration. In a trapezoidal channel, the transverse acceleration is equally important.

$$\frac{\partial}{\partial t}(\nabla_T \phi_1) + \nabla_T p_2 = 0 \tag{4.145}$$

where ∇_T is the Laplace operator in the transverse plane, (y, z). Then, upon integration, and recalling that $p_2 = \eta_2$ at $y = 0$, we obtain

$$\eta_2 = \frac{\partial^2 u_1}{\partial x \partial t} \psi(0, z) + H(x, t) \tag{4.146}$$

where $H(x, t)$ is some arbitrary function. Furthermore

$$u_2 = -\frac{\partial^2 u_1}{\partial x \partial t} \psi(y, z) + U(x, t) \tag{4.147}$$

where $U(x, t)$ is another arbitrary function. Similarly, the discharge and cross-sectional area can be determined to second order, as follows (Peregrine, 1968)

$$A = A_0 + \alpha b_0 \eta_1 + \alpha^2 \left(\frac{1}{2} b_1 \eta_1^2 + b_0 H + b_0 \psi_B \frac{\partial^2 u_1}{\partial x \partial t} \right) + \mathcal{O}(\alpha^3) \tag{4.148}$$

where

$$\psi_B = \frac{1}{b_0} \int_0^{0_0} \psi(0, z)\, dz \tag{4.149}$$

and

$$Q = \alpha b_0 u_1 + \alpha^2 \left(b_0 u_1 \eta_1^2 + A_0 U - A_0 \psi_A \frac{\partial^2 u_1}{\partial x \partial t} \right) + \mathcal{O}(\alpha^3) \tag{4.150}$$

where

$$\psi_A = \frac{1}{A_0} \int_{A_0} \psi(y, z)\, dy dz \tag{4.151}$$

These expressions may be substituted in the continuity and momentum equations to obtain approximations of second order. However, as shown for the Boussinesq equations in a rectangular channel, it is wiser to obtain a mixed order system of equations by combining the fist and second order variables. Thus, we introduce

$$\eta(x, t) = \alpha \eta_1 + \alpha^2 H \qquad \bar{u}(x, t) = \alpha u_1 + \alpha^2 \left(U - \psi_A \frac{\partial^2 u_1}{\partial x^2} \right) \tag{4.152}$$

Therefore, we can derive Boussinesq-type equations that describe finite, but fairly low dispersive waves. Specifically, the continuity and momentum equations can be written in dimensional form, as follows

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial \eta}{\partial x} = 0 \tag{4.153}$$

$$\frac{\partial}{\partial t}\left(\eta + \frac{1}{2}r\eta^2\right) + \frac{\partial}{\partial x}\left(c_0^2 u + \eta u\right) + (\psi_B - \psi_A)\frac{\partial^3 u}{\partial t^2 \partial x} = 0 \qquad (4.154)$$

where $r = b_1/b_0$. Finally, notice that only the difference $\psi_B - \psi_A$ appears in the equations, which is unique although ψ_B, ψ_A may contain arbitrary constants.

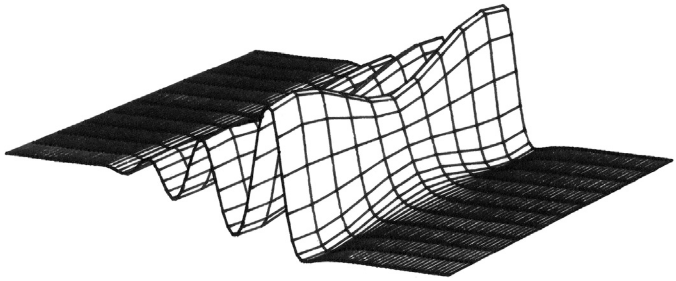

FIGURE 4.9 Undular bore in trapezoidal channel

Remarkably, Peregrine's model agrees very well with laboratory measurements, validating the a two-dimensional character of long waves in trapezoidal channels. The model produces a free-surface elevation that increases significantly near the banks of the channel, as shown in Fig. 4.9. A numerical solution of the Poisson equation is required in addition to the Boussinesq equations. However, this is minimal compared to the cost and complexity of solving the two-dimensional Boussinesq equations. The model behaves satisfactorily for small wave amplitudes although acceptable results are obtained for moderate amplitudes as well. At larger amplitudes, the model fails to simulate wave breaking on the channels banks, as this is outside the validity of the underlying theory.

4.8 THE SERRE EQUATIONS

In the analysis of nonlinear dispersive waves by Boussinesq-type equations, the dispersion characteristics are dependent on the choice of the velocity variable Nwogu (1993). It is also clear that although the Boussinesq system allows the inclusion of dispersion in a depth averaged model, it is limited by the restriction that nonlinearity and dispersion must be of the same order. This prevents the Boussinesq equations from describing further steepening of the waves, as they approach the breaking limit.

An alternative approach that overcomes the restrictions on nonlinearity was proposed by Serre (1953), who without an asymptotic approximation, integrated the two-dimensional, incompressible continuity equation, i.e. Eq. (I-5.13), and the Euler equations, i.e. Eq. (I-6.64), over the total depth of flow. To this end, the Euler equations are made dimensionless using the following variables

$$x_* = \frac{x}{L} \qquad y_* = \frac{y}{h_0} \qquad t_* = \frac{c_0 t}{L} \tag{4.155}$$

and

$$u_* = \frac{u}{\alpha c_0} \qquad v_* = \frac{v}{\alpha \beta^2 c_0} \qquad p_* = \frac{p}{\rho c_0^2} \qquad \eta_* = \frac{\eta}{\alpha h_0} \tag{4.156}$$

where $c_0 = \sqrt{g h_0}$. Although the selection of characteristic variables is similar to our previous choices, notice that now the parameters α and β appear explicitly in the non-dimensionalization process. Then, substitution in the continuity, momentum, and irrotationality equations leads to the following dimensionless system

$$\frac{\partial u_*}{\partial x} + \frac{\partial v_*}{\partial y} = 0 \tag{4.157}$$

$$\alpha \frac{\partial u_*}{\partial t} + \alpha^2 u_* \frac{\partial u_*}{\partial x} + \alpha^2 v_* \frac{\partial u_*}{\partial y} = -\frac{\partial p_*}{\partial x} \tag{4.158}$$

$$\alpha\beta^2 \frac{\partial v_*}{\partial t} + \alpha^2\beta^2 u_* \frac{\partial v_*}{\partial x} + \alpha^2\beta^2 v_* \frac{\partial v_*}{\partial y} = -1 - \frac{\partial p_*}{\partial y} \tag{4.159}$$

$$\frac{\partial u_*}{\partial y} - \beta^2 \frac{\partial v_*}{\partial x} = 0 \tag{4.160}$$

This system is subject to the dynamic and kinematic boundary conditions that are written in dimensionless form, as follows

$$p_* = 0, \qquad \frac{\partial \eta_*}{\partial t} + \alpha u_* \frac{\partial \eta_*}{\partial x} = v_* \qquad \text{at} \quad y_* = \alpha \eta_* \tag{4.161}$$

$$v_* = 0 \qquad \text{at} \quad y_* = -1 \tag{4.162}$$

where, for simplicity, we have assumed a horizontal bottom. Therefore, integration of the continuity equation over the depth, i.e. from -1 to $\alpha\eta$, leads to the standard, depth averaged continuity equation, i.e.

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}(\bar{u}h)=0 \tag{4.163}$$

where $h=1+\alpha\eta$, and the asterisks have been omitted for simplicity. Next, integration of the horizontal momentum equation leads to

$$\alpha h\frac{\partial \bar{u}}{\partial t}+\alpha^2\bar{u}h\frac{\partial \bar{u}}{\partial x}=-\alpha^2\frac{\partial}{\partial x}\int_{-1}^{\alpha\eta}\left(u^2-\bar{u}^2\right)dy-\int_{-1}^{\alpha\eta}\frac{\partial p}{\partial x}dy \tag{4.164}$$

The first term on the right hand side may be approximated by expanding the horizontal velocity in a Taylor series about the value at the bottom, u_b. Thus

$$u(x,y,t)=u_b(x,t)-\frac{1}{2}\beta^2(y+1)^2\frac{\partial^2 u_b}{\partial x^2}+\mathcal{O}(\beta^4) \tag{4.165}$$

where we have used the irrotationality condition at the bottom. Hence, integration over the vertical yields

$$\bar{u}(x,t)=u_b(x,t)-\frac{1}{6}\beta^2h^2\frac{\partial^2\bar{u}}{\partial x^2}+\mathcal{O}(\beta^4,\alpha\beta^4) \tag{4.166}$$

The first term on the right in Eq. (4.164) is $\mathcal{O}(\beta^4,\alpha\beta^4)$, and thus it can be ignored in the pursuit of a second-order analysis. Therefore, the key to the Serre equations lies in the evaluation of last term in Eq. (4.164), i.e. the integral of the pressure gradient. Using the Leibniz rule and the bottom condition, this term can be written, as follows

$$-\int_{-1}^{\alpha\eta}\frac{\partial p}{\partial x}dy=\frac{\partial}{\partial x}(h\bar{p}) \tag{4.167}$$

The pressure may be evaluated from the vertical momentum equation, as follows

$$-\frac{\partial p}{\partial y}=1+\alpha\beta^2\left(\frac{\partial v}{\partial t}+\alpha u\frac{\partial v}{\partial x}+\alpha v\frac{\partial v}{\partial y}\right) \tag{4.168}$$

This is not directly helpful, however, the vertical velocity may be approximated by integrating the continuity equation, and again approximating the horizontal velocity by a Taylor series, as follows

$$v(x,y,t)=-(y+1)\frac{\partial\bar{u}}{\partial x}+\mathcal{O}(\beta^2) \tag{4.169}$$

Substitution of this expression in Eq. (4.168) results in

$$-\frac{\partial p}{\partial y}=1-\alpha\beta^2\left\{(y+1)\left[\frac{\partial^2\bar{u}}{\partial x\partial t}+\alpha\bar{u}\frac{\partial^2\bar{u}}{\partial x^2}-\alpha\left(\frac{\partial\bar{u}}{\partial x}\right)^2\right]\right\}+\mathcal{O}(\beta^4,\alpha\beta^4)$$

Thus, the depth-averaged value of the pressure reads

$$h\bar{p} = \frac{1}{2}h^2 + \alpha\beta^2 \left[\frac{\partial^2 \bar{u}}{\partial x \partial t} + \alpha \bar{u} \frac{\partial^2 \bar{u}}{\partial x^2} - \alpha \left(\frac{\partial \bar{u}}{\partial x} \right)^2 \right] \int_{-1}^{\alpha\eta} \int_{-1}^{\alpha\eta} (y+1)\, dy dy$$

Therefore, upon substitution in Eq. (4.168), we obtain

$$\frac{\partial \bar{u}}{\partial t} + \alpha \bar{u} \frac{\partial \bar{u}}{\partial x} + \frac{\partial \eta}{\partial x} = \frac{\beta^2}{h} \frac{\partial}{\partial x} \left\{ \frac{h^3}{3} \left[\frac{\partial^2 \bar{u}}{\partial x \partial t} + \alpha \bar{u} \frac{\partial^2 \bar{u}}{\partial x^2} - \alpha \left(\frac{\partial \bar{u}}{\partial x} \right)^2 \right] \right\} + \mathcal{O}(\beta^4, \alpha\beta^4)$$

The final form of the Serre equations can be written in conservation form, in terms of dimensional variables, as follows

$$\begin{gathered} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (\bar{u} h) = 0 \\ \frac{\partial}{\partial t} (\bar{u} h) + \frac{\partial}{\partial x} \left(\bar{u}^2 h + \frac{g h^2}{2} \right) = \frac{\partial}{\partial x} \left\{ \frac{h^3}{3} \left(\frac{\partial^2 \bar{u}}{\partial x \partial t} + \bar{u} \frac{\partial^2 \bar{u}}{\partial x^2} - \left(\frac{\partial \bar{u}}{\partial x} \right)^2 \right) \right\} \end{gathered} \tag{4.170}$$

where $h = h_0 + \eta$ is the total depth, and no assumptions have been made on the magnitude of η. Notice that the left hand side of the momentum equation is identical to the zero-order shallow-water equations, in conservation form. This part of the momentum balance is capable of capturing fully nonlinear waves, albeit long, including the spontaneous formation of shocks. The right hand side of Eq. (4.170) represents wave dispersion. Notice, however, the appearance of additional nonlinear terms when compared to the Boussinesq equations.

The importance of the Serre equations cannot be overemphasized. For they offer a unified model for fully nonlinear dispersive waves in shallow water. These equation can therefore capture waves corresponding to Ursell numbers of any order, thus leading to superior solutions of classical problems such as the undular and hydraulic jump (Katopodes, 1986).

Figs. 4.10–4.12 show what is perhaps the best-known example of the transition from moderate to high Ursell numbers. The waves are generated by suddenly removing a dam separating water of different depths in a horizontal channel. The downstream side is initially occupied by still water at a depth of 0.1 m. Different wave amplitudes are generated by varying the depth at the upstream end of the channel so that the effect of the parameter α can be shown explicitly. The results of Fig. 4.11 correspond to a value of α of approximately 0.1, and the resulting undular bore shows very convincingly that a certain balance exists between the nonlinear and dispersive effects. Notice the train of waves traveling upstream with ever decreasing amplitude. This radiation of energy is made possible by the dispersion characteristics of the undular bore, and thus prevent it from steepening.

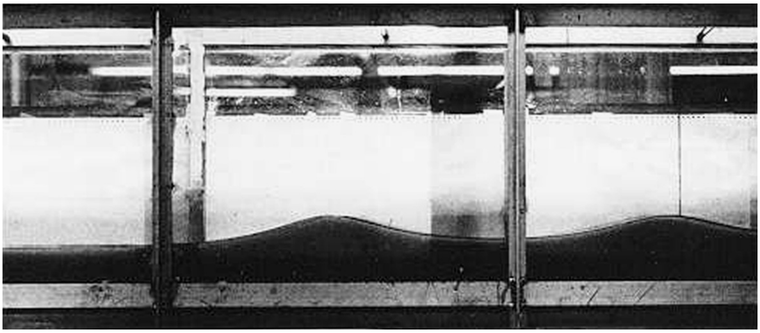

FIGURE 4.10 Undular bore; $\alpha = 0.1$

Fig. 4.11 shows a sudden release corresponding to a value of α of approximately 0.3, showing the weakening influence of the dispersion, and the gradual steepening of the leading undulation. This is accompanied by a reduction in the amplitude of the wave train, which disappears completely for $\alpha = 0.8$, as shown in Fig. 4.12. The free surface is now practically flat, and a turbulent bore or hydraulic jump has taken the place of the wave front. The energy radiation has been replaced by dissipation contained in a short region near the front, where nonlinear effects are evidently dominating the flow. It is beyond any doubt that the last situation cannot be modeled by the Boussinesq equations since the balance of dispersion and amplitude characteristics has been destroyed. On the other hand, the Serre equations are capable of capturing all three types of water release from the dam although the solution may be more complicated.

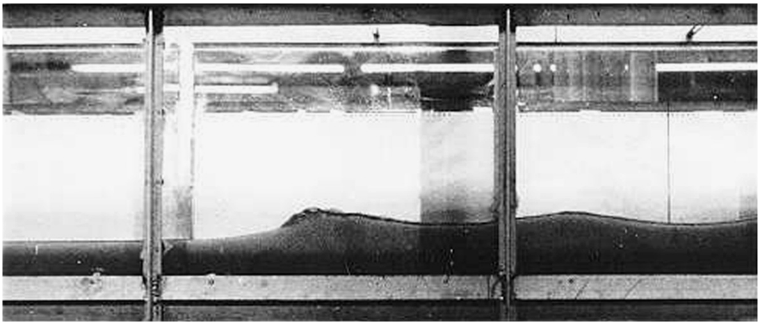

FIGURE 4.11 Undular bore at breaking limit; $\alpha = 0.3$

As mentioned earlier, there is nothing that prevents us from using the lower-order, nonlinear shallow-water theory for the last problem, and in fact we will find out that we can obtain excellent results for hydraulic bores and surges on the basis of Eqs. (4.135)–(4.136). Recall that since these equations state that the wavelength is infinitely large, the amplitude of the wave does not have to be small, and thus the lowest-order theory is not only satisfactory, but significantly more robust in practical applications.

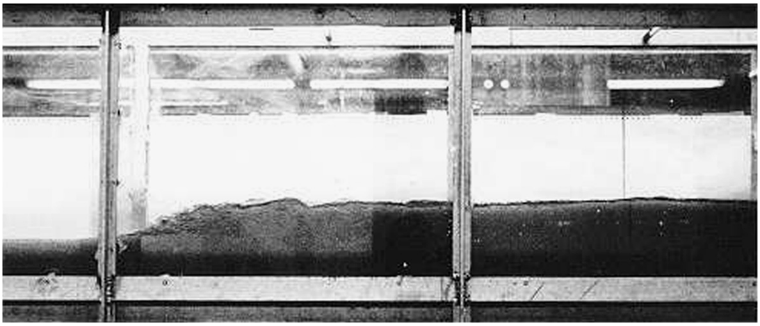

FIGURE 4.12 Breaking bore or surge; $\alpha = 0.8$

4.9 THE KORTEWEG-DE VRIES EQUATION

The wave models associated with the Boussinesq and Serre equations are rather complicated, therefore it is useful to seek simplifications that lead to easier solutions that can provide a better insight to the wave properties. For example, if the channel bottom is horizontal, $h = 1$ by definition, thus Eqs. (4.129) and (4.130) simplify to the following equations

$$\begin{aligned} \frac{\partial \eta}{\partial t} + \alpha \frac{\partial}{\partial x}(u\eta) + \frac{\partial u}{\partial \xi} &= \mathcal{O}(\alpha^2, \alpha\beta^2) \\ \frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \frac{\partial \eta}{\partial \xi} - \frac{1}{3}\beta^2 \frac{\partial^3 u}{\partial x^2 \partial t} &= \mathcal{O}(\alpha^2, \alpha\beta^2) \end{aligned} \tag{4.171}$$

where the hats have been dropped for simplicity. In addition, if we restrict the solution to waves propagating only in the positive x direction with nearly permanent shape, a transformation to an alternative coordinate system is possible by setting

$$\xi = x - ct \qquad \tau = \lambda t \tag{4.172}$$

where c is the wave speed, and $\lambda \ll 1$. Then, Eqs. (4.171) can be written as follows

$$\begin{aligned} \lambda \frac{\partial \eta}{\partial \tau} - c\frac{\partial \eta}{\partial \xi} + \alpha \frac{\partial}{\partial \xi}(u\eta) + \frac{\partial u}{\partial x} &= -\frac{1}{3}\beta^2 \frac{\partial^3 u}{\partial \xi^3} \\ \lambda \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial \xi} + \alpha u \frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial x} &= 0 \end{aligned} \tag{4.173}$$

Next, if the wave speed can be expanded in a power series, and only the first order term is retained i.e. $c = 1 + \alpha c^{(1)}$, then the momentum equation can be approximated, to first order, as follows

$$-\frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial \xi} = \mathcal{O}(\alpha, \lambda, \beta^2) \tag{4.174}$$

Thus, upon integration, we obtain the order relation

$$\eta = u + \mathcal{O}(\alpha, \lambda, \beta^2) \tag{4.175}$$

This is true, provided that $u = 0$ when $\eta = 0$ while both tend to zero, as $\xi \to \pm\infty$. Therefore, when Eqs. (4.173) are added, we obtain

$$\lambda \frac{\partial \eta}{\partial \tau} - \alpha c^{(1)} \frac{\partial \eta}{\partial \xi} + \frac{3}{2}\alpha\eta \frac{\partial \eta}{\partial \xi} + \frac{1}{6}\beta^2 \frac{\partial^3 \eta}{\partial \xi^3} = \mathcal{O}(\alpha, \lambda, \beta^2) \tag{4.176}$$

Finally, after converting back to the original independent variables, we can write

$$\frac{\partial \eta}{\partial t}+\frac{\partial \eta}{\partial x}+\frac{3}{2}\alpha\eta\frac{\partial \eta}{\partial x}+\frac{1}{6}\beta^2\frac{\partial^3 \eta}{\partial x^3}=\mathcal{O}(\alpha,\beta^2) \tag{4.177}$$

This is known as the *Korteweg-de Vries equation*, often abbreviated as KdV equation (Korteweg and de Vries, 1895). It accepts the same solutions as the Boussinesq equations, except the solutions are limited to uni-directional waves propagating on a horizontal bottom, and have a permanent shape. It can be shown that an alternative form is possible. Thus, by using the order relation given by Eq. (4.174), the KdV equation can also be written as follows

$$\frac{\partial \eta}{\partial t}+\frac{\partial \eta}{\partial x}+\frac{3}{2}\alpha\eta\frac{\partial \eta}{\partial x}+\frac{1}{6}\beta^2\frac{\partial^3 \eta}{\partial x^2 \partial t}=\mathcal{O}(\alpha,\beta^2) \tag{4.178}$$

This is known as the *Benjamin-Bona-Mahony equation* or BBM equation (Benjamin et al., 1972). It has the same limitations with the KdV equation, however, the BBM is more robust, thus it is also called the regularized long wave equation. It is definitely easier to solve numerically, as it has lower spatial continuity requirements.

The KdV and BBM equations are model equations for nonlinear dispersive waves, and are often used to validate numerical codes before attempting a solution of the Boussinesq or Serre equations. However, the solution of these unidirectional wave equations is by no means trivial. A topic worth noting presently is the treatment of boundary conditions. Clearly, these are third-order partial differential equations, therefore the wave amplitude and its first two derivatives must be specified. It is not so clear, however, how such conditions are obtained, and where they are to be specified. One of the few problems where the boundary conditions are transparent represents the solution of the solitary wave.

4.9.1 Solitary Wave

We now seek permanent solutions of the KdV equation. In the transformed coordinate system of Eq. (4.176), once the influence of time is eliminated, division by α leads to the following expression

$$-c^{(1)}\frac{d\eta}{d\xi}+\frac{3}{2}\eta\frac{d\eta}{d\xi}+\frac{1}{6\mathbf{U}}\frac{d^3\eta}{d\xi^3}=0 \tag{4.179}$$

This is an ordinary differential equation that can be integrated, as follows

$$-c^{(1)}\eta+\frac{3}{4}\eta^2+\frac{1}{6\mathbf{U}}\frac{d^2\eta}{d\xi^2}=A \tag{4.180}$$

where A is an integration constant. Following multiplication by $d\eta$, a second integration yields

$$\frac{1}{3}\mathbf{U}\left(\frac{d\eta}{d\xi}\right)^2=-\eta^3+2c^{(1)}\eta^2+A\eta+B \tag{4.181}$$

where B is another integration constant. These constants are identified by imposing boundary conditions to the flow problem. If we specify $\eta = 0$ as $\xi \to \pm\infty$, but allow for a non-zero slope of the free surface at infinity, we find that $A \geq 0$. If instead we require that

$$\eta = \frac{d\eta}{d\xi} = \frac{d^2\eta}{d\xi^2} = 0 \tag{4.182}$$

Then, both constants vanish, and the permanent solution for the free-surface elevation reads (Dingemans, 1973)

$$\eta(\xi) = 2c^{(1)}\,\mathrm{sech}^2\left(\sqrt{\frac{3c^{(1)}}{2\mathbf{U}}}\,\xi\right) \tag{4.183}$$

In this form, we recognize a wave whose maximum amplitude $\mathrm{H} = 2c^{(1)}$. It propagates with constant speed $c^{(1)}$, as its shape remains invariant, and its amplitude vanishes along with its slope and curvature at infinity. This is an analytical model for the *solitary wave*. In the original coordinate system, the free surface elevation is given by

$$\eta(x - ct) = \mathrm{H}\,\mathrm{sech}^2\left[K\,(x - ct)\right] \tag{4.184}$$

where

$$K = \sqrt{\frac{3\mathrm{H}}{4h_0^3}} \qquad\qquad c = \sqrt{g\,(h_0 + \mathrm{H})} \tag{4.185}$$

K is a measure of the width of the solitary wave, and c is the phase velocity. Fig. 4.13 shows the collision of two counter-propagating solitary waves in a laboratory experiment. Remarkably, the analytical solution of Eq. (4.184) reproduces experimental results with great accuracy.

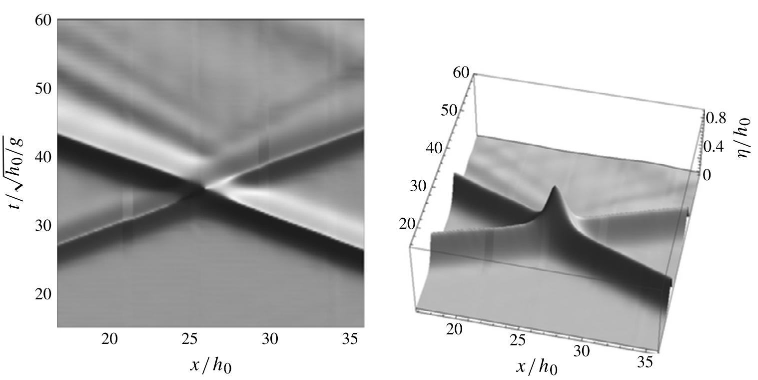

FIGURE 4.13 Two views of temporal variation of the water-surface profile of colliding solitary waves. The water depth is 6:0 cm. The waves have a relative amplitude of 0.4. The data were obtained with the laser-induced-fluorescent (LIF) imagery method (Chen and Yeh, 2014). Reproduced with permission by Cambridge University Press

4.10 HAMILTONIAN APPROACH TO WATER WAVES

Hamiltonian fluid mechanics is a subject of importance in classical mechanics that is well beyond the scope of this book. There are numerous advantages in the formulation of fluid problems by means of variational principles, as opposed to a balance of forces. In general, dynamical equations can be derived by means of the associated Hamiltonian that are better approximations to the complete solution than those derived by Newtonian mechanics. However, the only known advantage in free-surface flow is related to equations for fairly low long waves, such as the Boussinesq equations. The ambiguity of the definition of the velocity variable allows the possibility that a better set of governing equations may derived by other means. For the reader interested further in this subject, an excellent review if offered by Salmon (1988).

In theory, the Boussinesq equations are limited to a finite but small wave amplitude, due to the fact that their derivation is based on an expansion of the flow variables in a power series of a small parameter, $\alpha = a/h$. Therefore, these equations are valid for finite but fairly low waves. A second parameter, $\beta = h_0/L$ requires that these should also be fairly long waves although substantially shorter than flood or tidal waves. Furthermore, the two dimensionless parameters α and β^2 must remain of the same order, since both terms of order $\alpha\beta^2$ and β^4 are neglected. In practice, however, the Boussinesq equations represent a significant improvement over the zero-order shallow-water approximation by allowing for a moderate curvature of the free surface, variation of the velocity in the vertical, deviation of the pressure distribution from hydrostatic, and wave dispersion. In addition, waves of fairly large amplitude can be successfully simulated, so that in practical applications α can assume values almost up to $1/2$, with β^2 values of up to $1/50$.

The Boussinesq equations are known to behave erratically near the short-wave end of the spectrum, often indicating complex phase velocities (Katopodes et al., 1998). This often leads to complications in the numerical solution of these equations, therefore, it is important to seek alternative formulations that may potentially lead to more robust wave equations. This is done by re-deriving the equations of flow from variational principles, so that the total energy in the flow domain remains positive at all times. This approach was initiated by Broer (1974), and further developed by Miles (1977) and Benjamin and Olver (1982). The material in the next sections follows close the report of Katopodes and Dingemans (1989).

Let us recall the definition of the velocity potential, Φ, for ideal fluid flow, as described in Chapter I-6. For present purposes, we can write

$$u = \frac{\partial \Phi}{\partial x} \qquad\qquad v = \frac{\partial \Phi}{\partial y} \tag{4.186}$$

When the continuity equation and the irrotationality condition are combined, we obtain Laplace's equation for the velocity potential, i.e. Eq. (I-6.15), which we can rewrite here, as follows

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0; \qquad -h(x) \leq y \leq \eta \tag{4.187}$$

Next, the dynamic surface condition may be written as follows

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2\right] + g\eta = 0; \qquad y = \eta(x,t) \tag{4.188}$$

Similarly, the kinematic conditions imposed at the surface and bottom become

$$\frac{\partial \eta}{\partial t} + \frac{\partial \Phi}{\partial x}\frac{\partial \eta}{\partial x} - \frac{\partial \Phi}{\partial y} = 0; \qquad y = \eta(x,t) \tag{4.189}$$

and

$$\frac{\partial \Phi}{\partial x}\frac{dh}{dx} + \frac{\partial \Phi}{\partial y} = 0; \qquad y = -h(x) \tag{4.190}$$

For convenience, we will assume that both η and $\nabla\Phi$ vanish as $|x| \to \infty$. In general, periodicity of the solution at some finite length should serve the same purpose. Under these conditions, the total energy of the wave motion is exactly given by the *Hamiltonian*, as follows

$$\mathcal{H} = \frac{1}{2}\rho \int_{-\infty}^{\infty}\int_{-h}^{\eta}\left[\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2\right] dy dx + \frac{1}{2}\rho g \int_{-\infty}^{\infty} \eta^2\, dx \tag{4.191}$$

Therefore, the Hamiltonian represents the sum of kinetic and potential energy in the flow domain. Notice that Laplace's equation allows the velocity potential, Φ, to be uniquely determined, if boundary values of the potential are specified at the free surface, i.e. if

$$\phi(x,t) = \Phi[x,t,\eta(x,t)] \tag{4.192}$$

is given. It follows that $\mathcal{H}$ is a *functional* of η and ϕ, i.e.

$$\mathcal{H}\{\eta,\phi\} = \int_R H\left(\eta, \frac{\partial \eta}{\partial x}, \phi, \frac{\partial \phi}{\partial x}, x\right) dx \tag{4.193}$$

in which H is the *Hamiltonian density*, and R is some part of the real space. Notice that a functional maps a class of functions onto numbers, and that the derivative of the functional with respect to a function such as ϕ is given by

$$\frac{\delta \mathcal{H}}{\delta \phi} = \frac{\partial H}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial \phi_x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial H}{\partial \phi_{xx}}\right) - \cdots \tag{4.194}$$

where the subscripts denote partial differentiation with respect to x. Since time does not appear explicitly in the definition of the Hamiltonian and its derivative, the total energy is time invariant. Furthermore, it should be emphasized that the velocity

$$\frac{\partial \phi}{\partial x} = \left(\frac{\partial \Phi}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial \Phi}{\partial y} \right)_{y=\eta} \tag{4.195}$$

does not exactly coincide with either the tangential nor the horizontal velocity at the free surface. In fact, ϕ_x is slightly larger in magnitude than the surface velocity and, as it will be shown later, ϕ_x is identified with the velocity at $y = 0$ within the limits of the present approximation.

Next, consider the variation of Eq. (4.191) with respect to ϕ and η, which can be written as follows

$$\begin{aligned} \delta\mathcal{H} &= \mathcal{H}\{\phi + \delta\phi, \qquad \eta + \delta\eta\} - \mathcal{H}\{\phi, \eta\} \\ &= \frac{1}{2}\rho \int_{-\infty}^{\infty} \int_{-h}^{\eta+\delta\eta} \left(\Phi_x^2 + 2\Phi_x \delta\Phi_x + \Phi_y^2 + 2\Phi_y \delta\Phi_y \right) dy dx \\ &+ \frac{1}{2}\rho g \int_{-\infty}^{\infty} 2\eta\, \delta\eta\, dx - \frac{1}{2}\rho \int_{-\infty}^{\infty} \int_{-h}^{\eta} \left(\Phi_x^2 + \Phi_y^2 \right) dy dx \end{aligned} \tag{4.196}$$

Notice that part of the first integral on the right cancels out with the last integral. Then, following integration by parts, Eq. (4.196) becomes

$$\begin{aligned} \delta\mathcal{H} = \rho \int_{-\infty}^{\infty} \delta\eta \left[g\eta + \frac{1}{2}\left(\Phi_x^2 + \Phi_y^2 \right) \right]_{yz=\eta} dx \\ + \rho \int_{-\infty}^{\infty} \int_{-h}^{\eta} \left[\frac{\partial}{\partial x}(\Phi_x \delta\Phi) + \frac{\partial}{\partial y}\left(\Phi_y \delta\Phi \right) \right] dy dx \end{aligned} \tag{4.197}$$

where the last term has been simplified by means of Laplace's equation for Φ. The second integral in Eq. (4.197) can also be integrated by parts, which yields

$$\begin{aligned} \delta\mathcal{H} = \rho \int_{-\infty}^{\infty} \delta\eta \left[g\eta + \frac{1}{2}\left(\Phi_x^2 + \Phi_y^2 \right) \right]_{y=\eta} dx + \left[\rho \int_{-h}^{\eta} (\Phi_x \delta\Phi)\, dy \right]_{x=-\infty}^{x=\infty} \\ + \rho \int_{-\infty}^{\infty} \left\{ \left[\Phi_z - \eta_x \Phi_x \right]_{y=\eta} [\delta\Phi]_{y=\eta} - \left[\Phi_z + h_x \Phi_x \right]_{y=-h} [\delta\Phi]_{y=-h} \right\} dx \end{aligned} \tag{4.198}$$

Notice that the second term on the right vanishes as a result of the assumption of still water at large distances from the wave, i.e. $\nabla\Phi = 0$ as $|x| \to \infty$. Similarly, the last term vanishes as a result of the bottom boundary condition. Hence

$$\delta\mathcal{H} = \rho \int_{-\infty}^{\infty} \delta\eta \left[g\eta + \frac{1}{2}\left(\Phi_x^2 + \Phi_y^2\right)\right]_{y=\eta} dx \\ + \rho \int_{-\infty}^{\infty} \left[\left(\Phi_y - \eta_x \Phi_x\right)\delta\Phi\right]_{y=\eta} dx \tag{4.199}$$

From Eq. (4.195), the variation of the free-surface potential can be written as follows

$$\delta\phi = [\delta\Phi]_{y=\eta} + \left[\Phi_y\right]_{y=\eta} \delta\eta \tag{4.200}$$

When this is substituted in the expression for the variation of the Hamiltonian, we obtain

$$\delta\mathcal{H} = \rho \int_{-\infty}^{\infty} \delta\eta \left[g\eta + \frac{1}{2}\left(\Phi_x^2 + \Phi_y^2\right) - \Phi_y\left(\Phi_y - \eta_x \Phi_x\right)\right]_{y=\eta} dx \\ + \rho \int_{-\infty}^{\infty} \delta\phi \left[\Phi_y - \eta_x \Phi_x\right]_{y=\eta} dx \tag{4.201}$$

The free surface potential, ϕ, and the wave amplitude, η, are said to constitute a pair of *canonical variables*. Thus, by setting

$$\rho \frac{\partial\eta}{\partial t} = \frac{\delta\mathcal{H}}{\delta\phi} \\ \rho \frac{\partial\phi}{\partial t} = -\frac{\delta\mathcal{H}}{\delta\eta} \tag{4.202}$$

we recover the kinematic and dynamic free-surface boundary conditions. Eqs. (4.202) are called *canonical equations*, and their solution is equivalent to the solution of the complete surface wave problem consisting of Laplace's equation for the velocity potential. When the functional derivatives in Eq. (4.202) can be evaluated explicitly, partial differential equations for the wave amplitude and the free-surface velocity are obtained. Unfortunately, the formal expression of the Hamiltonian does not permit a direct evaluation of the functional derivatives. If, however, the formulation is restricted to fairly long and low waves, certain approximations that lead to satisfactory results are possible.

4.10.1 Approximation of the Kinetic Energy

The kinetic energy in Eq. (4.191) can be separated in two parts, corresponding to the flow region below and above the mean water level, as follows

$$E_k = E_{k_0} + E_{k_\eta} \tag{4.203}$$

where

$$E_{k_0} = \frac{1}{2}\rho \int_{-\infty}^{\infty} \int_{-h}^{0} \left[\left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial y} \right)^2 \right] dy dx \tag{4.204}$$

and

$$E_{k_\eta} = \frac{1}{2}\rho \int_{-\infty}^{\infty} \int_{0}^{\eta} \left[\left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial y} \right)^2 \right] dy dx \tag{4.205}$$

4.10.1.1 Kinetic Energy Below the Mean Water Level

Employing Green's theorem, Eq. (4.204) can be written as follows (Lamb, 1932, p. 66, 370)

$$E_{k_0} = \frac{1}{2}\rho \int_{-\infty}^{\infty} \left[\Phi \frac{\partial \Phi}{\partial y} \right]_{y=0} dx \tag{4.206}$$

This can be further written in terms of the stream function, Ψ, as follows

$$\begin{aligned} E_{k_0} &= -\frac{1}{2}\rho \int_{-\infty}^{\infty} \left[\Phi \frac{\partial \Psi}{\partial x} \right]_{y=0} dx \\ &= \frac{1}{2}\rho \int_{-\infty}^{\infty} \left[\Psi \frac{\partial \Phi}{\partial x} \right]_{y=0} dx \end{aligned} \tag{4.207}$$

where use was made of Green's reciprocal theorem (Rouse, 1959, p. 76). Certain approximations are now possible for fairly low, long waves. Let ψ_0 be the stream function and ϕ_0 the velocity potential at $y = 0$. Then, Ψ can be expanded formally in terms of ψ_0, as follows

$$\Psi[x, y] = \psi_0 + y \left[\frac{\partial \psi}{\partial y} \right]_{y=0} + \frac{1}{2} y^2 \left[\frac{\partial^2 \psi}{\partial y^2} \right]_{y=0} + \frac{1}{6} y^3 \left[\frac{\partial^3 \psi}{\partial y^3} \right]_{y=0} + \cdots \tag{4.208}$$

It is possible to eliminate all y derivatives from Eq. (4.208) based on the fundamental relations between the stream function and the velocity potential. By definition, $\Psi_y = \Phi_x$. Also, from Laplace's equation in terms of the stream function, $\Psi_{yy} = -\Psi_{xx}$, and thus $\Psi_{yyy} = -\Phi_{xxx}$. Then, following some simplification in the notation, Eq. (4.208) can be written as follows

$$\Psi = \psi_0 + y \frac{\partial \phi_0}{\partial x} - \frac{1}{2} y^2 \frac{\partial^2 \psi_0}{\partial x^2} - \frac{1}{6} y^3 \frac{\partial^3 \phi_0}{\partial x^3} + \cdots \tag{4.209}$$

At the bottom, the stream function, Ψ, can be arbitrarily assigned a value equal to zero. Then

$$\Psi\left[x,-h(x)\right]=0=\psi_0-h\frac{\partial\phi_0}{\partial x}-\frac{1}{2}h^2\frac{\partial^2\psi_0}{\partial x^2}+\cdots \tag{4.210}$$

Therefore, when we differentiate this expression twice, we obtain

$$\frac{\partial^2\psi_0}{\partial x^2}-h_{xx}\frac{\partial\phi_0}{\partial x}-2h_x\frac{\partial^2\phi_0}{\partial x^2}-h\frac{\partial^3\phi_0}{\partial x^3}+\cdots=0 \tag{4.211}$$

Thus, substitution of this result in Eq. (4.210), leads to an expression for ψ_0 in terms of derivatives of ϕ_0 only, as follows

$$\psi_0=\left(h+\frac{1}{2}h^2h_{xx}\right)\frac{\partial\phi_0}{\partial x}+h^2h_x\frac{\partial^2\phi_0}{\partial x^2}+\frac{1}{3}h^3\frac{\partial^3\phi_0}{\partial x^3}+\cdots \tag{4.212}$$

Finally, substitution of this result in Eq. (4.208) leads to the following expression for the kinetic energy below the mean water level

$$\begin{aligned}E_{k_0}=\frac{1}{2}\rho\int_{-\infty}^{\infty}\Bigg[&\left(h+\frac{1}{2}h^2h_{xx}\right)\left(\frac{\partial\phi_0}{\partial x}\right)^2\\&+h^2h_x\frac{\partial^2\phi_0}{\partial x^2}\frac{\partial\phi_0}{\partial x}+\frac{1}{3}h^3\frac{\partial^3\phi_0}{\partial x^3}\frac{\partial\phi_0}{\partial x}\Bigg]dx\end{aligned} \tag{4.213}$$

This can be further simplified by partial integration of the last term on the right hand side, which leads to

$$\begin{aligned}&\frac{1}{3}\int_{-\infty}^{\infty}h^3\widetilde{p}\phi_0x\frac{\partial\phi_0}{\partial x}\,dx\\&=\frac{1}{3}\left[h^3\frac{\partial\phi_0}{\partial x}\frac{\partial^2\phi_0}{\partial x^2}\right]_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\left[h^2h_x\frac{\partial^2\phi_0}{\partial x^2}\frac{\partial\phi_0}{\partial x}+\frac{1}{3}h^3\left(\frac{\partial^2\phi_0}{\partial x^2}\right)^2\right]dx\end{aligned} \tag{4.214}$$

The first term on the right vanishes because of the conditions at infinity. The second term on the right exactly cancels the second term on the right in Eq. (4.213), hence the kinetic energy expression becomes

$$E_{k_0}=\frac{1}{2}\rho\int_{-\infty}^{\infty}\left[\left(h+\frac{1}{2}h^2h_{xx}\right)\left(\frac{\partial\phi_0}{\partial x}\right)^2-\frac{1}{3}h^3\left(\frac{\partial^2\phi_0}{\partial x^2}\right)^2\right]dx \tag{4.215}$$

4.10.1.2 Kinetic Energy Above the Mean Water Level

In a similar fashion, the kinetic energy above the mean water level can be approximated by formally expanding the Laplacian about its value at $y = 0$. Hence

$$
\begin{aligned}
E_{k_\eta} = \frac{1}{2}\rho \int_{-\infty}^{\infty} \Bigg\{ & \eta \left[\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 \right]_{y=0} \\
& + \frac{1}{2}\eta^2 \frac{\partial}{\partial y} \left[\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 \right]_{y=0} + \cdots \Bigg\} dx
\end{aligned}
\tag{4.216}
$$

To this point, no approximations have been made regarding the wave length or amplitude. If the presentation is restricted to fairly long and fairly low waves, however, further simplification of the expression for the kinetic energy is possible. In order to obtain a systematic ordering of terms, it is useful to rewrite Eq. (4.216) in dimensionless form. To this end, the velocity potential is scaled by $\alpha c_0 L$, which is compatible with the magnitude of the horizontal velocity. For the vertical velocity to remain consistent with the scaling of long waves, differentiation in the vertical must represent an operation of order β. Thus in general

$$
\begin{aligned}
\frac{\partial \Phi}{\partial x} &= \alpha c_0 \frac{\partial \Phi_*}{\partial x_*} \\
\frac{\partial \Phi}{\partial y} &= \alpha \beta c_0 \frac{\partial \Phi_*}{\partial y_*}
\end{aligned}
\tag{4.217}
$$

Then, if a reference energy $E_0 = \rho \alpha^2 g h_0^2 L$ is used to scale the kinetic energy, Eq. (4.216) can be written in dimensionless form, as follows

$$
\begin{aligned}
E^*_{k_\eta} = \frac{1}{2} \int_{-\infty}^{\infty} \Bigg\{ & \alpha \eta_* \left[\left(\frac{\partial \Phi_*}{\partial x_*}\right)^2 + \beta^2 \left(\frac{\partial \Phi_*}{\partial y_*}\right)^2 \right]_{y_*=0} \\
& + \frac{1}{2}\alpha^2 \beta^2 \eta_*^2 \frac{\partial}{\partial z_*} \left[\left(\frac{\partial \Phi_*}{\partial x_*}\right)^2 + \beta^2 \left(\frac{\partial \Phi_*}{\partial y_*}\right)^2 \right]_{y_*=0} + \cdots \Bigg\} dx
\end{aligned}
\tag{4.218}
$$

4.10.1.3 Hamiltonian for Fairly Low Long Waves

For fairly low long waves, only linear terms are retained. Thus Eq. (4.218) may be simplified as follows

$$
E^*_{k_\eta} = \frac{1}{2} \int_{-\infty}^{\infty} \left[\alpha \eta_* \left(\frac{\partial \Phi_*}{\partial x_*}\right)^2_{y_*=0} + O(\alpha\beta^2, \alpha^2) \right] dx \tag{4.219}
$$

Thus, returning to dimensional variables, the kinetic energy above the mean water level can be approximated by the following expression

$$E_{k_\eta} = \frac{1}{2}\rho \int_{-\infty}^{\infty} \eta \left(\frac{\partial \phi_0}{\partial x} \right)^2 dx \tag{4.220}$$

Finally, upon substitution of Eqs. (4.215) and (4.220) in Eq. (4.191), the Hamiltonian for fairly low long waves on an uneven bottom can be written as follows

$$\mathcal{H} = \frac{1}{2}\rho \int_{-\infty}^{\infty} \left[g\eta^2 + \left(\eta + h + \frac{1}{2}h^2 h_{xx} \right) \phi_x^2 - \frac{1}{3}h^3 \phi_{xx}^2 \right] dx \tag{4.221}$$

where the potential, ϕ_0, at $y = 0$ was replaced by ϕ, the corresponding value at $y = \eta$. This is justified in the present approximation since the difference is of oder $\alpha\beta^2$, and can therefore be ignored.

4.10.1.4 Canonical Equations

The canonical equations associated with the Hamiltonian given by Eq. (4.221) are obtained as follows

$$\begin{aligned} \rho \frac{\partial \eta}{\partial t} &= \frac{\delta \mathcal{H}}{\delta \phi} \\ &= -\rho \frac{\partial}{\partial x} \left[\phi_x (\eta + h) \right] - \frac{1}{2}\rho \frac{\partial}{\partial x} \left(h^2 h_{xx} \phi_x \right) - \frac{1}{3}\rho \frac{\partial^2}{\partial x^2} \left(h^3 \phi_{xx} \right) \end{aligned} \tag{4.222}$$

and

$$\begin{aligned} \rho \frac{\partial \phi}{\partial t} &= -\frac{\delta \mathcal{H}}{\delta \eta} \\ &= -\rho g \eta - \frac{1}{2}\rho \phi_x^2 \end{aligned} \tag{4.223}$$

Thus, carrying out the differentiations, we obtain the following system of equations

$$\begin{aligned} &\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \left[\tilde{u} (\eta + h) \right] = -\frac{\partial}{\partial x} \left[\frac{1}{2}h^2 \frac{\partial^2 (\tilde{u} h)}{\partial x^2} - \frac{1}{6}h^3 \frac{\partial^2 \tilde{u}}{\partial x^2} \right] \\ &\frac{\partial \tilde{u}}{\partial t} + \tilde{u} \frac{\partial \tilde{u}}{\partial x} + g \frac{\partial \eta}{\partial x} = 0 \end{aligned} \tag{4.224}$$

These are Boussinesq-type equations, similar to the ones already derived in section 4.6, whose linear dispersion behavior was found to be erratic in the presence of short waves (Katopodes et al., 1998). The derivation through a Hamiltonian shows clearly that such behavior may have been anticipated by a simple observation of the fact that the corresponding approximate Hamiltonian, given by Eq. (4.221), becomes negative for short waves due to the presence of the term

$-\frac{1}{3}h^3\phi_{xx}^2$. Since the Hamiltonian in Eq. (4.221) is an approximation of the exact energy of the system, it is conceivable that another approximation may be possible, which results in a positive definite Hamiltonian, and thus the corresponding partial differential equations should be stable in the short wave end of the spectrum.

4.10.2 Horizontal Channel

For clarity of the presentation, the case of a horizontal bottom is examined first. The water depth is now constant, and equal to h_0. Then, Eq. (4.221) is simplified, as follows

$$\mathcal{H} = \frac{1}{2}\rho \int_{-\infty}^{\infty} \left[g\eta^2 + (\eta + h_0)\,\phi_x^2 - \frac{1}{3}h_0^3\phi_{xx}^2 \right] dx \tag{4.225}$$

Furthermore, the canonical equations can be evaluated, as follows

$$\rho\frac{\partial\eta}{\partial t} = \frac{\delta\mathcal{H}}{\delta\phi} = -\rho\frac{\partial}{\partial x}\left[\phi_x\,(\eta + h_0)\right] - \frac{1}{3}\rho h_0^3\phi_{xxxx} \tag{4.226}$$

and

$$\rho\frac{\partial\phi}{\partial t} = -\frac{\delta\mathcal{H}}{\delta\eta} = -\rho g\eta - \frac{1}{2}\rho\phi_x^2 \tag{4.227}$$

These yield the following pair of continuity and momentum equations

$$\begin{aligned} \frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}\left[\tilde{u}\,(\eta + h_0)\right] &= -\frac{1}{3}h_0^3\frac{\partial^3\tilde{u}}{\partial x^3} \\ \frac{\partial\tilde{u}}{\partial t} + \tilde{u}\frac{\partial\tilde{u}}{\partial x} + g\frac{\partial\eta}{\partial x} &= 0 \end{aligned} \tag{4.228}$$

These equations are also unstable since the Hamiltonian in Eq. (4.225) becomes negative in the presence of short waves. In the case of a horizontal bottom, however, it is possible to obtain a stable approximation. The regularity of the solution permits the kinetic energy to be represented by the Fourier transform of the free surface velocity potential. In addition, the linearity of the problem allows the identification of ϕ_0 with ϕ. Thus, following Broer (1974), the Fourier transform of $\phi(x)$ is computed, as follows

$$\phi_0(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\phi}_0(k)e^{\jmath kx}\,dk \tag{4.229}$$

Furthermore, the solution of the Laplace equation in the case of a horizontal bottom is given by

$$\Phi(x, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\phi}_0(k)\frac{\cosh\left[k(z + h_0)\right]}{\cosh kh_0}e^{\jmath kx}\,dk \tag{4.230}$$

Next, the derivative of the potential is computed, as follows

$$\left(\frac{\partial \Phi}{\partial z}\right)_{y=0} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\phi}_0(k) k \tanh kh_0 e^{\jmath kx}\, dk \tag{4.231}$$

in which $\jmath = \sqrt{-1}$. This can also be written as follows

$$\left(\frac{\partial \Phi}{\partial y}\right)_{y=0} = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\jmath k)^2 \hat{\phi}_0(k) \frac{\tanh kh_0}{k} e^{\jmath kx}\, dk \tag{4.232}$$

The inverse transform of $(\jmath k)^2 \hat{\phi}_0(k)$ is given by

$$(\jmath k)^2 \hat{\phi}_0(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 \phi_0}{\partial \xi^2} e^{-\jmath k\xi}\, d\xi \tag{4.233}$$

where ξ is a dummy variable in the physical space. Then, back substitution in the derivative of the potential leads to

$$\left(\frac{\partial \Phi}{\partial y}\right)_{y=0} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \frac{\partial^2 \phi_0}{\partial \xi^2} e^{-\jmath k\xi}\, d\xi \right\} \frac{\tanh kh_0}{k} e^{\jmath kx}\, dk \tag{4.234}$$

This can be written compactly in operator form, as follows

$$\left(\frac{\partial \Phi}{\partial y}\right)_{y=0} = -\mathcal{R}(x-\xi) \frac{\partial^2 \phi_0}{\partial x^2} \tag{4.235}$$

in which $\mathcal{R}$ is a positive, self-adjoint integral operator given by (Broer, 1974)

$$\mathcal{R}(x-\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\tanh kh_0}{k} e^{\jmath k(x-\xi)}\, dk \tag{4.236}$$

Notice that the operator, $\mathcal{R}$, performs a convolution process that embraces all points in space, and all possible wave numbers. Then, this process is repeated to sum up the kinetic energy in the entire flow domain. This defines $\mathcal{R}$ to be a global, as opposed to local, operator. It can also be shown that (Roseau, 1976, p. 189–191)

$$\mathcal{R}(x-\xi) = \frac{1}{\pi} \log\coth \frac{\pi\, |x-\xi|}{4h_0} \tag{4.237}$$

Therefore, the Fourier transform $\mathcal{R}$ is given by

$$\hat{\mathcal{R}} = \frac{\tanh kh_0}{k} = h_0 \left(1 - \frac{1}{3}(kh_0)^2 + \frac{2}{15}(kh_0)^4 - \cdots\right) \tag{4.238}$$

When the foregoing operator is substituted in the expression for the kinetic energy below the mean water level, the result reads

$$E_{k_0} = \frac{1}{2}\rho \int_{-\infty}^{\infty} \phi_0 \mathcal{R} \frac{\partial^2 \phi_0}{\partial x^2}\, dx \tag{4.239}$$

Thus, following integration by parts, we obtain

$$E_{k_0} = \frac{1}{2}\rho \int_{-\infty}^{\infty} \frac{\partial \phi_0}{\partial x} \mathcal{R} \frac{\partial \phi_0}{\partial x}\, dx \tag{4.240}$$

Similarly, the kinetic energy above the mean water level can be integrated over the solution domain, resulting in

$$E_{k_\eta} = \frac{1}{2}\rho \int_{-\infty}^{\infty} \left[\eta \left(\frac{\partial \phi_0}{\partial x} \right)^2 + \eta \left(\mathcal{R} \frac{\partial^2 \phi_0}{\partial x^2} \right)^2 + \cdots \right] dx \tag{4.241}$$

where the second term in the integrand is obviously of second order, and therefore, it can be omitted. Finally, for fairly low long waves, ϕ can be used instead of ϕ_0 in the variations of the Hamiltonian, which now reads

$$\mathcal{H} = \frac{1}{2}\rho \int_{-\infty}^{\infty} \left[\phi_x \, \mathcal{R} \phi_x + \eta \phi_x^2 + g\eta^2 \right] dx \tag{4.242}$$

Therefore, the corresponding canonical equations can be written as follows

$$\begin{aligned} \frac{\partial \eta}{\partial t} &= -\frac{\partial}{\partial x} (\mathcal{R}\phi_x) - \frac{\partial}{\partial x} (\eta \phi_x) \\ \frac{\partial \phi}{\partial t} &= -\frac{1}{2}\phi_x^2 - g\eta \end{aligned} \tag{4.243}$$

in which use was made of the identity

$$\frac{d}{d\phi_x} \{\phi_x \mathcal{R} \phi_x\} = 2\mathcal{R}\phi_x \tag{4.244}$$

4.10.3 Approximate Hamiltonian

The similarity of the spectrum function, $\hat{\mathcal{R}}$, to the dispersion relation for water waves suggests that an approximate operator corresponding to the first two terms in Eq. (4.238) can be used in place of the exact $\hat{\mathcal{R}}$. Recall that in physical space the operator reads

$$\mathcal{R}_1 = h_0 \left(1 + \frac{1}{3} h_0^2 \frac{\partial^2}{\partial x^2} \right) \tag{4.245}$$

Therefore, substitution in the canonical equations yields

$$\begin{aligned} \frac{\partial \eta}{\partial t} &= -h_0 \frac{\partial \tilde{u}}{\partial x} - \frac{1}{3} h_0^3 \frac{\partial^3 \tilde{u}}{\partial x^3} - \frac{\partial}{\partial x}(\eta \tilde{u}) \\ \frac{\partial \tilde{u}}{\partial t} &= -\tilde{u} \frac{\partial \tilde{u}}{\partial x} - g \frac{\partial \eta}{\partial x} \end{aligned} \tag{4.246}$$

These are of course identical to the dimensionless Boussinesq Equations for a horizontal bottom. The operator $\mathcal{R}_1$ is neither positive nor bounded, thus the resulting canonical equations suffer from short wave instability. Notice, however, that several choices exist that may lead to alternative equations of continuity and momentum with entirely different behavior in the frequency domain. To this end, an approximate Hamiltonian may be selected by a suitable choice of the integral operator, specifically one that is bounded, and thus should remain positive in all cases. The operator should also be simple in form, if the resulting equations are to be of practical use.

As an example, consider the following operator in an attempt to arrive at a positive Hamiltonian (Broer, 1974)

$$\hat{\mathcal{R}}_2 = h_0 \left[1 + \frac{1}{3} (kh_0)^2 \right]^{-1} = h_0 \left[1 - \frac{1}{3} (kh_0)^2 + \frac{1}{9} (kh_0)^4 + \cdots \right] \tag{4.247}$$

Its physical representation reads

$$\mathcal{R}_2^{-1} = h_0^{-1} - \frac{1}{3} h_0 \frac{\partial^2}{\partial x^2} \tag{4.248}$$

which is a positive bounded operator that, upon substitution in the canonical equations, yields

$$\begin{aligned} \frac{\partial \eta}{\partial t} - \frac{1}{3} h_0^2 \frac{\partial^3 \eta}{\partial x^2 \partial t} &= -h_0 \frac{\partial \tilde{u}}{\partial x} - \frac{\partial}{\partial x}(\eta \tilde{u}) + \frac{1}{3} h_0^2 \frac{\partial^3 (\eta \tilde{u})}{\partial x^3} \\ \frac{\partial \tilde{u}}{\partial t} &= -\tilde{u} \frac{\partial \tilde{u}}{\partial x} - g \frac{\partial \eta}{\partial x} \end{aligned} \tag{4.249}$$

These equations should not be subject to short wave instabilities, and therefore may represent an improvement over other Boussinesq-type equations. Notice that Eq. (4.249) contains a nonlinear dispersion term, that is, an order $\alpha\beta^2$ term that would be normally dropped in the approximation process. This particular term, however, serves to stabilize the equations, and therefore must be retained. It should also be noted that there exists still a possibility that Eq. (4.249) may become unstable. This is due to the presence of the $\eta\phi_x^2$ term in the Hamiltonian, which becomes negative as the free surface drops below the mean water level. In cases where the fluid velocity in such regions is larger than everywhere else, the entire Hamiltonian may become negative, and the corresponding equations are unstable.

4.10.4 The Free-Surface Approximation

In the previous section, the velocity potential at $y = 0$ was approximated by its value at the free surface. This was necessary because only the elevation and potential at the free surface are conjugate canonical variables, and thus the Hamiltonian density had to be re-written in terms of these variables. In the subsequent development, however, it was found that existing differences at the two levels were of higher order, and eventually neglected. The results based on the approximate operator, $\mathcal{R}$, also validated this action, leading to the classical Boussinesq equations in terms of the velocity at the mean water level. In the final result, however, higher order terms were added for stability purposes and, in hindsight, the decision to neglect similar terms at an earlier stage became questionable. To address this issue, the Hamiltonian is written as follows

$$\begin{aligned} E_{k_0} &= \frac{1}{2}\rho \int_{-\infty}^{\infty} \left[\phi_x + \frac{\partial}{\partial x}\left(\eta \mathcal{R}\phi_{xx}\right)\right] \mathcal{R} \left[\phi_x + \frac{\partial}{\partial x}\left(\eta \mathcal{R}\phi_{xx}\right)\right] dx \\ &= \frac{1}{2}\rho \int_{-\infty}^{\infty} \phi_x \mathcal{R}\phi_x - 2\eta \left(\mathcal{R}\phi_{xx}\right)^2 dx \end{aligned} \tag{4.250}$$

where, following integration by parts, the term containing η^2, being of order α^2, was dropped, but the order $\alpha\beta^2$ term was retained. Then, substitution in the expression for the Hamiltonian yields

$$\mathcal{H} = \frac{1}{2}\rho \int_{-\infty}^{\infty} \left[\phi_x \, \mathcal{R}\phi_x + \eta\phi_x^2 - \eta \left(\mathcal{R}\phi_{xx}\right)^2 + g\eta^2\right] dx \tag{4.251}$$

Since only the lowest order contribution of the third term on the right hand side is consistent with our development, we may approximate the Hamiltonian, as follows

$$\mathcal{H} = \frac{1}{2}\rho \int_{-\infty}^{\infty} \left[\phi_x \, \mathcal{R}\phi_x + \eta\phi_x^2 - \eta h_0^2 \phi_{xx}^2 + g\eta^2\right] dx \tag{4.252}$$

Therefore, the corresponding canonical equations are simplified, as follows

$$\begin{aligned} \frac{\partial \eta}{\partial t} &= -\frac{\partial}{\partial x}\left(\mathcal{R}\phi_x\right) - \frac{\partial}{\partial x}\left(\eta\phi_x\right) - \frac{\partial^2}{\partial x^2}\left(\eta h_0^2 \phi_{xx}\right) \\ \frac{\partial \phi}{\partial t} &= -\frac{1}{2}\phi_x^2 - g\eta + \frac{1}{2} h_0^2 \phi_{xx}^2 \end{aligned} \tag{4.253}$$

Presumably, these equations represent more accurate expressions of continuity and momentum in terms of the free surface velocity. However, the additional terms may result in difficulties with the numerical solution, thus the potential improvement should be viewed with caution.

4.10.5 Extension to Uneven Bottom

Although the Fourier transform method does not formally apply in the case of an uneven bottom, a heuristic extension of the operator, $\mathcal{R}$, is possible even when the depth is variable. For instance, if in the operator $\mathcal{R}_1$ the constant depth h_0 is replaced by h, and the necessary precautions on the continuity of the derivatives of h are taken, the following approximation is obtained

$$\mathcal{R}_3 = h\left(1 + \frac{1}{2}h\frac{\partial^2}{\partial x^2}h - \frac{1}{6}h^2\frac{\partial^2}{\partial x^2}\right) \tag{4.254}$$

Substitution in the canonical equations leads to the classical Boussinesq equations in terms of the free-surface velocity. To obtain an extension of the operator, $\mathcal{R}_2$, to an uneven bottom, we may consider by analogy the following operator

$$\mathcal{R}_3' = h\left[1 + \frac{1}{3}h\frac{\partial}{\partial x}\left(h\frac{\partial}{\partial x}\right)\right] = h\left[1 + \frac{1}{3}\left(hh_x\frac{\partial}{\partial x} + h^2\frac{\partial^2}{\partial x^2}\right)\right] \tag{4.255}$$

Some additional restrictions have to be placed on the variation of the bottom since in the approximation of the operator $\mathcal{R}_3'$ higher order terms containing derivatives of h have been neglected. The restrictions are not more severe than those placed already on the free surface variation, i.e. the bottom variation should be fairly small. This is different from the classical Boussinesq equations where no restrictions are placed on the derivatives of h, although it is implied that the variation of the bottom over a wave length distance cannot exceed the water depth itself. Based on these arguments, the suggested form of the stabilized operator, $\mathcal{R}_2$, extended to an uneven bottom should take the form

$$\mathcal{R}_4^{-1} = h^{-1}\left(1 - \frac{1}{2}h\frac{\partial^2}{\partial x^2}h + \frac{1}{6}h^2\frac{\partial^2}{\partial x^2}\right) \tag{4.256}$$

Finally, the canonical equations can be written as follows

$$\begin{aligned}\frac{\partial\eta}{\partial t} - \frac{1}{2}h\frac{\partial^3(\eta h)}{\partial x^2\partial t} + \frac{1}{6}h^2\frac{\partial^3\eta}{\partial x^2\partial t} &= -\frac{\partial\tilde{u}h}{\partial x} - \frac{\partial}{\partial x}(\tilde{u}\eta) + \frac{1}{2}h\frac{\partial^3(\eta\tilde{u}h)}{\partial x^3}\\ &\quad - \frac{1}{6}h^2\frac{\partial^3(\eta\tilde{u})}{\partial x^3}\\ \frac{\partial\tilde{u}}{\partial t} &= -\tilde{u}\frac{\partial\tilde{u}}{\partial x} - g\frac{\partial\eta}{\partial x}\end{aligned} \tag{4.257}$$

These equations should prove to be stable for short waves. However, there is still a possibility for instability due to the derivatives of h, which may become negative. This can be avoided in principle by devising another operator that can be written as a square, thus ensuring that it remains positive at all times.

4.10.6 Canonical Equations for the Average Velocity

The most popular of the Boussinesq-type models is based on the equations of flow in terms of the average velocity over the vertical. The absence of third spatial derivatives in these equations relaxes certain continuity requirements in numerical models, and makes the solution simpler (Katopodes and Wu, 1987). The model behaves better in the short wave regime than the corresponding model for the free-surface velocity, but unfortunately it cannot be derived from a Hamiltonian. There are no conjugate canonical variables from which the vertically averaged horizontal velocity and the surface elevation can result, thus the only possible way to stabilize the corresponding equations is through transformation of the equations in terms of the surface velocity.

In the case of a horizontal bottom, the two velocity variables are related by

$$\tilde{u} = \hat{u} - \frac{1}{3} h_0^2 \frac{\partial^2 \hat{u}}{\partial x^2} + \cdots \tag{4.258}$$

Therefore, the corresponding stabilized equations read

$$\begin{aligned}\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[\hat{u}\left(h_0 + \eta\right)\right] &= \frac{1}{3} h_0^2 \frac{\partial^3 \eta}{\partial x^2 \partial t} + \frac{1}{3} h_0^3 \frac{\partial^3 \hat{u}}{\partial x^3} + \frac{1}{3} h_0^2 \frac{\partial^3}{\partial x^3}\left(\eta \hat{u}\right) \\ \frac{\partial \hat{u}}{\partial t} + \hat{u}\frac{\partial \hat{u}}{\partial x} + g\frac{\partial \eta}{\partial x} &= \frac{1}{3} h_0^2 \frac{\partial^3 \hat{u}}{\partial x^2 \partial t}\end{aligned} \tag{4.259}$$

For an uneven bottom, the transformation equation can be written as follows

$$\tilde{u} = \hat{u} - \frac{1}{2} h \frac{\partial^2}{\partial x^2}\left(\hat{u} h\right) + \frac{1}{6} h^2 \frac{\partial^2 \hat{u}}{\partial x^2} + \cdots \tag{4.260}$$

Therefore, the recommended equations for continuity and momentum may be written as follows

$$\begin{aligned}\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[\hat{u}\left(h + \eta\right)\right] = \frac{1}{2} h \frac{\partial^3 (\eta h)}{\partial x^2 \partial t} - \frac{1}{6} h^2 \frac{\partial^3 \eta}{\partial x^2 \partial t} + \frac{1}{2} h \frac{\partial^3 (\hat{u} h)}{\partial x^3} \\ - \frac{1}{6} h^2 \frac{\partial^3 \hat{u}}{\partial x^3} + \frac{1}{2} h \frac{\partial^3 (\eta \hat{u} h)}{\partial x^3} - \frac{1}{6} h^2 \frac{\partial^3 (\eta \hat{u})}{\partial x^3}\end{aligned} \tag{4.261}$$

$$\frac{\partial \hat{u}}{\partial t} + \hat{u}\frac{\partial \hat{u}}{\partial x} + g\frac{\partial \eta}{\partial x} = \frac{1}{2} h \frac{\partial^3 (\hat{u} h)}{\partial x^2 \partial t} - \frac{1}{6} h^2 \frac{\partial^3 \hat{u}}{\partial x^2 \partial t} \tag{4.262}$$

The fundamental issue with Boussinesq-type equations is that the corresponding Hamiltonian either does not exist at all or is negative in the presence of very short waves such as those appearing in numerical computations. The preceding approach eliminates the possibility of a negative Hamiltonian. However, we have clearly lost the advantage of low-order spatial derivatives, and added some new nonlinear terms, both of which complicate significantly the numerical solution. Therefore, the robustness of these equations will have to be proven in practical applications (Sanders et al., 1998).

PROBLEMS

4-1. Examine the velocity variation induced by oscillatory waves in deep water and, in particular, for depths greater than the wave length.

4-2. What is the error in the vertical velocity estimation when the shallow-water theory is adopted with an accepted wave-speed error of 3%?

4-3. Derive the equations for the elliptical fluid particle paths corresponding to the motion under an oscillatory gravity wave.

4-4. What equations would you use to model a wave of amplitude equal to 0.05 m and a period of 20 min in 1000 m deep water? What if the same wave were in a depth of 10 m?

4-5. If you were to solve the nonlinear wave equations numerically, would it make any difference in the selection of your algorithm whether or not the dispersion terms were included?

4-6. Derive an equation for the variation of the vertical velocity according to the second-order approximation.

4-7. Do you think it is possible for a nonlinear wave to travel without change in shape?

4-8. Show that the derivatives of the velocity potential at the mean water level and the free surface differ only by a term that is of order $\alpha\beta^2$.

4-9. Find the conditions for a stationary solution of the Rossby wave in one dimension. What length scale is required in order for this to be possible in the Earth's atmosphere?

4-10. Show that it is possible to generate Rossby waves at the Equator, despite the fact that the Coriolis parameter vanishes there.

4-11. Would it be possible to generate Rossby waves on the surface of a planet that resembles a cylinder?

4-12. Show that the group velocity of Rossby waves is eastward for short waves and westward for long waves.

4-13. Show that the group velocity of eastward propagating Rossby short waves is smaller than for westward propagating long waves.

4-14. Determine the speed of a westward Rossby wave at 45° latitude for a disturbance with a wave length of 1000 km.

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