CHAPTER 10

EQUATIONS OF FLUID MECHANICS

10.1 INTRODUCTION

In this chapter, we consider another important branch of continuum mechanics: the mechanics of nonviscous and Newtonian viscous fluids. Like the classical elasticity theory, this branch also uses linear constitutive equations. Many common fluids including water and air satisfy such constitutive equations. Nonviscous and Newtonian viscous fluids therefore serve as excellent models for studying the mechanical behavior of a wide variety of common liquids and gases.

The scope of fluid mechanics is vast. We restrict ourselves to the derivation of the governing equations for nonviscous and Newtonian viscous fluid flows and their immediate consequences. Some simple and standard applications are also discussed.

10.2 VISCOUS AND NONVISCOUS FLUIDS

The fundamental characteristic property of a fluid, which distinguishes it from a solid, is its inability to sustain shear stresses when it is at rest or in

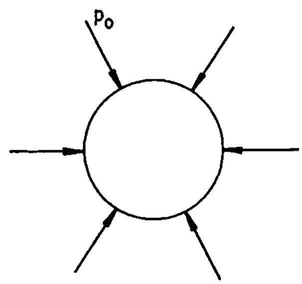

Figure 10.1. Hydrostatic pressure.

uniform motion. More specifically, whereas shear stresses can occur in a solid even when the solid is in static equilibrium or in uniform motion, shear stresses cannot occur in a fluid unless it undergoes a nonuniform motion. Experimental observations show that, in a fluid at rest or in uniform motion, the stress on a surface element consists only of a pressure (negative normal stress) and that this pressure has the same magnitude on all surface elements through a point. This pressure is usually called *hydrostatic pressure* or simply *static pressure*. Like density, static pressure is a physical property of the fluid and generally varies from one point to another.

Thus, on the basis of experimental observations, it is postulated that on a surface element in a fluid at rest or in uniform motion the stress vector s is given by

$$\mathbf{s} = -p_0 \mathbf{n} \tag{10.2.1}$$

where p_0 is hydrostatic pressure; see Figure 10.1.

The Cauchy's law, (7.4.8), then yields (see Example 7.4.4)

$$\mathbf{T} = -p_0 \mathbf{I} \tag{10.2.2}$$

Taking the trace of this equation, we find that

$$p_0 = \bar{p} \tag{10.2.3}$$

where $\bar{p} = -(1/3)(\operatorname{tr} \mathbf{T})$ is the mean pressure. Thus, the hydrostatic pressure is equal to the mean pressure.

Experimental observation also shows that fluids in nonuniform motion exert shear stresses in addition to normal stresses. This means that, when a fluid undergoes a nonuniform motion, the stress vector on a surface element in the fluid acts *obliquely* to the surface; see Figure 10.2. As such, for fluids in nonuniform motion relation (10.2.1) *does not* hold, in general.

However, in many practical problems, it is found that the effects of shear stresses are small and can even be neglected. But, important realistic situations occur where these effects are appreciable and cannot be ignored.

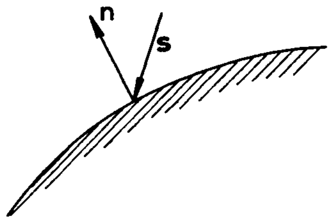

Figure 10.2. Stress vector acting obliquely to a surface.

In view of this observation, fluids are generally classified into two classes: (i) fluids that exert no or negligible shear stresses, and (ii) fluids that exert nonnegligible shear stresses. A fluid in which shear stresses are *not* negligible is called a *viscous fluid*. On the other hand, a fluid in which no or negligible shear stresses occur is called a *nonviscous* (or *inviscid* or *frictionless*) *fluid*. It should be pointed out that all fluids are viscous to a certain degree and that the concept of a nonviscous fluid is just an ideal one. Nevertheless, the study of nonviscous fluid flows is of great practical utility in engineering applications dealing with common liquids and gases (like water and air, for example) and an essential part of fluid mechanics.

10.3

STRESS TENSOR FOR A NONVISCOUS FLUID

From the definition of a nonviscous fluid, it follows that the stress vector on a surface element in such a fluid is of the form

$$\mathbf{s} = -p\mathbf{n} \tag{10.3.1}$$

even when there is a nonuniform motion, where p is a scalar function that reduces to the hydrostatic pressure p_0 when there is no motion or when the motion is uniform. Relation (10.3.1) essentially implies that at a point of a nonviscous fluid every surface element is a principal plane of stress and every direction is a principal direction of stress, with $-p$ as the principal stress.

Consequently, Cauchy's law (7.4.8) gives the following equation for the stress tensor (as shown in Example 7.4.4):

$$\mathbf{T} = -p\mathbf{I} \tag{10.3.2}$$

We note that $\mathbf{T}$, given by (10.3.2), is symmetric and isotropic. Thus, for a nonviscous fluid the symmetry of $\mathbf{T}$ is a physical property as in the case of an elastic solid governed by Hooke's law (9.2.3). Further, unlike in the case of an elastic solid, the isotropy of $\mathbf{T}$ is also a physical property of a nonviscous fluid. This means that there is no preferred direction in a nonviscous fluid. Relation (10.3.2), which describes the intrinsic material property of a nonviscous fluid, is taken as the material law for such a fluid.

It follows from (10.3.2) that

$$p = \bar{p} \tag{10.3.3}$$

where $\bar{p}$ is, as usual, the mean pressure. Thus, p present in the material law (10.3.2) represents the mean pressure in a nonviscous fluid in nonuniform motion. Like the static pressure p_0, the mean pressure p generally varies from one point to another and at any point it has the same value in all directions. When the fluid is brought to uniform motion or rest, p reduces to p_0, as already stated. Therefore p is called *dynamic pressure* or just *pressure*.

10.4 GOVERNING EQUATIONS FOR A NONVISCOUS FLUID FLOW

In Section 8.7, it was pointed out that the field equations (8.7.1) to (8.7.3) hold for all continua. Hence, these equations automatically hold for nonviscous fluids. Let us recall the spatial forms of these equations and record them here for ready reference.

Equation of continuity:

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{v} = 0 \tag{10.4.1}$$

Equations of motion:

$$\operatorname{div} \mathbf{T}^T + \rho \mathbf{b} = \rho \frac{D\mathbf{v}}{Dt} \tag{10.4.2a}$$

$$\mathbf{T} = \mathbf{T}^T \tag{10.4.2b}$$

Equation of energy:

$$\rho \frac{D\varepsilon}{Dt} = \mathbf{T} \cdot \nabla \mathbf{v} - \operatorname{div} \mathbf{q} + \rho h \tag{10.4.3}$$

To these field equations, let us append the material law (10.3.2), which is valid only for nonviscous fluids.

Material law:

$$\mathbf{T} = -p\mathbf{I} \tag{10.4.4}$$

As noted earlier, $\mathbf{T}$ is symmetric by virtue of this material law; therefore, equation (10.4.2b) is identically satisfied.

It is easily seen that 11 scalar equations are involved in the governing equations (10.4.1)–(10.4.4), whereas 15 unknown field functions are present in these equations, the unknowns being 3 velocity components v_i, 3 heat flux components q_i, 6 stress components $\tau_{ij} = (\tau_{ji})$ and 3 scalars ρ, p and ε. (The body force $\mathbf{b}$ and heat supply h are taken as known functions as usual.) Accordingly, equations (10.4.1) to (10.4.4) are inadequate to determine all the field functions. Therefore, we have to either reduce the number of unknowns or increase the number of governing equations in order to close the system of governing equations. These two possibilities lead us to consider the following two important particular cases.

10.4.1. INCOMPRESSIBLE FLUIDS

In Section 6.3, we noted that, for every motion of an incompressible continuum,

$$\operatorname{div} \mathbf{v} = 0 \tag{10.4.5}$$

In view of the equation of continuity (10.4.1), this condition is equivalent to the condition

$$\frac{D\rho}{Dt} \equiv 0 \tag{10.4.6}$$

which implies that ρ retains its initial value ρ_0 throughout the motion. A fluid for which this property of ρ holds is called an *incompressible fluid*, and (10.4.5) is employed as the equation of continuity for the motion of such a fluid.

In this case, equations (10.4.2) can be together rewritten as

$$\operatorname{div} \mathbf{T} + \rho_0 \mathbf{b} = \rho_0 \frac{D\mathbf{v}}{Dt} \tag{10.4.7}$$

Thus, for an incompressible fluid, equations (10.4.1) and (10.4.2) are replaced by equations (10.4.5) and (10.4.7).

From equations (10.4.4) and (10.4.5), we get

$$\mathbf{T} \cdot \nabla \mathbf{v} = -p(\mathbf{I} \cdot \nabla \mathbf{v}) = -p \operatorname{div} \mathbf{v} = 0 \tag{10.4.8}$$

Hence, for an incompressible nonviscous fluid, the energy equation (10.4.3) reduces to

$$\rho_0 \frac{D\varepsilon}{Dt} = -\operatorname{div} \mathbf{q} + \rho h \tag{10.4.9}$$

We note that 10 scalar equations are involved in (10.4.4), (10.4.5) and (10.4.7) and the number of unknown field functions is also 10, the functions being p, v_i and τ_{ij}. Equation (10.4.9) does not contain these field functions and is therefore not needed in their determination. Equations (10.4.4), (10.4.5) and (10.4.7) thus serve as a closed system of governing differential equations for the study of nonviscous, incompressible fluid flows. For such flows, energy considerations are generally not important and equation (10.4.9), which is purely an equation of balance of thermal energy, may be discarded.

10.4.2. COMPRESSIBLE FLUIDS

A fluid is said to be *compressible* if it is not incompressible; that is, if ρ is not constant during the motion. In this case, some constitutive relations from thermodynamics are appended to the system of equations (10.4.1) to (10.4.4) to close the system.

Since the density of a compressible fluid varies during the motion, the equation of continuity (10.4.1) shows that div $\mathbf{v} \neq 0$. Hence, fluid elements undergo a change in their volumes during the motion. A change in volume of a material is generally caused by a change in normal stress and a change in temperature. Thus, for a compressible fluid, it is postulated that p, ρ and T are related in a definite way, where T is absolute temperature; that is, there exists a functional relationship of the form

$$f(p, \rho, T) = 0 \tag{10.4.10}$$

where f is a known function. An equation of this type is called a *kinetic equation of state*. The actual form of the equation depends on the physical properties of the fluid being dealt with.

The equation of energy (10.4.3) shows that ε is related to ρ, $\mathbf{T}$, $\mathbf{v}$ and $\mathbf{q}$ in a definite way. From the material law (10.4.4) and the equation of continuity (10.4.1), we find that

$$\mathbf{T} \cdot \nabla\mathbf{v} = -p\mathbf{I} \cdot (\nabla\mathbf{v}) = -p \operatorname{div} \mathbf{v} = \frac{p}{\rho}\frac{D\rho}{Dt} \tag{10.4.11}$$

Thus, the stress-power $\mathbf{T} \cdot \nabla\mathbf{v}$ is directly related with p and ρ in a definite way. Consequently, it follows that ε is related to ρ, p and $\mathbf{q}$ in a definite way. If we employ *Fourier's law of heat conduction* (8.7.20), namely,

$$\mathbf{q} = -k\,\nabla T \tag{10.4.12}$$

then $\mathbf{q}$ is determined by T. Also, the kinetic equation of state, (10.4.10), determines p as a function of ρ and T. Thus, ultimaltely ε can be related to ρ and T in a definite way. Hence, it is postulated that ε may be taken as a known function of ρ and T; that is,

$$\varepsilon = \varepsilon(\rho, T) \tag{10.4.13}$$

An equation of this type is called a *caloric equation of state*. The actual form of the function $\varepsilon(\rho, T)$ depends on the physical properties of the fluid being dealt with.

If equations (10.4.10), (10.4.12) and (10.4.13) are appended to equations (10.4.1) to (10.4.4), then we have in all 16 scalar equations with 16 unknown field functions, the field functions being 3 velocity components v_i, 3 heat-flux components q_i, 6 stress components τ_{ij} and 4 scalars ρ, p, T and ε. Equations (10.4.1) to (10.4.4), (10.4.10), (10.4.12) and (10.4.13) therefore serve as a closed system of governing differential equations for the study of nonviscous compressible fluid flows. Note that whereas (10.4.1) to (10.4.3) are field equations, (10.4.4), (10.4.10), (10.4.12) and (10.4.13) are all constitutive equations. Unlike in the case of an incompressible fluid, where the material law (10.4.4) is the only constitutive equation, the constitutive equations for a compressible fluid consist of the equations of state (10.4.10) and (10.4.13) and a law of heat conduction (10.4.12), in addition to the material law (10.4.4). Also, in this case, the energy equation (10.4.3) is coupled with other governing equations.

Thus, for a nonviscous fluid flow, two sets of equations govern the flow—one set is applicable to incompressible fluids, and the other to compressible fluids. For an incompressible fluid, the density does not change with motion, and the pressure p, the velocity field $\mathbf{v}$ and the stress $\mathbf{T}$ serve as the field functions. On the other hand, for the case of a compressible fluid in which the density changes with motion, the field functions are the density ρ, the pressure p, the temperature T, the internal energy ε, the heat flux $\mathbf{q}$, the velocity $\mathbf{v}$ and the stress $\mathbf{T}$. Obviously, the study of compressible fluid flows is more difficult than that of incompressible fluid flows.

Experimental observations show that in the case of liquids and low-speed gases, the changes in density during the motion are very small and hence can be neglected. Liquids and low-speed gases are therefore generally considered as incompressible fluids. In the case of gases moving with high speed, the variations in density during the motion are considerably large and cannot be neglected. High-speed gases are therefore regarded as compressible fluids. In reality, no fluid is incompressible; every fluid is compressible to a certain degree. Like the concept of nonviscous fluid, the concept of an incompressible fluid is just an ideal one.

It is to be emphasized that fluids are classified as compressible or incompressible accordingly as the density changes or does not change *during the motion*. In a fluid, the density may change from one part of the fluid to the other, and such a change of density may occur in both compressible and incompressible fluids. A fluid in which the density is the same everywhere in the fluid (so that $\nabla\rho \equiv \mathbf{0}$) is called a *homogeneous fluid*. On the other hand, a fluid in which there is a change in density from one part of the fluid to the other (so that $\nabla\rho \neq \mathbf{0}$ in general) is called a *nonhomogeneous fluid*. Salt water with nonuniform concentrations with depth is a standard example of a nonhomogeneous fluid. The governing equations considered in the previous paragraphs hold for both homogeneous and nonhomogeneous nonviscous fluids.

A point that is to be clearly understood is the following. Recall that the initial density ρ_0 has been defined by $\rho_0 = \rho(\mathbf{x}^0, 0)$. Evidently, ρ_0 varies with $\mathbf{x}^0$ in general. It is only for homogeneous fluids that we have ρ_0 = constant.

10.4.3. STANDARD FORMS OF EQUATIONS OF STATE

It was pointed out that the equations of state, namely, (10.4.10) and (10.4.13), serve as constitutive equations for compressible fluids and the exact forms of these equations depend on the physical properties of the fluid being dealt with. Here we give some standard forms of these equations that are needed in our further discussion.

The caloric equation of state (10.4.13) is intended to specify ε in terms of ρ and T. One of the simplest forms of this equation is

$$\varepsilon = c_v T \tag{10.4.14}$$

where c_v is either a constant or a function of ρ. Then c_v is called the *specific heat at constant volume.*

The kinetic equation of state (10.4.10) is intended to specify a functional relationship between p, ρ and T. One of the simplest forms of this equation is the *Boyle's law* given by

$$p = \rho RT \tag{10.4.15}$$

where R is a constant for a particular fluid under consideration.

A compressible fluid for which (10.4.14) with c_v = constant and (10.4.15) hold is called a *perfect gas*, and R is then known as *gas constant.* Common gases like air, oxygen and nitrogen are regarded as perfect gases. For air, the value of R is approximately equal to 2.87×10^6.

An important particular case of equation (10.4.15) is

$$\frac{p}{\rho} = \text{constant} \tag{10.4.16a}$$

$$T = \text{constant} \tag{10.4.16b}$$

A compressible fluid for which (10.4.14) with c_v = constant and (10.4.16) hold is called an *isothermal perfect gas.*

An important generalization of equations (10.4.16) is

$$\frac{p}{\rho^{\gamma}} = \text{constant} \tag{10.4.17a}$$

$$\frac{T}{\rho^{\gamma-1}} = \text{constant} \tag{10.4.17b}$$

where

$$\gamma = \left(1 + \frac{R}{c_v}\right) > 1 \tag{10.4.18}$$

A compressible fluid for which (10.4.14) with c_v = constant and (10.4.17) hold is called an *isentropic perfect gas*; the constant γ is called the *adiabatic constant*. If

$$c_p = c_v + R \tag{10.4.19}$$

then c_p is called the *specific heat at constant pressure*, and the following relations can be verified:

$$c_v = \frac{R}{\gamma - 1} \tag{10.4.20a}$$

$$c_p = \frac{\gamma R}{\gamma - 1} \tag{10.4.20b}$$

$$\frac{c_p}{c_v} = \gamma \tag{10.4.20c}$$

We often deal with what are called *barotropic fluids*. By a barotropic fluid, we mean a compressible fluid for which the kinetic equation of state, (10.4.10), is independent of T and can be expressed in *either* of the following forms:

$$p = p(\rho) \tag{10.4.21a}$$

$$\rho = \rho(p) \tag{10.4.21b}$$

A nonviscous barotropic fluid is often referred to as an *Eulerian fluid* or *elastic fluid*.

Equations of state that are more general than those just summarized are also found in the literature. For easy reference, the governing equations for incompressible and compressible nonviscous fluid flows are summarized in Tables 10.1 and 10.2 with the same equation numbers as in the text.

Table 10.1. Governing Equations for Nonviscous Incompressible Fluid Flows

1. Equation of continuity:

$$\operatorname{div} \mathbf{v} = 0 \tag{10.4.5}$$

2. Equation of motion:

$$\operatorname{div} \mathbf{T} + \rho_0 \mathbf{b} = \rho_0 \frac{D\mathbf{v}}{Dt} \tag{10.4.7}$$

3. Material law:

$$\mathbf{T} = -p\mathbf{I} \tag{10.4.4}$$

Total number of scalar equations: 10
Total number of unknown field functions (p, v_i and τ_{ij}): 10

Table 10.2. Governing equations for Nonviscous Compressible Fluid Flows

1. Equation of continuity:

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{v} = 0 \tag{10.4.1}$$

2. Equation of motion:

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \rho \frac{D\mathbf{v}}{Dt} \tag{10.4.2}$$

3. Equation of energy:

$$\rho \frac{D\varepsilon}{Dt} = \mathbf{T} \cdot \nabla \mathbf{v} - \operatorname{div} \mathbf{q} + \rho h \tag{10.4.3}$$

4. Material law:

$$\mathbf{T} = -p\mathbf{I} \tag{10.4.4}$$

5. Fourier's law of heat conduction:

$$\mathbf{q} = -k \nabla T \tag{10.4.12}$$

6. Equations of state

$$f(p, \rho, T) = 0 \tag{10.4.10}$$

$$\varepsilon = \varepsilon(\rho, T) \tag{10.4.13}$$

Total number of scalar equations: 16
Total number of unknown field functions ($\rho, p, T, \varepsilon, q_i, v_i$ and τ_{ij}): 16

EXAMPLE 10.4.1 Assuming that the coefficient of thermal conductivity k is constant, show that the energy equation for a nonviscous compressible fluid can be expressed in the following alternative forms:

$$\frac{D\varepsilon}{Dt} = \frac{p}{\rho^2} \frac{D\rho}{Dt} + \frac{k}{\rho} \nabla^2 T + h \tag{10.4.22}$$

$$\frac{D}{Dt}\left[\varepsilon + \frac{p}{\rho}\right] = \frac{k}{\rho} \nabla^2 T + \frac{1}{\rho} \frac{Dp}{Dt} + h \tag{10.4.23}$$

Solution Substituting for $\mathbf{T} \cdot \nabla \mathbf{v}$ from (10.4.11) and for $\mathbf{q}$ from (10.4.12) in the energy equation (10.4.3) we immediately arrive at equation (10.4.22).

With the use of the identity

$$\frac{D}{Dt}\left(\frac{p}{\rho}\right) = \frac{1}{\rho}\frac{Dp}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} \tag{10.4.24}$$

equation (10.4.22) reduces to equation (10.4.23). ■

Note: The quantity $\varepsilon + (p/\rho)$ present on the lefthand side of (10.4.23) is called *specific enthalpy*.

EXAMPLE 10.4.2 A flow in which there is no heat flux is called an *adiabatic flow*. Assuming that the fluid is nonviscous and there is no heat supply, prove the following for an adiabatic flow:

(i) For a baratropic fluid

$$\varepsilon + \frac{p}{\rho} = \int \frac{dp}{\rho} + \text{constant} \tag{10.4.25}$$

(ii) For an isentropic perfect gas

$$\varepsilon + \frac{p}{\rho} = \frac{\gamma}{(\gamma - 1)}\frac{p}{\rho} + \text{constant} \tag{10.4.26}$$

Solution (i) For an adiabatic flow, we have $\mathbf{q} = \mathbf{0}$, so that $\nabla T = \mathbf{0}$ by the Fourier's law of heat conduction (10.4.12). Consequently, $\operatorname{div}(\nabla T) = \nabla^2 T = 0$. Then in the absence of h, we obtain, from (10.4.23),

$$\frac{D}{Dt}\left[\varepsilon + \frac{p}{\rho}\right] = \frac{1}{\rho}\frac{Dp}{Dt} \tag{10.4.27}$$

For a barotropic fluid, we have $\rho = \rho(p)$ so that

$$\frac{d}{dp}\left\{\int \frac{1}{\rho}dp\right\} = \frac{1}{\rho} \tag{10.4.28}$$

Hence

$$\frac{1}{\rho}\frac{Dp}{Dt} = \frac{d}{dp}\left\{\int \frac{1}{\rho}dp\right\}\frac{Dp}{Dt} = \frac{D}{Dt}\left\{\int \frac{1}{\rho}dp\right\} \tag{10.4.29}$$

Using this in (10.4.27) and integrating the resulting equation, we obtain (10.4.25).

(ii) Recall that for an isentropic perfect gas, $p/\rho^\gamma = \text{constant} = \beta$, say. Then

$$\int \frac{dp}{\rho} = \beta\gamma \int \rho^{\gamma-2}\, d\rho = \frac{\beta\gamma}{(\gamma - 1)}\rho^{\gamma-1} + \text{constant} = \frac{\gamma}{(\gamma - 1)}\frac{p}{\rho} + \text{constant} \tag{10.4.30}$$

Using this in (10.4.25), we obtain (10.4.26). ■

EXAMPLE 10.4.3 Show that for a nonviscous perfect gas flow, the energy equation can be expressed in the following alternative forms:

$$c_v \frac{DT}{Dt} = \frac{p}{\rho^2}\frac{D\rho}{Dt} + \frac{k}{\rho}\nabla^2 T + h \tag{10.4.31}$$

$$c_p \frac{DT}{Dt} = \frac{1}{\rho}\frac{Dp}{Dt} + \frac{k}{\rho}\nabla^2 T + h \tag{10.4.32}$$

Solution For a perfect gas, we get from (10.4.14), (10.4.15) and (10.4.20),

$$\varepsilon = c_v T = \frac{c_v}{R}\frac{p}{\rho} = \left(\frac{c_v}{R} - 1\right)\frac{p}{\rho} = c_p T - \frac{p}{\rho} \tag{10.4.33}$$

Substituting $\varepsilon = c_v T$ in (10.4.22), we immediately get (10.4.31). Substituting $\varepsilon = c_p T - (p/\rho)$ in (10.4.23), we obtain (10.4.32). ■

EXAMPLE 10.4.4 (a) Show that for a nonviscous incompressible fluid flow, the equation of balance of mechanical energy is given by

$$\frac{DK}{Dt} = \int_V \rho(\mathbf{b} \cdot \mathbf{v})\, dV - \int_S p\mathbf{v} \cdot \mathbf{n}\, dS \tag{10.4.34}$$

(b) If the body force is conservative so that $\mathbf{b} = -\nabla\chi$ for some scalar potential $\chi = \chi(\mathbf{x})$, show that (10.4.34) can be put in the form

$$\frac{D}{Dt}(K + \Omega) = -\int_S p\mathbf{v} \cdot \mathbf{n}\, dS \tag{10.4.35}$$

where

$$\Omega = -\int_V \rho\chi\, dV \tag{10.4.36}$$

is called the *potential energy* of the fluid contained in V.

Deduce that (i) if the fluid moves tangentially over the surface S, then $K + \Omega$ remains constant during the motion, and (ii) if $K + \Omega$ remains constant during the motion, then the stream lines lie on surfaces of constant pressure.

Solution (a) For a continuum, the equation of balance of mechanical energy is given by (8.6.27). For a nonviscous incompressible fluid, $\mathbf{T} \cdot \nabla\mathbf{v} = 0$, see (10.4.8). Recall also that $\mathbf{s} = -p\mathbf{n}$ on S and $\mathbf{T} \cdot \nabla\mathbf{v} = \mathbf{T} \cdot \mathbf{D}$. Using these results in (8.6.27), we immediately obtain equation (10.4.34).

Note: For an incompressible nonviscous fluid, the equation of balance of thermal energy is given by (10.4.9), which has no common term with the equation of balance of mechanical energy, (10.4.34). Hence the motion and the thermal state of such a fluid do not influence one another.

(b) For $\mathbf{b} = -\nabla\chi$, we have

$$\int_V \rho\mathbf{b}\cdot\mathbf{v}\,dV = -\int_V \rho(\nabla\chi\cdot\mathbf{v})\,dV = -\int_V \rho\left(\nabla\chi\cdot\frac{D\mathbf{x}}{Dt}\right)dV = -\int_V \rho\frac{D\chi}{Dt}\,dV \tag{10.4.37}$$

which, by use of (8.2.10), becomes

$$\int_V \rho\mathbf{b}\cdot\mathbf{v}\,dV = -\frac{D}{Dt}\int_V \rho\chi\,dV \tag{10.4.38}$$

Using (10.4.38) and (10.4.36) in (10.4.34) we immediately arrive at (10.4.35).

If the fluid moves tangentially over the surface S, we have $\mathbf{v}\cdot\mathbf{n} = 0$ on S. Consequently, (10.4.35) yields

$$\frac{D}{Dt}(K + \Omega) = 0 \tag{10.4.39}$$

This implies that $K + \Omega$ is constant during the motion.

Conversely, if $K + \Omega$ is constant during the motion, then (10.4.39) holds and (10.4.35) yields

$$\int_S p(\mathbf{v}\cdot\mathbf{n})\,dS = 0 \tag{10.4.40}$$

which, by use of the divergence theorem (3.6.1), becomes

$$\int_V \operatorname{div}(p\mathbf{v})\,dV = 0 \tag{10.4.41}$$

Using the identity (3.4.14) and the equation of continuity (10.4.5), expression (10.4.41) reduces to

$$\int_V (\nabla p\cdot\mathbf{v})\,dV = 0 \tag{10.4.42}$$

Since V is an arbitrary volume, it follows that $\nabla p\cdot\mathbf{v} = 0$ at all points of the fluid; that is, $\mathbf{v}$ is orthogonal to ∇p everywhere in the fluid. Since ∇p is orthogonal to surfaces of constant p (see Section 3.4), it follows that $\mathbf{v}$ is tangential to these surfaces. Thus, stream lines lie on surfaces of constant pressure at every point of the fluid. ■

10.5 INITIAL AND BOUNDARY CONDITIONS

A boundary value problem in fluid mechanics consists in determining all the unknown field functions by solving the governing equations under appropriate initial and boundary conditions. For incompressible nonviscous fluid flows, the unknown functions are p, $\mathbf{v}$ and $\mathbf{T}$ and the governing equations are (10.4.4), (10.4.5) and (10.4.7). In this case, the following initial and boundary conditions are prescribed:

$$\mathbf{v} = \mathbf{v}^{(0)} \quad \text{in} \quad V \quad \text{at} \quad t = 0 \tag{10.5.1}$$

$$\mathbf{v} = \mathbf{v}^* \quad \text{on} \quad S_v \quad \text{for} \quad t \geq 0 \tag{10.5.2a}$$

$$\mathbf{Tn} = \mathbf{s}^* \quad \text{on} \quad S_\tau \quad \text{for} \quad t \geq 0 \tag{10.5.2b}$$

where V is a *material volume* of the fluid bounded by a closed surface S, and S_v and S_τ are parts of S that are complementary to each other. As special cases S_v or S_τ can be equal to S. Also, $\mathbf{v}^{(0)}$, $\mathbf{v}^*$ and $\mathbf{s}^*$ are prescribed quantities in their respective domains.

The initial conditions (10.5.1) mean that the velocity of the fluid is known at every point of the initial volume. For a fluid motion starting from rest at time $t = 0$, $\mathbf{v}^{(0)} = \mathbf{0}$. The boundary conditions (10.5.2) mean that the velocity is prescribed on S_v and stress is specified on S_τ. The conditions (10.5.1) and (10.5.2ab) are similar to conditions (9.5.4) and (9.5.3), but there is one difference. In the case of linear elastic solids, S can be treated as the boundary surface in an initial configuration. In the case of fluids, S is the boundary surface in the current configuration.

For nonviscous compressible fluid flows, the unknown functions are ρ, p, T, ε, $\mathbf{v}$, $\mathbf{q}$ and $\mathbf{T}$; and the governing equations are (10.4.1)–(10.4.4), (10.4.10), (10.4.12) and (10.4.13). Since p, ε and $\mathbf{q}$ can be regarded as known functions of ρ and T through the equations of state (10.4.10) and (10.4.13) and the law of heat conduction (10.4.12), the initial conditions are usually specified in terms of $\mathbf{v}$, ρ and T and the boundary conditions in terms of $\mathbf{v}$, $\mathbf{T}$ and T. The conditions for $\mathbf{v}$ and $\mathbf{T}$ are usually taken as those given in (10.5.1) and (10.5.2ab), whereas the conditions for ρ and T are as follows:

$$\rho = \rho_0 \quad \text{in} \quad V \quad \text{at} \quad t = 0 \tag{10.5.3}$$

$$T = T_0 \quad \text{in} \quad V \quad \text{at} \quad t = 0 \tag{10.5.4}$$

$$\left.\begin{aligned} T &= T^* \quad \text{on} \quad S_T \\ \nabla T \cdot \mathbf{n} &= q^* \quad \text{on} \quad S_q \end{aligned}\right\} \quad \text{for} \quad t \geq 0 \tag{10.5.5}$$

where S_T and S_q are parts of S that are complementary to each other (as special cases S_T and S_q can be equal to S). Also, ρ_0 is the initial density and T_0 is the initial temperature, and T^* and q^* are prescribed quantities in their respective domains in S.

The boundary conditions (10.5.5) mean that the temperature is prescribed on the part S_T of S and the normal derivative of temperature on the remaining part S_q of S. If $T^* = T_0$, then the points of S_T experience no change from the initial temperature, and S_T is then called an *isothermal surface*. On the other hand, if $q^* = 0$, then no heat flux occurs across S_q, and S_q is then called an *adiabatic surface*.

The boundary conditions (10.5.2) require special attention when the boundary surface is a rigid impermeable solid surface contacting a fluid and when it is a fluid surface exposed to atmosphere. A rigid solid surface contacting a fluid could be the wall of a container or the surface of a body immersed in the fluid. Since the fluid is nonviscous, shear stress on every surface element is 0; hence on a rigid surface S, no resistance to the motion of the fluid relative to S occurs and the fluid is free to slip over S. Therefore, no constraint can be imposed on the tangential component of $\mathbf{v}$ at a point of S. However, for the fluid and S to be in geometrical contact, the velocity of the fluid at the boundary must be such that its component normal to the boundary is equal to the normal component of velocity of the boundary. That is,

$$\mathbf{v} \cdot \mathbf{n} = \mathbf{v}_s \cdot \mathbf{n} \tag{10.5.6}$$

on S, where $\mathbf{v}_s$ is the velocity of the rigid boundary and $\mathbf{n}$ is the unit normal to the boundary. In the particular case when the boundary is at rest, the condition (10.5.6) becomes

$$\mathbf{v} \cdot \mathbf{n} = 0 \tag{10.5.7}$$

on S.

The condition (10.5.6) can be rewritten in an alternative form by using the geometrical equation of the boundary surface. Let $F(\mathbf{x}, t) = 0$ be the geometrical equation of the boundary surface. Since particles that initially lie on a boundary continue to lie on a boundary, $F(\mathbf{x}, t) = 0$ is a material surface, and the speed with which this surface advances perpendicular to itself is given by (6.2.62). That is,

$$\mathbf{v}_s \cdot \mathbf{n} = -\frac{(\partial F/\partial t)}{|\nabla F|} \tag{10.5.8}$$

Consequently, the condition (10.5.6) can be rewritten as

$$\frac{\partial F}{\partial t} + (\mathbf{v} \cdot \mathbf{n})|\nabla F| = 0 \tag{10.5.9}$$

on the boundary, $F(\mathbf{x}, t) = 0$.

When S is the boundary surface of a fluid that is exposed to atmosphere, the fluid pressure on S must balance the atmospheric pressure p_a.

Thus, on such a surface we impose the following boundary condition:

$$p = p_a \tag{10.5.10}$$

on S. This condition is valid only if the *surface tension* effects are neglected.

EXAMPLE 10.5.1 For a certain flow of a nonviscous fluid in a region bounded by the fixed solid boundary $y = 0$, the velocity field is given by $\mathbf{v} = \nabla\phi$, where $\phi = x^3 - 3xy^2$. Verify that the boundary condition $\mathbf{v} \cdot \mathbf{n} = 0$ is satisfied and that at all points except the origin the fluid slips on the boundary.

Note that here and in the following discussion, (x, y) or (x, y, z) denote the Cartesian coordinates.

Solution From the given expression for $\mathbf{v}$, we find that

$$v_1 = 3(x^2 - y^2), \qquad v_2 = -6xy, \qquad v_3 = 0 \tag{10.5.11}$$

From these components we note that on the boundary $y = 0$, we have $\mathbf{v} \cdot \mathbf{n} = v_2 = 0$. Thus, the boundary condition $\mathbf{v} \cdot \mathbf{n} = 0$ is verified.

Since $v_3 = 0$, v_1 is the only tangential component of velocity that occurs on the boundary $y = 0$. For $y = 0$, (10.5.11) yields $v_1 = 3x^2$, and thus v_1 vanishes only at the origin. Hence, except at the origin, everywhere else on the boundary $y = 0$ the fluid slips with speed $3x^2$ in the x direction. ■

EXAMPLE 10.5.2 A long cylinder is fixed rigidly in the flow of a nonviscous fluid. Obtain the boundary condition for velocity on the lateral surface of the cylinder.

Solution Let us choose the coordinate axes such that the z axis is along the axis of the cylinder. Then at any point P on the lateral surface of the cylinder, the normal $\mathbf{n}$ is parallel to the xy plane so that $n_1 = \cos\theta$, $n_2 = \sin\theta$, $n_3 = 0$, where θ is the angle between $\mathbf{n}$ and the x direction;

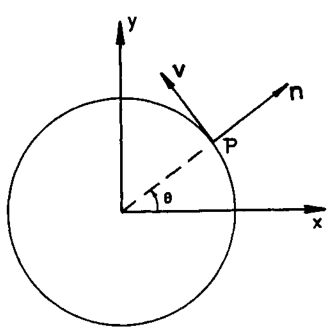

Figure 10.3. Example 10.5.2.

see Figure 10.3. Since the cylinder is stationary, the boundary condition (10.5.7) yields

$$\mathbf{v} \cdot \mathbf{n} \equiv v_1 \cos\theta + v_2 \sin\theta = 0 \tag{10.5.12}$$

This is the condition to be satisfied by the velocity components of the fluid at a point P on the lateral surface of the cylinder. ■

10.6

EULER'S EQUATION OF MOTION

In Section 10.4, it was shown that for a nonviscous fluid flow there are ten field functions (namely, p, v_i and τ_{ij}) for the incompressible case and sixteen field functions (namely, ρ, p, ε, T, v_i, q_i and τ_{ij}) for the compressible case, and as many governing equations. By combining the equations of motion (10.4.2) and the material law (10.4.4), it is possible to eliminate the stress components from the governing equations and thereby reduce the number of field functions and their governing equations. In this section, we derive the equation of motion through this method.

We start with the material law (10.4.4). Taking the divergence on both sides of this tensor equation and using the identity (3.5.35), we obtain

$$\operatorname{div} \mathbf{T} = -\nabla p \tag{10.6.1}$$

Substituting this into the equation of motion (10.4.2) we obtain the following equation of motion expressed in terms of ρ, p and $\mathbf{v}$:

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{b} \tag{10.6.2}$$

This equation was first obtained by Euler in 1755 and is known as *Euler's equation of motion*. This equation holds for both incompressible and compressible fluids and is a basic governing differential equation in the theory of nonviscous fluid flows. In the case of incompressible fluid flows, $\rho = \rho_0$ and the equation contains $\mathbf{v}$ and p as the field functions. In the case of compressible fluid flows, the equation includes $\mathbf{v}$, ρ and p as the field functions. Thus the stresses are completely eliminated from the equation of motion.

For a fluid at rest (static equilibrium) or in uniform motion, Euler's equation (10.6.2) reduces to the *equation of equilibrium*:

$$-\frac{1}{\rho}\nabla p + \mathbf{b} = \mathbf{0} \tag{10.6.3}$$

where p denotes the *static pressure* and ρ denotes *time-independent density*.

By using the identities (6.2.10) and (6.2.11), that is,

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{w} \times \mathbf{v} + \frac{1}{2}\nabla v^2 \tag{10.6.4}$$

where $\mathbf{w} = \operatorname{curl} \mathbf{v}$ is the vorticity vector, equation (10.6.2) can be rewritten in the following alternative forms:

$$-\frac{1}{\rho}\nabla p + \mathbf{b} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \tag{10.6.5a}$$

$$= \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\nabla v^2 + \mathbf{w} \times \mathbf{v} \tag{10.6.5b}$$

Thus the Euler's equation of motion is a *nonlinear* partial differential equation of the first order, the nonlinearity being caused by the acceleration vector. This implies that in principle it is not possible to simply superpose solutions of Euler's equation.

In the case of an incompressible fluid flow, equation (10.6.2) may be solved under appropriate initial and boundary conditions described in Section 10.5 to find a solution for the velocity field $\mathbf{v}$ for a prescribed pressure p; generally p is specified through boundary data. Equation of continuity (10.4.5) serves as a constraint on $\mathbf{v}$.

For compressible fluid flows, p is usually specified by a kinetic equation of state, and equation (10.6.2) and (10.4.1) are to be solved *simultaneously* under appropriate initial and boundary conditions described in Section 10.5 to find $\mathbf{v}$ and ρ. Once $\mathbf{v}$ and ρ are so determined, the energy equation (10.4.3) may be employed under appropriate initial and boundary conditions to determine T. Note that ε is usually specified in terms of ρ and T by a caloric equation of state.

EXAMPLE 10.6.1 A rectangular tank containing a nonviscous liquid of constant density moves horizontally to the right with a constant acceleration. Gravitational force is the only external force. Find the pressure distribution in the liquid and the geometrical shape of the upper surface of the liquid.

Solution Choose the directions of the coordinate axes as indicated in Figure 10.4. Then, $D\mathbf{v}/Dt = a\mathbf{e}_1$ and $\mathbf{b} = -g\mathbf{e}_3$, where $a = |D\mathbf{v}/Dt|$ is a constant and g is the (constant) acceleration due to gravity. Euler's equation (10.6.2) now yields the following three equations for the three Cartesian (x, y, z) components of ∇p:

$$\frac{\partial p}{\partial x} = -a\rho \tag{10.6.6a}$$

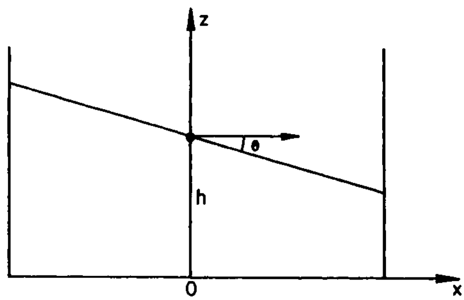

Figure 10.4. Example 10.6.1.

$$\frac{\partial p}{\partial y} = 0 \tag{10.6.6b}$$

$$\frac{\partial p}{\partial z} = -g\rho \tag{10.6.6c}$$

Equation (10.6.6b) shows that p is independent of y, and hence (10.6.6a) gives

$$p = -\rho a x + f(z) \tag{10.6.7}$$

where $f(z)$ is an arbitrary function of z.

Equations (10.6.6c) and (10.6.7) give $f(z) = -\rho g z + C$, where C is a constant. Putting this $f(z)$ into (10.6.7), we get

$$p = -\rho(ax + gz) + C \tag{10.6.8}$$

At the point where the z axis meets the upper surface of the liquid, we have $p = p_a$, where p_a is the atmospheric pressure. If this point is at a height h above the origin, (10.6.8) gives $C = p_a + \rho g h$. Thus the pressure distribution in the liquid is

$$p = p_a - \rho(ax + gz - gh) \tag{10.6.9}$$

For $p = p_a$, equation (10.6.9) becomes

$$z = -\left(\frac{a}{g}\right)x + h$$

This is the equation of the upper surface of the liquid. Evidently, this surface is a plane making an acute angle $\theta = \tan^{-1}(a/g)$ with the horizontal. ■

Note that in the limiting case when $a \to 0$, the liquid moves with constant velocity and the upper surface of the liquid becomes a horizontal plane.

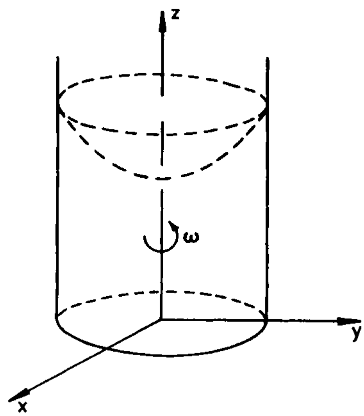

Figure 10.5. Example 10.6.2.

EXAMPLE 10.6.2 A column of a nonviscous liquid of constant density contained in a vertical circular vessel rotates like a rigid body about the axis of the vessel with a constant angular velocity $\boldsymbol{\omega}$. Gravitational force is the only external force. Find the pressure distribution in the liquid and the geometrical form of the upper surface of the liquid.

Solution For a fluid rotating like a rigid body, the acceleration is given by (6.2.20). Taking the axis of rotation as the z axis directed positive upward (see Figure 10.5) and noting $\boldsymbol{\omega} = \omega \mathbf{e}_3$ is a constant angular velocity, this relation becomes

$$\frac{D\mathbf{v}}{Dt} = \omega \mathbf{e}_3 \times (\omega \mathbf{e}_3 \times \mathbf{x}) = -\omega^2 \mathbf{r} \tag{10.6.10}$$

where $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2$.

Using (10.6.10) and the fact that $\mathbf{b} = -g\mathbf{e}_3$, Euler's equation of motion (10.6.2) reduces to

$$\nabla p = \rho(\omega^2 \mathbf{r} - g\mathbf{e}_3) \tag{10.6.11}$$

Taking dot product with $\hat{\mathbf{r}} = \mathbf{r}/r$ (where $r = |\mathbf{r}|$) on both sides of this equation and noting that $\nabla p \cdot \hat{\mathbf{r}} = \partial p/\partial r$, we obtain

$$\frac{\partial p}{\partial r} = \rho \omega^2 r \tag{10.6.12}$$

On the other hand, if we take dot product with $\mathbf{e}_3$ on both sides of (10.6.11), we find

$$\frac{\partial p}{\partial z} = -\rho g \tag{10.6.13}$$

Due to the axial symmetry of the problem, we assume that p depends only on r and z so that

$$dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}dz \tag{10.6.14}$$

Using (10.6.12) and (10.6.13) in (10.6.14) and integrating the resulting equation, we obtain

$$p = \tfrac{1}{2}\rho\omega^2 r^2 - \rho g z + C \tag{10.6.15}$$

where C is a constant of integration. At the point where the z axis meets the upper surface of the liquid, we have $p = p_a$, where p_a is the atmospheric pressure. If this point is at a height h above the origin, (10.6.15) gives $C = p_a + \rho g h$. Putting this value of C into (10.6.15) we obtain the pressure distribution in the liquid as

$$p = p_a + \tfrac{1}{2}\omega^2 r^2 - g\rho(z - h) \tag{10.6.16}$$

For $p = p_a$, equation (10.6.16) becomes

$$z = h - \frac{\omega^2}{2g}r^2 \tag{10.6.17}$$

This is the equation of the upper surface of the rotating liquid; the surface is a paraboloid of revolution with the z axis as its axis and vertex downward. ■

EXAMPLE 10.6.3 Treating the atmosphere around the earth as a non-viscous perfect gas at rest under constant gravitational field, show that the pressure at height z above the ground level is given by

$$p = p_1 \exp\left[-\frac{g}{R}\int_0^z \frac{dz}{T}\right] \tag{10.6.18}$$

where p_1 is the pressure at the earth's surface. If T = constant = T_0, deduce that

$$\frac{p}{p_1} = \frac{\rho}{\rho_1} = \exp\left(-\frac{gz}{RT_0}\right) \tag{10.6.19}$$

where ρ_1 is the density at the earth's surface.

Solution With the positive z axis vertical upward, we have $\mathbf{b} = -g\mathbf{e}_3$, since the gravitational force is the only body force. The equation of equilibrium (10.6.3) now yields

$$\frac{\partial p}{\partial x} = 0, \qquad \frac{\partial p}{\partial y} = 0, \qquad \frac{\partial p}{\partial z} = -g\rho$$

The first two equations show that p is independent of x and y. Hence the last equation becomes

$$\frac{dp}{dz} = -\rho g \tag{10.6.20}$$

Since the fluid under consideration is a perfect gas, we have $p = \rho RT$ by the equation state (10.4.15), and equation (10.6.20) can be rewritten as

$$\frac{dp}{dz} + \frac{g}{R}\frac{p}{T} = 0 \tag{10.6.21}$$

Integration of this equation yields the pressure at a height z above the ground level ($z = 0$) as

$$p = A \exp\left(-\frac{g}{R}\int_0^z \frac{dz}{T}\right) \tag{10.6.22}$$

where A is a constant of integration. Using the conditions $p = p_1$ at $z = 0$, we find $A = p_1$ and (10.6.18) follows immediately.

If $T = T_0$, then (10.6.18) becomes

$$p = p_1 \exp\left(-\frac{gz}{RT_0}\right) \tag{10.6.23}$$

In view of the equation of state $p = \rho RT = \rho RT_0$, (10.6.23) gives

$$\rho = \frac{p_1}{RT_0}\exp\left(-\frac{gz}{RT_0}\right) \tag{10.6.24}$$

The equation $p = \rho RT$ also gives $p_1 = \rho_1 RT_0$. Hence (10.6.24) becomes

$$\rho = \rho_1 \exp\left(-\frac{gz}{RT_0}\right) \tag{10.6.25}$$

Expressions (10.6.23) and (10.6.25) yield (10.6.19). ■

EXAMPLE 10.6.4 In a part of the atmosphere lying nearer the earth's surface, the temperature distribution is known to be of the form $T = T_1 - \alpha z$, where T_1 is the temperature at the ground level, z is the vertical distance from the ground level and α is a constant. Assuming that the atmosphere is a nonviscous perfect gas at rest under the gravitational field, prove the following:

(i)
$$\frac{p}{p_1} = \left(\frac{T}{T_1}\right)^{(g/R\alpha)} \tag{10.6.26}$$

(ii) $$\frac{\rho}{\rho_1} = \left(\frac{T}{T_1}\right)^{(g/R\alpha)-1} \tag{10.6.27}$$

(ii) $$\frac{\rho}{\rho_1} = \left(\frac{p}{p_1}\right)^{1-(R\alpha/g)} \tag{10.6.28}$$

where p_1 and ρ_1 are the pressure and the density at the ground level.

Solution Recall that (10.6.18) is the equation for the pressure p in a nonviscous perfect gas at rest under the gravitational field. Setting $T = T_1 - \alpha z$ in this equation and simplifying it with the condition, $T = T_1$ at $z = 0$, we get

$$p = p_1\left(1 - \frac{\alpha z}{T_1}\right)^{(g/R\alpha)} = p_1\left(\frac{T}{T_1}\right)^{(g/R\alpha)}$$

which is (10.6.26).

For $z = 0$, the equation of state $p = \rho RT$ yields $RT_1 = p_1/\rho_1$. Hence the equation $p = \rho RT$ can be rewritten as

$$p = \frac{\rho RTT_1}{T_1} = \frac{\rho p_1 T}{\rho_1 T_1}$$

so that

$$\frac{p}{p_1} = \frac{\rho}{\rho_1}\left(\frac{T}{T_1}\right) \tag{10.6.29}$$

Elimination of ρ/ρ_1 from (10.6.26) and (10.6.29) gives the result (10.6.27). Also, (10.6.28) follows by the elimination of T/T_1 from (10.6.26) and (10.6.29). ■

EXAMPLE 10.6.5 For an adiabatic flow of a nonviscous perfect gas with no heat supply, show that

$$\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} = \frac{\gamma p}{\rho}\frac{D\rho}{Dt} \tag{10.6.30}$$

Deduce that

(i) in the absence of body force,

$$\frac{D}{Dt}\left(c_p T + \frac{1}{2}v^2\right) = \frac{1}{\rho}\frac{\partial p}{\partial t} \tag{10.6.31}$$

(ii) in the absence of body force and for steady flow,

$$\frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}v^2 = \text{constant} \tag{10.6.32}$$

Solution Recall that for a flow of a nonviscous perfect gas, the energy equation has been expressed in two alternative forms as given by equations (10.4.31) and (10.4.32). In the case of an adiabatic flow with no heat supply, these equations become

$$c_v \frac{DT}{Dt} = \frac{p}{\rho^2} \frac{D\rho}{Dt} \tag{10.6.33}$$

$$c_p \frac{DT}{Dt} = \frac{1}{\rho} \frac{Dp}{Dt} \tag{10.6.34}$$

Since $p = \rho RT$ for a perfect gas, we find

$$\frac{DT}{Dt} = \frac{1}{R} \frac{D}{Dt}\left(\frac{p}{\rho}\right) = \frac{1}{\rho R}\left[\frac{Dp}{Dt} - \frac{p}{\rho} \frac{D\rho}{Dt}\right]$$

Using this in (10.6.33) and taking note of (10.4.18) we get the equation

$$\frac{p\gamma}{\rho} \frac{D\rho}{Dt} = \frac{Dp}{Dt} \tag{10.6.35}$$

Equations (10.6.34) and (10.6.35) together constitute equation (10.6.30).

In the absence of body force, Euler's equation of motion (10.6.2) gives

$$\mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} = -\mathbf{v} \cdot \left(\frac{1}{\rho} \nabla p\right)$$

so that

$$\frac{D}{Dt}(v^2) = 2\mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} = -2\mathbf{v} \cdot \left(\frac{1}{\rho} \nabla p\right) \tag{10.6.36}$$

Also,

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + (\mathbf{v} \cdot \nabla) p \tag{10.6.37}$$

Using (10.6.36) and (10.6.37) in (10.6.34), we obtain (10.6.31).

For steady flow, $\partial p/\partial t = 0$, and equation (10.6.31) yields

$$c_p T + \tfrac{1}{2} v^2 = \text{constant} \tag{10.6.38}$$

Since $c_p = \gamma R/(\gamma - 1)$ by (10.4.20b) and $p = \rho RT$, we get

$$c_p T = \frac{\gamma}{\gamma - 1} \frac{p}{\rho} \tag{10.6.39}$$

Results (10.6.38) and (10.6.39) together yield (10.6.32). ■

EXAMPLE 10.6.6 Show that the equation of motion of a nonviscous fluid can be expressed in terms of $\mathbf{x}^0$ and t as follows:

$$\nabla^0 p = \rho \mathbf{F}^T \left[\mathbf{b} - \frac{D^2\mathbf{x}}{Dt^2} \right] \tag{10.6.40}$$

Solution Treating p as a function of x_i^0 and t, we have $p_{;i} = p_{,k} x_{k;i}$. That is,

$$\nabla^0 p = \mathbf{F}^T(\nabla p) \tag{10.6.41}$$

Substituting for ∇p from Euler's equation (10.6.2) in (10.6.41) and recalling that $D\mathbf{v}/Dt = D^2\mathbf{x}/Dt^2$, we get the equation (10.6.40). ■

Note: Equation (10.6.40) is the equation of motion for a nonviscous fluid in the *material (Lagrangian) form*.

10.7 EQUATION OF MOTION OF AN ELASTIC FLUID

We now specialize Euler's equation of motion (10.6.2) to the case of an *elastic fluid* and analyze some consequences of it. Recall from Section 10.4 that an elastic fluid is a nonviscous compressible fluid for which the kinetic equation of state is given by (10.4.21a) or (10.4.21b).

Let us set

$$P = \int \frac{1}{\rho} dp \tag{10.7.1}$$

Using (10.4.21b), that is, $\rho = \rho(p)$, we find by virtue of the result (ii) of Example 3.4.1 that

$$\nabla P = \frac{1}{\rho} \nabla p \tag{10.7.2}$$

Consequently, Euler's equation of motion (10.6.2) becomes

$$\frac{D\mathbf{v}}{Dt} = -\nabla P + \mathbf{b} \tag{10.7.3}$$

This is the equation of motion for an elastic fluid. Evidently, this equation holds for a fluid of constant density also; in this case, $P = p/\rho$, where ρ is the constant density.

Equation (10.7.3) can be expressed in an alternative form. Since $p = p(\rho)$ for an elastic fluid, by (10.4.21a), we have

$$\nabla p = \left(\frac{dp}{d\rho}\right) \nabla \rho \tag{10.7.4}$$

Setting

$$c_s = \left(\frac{dp}{d\rho}\right)^{1/2} \tag{10.7.5}$$

we find from (10.7.2) and (10.7.4) that

$$\nabla P = \frac{c_s^2}{\rho}(\nabla\rho) \tag{10.7.6}$$

The physical meaning of c_s, which is evidently *not* a constant, will be given in Example 10.7.5.

Using (10.7.6) in (10.7.3), we get the following alternative version of (10.7.3):

$$\frac{D\mathbf{v}}{Dt} = -\frac{c_s^2}{\rho}(\nabla\rho) + \mathbf{b} \tag{10.7.7}$$

In many practical problems, the body force $\mathbf{b}$ is conservative; that is, $\mathbf{b} = -\nabla\chi$ for some scalar function χ (called the *potential of* $\mathbf{b}$). Equations (10.7.3) and (10.7.7) then become

$$\frac{D\mathbf{v}}{Dt} = -\nabla(P + \chi) \tag{10.7.8}$$

$$= -\frac{c_s^2}{\rho}(\nabla\rho) - \nabla\chi \tag{10.7.9}$$

10.7.1 CIRCULATION THEOREM

From equation (10.7.8) we note that the acceleration is the gradient of a scalar function. Hence, by Kelvin's circulation theorem proven in Section 6.6, it readily follows that the motion for which (10.7.8) is the governing differential equation is circulation preserving. Thus, *every motion of an elastic fluid under conservative body force is circulation preserving.* Consequently, all the properties of circulation-preserving motions, like the permanence of irrotational motion and transportation of vortex lines, as discussed in Section 6.6, hold for such a motion.

10.7.2 VORTICITY EQUATION

For a continuum, the vorticity vector $\mathbf{w}$ is governed by Beltrami's vorticity equation (8.2.16). For an elastic fluid, we find from (10.7.3) that

$$\operatorname{curl}\left(\frac{D\mathbf{v}}{Dt}\right) = \operatorname{curl}\mathbf{b}$$

Substituting this in (8.2.16), we obtain the equation

$$\frac{D}{Dt}\left(\frac{\mathbf{w}}{\rho}\right) = \left(\frac{\mathbf{w}}{\rho}\cdot\nabla\right)\mathbf{v} + \frac{1}{\rho}\operatorname{curl}\mathbf{b} \tag{10.7.10}$$

This is the *vorticity equation for an elastic fluid.* This equation is by Helmholtz (1868) and Nanson (1874). If the external force is conservative, the second term on the righthand side of the equation vanishes identically.

10.7.3 EQUATIONS OF EQUILIBRIUM

For an elastic fluid at rest (static equilibrium) or in uniform motion, equations (10.7.3) and (10.7.7) become the *equations of equilibrium*:

$$\nabla P = \mathbf{b} \tag{10.7.11}$$

$$\frac{c_s^2}{\rho}(\nabla\rho) = \mathbf{b} \tag{10.7.12}$$

For $\mathbf{b} = -\nabla\chi$, equation (10.7.11) yields

$$P + \chi \equiv \int\frac{dp}{\rho} + \chi = \text{constant} \tag{10.7.13}$$

EXAMPLE 10.7.1 For an elastic fluid in static equilibrium under the earth's gravitational field (taken as constant), show that

$$\int\frac{dp}{\rho} + gz = \text{constant} \tag{10.7.14}$$

where z is the height above the surface of the earth. Hence deduce the following.

(i) For an isothermal perfect gas,

$$gz + \frac{p}{\rho}\log p = \text{constant} \tag{10.7.15}$$

(ii) For an isentropic perfect gas,

$$gz + \frac{\gamma}{(\gamma - 1)}\frac{p}{\rho} = \text{constant} \tag{10.7.16}$$

Solution Let us choose the positive z axis vertically upward from the earth's surface. Then, if gravity is the only body force that is taken as constant, we have $\mathbf{b} = -g\mathbf{e}_3 = -\nabla(gz)$, so that $\chi = gz$ is the potential of $\mathbf{b}$. Substituting $\chi = gz$ in the equation of equilibrium (10.7.13), we readily get (10.7.14).

For an isothermal perfect gas, we have by (10.4.16), $(p/\rho) = \text{constant} = A$, say, so that

$$\int \frac{dp}{\rho} = A \int \frac{dp}{p} = A \log p + \text{constant} = \frac{p}{\rho} \log p + \text{constant} \quad (10.7.17)$$

Using this in (10.7.14), we obtain (10.7.15).

For an isentropic perfect gas, we have, by (10.4.17a), $p = \beta\rho^\gamma$, where β and $\gamma(>1)$ are constants, so that

$$\int \frac{dp}{\rho} = \beta \int \frac{1}{\rho} \gamma\rho^{\gamma-1}\, d\rho = \beta\gamma \frac{\rho^{\gamma-1}}{(\gamma - 1)} + \text{constant} = \frac{\gamma}{(\gamma - 1)} \frac{p}{\rho} + \text{constant} \quad (10.7.18)$$

The use of this result in (10.7.14) gives (10.7.16). ■

EXAMPLE 10.7.2 A vertical rectangular plate of width a and height b is exposed to an atmosphere that is an isothermal perfect gas in equilibrium under the earth's gravitational field (which is taken as constant). Show that the magnitude of the total force acting on a face of the plate is given by

$$f = \frac{RTa}{g} p(z_0)\left[1 - \exp\left(-\frac{g}{RT} b\right)\right] \quad (10.7.19)$$

where $p(z_0)$ is the pressure at the lower edge of the plate, and T is the constant temperature.

Solution Since the atmosphere to which the plate is exposed is an isothermal perfect gas in equilibrium under the earth's gravitational field, equation (10.7.15) holds. Making use of $p = \rho RT$ in (10.7.15), we find

$$gz + RT \log p = C \quad (10.7.20)$$

where C is a constant. If the lower edge of the plate is at a height z_0 above the earth's surface, see Figure 10.6, (in the particular case, z_0 can be 0), equation (10.7.20) gives

$$C = gz_0 + RT \log p(z_0)$$

Substituting this value of C into (10.7.20) and rewriting the resulting equation yields

$$p = p(z_0) \exp\left[-\frac{g}{RT}(z - z_0)\right] \quad (10.7.21)$$

The magnitude of the total force exerted on a face of the plate due to the pressure p is given by

$$f = \int_s p\, dS \quad (10.7.22)$$

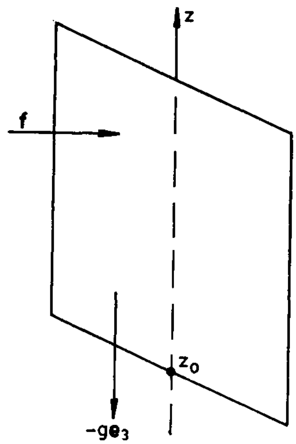

Figure 10.6. Example 10.7.2.

where S is the surface area of a face of the plate. Substituting for p from (10.7.21) into (10.7.22) and noting that p depends only on the vertical distance (as is evident from (10.7.21)), we obtain

$$f = \int_{z_0}^{z_0+b} ap(z)\,dz = \frac{aRT}{g}p(z_0)\left[1 - \exp\left(-\frac{gb}{RT}\right)\right]$$

which is the desired result (10.7.19). ■

EXAMPLE 10.7.3 In a steady flow of an elastic fluid, show that $\mathbf{v}$ satisfies the equation

$$c_s^2(\operatorname{div}\mathbf{v}) + \mathbf{b}\cdot\mathbf{v} = \mathbf{v}\cdot(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{v}\cdot\nabla(\tfrac{1}{2}v^2) \qquad (10.7.23)$$

Deduce that, for potential flow, the velocity potential ϕ satisfies the equation

$$(c_s^2\delta_{km} - \phi_{,k}\phi_{,m})\phi_{,km} + b_k v_k = 0 \qquad (10.7.24)$$

Equations (10.7.23) and (10.7.24) are usually referred to as *gas dynamical equations.*

Solution Recalling that

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)$$

the equation of motion (10.7.7) for an elastic fluid can be rewritten as

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{c_s^2}{\rho}(\nabla\rho) + \mathbf{b} \qquad (10.7.25)$$

Similarly, the equation of continuity (10.4.1) can be rewritten as (see equation (8.2.5a))

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \operatorname{div} \mathbf{v} = 0 \tag{10.7.26}$$

For steady flow, $\partial \mathbf{v}/\partial t = \mathbf{0}$ and $\partial \rho/\partial t = 0$. In this case (10.7.25) and (10.7.26) give

$$\mathbf{v} \cdot (\mathbf{v} \cdot \nabla)\mathbf{v} = c_s^2 \operatorname{div} \mathbf{v} + \mathbf{b} \cdot \mathbf{v} \tag{10.7.27}$$

By use of identity (3.4.29), we find that

$$\mathbf{v} \cdot (\mathbf{v} \cdot \nabla)\mathbf{v} = \tfrac{1}{2}\nabla(v^2) \cdot \mathbf{v} \tag{10.7.28}$$

Identity (10.7.28) together with equation (10.7.27) yields the result (10.7.23).

In the suffix notation, equation (10.7.27) reads

$$v_k v_m v_{k,m} = c_s^2 v_{k,k} + b_k v_k$$

which can be rewritten as

$$(c_s^2 \delta_{km} - v_k v_m) v_{k,m} + b_k v_k = 0 \tag{10.7.29}$$

For potential flow, we have $v_i = \phi_{,i}$, where ϕ is the velocity potential. Equation (10.7.29) then reduces to (10.7.24). ■

EXAMPLE 10.7.4 Show that for an elastic fluid moving under conservative body force the equation of motion can be expressed in the form

$$\mathbf{F}^T \mathbf{v} = \mathbf{v}^0 - \nabla^0 \left\{ \int_0^t \left(\int \frac{dp}{\rho} + \chi - \frac{1}{2} v^2 \right) dt \right\} \tag{10.7.30}$$

where $\mathbf{v}^0$ is the initial velocity. This equation is known as *Weber's equation*, after Weber who derived it in 1868.

Solution The equation of motion of an elastic fluid in the presence of conservative body force is given by (10.7.8). Operating by the tensor $\mathbf{F}^T$ on both sides of this equation, we obtain

$$\mathbf{F}^T \nabla(P + \chi) = -\mathbf{F}^T \frac{D\mathbf{v}}{Dt} = -\frac{D}{Dt}(\mathbf{F}^T \mathbf{v}) + \frac{D\mathbf{F}^T}{DT}\mathbf{v} \tag{10.7.31}$$

But

$$\frac{D\mathbf{F}^T}{Dt}\mathbf{v} = \{(\nabla \mathbf{v})\mathbf{F}\}^T \mathbf{v}$$

by (6.2.44).

$$= \mathbf{F}^T (\nabla \mathbf{v})^T \mathbf{v} = \tfrac{1}{2}\mathbf{F}^T \nabla(\mathbf{v} \cdot \mathbf{v}) \tag{10.7.32}$$

Substituting this into (10.7.31) and noting that $\mathbf{F}^T(\nabla\phi) = \nabla^0\phi$ for any function ϕ, we get the equation

$$\frac{D}{Dt}(\mathbf{F}^T\mathbf{v}) = -\nabla^0(P + \chi - \tfrac{1}{2}v^2) \tag{10.7.33}$$

Integrating this equation with respect to t from 0 to t, we find

$$[\mathbf{F}^T\mathbf{v}]_0^t = -\nabla^0\left\{\int_0^t (P + \chi - \tfrac{1}{2}v^2)\,dt\right\} \tag{10.7.34}$$

If we use the fact that $\mathbf{F} = \mathbf{I}$ and $\mathbf{v} = \mathbf{v}^0$ at $t = 0$, and note that $P = \int dp/\rho$, equation (10.7.34) reduces to equation (10.7.30). ■

EXAMPLE 10.7.5 In the linearized case of an elastic fluid motion in the absence of body forces, show that ρ, p and div $\mathbf{v}$ all satisfy the same wave equation

$$c_s^2\nabla^2 f = \frac{\partial^2 f}{\partial t^2} \tag{10.7.35}$$

Solution In the absence of body force, equations (10.7.2), (10.7.3) and (10.7.7) yield

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p = -c_s^2\,\nabla\rho \tag{10.7.36}$$

In the linearized case, we make the approximation

$$\rho\frac{D\mathbf{v}}{Dt} = \rho\left[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] \approx \rho_0\frac{\partial\mathbf{v}}{\partial t} \tag{10.7.37}$$

Hence (10.7.36) becomes

$$\rho_0\frac{\partial\mathbf{v}}{\partial t} = -\nabla p = -c_s^2\,\nabla\rho \tag{10.7.38}$$

so that

$$\rho_0\frac{\partial}{\partial t}(\operatorname{div}\mathbf{v}) = -\nabla^2 p = -c_s^2\nabla^2\rho \tag{10.7.39}$$

In obtaining the term on the far right, we have neglected $(\nabla c_s^2)\cdot\nabla\rho$.

In the linearized case, the equation of continuity (10.7.26) becomes

$$\frac{\partial\rho}{\partial t} + \rho_0\operatorname{div}\mathbf{v} = 0 \tag{10.7.40}$$

Since $\rho = \rho(p)$ for an elastic fluid, (10.7.40) can be rewritten as

$$\frac{d\rho}{dp}\left(\frac{\partial p}{\partial t}\right) + \rho_0 \operatorname{div} \mathbf{v} = 0 \tag{10.7.41}$$

or, on using (10.7.5),

$$\frac{1}{c_s^2}\frac{\partial p}{\partial t} + \rho_0 \operatorname{div} \mathbf{v} = 0. \tag{10.7.42}$$

Substituting for div **v** from (10.7.40) and (10.7.42) in (10.7.39) we get

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{1}{c_s^2}\frac{\partial^2 p}{\partial t^2} = \nabla^2 p = c_s^2 \nabla^2 \rho \tag{10.7.43}$$

These show that both ρ and p satisfy the wave equation (10.7.35).

Taking the Laplacian of (10.7.40) and substituting for $\nabla^2 \rho$ from (10.7.39) in the resulting equation, we find

$$c_s^2 \nabla^2(\operatorname{div} \mathbf{v}) = \frac{\partial^2}{\partial t^2}(\operatorname{div} \mathbf{v}) \tag{10.7.44}$$

This shows that (div **v**) also satisfies the wave equation (10.7.35). ■

Note: The wave equation (10.7.35) governing p, ρ and div **v** plays a fundamental role in the field of acoustics; it is called the *acoustic wave equation* for *small disturbances*. The speed $c_s = (dp/d\rho)^{1/2}$ associated with this equation represents the *local speed of sound*. It should be noted that c_s is *not* a constant, in general. For an isothermal nonviscous perfect gas (for which (p/ρ) = constant), we find that $c_s = (p/\rho)^{1/2}$, which is a constant. For an isentropic, nonviscous perfect gas (for which $p = \beta\rho^\gamma$) we obtain $c_s = (\gamma p/\rho)^{1/2}$. Experiments on common gases show that $c_s = (\gamma p/\rho)^{1/2}$ is a good approximation for computing the local speed of sound. With $\gamma = 1.4$, the value of c_s for air at 0°C is known to be 331.3 m/s.

10.8 BERNOULLI'S EQUATIONS

It has been shown that for an elastic fluid moving under conservative body force the equation of motion is given by (10.7.8). Using relation (10.6.4), equation (10.7.8) can be rewritten as

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{w} \times \mathbf{v} = -\nabla H \tag{10.8.1}$$

where

$$H = P + \chi + \tfrac{1}{2}v^2 \tag{10.8.2}$$

It is possible to integrate the equation of motion (10.8.1) in three important special cases, considered next.

10.8.1 CASE I

Suppose the flow is of potential kind; that is, $\mathbf{v} = \nabla\phi$ for some scalar ϕ. Then $\mathbf{w} = \mathbf{0}$, and equation (10.8.1) becomes

$$\nabla\left(H + \frac{\partial\phi}{\partial t}\right) = 0 \tag{10.8.3}$$

This equation implies that

$$H + \frac{\partial\phi}{\partial t} = f(t) \tag{10.8.4}$$

where $f(t)$ is an arbitrary function of t. Substitution for H from (10.8.2) in (10.8.4) gives

$$P + \frac{1}{2}(\nabla\phi)^2 + \chi + \frac{\partial\phi}{\partial t} = f(t) \tag{10.8.5}$$

10.8.2 CASE II

Suppose the flow is steady, and the stream lines and vortex lines are noncoincident; that is, $\partial/\partial t \equiv 0$ and $\mathbf{v} \times \mathbf{w} \neq \mathbf{0}$. Then equation (10.8.1) becomes

$$\mathbf{w} \times \mathbf{v} = -\nabla H \tag{10.8.6}$$

We note that ∇H is a vector normal to the surface of constant H, see Section 3.4. Also, $\mathbf{v} \times \mathbf{w}$ is a vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$. Hence, from equation (10.8.6) it follows that $\mathbf{v}$ and $\mathbf{w}$ are tangential to a surface of constant H. Since $\mathbf{v}$ and $\mathbf{w}$ are tangential to the stream lines and vortex lines, respectively, it follows that these lines lie on a surface of constant H. In other words, H is constant along stream lines and vortex lines; that is, by (10.8.2),

$$P + \tfrac{1}{2}v^2 + \chi = \text{constant} \tag{10.8.7}$$

along stream lines and vortex lines.

A surface of constant H covered by a network of stream lines and vortex lines is known as *Lamb surface*, after H. Lamb (1878); this is illustrated in Figure 10.7.

10.8.3 CASE III

Suppose the flow is steady and is either an irrotational or a Beltrami flow. (A flow is called a *Beltrami flow* if the velocity vector is a Beltrami vector, as defined in Example 3.4.7.) Then $\partial/\partial t = 0$, and either $\mathbf{w} = \mathbf{0}$ or

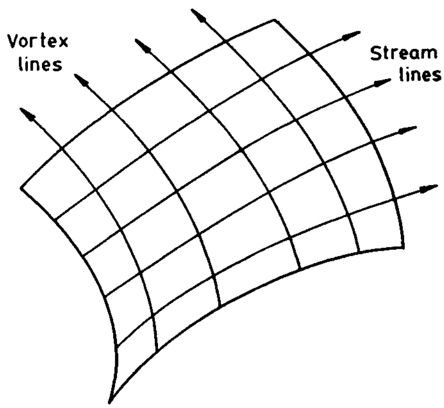

Figure 10.7. Lamb surface.

$\mathbf{v} \times \mathbf{w} = \mathbf{0}$. Then equation (10.8.1) gives $\nabla H = \mathbf{0}$. Since $\partial H/\partial t = 0$, it follows that

$$H \equiv P + \tfrac{1}{2}v^2 + \chi = \text{constant} \tag{10.8.8}$$

everywhere in the fluid.

Thus, under the assumed conditions, relations (10.8.5), (10.8.7) and (10.8.8) are integrals of the equation of motion of an elastic fluid. These relations are known as the *Bernoulli's equations*, after Daniel Bernoulli (1738). The function H, defined by (10.8.2), is known as the *Bernoulli function*.

Note: Since the Bernoulli's equations hold for an elastic fluid for which $\rho = \rho(p)$, they automatically hold in the special case of ρ = constant as well.

EXAMPLE 10.8.1 For a certain flow of a nonviscous fluid of constant density under the earth's gravitational field, the velocity distribution is given by $\mathbf{v} = \nabla\phi$, where $\phi = x^3 - 3xy^2$. Find the pressure distribution.

Solution From the given $\mathbf{v}$, we find that $\operatorname{curl} \mathbf{v} = \mathbf{0}$ and $\partial\mathbf{v}/\partial t = \mathbf{0}$. Further, since the body force is the gravitational force, it is conservative with $\chi = gz$, where z is measured vertically upward. Accordingly, the Bernoulli's equation (10.8.8) with $P = p/\rho$ holds; that is,

$$\frac{p}{\rho} + \frac{1}{2}v^2 + gz = C \tag{10.8.9}$$

where C is a constant.

From the given $\mathbf{v}$, we also find that

$$v_1 = \frac{\partial\phi}{\partial x} = 3(x^2 - y^2), \qquad v_2 = \frac{\partial\phi}{\partial y} = -6xy, \qquad v_3 = 0$$

so that

$$v^2 = v_1^2 + v_2^2 = 9(x^2 + y^2)^2$$

Hence equation (10.8.9) becomes

$$\frac{p}{\rho} + \frac{9}{2}(x^2 + y^2)^2 + gz = C$$

From this result, it is evident that $C = p^0/\rho$, where p^0 is the pressure at the origin. Thus,

$$p = p^0 - \rho[\tfrac{9}{2}(x^2 + y^2)^2 + gz] \tag{10.8.10}$$

is the required pressure distribution. ■

EXAMPLE 10.8.2 A liquid flows out of a very large reservoir through a small opening (Figure 10.8). Assuming that the liquid is nonviscous and of constant density and that the flow is steady and irrotational, find the exit speed of the liquid jet. Assume that there is no external force apart from the gravitational force.

Solution For the given flow, the Bernoulli's equation (10.8.8) with $P = p/\rho$ and $\chi = gz$, where z is measured vertically upward, holds; that is,

$$\frac{p}{\rho} + \frac{1}{2}v^2 + gz = C \tag{10.8.11}$$

where C is a constant.

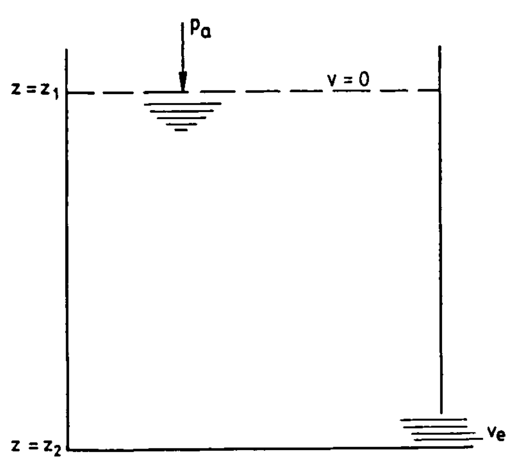

Figure 10.8. Example 10.8.2.

Since the reservoir is very large, at any given instant of time, the top surface of the liquid may be assumed to be at rest. Then at a point A on this surface (say, $z = z_1$) we have $v = 0$ and $p = p_a$, where p_a is the atmospheric pressure. Hence (10.8.11) gives

$$\frac{p_a}{\rho} + gz_1 = C \tag{10.8.12}$$

Also, at a point B on the surface of the jet (say, $z = z_2$) we have $v = v_e$ and $p = p_a$, where v_e is the exit speed of the jet. Hence (10.8.11) now gives

$$\frac{p_a}{\rho} + \frac{1}{2} v_e^2 + gz_2 = C \tag{10.8.13}$$

Relations (10.8.12) and (10.8.13) yield the following expression for the exit speed:

$$v_e = \sqrt{2g(z_1 - z_2)} \tag{10.8.14}$$

This expression is known as the *Torricelli's formula*, after Torricelli (1608–1647). We note that v_e is actually equal to the speed acquired by a body falling freely (under gravity) through a height $(z_1 - z_2)$. ■

EXAMPLE 10.8.3 A liquid of constant density flows down under gravity in a long inclined pipe with a slowly tapering circular cross section. The length of the pipe is L, the angle of inclination is α and the radius of the upper end is N times the radius of the lower end, where $N > 1$. Also, the lower end is maintained at the atmospheric pressure p_a whereas the upper end is at pressure Mp_a, where M is a constant greater than 1. Assuming that the liquid is nonviscous and that the flow is steady and irrotational, find the exit speed of the liquid at the lower end.

Solution For the given flow, Bernoulli's equation (10.8.8) with $P = (p/\rho)$ and $\chi = gz$ holds, the positive z axis being taken upward as shown in Figure 10.9. Thus,

$$\frac{p}{\rho} + \frac{1}{2}v^2 + gz = C \tag{10.8.15}$$

where C is a constant.

Let v_1 be the speed of entry at the upper end ($z = L\sin\alpha$) and v_e be the speed of exit at the lower end ($z = 0$). Then, for $z = 0$ we have $v = v_e$ and $p = p_a$, and for $z = L\sin\alpha$ we have $v = v_1$ and $p = Mp_a$. Equation (10.8.15) therefore gives

$$\frac{1}{\rho} p_a + \frac{1}{2} v_e^2 = C \tag{10.8.16}$$

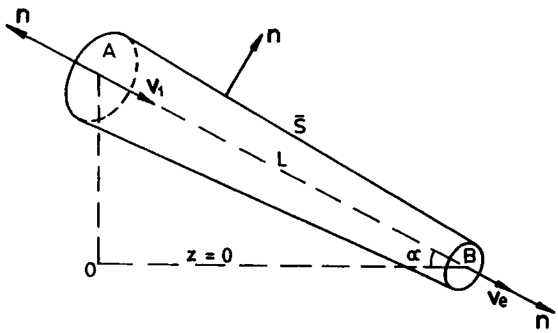

Figure 10.9. Example 10.8.3.

and

$$\frac{M}{\rho} p_a + \frac{1}{2} v_1^2 + gL \sin \alpha = C \tag{10.8.17}$$

so that

$$(M - 1) \frac{p_a}{\rho} + \frac{1}{2} (v_1^2 - v_e^2) + gL \sin \alpha = 0 \tag{10.8.18}$$

This does not give v_e unless v_1 is known. To find v_1, we use the equation of continuity, $\operatorname{div} \mathbf{v} = 0$, which holds at every point of the liquid. If V is the volume of the liquid contained in the pipe (at any chosen instant of time), the equation of continuity yields

$$\int_V (\operatorname{div} \mathbf{v}) \, dV = 0$$

By divergence theorem, this becomes

$$\int_S \mathbf{v} \cdot \mathbf{n} \, dS = 0 \tag{10.8.19}$$

where S is the surface enclosing V. We note that S is made up of the upper and lower end surfaces of the pipe, say, A and B, and the lateral surface of the pipe, say, $\bar{S}$. Since $\bar{S}$ is a solid surface that is at rest, we have $\mathbf{v} \cdot \mathbf{n} = 0$ on $\bar{S}$ by (10.5.7). Therefore equation (10.8.19) gives, on noting that $\mathbf{v} \cdot \mathbf{n} = -v_1$ on A and $\mathbf{v} \cdot \mathbf{n} = v_e$ on B,

$$-\int_A v_1 \, dS + \int_B v_e \, dS = 0 \tag{10.8.20}$$

We note that A and B are both circular areas such that if πa^2 is the area of B, then $\pi N^2 a^2$ is the area of A. If we take it that the speeds on A and B

are uniform, then (10.8.20) gives $(\pi N^2 a^2)v_1 = (\pi a^2)v_e$, so that

$$v_1 = \frac{1}{N^2} v_e \tag{10.8.21}$$

Substituting this back into (10.8.18), we get

$$v_e^2 = \frac{2N^4}{N^4 - 1}\left[(M-1)\frac{p_a}{\rho} + gL\sin\alpha\right] \tag{10.8.22}$$

which gives the exit speed at the lower end of the pipe. ■

EXAMPLE 10.8.4 In a steady irrotational flow of a fluid of constant density under zero body force, show that the pressure is maximum at a stagnation point. (*A stagnation point* is a point at which the velocity is 0.)

Solution For a given flow, the Bernoulli's equation (10.8.8) with $P = p/\rho$ and $\chi = 0$, holds; that is,

$$\frac{p}{\rho} + \frac{1}{2}v^2 = C$$

where C is a constant. Consequently, since ρ is constant,

$$\frac{dp}{dv} = -\rho v, \qquad \frac{d^2p}{dv^2} = -\rho$$

Evidently, $dp/dv = 0$ for $v = 0$, and $d^2p/dv^2 < 0$ for all v. Hence p is maximum when $v = 0$. ■

EXAMPLE 10.8.5 Consider the steady and irrotational flow of an elastic fluid for which the pressure-density relationship is $p = \beta\rho^\gamma$, where β and $\gamma (> 1)$ are constants. Assuming that the body force is absent, show that

(i)
$$\frac{1}{2}v^2 + \frac{\gamma}{(\gamma-1)}\frac{p}{\rho} = \text{constant} \tag{10.8.23}$$

(ii)
$$v^2 = \frac{2\gamma}{(\gamma-1)}\frac{p}{\rho}\left[\left(\frac{p_0}{p}\right)^{(\gamma-1)/\gamma} - 1\right] \tag{10.8.24}$$

$$= \frac{2c_s^2}{\gamma-1}\left[\left(\frac{p_0}{p}\right)^{(\gamma-1)/\gamma} - 1\right] \tag{10.8.25}$$

(iii)
$$p_0 = p\left[1 + \frac{1}{2}(\gamma-1)\frac{v^2}{c_s^2}\right]^{\gamma/(\gamma-1)} \tag{10.8.26}$$

where p_0 is the pressure at 0 speed.

Solution (i) Under the given conditions, Bernoulli's equation (10.8.8) holds with $\chi = 0$; that is,

$$P + \tfrac{1}{2}v^2 = \text{constant} \tag{10.8.27}$$

Also,

$$\begin{aligned} P \equiv \int \frac{dp}{\rho} &= \beta\gamma \int \rho^{\gamma-2}\, d\rho \\ &= \beta\gamma \frac{\rho^{\gamma-1}}{(\gamma-1)} + \text{constant} \\ &= \frac{\gamma}{(\gamma-1)} \frac{p}{\rho} + \text{constant} \end{aligned} \tag{10.8.28}$$

Using this in (10.8.27), we readily get (10.8.23).

(ii) For $v = 0$, we have $p = p_0$. Also, the relation $p = \beta\rho^\gamma$ gives $\rho = ((1/\beta)p_0)^{(1/\gamma)}$ for $p = p_0$. Consequently, expression (10.8.23) yields

$$\frac{1}{2}v^2 + \frac{\gamma}{(\gamma-1)}\frac{p}{\rho} = \frac{\gamma}{(\gamma-1)} p_0 \left(\frac{\beta}{p_0}\right)^{1/\gamma} \tag{10.8.29}$$

Substituting $\beta = p/\rho^\gamma$ on the righthand side of (10.8.29) and simplifying the resulting expression, we obtain (10.8.24).

Using the relation $p = \beta\rho^\gamma$, we find from (10.7.5) that

$$c_s^2 = \frac{d}{d\rho}(\beta\rho^\gamma) = \beta\gamma\rho^{\gamma-1} = \gamma\frac{p}{\rho} \tag{10.8.30}$$

By use of (10.8.30), expression (10.8.24) immediately reduces to (10.8.25).

(iii) Expression (10.8.25) may be rewritten as

$$1 + \frac{(\gamma-1)}{2}\frac{v^2}{c_s^2} = \left(\frac{p_0}{p}\right)^{(\gamma-1)/\gamma}$$

so that

$$\frac{p_0}{p} = \left[1 + \frac{(\gamma-1)}{2}\frac{v^2}{c_s^2}\right]^{\gamma/(\gamma-1)}$$

which is (10.8.26). ■

Note: In Example 10.6.5, expression (10.8.23) was obtained for a perfect gas with the use of the energy equation.

10.9

WATER WAVES

In Section 9.13 we considered Rayleigh waves that propagate near the surface of an elastic half-space. We now consider waves that propagate near the surface of a nonviscous, incompressible fluid, in particular, water. Water wave motions are of great importance; they range from waves generated by wind or solar heating in the oceans to flood waves in rivers, from waves caused by a moving ship in a channel to tsunami waves (tidal waves) generated by earthquakes, and from solitary waves on the surface of a canal caused by a disturbance to waves generated by underwater explosions, to mention only a few. We restrict ourselves to two simple cases, which are the starting points for the study of linear and nonlinear water waves.

We consider a body of water (nonviscous, incompressible fluid) occupying the region $-h < z < 0$ with the plane $z = -h$ as the *bottom boundary* and the plane $z = 0$ as the *upper boundary*, in the *undisturbed* (initial) state; see Figure 10.10. We suppose that the bottom boundary is a rigid solid surface and the upper boundary is the surface exposed to a constant atmospheric pressure p_a. We consider plane waves propagating in the x direction whose amplitude varies in the z direction, with the gravitational force as the only body force. Since the motion is supposed to start from rest, it is necessarily irrotational as a consequence of Kelvin's circulation theorem (see Section 10.7), and we take the velocity field to be the gradient of a potential $\phi = \phi(\mathbf{x}, t)$; that is, $\mathbf{v} = \nabla\phi$. The equation of continuity

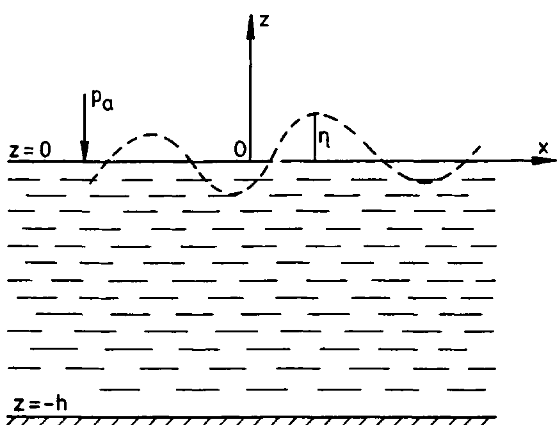

Figure 10.10. Water waves.

(10.4.5) then implies that ϕ is a harmonic function at every point of the fluid; that is,

$$\nabla^2\phi \equiv \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{10.9.1}$$

Since the gravitational force is the only body force, which acts in the negative z direction, we have $\mathbf{b} = -g\mathbf{e}_3 = -\nabla(gz)$, and Bernoulli's equation (10.8.5) yields the following equation for pressure p at every point of the fluid:

$$\frac{p}{\rho} + \frac{1}{2}(\nabla\phi)^2 + gz + \frac{\partial\phi}{\partial t} = f(t), \qquad t \geq 0 \tag{10.9.2}$$

Since the upper surface is exposed to the constant atmospheric pressure p_a, we have $p = p_a$ on this surface. In the undisturbed state, the equation of this surface is $z = 0$. After the motion is set up, let the equation of this surface, denoted S, be $z = \eta(x, t)$ where η is an *unknown* function of x and t that tends to 0 as $t \to 0$. The function $\eta(x, t)$ is referred to as the *surface elevation*. Thus, we have $p = p_a$ on S. Using (10.9.2), this condition reads

$$\frac{p_a}{\rho} + \frac{1}{2}(\nabla\phi)^2 + gz + \frac{\partial\phi}{\partial t} = f(t)$$

on S for $t \geq 0$. Absorbing (p_a/ρ) and $f(t)$ into $\partial\phi/\partial t$, this condition may be rewritten as

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + gz = 0 \tag{10.9.3}$$

on S for $t \geq 0$.

Since S is a boundary surface, it contains the same fluid particles for all time; that is, S is a material surface. Hence it follows from (10.9.3) that

$$\frac{D}{Dt}\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + gz\right] = 0$$

on S for $t \geq 0$, or equivalently

$$\left(\frac{\partial}{\partial t} + \nabla\phi\cdot\nabla\right)\left\{\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + gz\right\}$$
$$\equiv \frac{\partial^2\phi}{\partial t^2} + 2\,\nabla\phi\cdot\nabla\left(\frac{\partial\phi}{\partial t}\right) + \frac{1}{2}\nabla\phi\cdot\nabla(\nabla\phi)^2 + g\frac{\partial\phi}{\partial z} = 0 \tag{10.9.4}$$

on S for $t \geq 0$.

Since the lower boundary $z = -h$ is a rigid solid surface at rest, the condition to be satisfied at this boundary is, by (10.5.7),

$$\frac{\partial \phi}{\partial z} = 0 \tag{10.9.5}$$

for $z = -h$, $t \geq 0$.

Thus, for the problem considered, Laplace's equation (10.9.1) serves as the partial differential equation satisfied by ϕ, and (10.9.4) and (10.9.5) serve as the upper and lower boundary conditions for ϕ. Once ϕ is determined, equation (10.9.3) gives the surface elevation, $z = \eta$.

Because of the presence of nonlinear terms in the boundary condition (10.9.4), the determination of ϕ in the general case is a difficult task. We restrict ourselves to two particular cases.

10.9.1 SMALL AMPLITUDE WAVES

We first consider the case where the motion is linear so that nonlinear terms in velocity components may be neglected. In this case no distinction is made between the initial and the current states of the upper boundary, and the boundary conditions (10.9.3) and (10.9.4) are taken in the linearized forms

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \tag{10.9.6}$$

for $z = 0$, $t > 0$;

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0 \tag{10.9.7}$$

for $z = 0$, $t > 0$.

These conditions yield

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \tag{10.9.8}$$

for $z = 0$, $t > 0$.

For a plane wave propagating in the x direction with frequency $\omega/2\pi$ and wavelength $2\pi/k$, we seek ϕ in the form

$$\phi = \operatorname{Re} \Phi(z) \exp[i(\omega t - kx)] \tag{10.9.9}$$

where $\Phi(z)$ is a function to be determined.

We find that ϕ given by (10.9.9) satisfies Laplace's equation (10.9.1) provided

$$\frac{d^2\Phi}{dz^2} - k^2\Phi = 0 \tag{10.9.10}$$

The general solution of this ordinary linear differential equation is

$$\Phi(z) = Ae^{kz} + Be^{-kz} \tag{10.9.11}$$

where A and B are arbitrary constants. Using the boundary condition (10.9.5) we get $Ae^{-kh} = Be^{kh}$. Consequently, solution (10.9.11) takes the form

$$\Phi = C \cosh k(z + h) \tag{10.9.12}$$

where $C(=2Ae^{-kh} = 2Be^{kh})$ is an arbitrary constant so that the solution (10.9.9) becomes

$$\phi = \operatorname{Re} C \cosh k(z + h) \exp[i(\omega t - kx)] \tag{10.9.13}$$

Using (10.9.6), we find

$$\eta = \operatorname{Re} a \exp[i(\omega t - kx)] \tag{10.9.14}$$

where $a = (C\omega/ig) \cosh kh = \max|\eta|$ is the *amplitude*. Thus, the solution for ϕ has the final form

$$\phi = \operatorname{Re}\left(\frac{iag}{\omega}\right) \frac{\cosh k(z + h)}{\cosh kh} \exp[i(\omega t - kx)] \tag{10.9.15}$$

Using (10.9.15) in (10.9.7), we obtain the following dispersion relation between the frequency and wave number:

$$\omega^2 = gk \tanh kh \tag{10.9.16}$$

This *dispersion relation* can be rewritten in terms of the *phase velocity* $c = \omega/k$ as

$$c^2 = \frac{\omega^2}{k^2} = \frac{g}{k} \tanh(kh) \tag{10.9.17}$$

This equation shows that the phase velocity c depends on the gravity g and depth h *as well as* the wavelength $2\pi/k$. Hence, the waves are dispersive in nature (like the Love waves considered in Section 9.13). This means that, as the time passes, the waves would disperse (spread out) into different groups such that each group would consist of waves having approximately the same wavelength. The quantity $d\omega/dk$ represents the velocity of such a group in the direction of propagation and is called *group velocity*, denoted $C(k)$. From (10.9.16) we find

$$C(k) = \frac{d\omega}{dk} = \left(\frac{g}{2\omega}\right)(\tanh kh + kh \operatorname{sech}^2 kh) \tag{10.9.18}$$

which on using (10.9.17) becomes

$$C(k) = \frac{1}{2}c\left\{1 + \frac{2kh}{\sinh(2kh)}\right\} \tag{10.9.19}$$

Evidently, the group velocity is different from the phase velocity.

In the case where the wavelength $2\pi/k$ is large compared with the depth, such waves are called *shallow water waves*, $kh \ll 1$ so that $\tanh kh \approx kh$ and $\sin 2kh \approx 2kh$. In such a situation, results (10.9.17) and (10.9.19) yield

$$C(k) \approx c \approx \sqrt{gh} \tag{10.9.20}$$

Thus, shallow water waves are nondispersive, and their speed varies as the square root of the depth.

In the other limiting case where the wavelength is very small compared with the depth, such waves are called *deep water waves*, $kh \gg 1$. In the limit as $kh \to \infty$, $[\cosh k(z + h)]/(\cosh kh) \to e^{kz}$, and the corresponding solutions for ϕ and η become

$$\phi = \mathrm{Re}\left(\frac{iag}{\omega}\right)\exp[kz + i(\omega t - kx)] = \left(\frac{ag}{\omega}\right)e^{kz}\sin(kx - \omega t) \tag{10.9.21}$$

$$\eta = \mathrm{Re}\, a\exp[i(\omega t - kx)] = a\cos(kx - \omega t) \tag{10.9.22}$$

Results (10.9.16), (10.9.17) and (10.9.19) yield

$$\omega^2 = gk \tag{10.9.23a}$$

$$c = (g/k)^{1/2} = (g\lambda/2\pi)^{1/2} \tag{10.9.23b}$$

$$C(k) = \tfrac{1}{2}c \tag{10.9.23c}$$

Thus, deep water waves continue to be dispersive and their phase velocity now is proportional to the square root of their wavelengths. Also, the group velocity is equal to one-half of the phase velocity.

10.9.2. FINITE AMPLITUDE WAVES (STOKES'S WAVES)

We now consider the case where the motion is nonlinear and the amplitude is not small. Let us recall (10.9.3) and (10.9.4) and write them for ready reference in the form

$$\eta = -\frac{1}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2\right]_{z=\eta} \tag{10.9.24}$$

$$[\phi_{tt} + g\phi_z]_{z=\eta} + 2[\nabla\phi\cdot\nabla\phi_t]_{z=\eta} + \tfrac{1}{2}[\nabla\phi\cdot\nabla(\nabla\phi)^2]_{z=\eta} = 0 \tag{10.9.25}$$

where, for simplicity, we have written ϕ_t for $\partial\phi/\partial t$, ϕ_z for $\partial\phi/\partial z$, ϕ_{tt} for $\partial^2\phi/\partial t^2$, etc.

A systematic procedure can be employed to rewrite these boundary conditions by using Taylor's series expansions of the potential ϕ and its derivatives in the typical form

$$\phi(x, y, z = \eta, t) = [\phi]_{z=0} + \eta[\phi_z]_{z=0} + \tfrac{1}{2}\eta^2[\phi_{zz}]_{z=0} + \cdots \tag{10.9.26}$$

$$\phi_z(x, y, z = \eta, t) = [\phi_z]_{z=0} + \eta[\phi_{zz}]_{z=0} + \tfrac{1}{2}\eta^2[\phi_{zzz}]_{z=0} + \cdots \tag{10.9.27}$$

Substituting these and similar Taylor's expansions into (10.9.24) gives

$$\begin{aligned}
\eta &= -\frac{1}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2\right]_{z=0} + \eta\left[-\frac{1}{g}\left\{\phi_t + \frac{1}{2}(\nabla\phi)^2\right\}_z\right]_{z=0} + \cdots \\
&= -\frac{1}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2\right]_{z=0} + \frac{1}{g^2}\left[\left\{\phi_t + \frac{1}{2}(\nabla\phi)^2\right\}\left\{\phi_t + \frac{1}{2}(\nabla\phi)^2\right\}_z\right]_{z=0} + \cdots \\
&= -\frac{1}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{g}\phi_t\phi_{zt}\right]_{z=0} + O(\phi^3)
\end{aligned} \tag{10.9.28}$$

Similarly, condition (10.9.25) gives

$$\begin{aligned}
&[\phi_{tt} + g\phi_z]_{z=0} + \eta[(\phi_{tt} + g\phi_z)_z] + \tfrac{1}{2}\eta^2[(\phi_{tt} + g\phi_z)_{zz}]_{z=0} + \cdots \\
&\quad + 2[\nabla\phi \cdot \nabla\phi_t]_{z=0} + 2\eta[\{\nabla\phi \cdot \nabla\phi_t\}_z]_{z=0} + \eta^2[\{\nabla\phi \cdot \nabla\phi_t\}_{zz}]_{z=0} + \cdots \\
&\quad + \tfrac{1}{2}[\{\nabla\phi \cdot \nabla(\nabla\phi)^2\}]_{z=0} + \tfrac{1}{2}\eta[\{\nabla\phi \cdot \nabla(\nabla\phi)^2\}_z]_{z=0} + \tfrac{1}{4}\eta^2[\{\nabla\phi \cdot \nabla(\nabla\phi)^2\}_{zz}]_{z=0} \\
&\quad + \cdots = 0
\end{aligned} \tag{10.9.29}$$

We substitute (10.9.28) for η into (10.9.29) to obtain

$$\begin{aligned}
&[\phi_{tt} + g\phi_z]_{z=0} - \frac{1}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{g}\phi_t\phi_{zt}\right]_{z=0}[(\phi_{tt} + g\phi_z)_z]_{z=0} \\
&\quad + \frac{1}{2g^2}\left[\left\{\phi_t + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{g}\phi_t\phi_{zt}\right\}^2\right]_{z=0}[(\phi_{tt} + g\phi_z)_{zz}]_{z=0} \\
&\quad + 2[(\nabla\phi) \cdot \nabla\phi_t]_{z=0} - \frac{2}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{g}\phi_t\phi_{zt}\right]_{z=0}[(\nabla\phi \cdot \nabla\phi_t)_z]_{z=0} \\
&\quad + \frac{1}{2}[\nabla\phi \cdot \nabla(\nabla\phi)^2]_{z=0} - \frac{2}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{g}\phi_t\phi_{zt}\right]_{z=0}[\{\nabla\phi \cdot \nabla(\nabla\phi)^2\}_z]_{z=0} \\
&= 0
\end{aligned} \tag{10.9.30}$$

The first-, second- and third-order boundary conditions on $z = 0$ are, respectively, given by

$$(\phi_{tt} + g\phi_z) = 0 + O(\phi^2) \tag{10.9.31}$$

$$(\phi_{tt} + g\phi_z) + 2[\nabla\phi \cdot \nabla\phi_t] - \frac{1}{g}\phi_t(\phi_{tt} + g\phi_z)_z = 0 + O(\phi^3) \tag{10.9.32}$$

$$\begin{aligned}(\phi_{tt} + g\phi_z) &+ 2[\nabla\phi \cdot \nabla\phi_t] + \frac{1}{2}[\nabla\phi \cdot \nabla(\nabla\phi)^2] - \frac{1}{g}\phi_t[\phi_{tt} + g\phi_z + 2(\nabla\phi \cdot \nabla\phi_t)]_z \\ &- \frac{1}{g}\left[\frac{1}{2}(\nabla\phi)^2 - \frac{1}{g}\phi_t\phi_{zt}\right][\phi_{tt} + g\phi_z]_z + \frac{1}{2g^2}[\phi_t]^2[(\phi_{tt} + g\phi_z)_{zz}] \\ &= 0 + O(\phi^4)\end{aligned} \tag{10.9.33}$$

where $O(\ )$ indicates the order of magnitude of the neglected terms. These results can be used to determine the third-order expansion of plane progressive waves.

As indicated before, the first-order plane wave potential ϕ in deep water is given by (10.9.21). Direct substitution of the first-order velocity potential (10.9.21) in the second-order boundary condition (10.9.32) reveals that the second-order terms in (10.9.32) vanish. Thus the first-order potential is a solution of the second-order boundary-value problem, and we can state that

$$\phi = \left(\frac{ga}{\omega}\right)e^{kz}\sin(kx - \omega t) + O(a^3) \tag{10.9.34}$$

Substitution of this result into (10.9.28) leads to the second-order result for η in the form

$$\begin{aligned}\eta &= a\cos(kx - \omega t) - \tfrac{1}{2}ka^2 + ka^2\cos^2(kx - \omega t) + \cdots \\ &= a\cos(kx - \omega t) + \tfrac{1}{2}ka^2\cos 2(kz - \omega t) + \cdots\end{aligned} \tag{10.9.35}$$

The second term in (10.9.35), which represents the second-order correction to the surface profile, is positive at the *crests* $kx - \omega t = 0, 2\pi, 4\pi, \ldots,$ and negative at the *troughs* $kx - \omega t = \pi, 3\pi, 5\pi, \ldots$. But the crests are steeper, and the troughs flatter as a result of the nonlinear effect. The notable feature of solution (10.9.35) is that the wave profile is no longer sinusoidal. The actual shape of the wave profile is a curve known as a *trochoid* (see Figure 10.11), whose crests are steeper and the troughs are flatter than the sinusoidal wave.

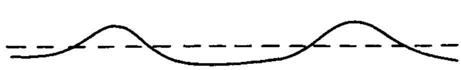

Figure 10.11. The surface wave profile.

Substituting the wave potential (10.9.34) in the third-order boundary condition (10.9.33) reveals that all nonlinear terms vanish identically except but one term, $(1/2)\nabla\phi \cdot \nabla(\nabla\phi)^2$. Thus the boundary condition for the third-order plane-wave solution is given by

$$\phi_{tt} + g\phi_z + \tfrac{1}{2}\nabla\phi \cdot \nabla(\nabla\phi)^2 = 0 + O(\phi^4) \qquad (10.9.36)$$

If the first-order solution (10.9.34) is substituted into the third-order boundary condition on $z = 0$, the *dispersion relation* with second-order effect is obtained in the form

$$\omega^2 = gk(1 + a^2k^2) + O(k^3a^3) \qquad (10.9.37)$$

Note that this relation involves the amplitude in addition to frequency and wave number. This *nonlinear dispersion relation* can be expressed in terms of the phase velocity as

$$c = \frac{\omega}{k} = \left(\frac{g}{k}\right)^{1/2}(1 + k^2a^2)^{1/2} \approx \left(\frac{g}{k}\right)^{1/2}\left(1 + \frac{1}{2}a^2k^2\right) \qquad (10.9.38)$$

Thus the phase velocity depends on the wave amplitude, and waves of large amplitude travel faster than smaller ones. The dependence of c on amplitude is known as the *amplitude dispersion* in contrast to the frequency dispersion as given by (10.9.23a, b). The nonlinear solutions for plane waves based on systematic power series in the wave amplitude are known as *Stokes's expansions.*

We conclude this section by discussing the phenomenon of breaking of water waves which is one of the most common observable phenomena in an ocean beach. A wave coming from deep ocean changes shape as it moves across a shallow beach. Its amplitude and wavelength also are modified. The wave train is very smooth some distance offshore, but as it moves inshore, the front of the wave steepens noticeably until, finally, it breaks. After breaking, waves continue to move inshore as a series of bores or hydraulic jumps, whose energy is gradually dissipated by means of the water turbulence. Of the phenomena common to waves on beaches, breaking is the physically most significant and mathematically least known. In fact, it is one of the most intriguing longstanding problems of water wave theory.

For waves of small amplitude in deep water, maximum particle velocity is $v = a\omega = ack$. But the basic assumption of small amplitude theory implies that $v/c = ak \ll 1$. Therefore, wave breaking can never be predicted by the small-amplitude wave theory, and the possibility arises only in the theory of finite-amplitude waves. It is to be noted that the Stokes's expansions are limited to relatively small amplitude and cannot predict the

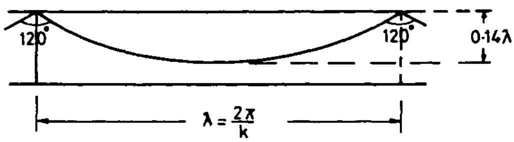

Figure 10.12. The steepest wave profile.

wavetrain of maximum height at which the crests are found to be very sharp. For a wave profile of a constant shape moving at a uniform velocity, it can be shown that the maximum total crest angle as the wave begins to break is 120°; see Figure 10.12.

10.10

STRESS TENSOR FOR A VISCOUS FLUID

In Sections 10.3 through 10.9 we dealt with equations governing the motion of nonviscous fluids and some of their consequences. Let us now turn our attention to viscous fluids. As mentioned in Section 10.2, a viscous fluid is characterized by the property that when the fluid undergoes nonuniform motion, a fluid surface experiences not only normal stress but shear stress as well. In view of this property, it is postulated that the stress tensor at a point in a viscous fluid is made up of two parts: one part due to normal stress and the other due to shear stress. Since the distinction between viscous and nonviscous fluids completely disappears when the fluid is at rest or in uniform motion, we make an additional postulate that the part due to normal stress is analogous to the stress tensor occurring in a nonviscous fluid as given by (10.3.2) and that the part due to shear stress is represented by a new tensor that vanishes when the motion is absent or uniform. We refer to this new tensor as the *viscous stress tensor*, denoted $\mathbf{T}^{(v)}$. Recall that in Section 6.3 we introduced the stretching tensor (strain-rate tensor) $\mathbf{D}$ and made the observation that $\mathbf{D} = \mathbf{0}$ if and only if the motion is absent or uniform. As such, the requirement that $\mathbf{T}^{(v)}$ should vanish when the motion is absent or uniform can be met by imposing the restriction: $\mathbf{T}^{(v)} = \mathbf{0}$ for $\mathbf{D} = \mathbf{0}$. Thus, the following generalization of the expression (10.3.2), proposed by George G. Stokes in 1845, is adopted in dealing with viscous fluids:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{T}^{(v)} \tag{10.10.1}$$

with $\mathbf{T}^{(v)} = \mathbf{0}$, for $\mathbf{D} = \mathbf{0}$, or in components,

$$\tau_{ij} = -p\delta_{ij} + \tau_{ij}^{(v)}, \qquad (10.10.1)'$$

with $\tau_{ij}^{(v)} = 0$ for $d_{ij} = 0$, where $\tau_{ij}^{(v)}$ are components of $\mathbf{T}^{(v)}$ and p is a scalar analogous to (but not necessarily identical with) the p of (10.3.2), which is also referred to as *dynamic pressure.*

The most direct way of accommodating the condition $\mathbf{T}^{(v)} = \mathbf{0}$ for $\mathbf{D} = \mathbf{0}$ is to postulate that $\mathbf{T}^{(v)}$ is a *homogeneous function of* $\mathbf{D}$. When this postulate is made, the fluid being dealt with is called a *Stokesian fluid*, after Stokes. In the particular case when $\mathbf{T}^{(v)}$ is taken as a *homogeneous linear function of* $\mathbf{D}$, the fluid is called a *linear viscous fluid.* Thus, for linear viscous fluids, it is postulated that

$$\tau_{ij}^{(v)} = a_{ijkm} d_{km} \tag{10.10.2}$$

where the coefficients a_{ijkm} are independent of d_{ij}. Note that the relations (10.10.2) are analogous to the relations (9.2.3) representing the generalized Hooke's law.

Since $\tau_{ij}^{(v)}$ and d_{ij} are components of tensors $\mathbf{T}^{(v)}$ and $\mathbf{D}$, respectively, it follows from (10.10.2) and the quotient law proven in Example 2.4.7 that a_{ijkm} are components of a fourth-order tensor. This tensor is analogous to the elasticity tensor and is called the *viscosity tensor.* Like the elasticity tensor, the viscosity tensor depends on the physical properties of the material, and its 81 components a_{ijkm} are called *coefficients of viscosity.* From (10.10.1)′ and (10.10.2), we find that these coefficients have dimensions: (force × time)/area.

An isotropic linear elastic solid has been defined as the material for which the elasticity tensor is isotropic. Similarly, an isotropic linear viscous fluid is defined as the material for which the viscosity tensor is isotropic. This means that, like in an isotropic elastic solid, there is no preferred direction in an isotropic viscous fluid and the relations (10.10.2) represent a physical law that does not depend on the orientation of the axes. By virtue of the general representation (2.6.1) for a fourth-order isotropic tensor, it follows that, for a linear isotropic viscous fluid, a_{ijkm} are of the form

$$a_{ijkm} = \alpha\delta_{ij}\delta_{km} + \beta\delta_{ik}\delta_{jm} + \gamma\delta_{im}\delta_{jk} \tag{10.10.3}$$

where α, β and γ are scalars.

Substituting for a_{ijkm} from (10.10.3) in (10.10.2) and recalling that $d_{ij} = d_{ji}$, we obtain

$$\tau_{ij}^{(v)} = \lambda\delta_{ij} d_{kk} + 2\mu d_{ij} \tag{10.10.4}$$

where we have set $\lambda = \alpha$ and $(1/2)(\beta + \gamma) = \mu$.

The relations (10.10.4) are analogous to the stress-strain relations (9.2.6). It is to be noted that the symbols λ and μ appearing in (10.10.4) are *not* the same as those appearing in (9.2.6). While in (9.2.6), λ and μ denote the elastic moduli, in (10.10.4) these are the *coefficients of viscosity*. Thus, for a linear isotropic viscous fluid, there are only *two* viscosity coefficients.

Substituting for $\tau_{ij}^{(v)}$ from (10.10.4) into (10.10.1)′, we obtain

$$\tau_{ij} = (-p + \lambda d_{kk})\delta_{ij} + 2\mu d_{ij} \tag{10.10.5}$$

In the direct notation, these relations read

$$\mathbf{T} = \{-p + \lambda(\operatorname{tr}\mathbf{D})\}\mathbf{I} + 2\mu\mathbf{D} \tag{10.10.6}$$

This is the *stress-strain rate relation* valid for a linear, isotropic viscous fluid. This relation was obtained by Stokes in 1845 and is also known as *Stokes's law*. This law is adopted as the material law in the theory of linear, isotropic, viscous fluid flows. Note that, since $\mathbf{D}$ is symmetric, so is $\mathbf{T}$ by virtue of this law. Thus, as in the cases of a linear, isotropic elastic solid and a nonviscous fluid, the symmetry of $\mathbf{T}$ is a physical property of a linear, isotropic viscous fluid. A large class of liquids and gases possess this property. As such, the theory of viscous fluid flows obeying Stokes's law is very successful from practical point of view also. Henceforth we will refer to a linear, isotropic, viscous fluid simply as a *viscous fluid*.

It is sometimes convenient to have the Stokes's law (10.10.6) expressed in terms of $\mathbf{T}$ and $\mathbf{v}$. Substitution for $\mathbf{D}$ from (6.3.3) in (10.10.6) yields

$$\mathbf{T} = (-p + \lambda \operatorname{div}\mathbf{v})\mathbf{I} + \mu(\nabla\mathbf{v} + \nabla\mathbf{v}^T) \tag{10.10.7}$$

or, in components,

$$\tau_{ij} = (-p + \lambda v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i}) \tag*{(10.10.7)′}$$

The *stress-velocity relation* (10.10.7) is analogous to the stress-displacement relation (9.8.4).

We note that in the limiting case when $d_{ij} \to 0$, that is, when the motion is absent or uniform, (10.10.6) reduces to the material law (10.3.2) valid for a nonviscous fluid; p then represents the mean pressure.

Comparison of Stokes's law (10.10.6) with Hooke's law (9.2.6) reveals an important difference between elastic solids and viscous fluids. Whereas in the case of an elastic solid the Hooke's law (9.2.6) shows that the stresses are all 0 when there is no deformation, in the case of a viscous fluid Stokes's law (10.10.6) shows that nonzero stresses specified by $-p\mathbf{I}$ do occur even when there is no deformation. The presence of the *residual stress* $-p\mathbf{I}$ makes a fundamental difference between elasticity and fluid mechanics.

EXAMPLE 10.10.1 By using Stokes's law, show that at every point of a viscous fluid, the principal directions of $\mathbf{T}$, $\mathbf{T}^{(v)}$ and $\mathbf{D}$ are all coincident.

Solution At a point of a viscous fluid let $\mathbf{a}$ be an eigenvector of $\mathbf{D}$. Then $\mathbf{a}$ is along a principal direction of $\mathbf{D}$ and $\mathbf{Da} = \Lambda\mathbf{a}$ for some scalar Λ. Stokes's law (10.10.6) then gives

$$\mathbf{Ta} = \bar{\Lambda}\mathbf{a} \tag{10.10.8}$$

where $\bar{\Lambda} = -p + \lambda(\operatorname{tr}\mathbf{D}) + 2\mu\Lambda$, which is a scalar. Hence $\mathbf{a}$ is an eigenvector of $\mathbf{T}$ as well. Consequently, $\mathbf{a}$ is along a principal direction of $\mathbf{T}$. Further, by use of (10.10.1) and (10.10.8), we get

$$\mathbf{T}^{(v)}\mathbf{a} = (\mathbf{T} + p\mathbf{I})\mathbf{a} = (\bar{\Lambda} + p)\mathbf{a} \tag{10.10.9}$$

from which it is evident that $\mathbf{a}$ is along a principal direction of $\mathbf{T}^{(v)}$ also. ■

Note: This result is analogous to that proven in Example 9.2.1.

EXAMPLE 10.10.2 Show that Stokes's law (10.10.6) is equivalent to the following relations taken together:

$$\mathbf{T}^{(d)} = 2\mu\mathbf{D}^{(d)} \tag{10.10.10a}$$

$$\operatorname{tr}\mathbf{T} = -3p + (3\lambda + 2\mu)(\operatorname{tr}\mathbf{D}) \tag{10.10.10b}$$

Solution Let us recall that $\mathbf{T}^{(d)}$ and $\mathbf{D}^{(d)}$ are the deviator parts of $\mathbf{T}$ and $\mathbf{D}$, respectively, so that

$$\mathbf{T}^{(d)} = \mathbf{T} - \tfrac{1}{3}(\operatorname{tr}\mathbf{T})\mathbf{I} \tag{10.10.11a}$$

$$\mathbf{D}^{(d)} = \mathbf{D} - \tfrac{1}{3}(\operatorname{tr}\mathbf{D})\mathbf{I} \tag{10.10.11b}$$

Let us now consider Stokes's law (10.10.6). Taking the trace throughout in (10.10.6), we get (10.10.10b). Substituting for $\mathbf{T}$ and ($\operatorname{tr}\mathbf{T}$) from (10.10.6) and (10.10.10b) in (10.10.11a) and using (10.10.11b), we obtain (10.10.10a). Thus, (10.10.6) yields (10.10.10).

Conversely, with the use of (10.10.11), relations (10.10.10) yield

$$\begin{aligned}\mathbf{T} &= \mathbf{T}^{(d)} + \tfrac{1}{3}(\operatorname{tr}\mathbf{T})\mathbf{I}\\ &= 2\mu\{\mathbf{D} - \tfrac{1}{3}(\operatorname{tr}\mathbf{D})\mathbf{I}\} + \tfrac{1}{3}\{-3p + (3\lambda + 2\mu)(\operatorname{tr}\mathbf{D})\}\mathbf{I}\\ &= -p\mathbf{I} + \lambda(\operatorname{tr}\mathbf{D})\mathbf{I} + 2\mu\mathbf{D}\end{aligned}$$

which is (10.10.6). ■

Note: This result is analogous to that proven in Example 9.2.2.

EXAMPLE 10.10.3 Let

$$\Phi = (\lambda + \tfrac{2}{3}\mu)(\operatorname{tr}\mathbf{D})^2 + 2\mu|\mathbf{D}^{(d)}|^2 \tag{10.10.12}$$

Prove the following:

(i) $$\frac{\partial \Phi}{\partial d_{ij}} = 2v_{ij} \qquad (10.10.13)$$

(ii) $$\Phi = \mathbf{T}^{(v)} \cdot \mathbf{D} \qquad (10.10.14)$$

(iii) $$\Phi = \mathbf{T} \cdot \mathbf{D} + (\operatorname{tr} \mathbf{D})p \qquad (10.10.15)$$

(iv) $$\Phi = \lambda(\operatorname{tr} \mathbf{D})^2 + 2\mu|\mathbf{D}|^2 \qquad (10.10.16)$$

(v) $$\Phi = \lambda(\operatorname{div} \mathbf{v})^2 + \mu(\nabla\mathbf{v} + \nabla\mathbf{v}^T) \cdot \nabla\mathbf{v} \qquad (10.10.17)$$

(vi) $$\Phi = \lambda(\operatorname{div} \mathbf{v})^2 + 2\mu\left\{\operatorname{div}\frac{D\mathbf{v}}{Dt} - \frac{D}{Dt}(\operatorname{div} \mathbf{v})^2 + \frac{1}{2}\mathbf{w}^2\right\} \qquad (10.10.18)$$

(vii) $$\int_V \Phi\, dV = \int_V \left\{\lambda(\operatorname{div} \mathbf{v})^2 - 2\mu\frac{D}{Dt}(\operatorname{div} \mathbf{v})^2 + \mu\mathbf{w}^2\right\} dV + 2\mu\int_S \frac{D\mathbf{v}}{Dt} \cdot \mathbf{n}\, dS \qquad (10.10.19)$$

Expression (10.10.19) is known as the *Bobyleff-Forsythe formula.*

Solution (i) We first note that the given function Φ has the following expression in the suffix notation

$$\Phi = (\lambda + \tfrac{2}{3}\mu)(d_{kk})^2 + 2\mu d_{km}^{(d)} d_{km}^{(d)} \qquad (10.10.20)$$

Differentiating both sides of this expression w.r.t. d_{ij} and noting that

$$\frac{\partial}{\partial d_{ij}}(d_{kk}) = \delta_{ij}$$

and

$$\frac{\partial}{\partial d_{ij}}(d_{km}^{(d)}) = \frac{\partial}{\partial d_{ij}}\left(d_{km} - \frac{1}{3}\delta_{km} d_{rr}\right) = (\delta_{ik}\delta_{jm} - \tfrac{1}{3}\delta_{km}\delta_{ij})$$

we obtain

$$\frac{\partial \Phi}{\partial d_{ij}} = 2(\lambda + \tfrac{2}{3}\mu)d_{kk}\delta_{ij} + 4\mu(\delta_{ik}\delta_{jm} - \tfrac{1}{3}\delta_{km}\delta_{ij})d_{km} = 2(\lambda\delta_{ij}d_{kk} + 2\mu d_{ij}) \qquad (10.10.21)$$

Expression (10.10.4) now yields (10.10.13).

(ii) Using (10.10.4) and (10.10.11b) we obtain

$$\begin{aligned}\mathbf{T}^{(v)} \cdot \mathbf{D} &= \{\lambda(\operatorname{tr} \mathbf{D})\mathbf{I} + 2\mu\mathbf{D}\} \cdot \mathbf{D} \\ &= \lambda(\operatorname{tr} \mathbf{D})^2 + 2\mu\mathbf{D}^{(d)} \cdot \mathbf{D}^{(d)} + \tfrac{2}{3}\mu(\operatorname{tr} \mathbf{D})^2 \\ &= (\lambda + \tfrac{2}{3}\mu)(\operatorname{tr} \mathbf{D})^2 + 2\mu|\mathbf{D}^{(d)}|^2\end{aligned}$$

Expression (10.10.12) now yields (10.10.14).

(iii) Using (10.10.1) and (10.10.14) we get

$$\mathbf{T} \cdot \mathbf{D} = (-p\mathbf{I} + \mathbf{T}^{(v)}) \cdot \mathbf{D} = -(\text{tr}\, \mathbf{D})p + \Phi$$

which is (10.10.15).

(iv) Expressions (10.10.15) and (10.10.6) yield

$$\begin{aligned}\Phi = \mathbf{T} \cdot \mathbf{D} + (\text{tr}\, \mathbf{D})p &= \{\lambda(\text{tr}\, \mathbf{D})\mathbf{I} + 2\mu\mathbf{D}\} \cdot \mathbf{D} \\ &= \lambda(\text{tr}\, \mathbf{D})^2 + 2\mu|\mathbf{D}|^2\end{aligned}$$

which is (10.10.16).

(v) Since $\mathbf{D} = \text{sym}\, \nabla\mathbf{v} = (1/2)(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$, we have $\text{tr}\, \mathbf{D} = \text{div}\, \mathbf{v}$ and

$$\begin{aligned}(\nabla\mathbf{v} + \nabla\mathbf{v}^T) \cdot \nabla\mathbf{v} = 2\mathbf{D} \cdot (\nabla\mathbf{v}) &= 2\mathbf{D} \cdot (\text{sym}\, \nabla\mathbf{v} + \text{skw}\, \nabla\mathbf{v}) \\ &= 2\mathbf{D} \cdot \mathbf{D} = 2|\mathbf{D}|^2\end{aligned}$$

Consequently, (10.10.16) yields (10.10.17).

(vi) By using result (i) of Example 6.3.1 in (10.10.16) and noting that $\text{tr}\, \mathbf{D} = \text{div}\, \mathbf{v}$, we immediately get (10.10.18).

(vii) Integrating both sides of (10.10.18) over a volume V and employing the divergence theorem (3.6.1), we obtain (10.10.19). ■

Note: The function Φ is analogous to the strain-energy function of the elasticity theory and is called the *viscous dissipative function*. This function is nonnegative for $\lambda + (2/3)\mu \geq 0$ and $\mu \geq 0$, as is evident from (10.10.12).

EXAMPLE 10.10.4 In a certain flow obeying Stokes's law, the velocity field is given by

$$v_1 = 4x_1x_2x_3, \qquad v_2 = x_3^2, \qquad v_3 = -2x_2x_3^2 \tag{10.10.22}$$

Find the strain-rate tensor and the stress tensor.

Solution Substituting for v_i from (10.10.22) in (6.3.4), we obtain

$$\begin{aligned}&d_{11} = 4x_2x_3, \qquad d_{22} = 0, \qquad d_{33} = -4x_2x_3 \\ &d_{12} = 2x_1x_3, \qquad d_{13} = 2x_1x_2, \qquad d_{23} = 2x_3(1 - x_3)\end{aligned} \tag{10.10.23}$$

These are the components of the strain-rate tensor associated with the given velocity field.

Substituting for d_{ij} from (10.10.23) into the Stokes's law (10.10.5), we obtain

$$\begin{aligned}&\tau_{11} = p + 8\mu x_2x_3, \qquad \tau_{12} = 4\mu x_1x_3 \\ &\tau_{22} = -p, \qquad \tau_{23} = 4\mu x_3(1 - x_3) \\ &\tau_{33} = -p - 8\mu x_2x_3, \qquad \tau_{13} = 4\mu x_1x_2\end{aligned} \tag{10.10.24}$$

These are the components of the stress tensor associated with the given velocity field. ■

Note that unless p is specified, normal stresses are not completely determined.

10.11
SHEAR VISCOSITY AND BULK VISCOSITY

In order to obtain the physical meanings of the coefficients of viscosity, let us consider the flow of a viscous fluid for which the velocity field is as follows:

$$v_1 = v_1(x_3), \qquad v_2 = 0, \qquad v_3 = 0 \tag{10.11.1}$$

For this velocity field, Stokes's law (10.10.7) gives

$$\tau_{11} = \tau_{22} = \tau_{33} = -p, \qquad \tau_{12} = \tau_{23} = 0 \tag{10.11.2}$$

$$\tau_{31} = \mu \frac{dv_1}{dx_3} \tag{10.11.3}$$

A velocity field of the form (10.11.1) occurs, for example, when a viscous fluid bounded between two parallel horizontal plates is made to move due to a uniform movement of the upper plate in the x_1 direction, keeping the lower plate stationary (Figure 10.13). Relation (10.11.3) shows that there occurs a shear stress in the direction of the flow on planes parallel to the flow and that, this shear stress is directly proportional to the velocity gradient in the perpendicular direction, with μ serving as the proportionality factor. The coefficient μ thus represents the shear stress on a plane element parallel to the direction of flow due to a unit of velocity gradient in the perpendicular

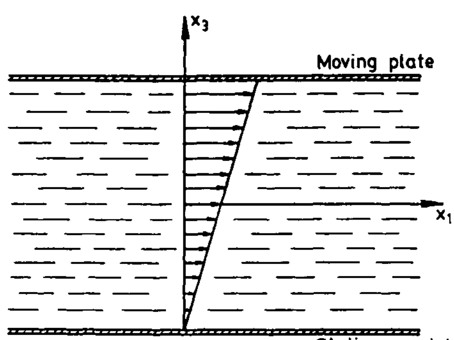

Figure 10.13. Shear viscosity.

direction. This coefficient is known as the *coefficient of shear viscosity*; it is analogous to the shear modulus in the elasticity theory. Experimental observations on common fluids show that μ is a nonnegative number.

Relation (10.11.3) that defines μ and is a particular case of Stokes's law (10.10.7) was proposed by Newton in 1687 on experimental grounds. This relation is known as *Newton's law of viscosity*. Stokes's law (10.10.6) was developed as a generalization of Newton's law. For this reason linear viscous fluids (obeying Stokes's law) are referred to as *Newtonian viscous fluids*.

Returning to the general case, we find, from relation (10.10.4), that

$$\tau_{kk}^{(v)} = 3\kappa d_{kk} \qquad (10.11.4)$$

where

$$\kappa = \lambda + \tfrac{2}{3}\mu \qquad (10.11.5)$$

Thus, κ is the proportionality factor relating the mean viscous stress $(1/3)\tau_{kk}^{(v)}$ to the rate of dilatation d_{kk}. (Recall the geometrical meaning of d_{kk} from Section 6.3.) The coefficient κ is thus analogous to the bulk modulus of the elasticity theory and is called the *coefficient of bulk viscosity*. Experimental observations on common fluids indicate that this coefficient may also be taken to be nonnegative.

From (10.11.5), it follows that λ represents the bulk viscosity diminished by two-thirds of the shear viscosity. It is called the *coefficient of dilatational viscosity* or *second coefficient of viscosity*. Thus, the coefficients λ and μ appearing in Stokes's law have definite physical meanings. Experiments show that these coefficients generally change with temperature and pressure. However, these changes are usually small and in most problems λ and μ are treated as constants.

10.11.1 STOKES'S CONDITION

Taking the trace of Stokes's law (10.10.7) we find that

$$p = \bar{p} + \kappa(\operatorname{div} \mathbf{v}) \qquad (10.11.6)$$

where $\bar{p} = -(1/3)(\operatorname{tr} \mathbf{T})$ is as usual the mean pressure. Comparing (10.11.6) with (10.3.3), we find that the dynamic pressure p of the viscous case is not generally identical with the corresponding p of the nonviscous case. Whereas the p of the nonviscous case is equal to the mean pressure, the p of the viscous case represents the mean pressure plus the dilatation rate multiplied by the bulk viscosity. We find that $p = \bar{p}$ if and only if *one* of the following two conditions is satisfied:

(i) $$\operatorname{div} \mathbf{v} = 0 \qquad (10.11.7)$$

(ii) $$\kappa = 0 \qquad (10.11.8)$$

Condition (10.11.7) holds whenever the volume of each element of the fluid remains constant during the motion. As mentioned in Section 6.3, such a motion is called an *isochoric motion* and materials for which every motion is isochoric are called *incompressible materials*. Thus, for an incompressible fluid, the geometrical condition (10.11.7) always holds. In fact (10.11.7) is the equation of continuity for an incompressible fluid as already noted in the nonviscous case.

In a compressible fluid, volumes of material elements generally change with time and condition (10.11.7) is an impossibility. However, in this case condition (10.11.8) can hold. Stokes studied compressible fluid flows by assuming condition (10.11.8), and this condition is known as *Stokes's condition*. Nearly all studies on compressible fluid flows make use of this condition, and a large number of theoretical results so obtained are verified even experimentally.

Thus, for incompressible viscous fluids and for compressible viscous fluids with zero bulk viscosity, the dynamic pressure p and the mean pressure $\bar{p}$ are identical as in the case of nonviscous fluids. For compressible viscous fluids not obeying Stokes's condition (10.11.8), the two pressures are not the same. Dense gases belong to such a class of compressible fluids.

It is to be pointed out that Stokes's law (10.10.6), upon which the theory of incompressible and compressible Newtonian viscous fluid flows is based, is a linear relationship between **T** and **D** and that the linearity of the relationship is merely a hypothesis. In recent years, relationships between **T** and **D** that are linear but more general than (10.10.6) and those that are even nonlinear in nature have also been postulated, and based upon such relationships some nonclassical theories of fluid flows have been developed. Fluids for which material laws other than (10.10.6) are postulated are referred to as *non-Newtonian fluids*, and the study of non-Newtonian fluid flows is called *rheology*. This topic is not treated in this text.

EXAMPLE 10.11.1 For a compressible fluid with zero bulk viscosity, show that Stokes's law (10.10.6) can be written in the following equivalent forms:

(i) $$\mathbf{T} = -(p + \tfrac{2}{3}\mu \operatorname{tr}\mathbf{D})\mathbf{I} + 2\mu\mathbf{D} \tag{10.11.9}$$

(ii) $$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}^{(d)} \tag{10.11.10}$$

(iii) $$\mathbf{T}^{(d)} = 2\mu\mathbf{D}^{(d)} \quad \text{and} \quad \operatorname{tr}\mathbf{T} = -3p \tag{10.11.11}$$

Solution When $\kappa = 0$, we have $\lambda = -(2/3)\mu$ by (10.11.5) and Stokes's law (10.10.6) becomes (10.11.9).

Relation (10.11.9) may be rewritten as

$$\mathbf{T} = -p\mathbf{I} + 2\mu\{\mathbf{D} - \tfrac{1}{3}(\operatorname{tr}\mathbf{D})\mathbf{I}\} = -p\mathbf{I} + 2\mu\mathbf{D}^{(d)}$$

which is (10.11.10). Thus (10.11.9) yields (10.11.10).

From relation (10.11.10), we get, on noting that $\operatorname{tr}\mathbf{D}^{(d)} = 0$,

$$\operatorname{tr}\mathbf{T} = -3p \tag{10.11.12}$$

Hence

$$\mathbf{T}^{(d)} = \mathbf{T} - \tfrac{1}{3}(\operatorname{tr}\mathbf{T})\mathbf{I} = \{-p\mathbf{I} + 2\mu\mathbf{D}^{(d)}\} - \tfrac{1}{3}(-3p)\mathbf{I} = 2\mu\mathbf{D}^{(d)} \tag{10.11.13}$$

Relations (10.11.12) and (10.11.13) constitute (10.11.11). Thus, (10.11.10) yields (10.11.11).

Relation (10.11.13) yields

$$\mathbf{T} - (\tfrac{1}{3}\operatorname{tr}\mathbf{T})\mathbf{I} = 2\mu\{\mathbf{D} - \tfrac{1}{3}(\operatorname{tr}\mathbf{D})\mathbf{I}\}$$

which on utilizing (10.11.12) becomes

$$\mathbf{T} = -(p + \tfrac{2}{3}\mu\operatorname{tr}\mathbf{D})\mathbf{I} + 2\mu\mathbf{D}$$

This is the relation (10.11.9). Thus, the relations (10.11.11) yield the relation (10.11.9).

This proves that the relations (10.11.9) to (10.11.11) are equivalent and hold for a compressible fluid obeying Stokes's condition $\kappa = 0$. ■

EXAMPLE 10.11.2 For a compressible viscous fluid with zero bulk viscosity, show that the viscous stress tensor is identical with the stress deviator tensor.

Solution When $\kappa = 0$, we have $p = \bar{p} = -(1/3)\operatorname{tr}\mathbf{T}$. Consequently, we find from equations (10.10.1) and (10.10.11a) that

$$\mathbf{T}^{(v)} = p\mathbf{I} + \mathbf{T} = -\tfrac{1}{3}(\operatorname{tr}\mathbf{T})\mathbf{I} + \mathbf{T} = \mathbf{T}^{(d)} \quad ■ \tag{10.11.14}$$

EXAMPLE 10.11.3 At a certain point of an incompressible viscous fluid, the stress matrix is

$$[\mathbf{T}] = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find the pressure and the viscous stress tensor.

Solution For an incompressible fluid, we have $p = -(1/3)\tau_{kk}$. The given stress matrix yields $p = 1$.

Using the given τ_{ij} and the fact that $p = 1$ in (10.10.1), we get

$$[\mathbf{T}^{(v)}] = [\mathbf{T}] + p[\mathbf{I}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

This is the matrix of the required viscous stress tensor. ■

10.12 GOVERNING EQUATIONS FOR A VISCOUS FLUID FLOW

With the material law applicable to viscous fluids formulated and analyzed in Sections 10.10 and 10.11, let us summarize all the equations that serve as governing equations for flows of such fluids. Since the equation of continuity (10.4.1), the equations of motion (10.4.2a, b) and the equation of energy (10.4.3) hold for all continua, they hold not only for nonviscous fluids but also for viscous fluids. Let us record these equations for easy reference.

Equation of Continuity:

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{v} = 0 \tag{10.12.1}$$

Equations of Motion:

$$\operatorname{div} \mathbf{T}^T + \rho \mathbf{b} = \rho \frac{D\mathbf{v}}{Dt} \tag{10.12.2a}$$

$$\mathbf{T} = \mathbf{T}^T \tag{10.12.2b}$$

Equation of Energy:

$$\rho \frac{D\varepsilon}{Dt} = \mathbf{T} \cdot \nabla \mathbf{v} - \operatorname{div} \mathbf{q} + \rho h \tag{10.12.3}$$

Next, in place of the material law (10.4.4), which is valid only for nonviscous fluids, let us consider Stokes's law as given by (10.10.7) and record it.

Material Law:

$$\mathbf{T} = (-p + \lambda \operatorname{div} \mathbf{v})\mathbf{I} + \mu(\nabla \mathbf{v} + \nabla \mathbf{v}^T) \tag{10.12.4}$$

As noted earlier, $\mathbf{T}$ is symmetric by virtue of this material law. Hence equation (10.12.2b) is identically satisfied (as in the case of nonviscous fluids).

It is easily seen that 11 scalar equations are involved in the governing equations (10.12.1)–(10.2.4) whereas 15 unknown field functions are present in these equations, the unknowns being 3 velocity components v_i, 3 heat flux components q_i, 6 stress components $\tau_{ij}(=\tau_{ji})$ and 3 scalars ρ, p and ε (as in the case of nonviscous fluids). The body force **b** and heat supply h are taken as known functions as usual. Accordingly, equations (10.12.1) to (10.12.4) are inadequate to determine all the field functions. Therefore, we have to either reduce the number of unknowns or increase the number of governing equations in order to close the system of governing equations. These two possibilities lead us to consider the cases of incompressible and compressible fluids as in the theory of nonviscous fluid flows.

The classification of fluids into the categories of compressible fluids and incompressible fluids holds for both nonviscous and viscous fluids. Note that the property of compressibility is associated directly with the change in density of fluid elements during the motion, and the property of viscosity is linked with the ability of the fluid to sustain shear stresses, again during the motion. Since the change in density is always accompanied by change in volume of elements, and shear stresses are always accompanied by change in shape of elements, the properties of compressibility and viscosity are not interlinked (at least theoretically). This means that a fluid can be (i) nonviscous and incompressible, (ii) nonviscous and compressible, (iii) viscous and incompressible and (iv) viscous and compressible. Thus, there exist four different classes of fluids. The first two classes of fluids, namely, incompressible and compressible nonviscous fluids, were analyzed in Section 10.4. Let us now consider the cases of incompressible and compressible viscous fluids to complete the task of obtaining a closed system of governing equations for flow of such fluids.

10.12.1 INCOMPRESSIBLE FLUIDS

In the case of incompressible fluids, for which ρ retains its initial value ρ_0 during the motion, the equations of continuity, motion and energy, namely, (10.12.1), (10.12.2) and (10.12.3), reduce to the following equations (respectively):

$$\operatorname{div} \mathbf{v} = 0 \tag{10.12.5}$$

$$\operatorname{div} \mathbf{T} + \rho_0 \mathbf{b} = \rho_0 \frac{D\mathbf{v}}{Dt} \tag{10.12.6}$$

$$\rho_0 \frac{D\varepsilon}{Dt} = -\operatorname{div} \mathbf{q} + \rho h \tag{10.12.7}$$

Also, in this case, the material law (10.12.4) reduces to

$$\mathbf{T} = -p\mathbf{I} + \mu(\nabla\mathbf{v} + \nabla\mathbf{v}^T) \tag{10.12.8}$$

We note that ten scalar equations are involved in (10.12.5), (10.12.6) and (10.12.8) and the number of unknown field functions is also ten, the field functions being p, v_i and τ_{ij}. Equation (10.12.7), which is purely an equation of thermal energy does not contain these field functions and is therefore not needed in their determination. Equations (10.12.5), (10.12.6) and (10.12.8) thus serve as a closed system of governing equations for incompressible viscous fluid flows.

10.12.2 COMPRESSIBLE FLUIDS

As in the case of nonviscous compressible fluid flows, the equations of state and law of heat conduction are appended to the material law for obtaining a full set of constitutive equations for viscous compressible fluid flows. These additional constitutive equations are the same as those considered in Section 10.4. These equations are the *kinetic equation of state* (10.4.10), namely,

$$f(p, \rho, T) = 0 \tag{10.12.9}$$

the *caloric equation of state* (10.4.13), namely,

$$\varepsilon = \varepsilon(\rho, T) \tag{10.12.10}$$

and Fourier's law of heat conduction (10.4.12), namely,

$$\mathbf{q} = -k\,\nabla T \tag{10.12.11}$$

If equations (10.12.9) to (10.12.11) are appended to equations (10.12.1) to (10.12.4), then we will have in all sixteen scalar equations for sixteen field functions, the field functions being three velocity components v_i, three heat-flux components q_i, six stress components τ_{ij} and four scalars ρ, p, T and ε. Equations (10.12.1) to (10.12.4) and (10.12.9) to (10.12.11) therefore serve as a closed system of governing differential equations for compressible viscous fluid flows. Note that, unlike the case of incompressible fluid flows, energy equation (10.12.3) is now fully coupled with other governing equations.

Thus, as in the case of nonviscous fluid flows, there are two sets of governing equations for viscous fluid flows. One of these sets is applicable to incompressible fluid flows and the other set to compressible fluid flows. Comparison of the governing equations of viscous fluid flows with those of nonviscous fluid flows reveals that the material law is the only governing equation that brings out the difference between the theory of viscous fluid flows and the theory of nonviscous fluid flows; (10.12.4) is the material law employed for viscous fluids and (10.4.4) is the material law for nonviscous fluids. The standard forms of equations of state, given by (10.4.14) to (10.4.21) are employed in the theory of viscous fluids also.

In Section 10.11, it was pointed out that compressible viscous fluids themselves fall into two classes depending on whether condition (10.11.8) holds or not. When condition (10.11.8) holds, we have $\lambda = -(2/3)\mu$ by (10.11.5) and the material law (10.12.4) reduces to

$$\mathbf{T} = -p\mathbf{I} + \mu\{(-\tfrac{2}{3})(\operatorname{div}\mathbf{v})\mathbf{I} + \nabla\mathbf{v} + (\nabla\mathbf{v})^T\} \qquad (10.12.12)$$

This relation together with (10.12.1) to (10.12.3) and (10.12.9) to (10.12.11) constitute the governing equations for compressible viscous fluid flows for which Stokes's condition (10.11.8) holds.

Thus, the theory of viscous fluid flows is generally studied in three different cases: (i) the incompressible case governed by equations (10.12.5), (10.12.6) and (10.12.8); (ii) compressible case with Stokes's condition, governed by equations (10.12.1) to (10.12.3), (10.12.12) and (10.12.9) to (10.12.11); and (iii) general compressible case governed by equations (10.2.1) to (10.12.4) and (10.12.9) to (10.12.11). The governing equations for viscous fluid flows valid in different cases are summarized in Tables 10.3 and 10.4 with the same equation numbers as in the text.

EXAMPLE 10.12.1 Show that the energy equation (10.12.3) can be rewritten in the form

$$\rho\frac{D\varepsilon}{Dt} = -p(\operatorname{div}\mathbf{v}) - \operatorname{div}\mathbf{q} + \rho h + \Phi \qquad (10.12.13)$$

where Φ is the viscous dissipative function considered in Example 10.10.3.

Solution By use of the material law (10.12.4), we find that

$$\mathbf{T}\cdot\nabla\mathbf{v} = -p(\operatorname{div}\mathbf{v}) + \lambda(\operatorname{div}\mathbf{v})^2 + \mu(\nabla\mathbf{v} + \nabla\mathbf{v}^T)\cdot\nabla\mathbf{v} \qquad (10.12.14)$$

Using (10.10.17), this becomes

$$\mathbf{T}\cdot\nabla\mathbf{v} = -p(\operatorname{div}\mathbf{v}) + \Phi \qquad (10.12.15)$$

Substituting this in (10.12.3), we obtain (10.12.13). ■

Table 10.3. Governing Equations for Incompressible Viscous Fluid Flows

1. Equation of continuity:

$$\operatorname{div}\mathbf{v} = 0 \qquad (10.12.5)$$

2. Equation of motion:

$$\operatorname{div}\mathbf{T} + \rho_0\mathbf{b} = \rho_0\frac{D\mathbf{v}}{Dt} \qquad (10.12.6)$$

3. Material law:

$$\mathbf{T} = -p\mathbf{I} + \mu(\nabla\mathbf{v} + \nabla\mathbf{v}^T) \qquad (10.12.8)$$

Total number of scalar equations: 10
Total number of unknown field functions (namely, p, v_i and τ_{ij}): 10

Table 10.4. Governing Equations for Compressible Viscous Fluid Flows

1. Equation of continuity:

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{v} = 0 \tag{10.12.1}$$

2. Equation of motion:

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \rho \frac{D\mathbf{v}}{Dt} \tag{10.12.2}$$

3. Equation of energy:

$$\rho \frac{D\varepsilon}{Dt} = \mathbf{T} \cdot \nabla \mathbf{v} - \operatorname{div} \mathbf{q} + \rho h \tag{10.12.3}$$

4. Material law:

$$\mathbf{T} = \{-p + \lambda(\operatorname{div} \mathbf{v})\}\mathbf{I} + \mu(\nabla \mathbf{v} + \nabla \mathbf{v}^T), \quad \text{for } \kappa \neq 0 \tag{10.12.4}$$

$$\mathbf{T} = -p\mathbf{I} + \mu\{(-\tfrac{2}{3})(\operatorname{div} \mathbf{v})\mathbf{I} + (\nabla \mathbf{v} + \nabla \mathbf{v}^T)\}, \quad \text{for } \kappa = 0 \tag{10.12.12}$$

5. Equations of state:

$$f(p, \rho, T) = 0 \tag{10.12.9}$$

$$\varepsilon = \varepsilon(\rho, T) \tag{10.12.10}$$

6. Fourier's law of heat conduction:

$$\mathbf{q} = -k \nabla T \tag{10.12.11}$$

Total number of scalar equations: 16
Total number of unknown field functions (namely, $\rho, p, T, \varepsilon, q_i, v_i$ and τ_{ij}): 16

Note: By using Fourier's law (10.12.11), equation (10.12.13) can be expressed in terms of T as follows:

$$\rho \frac{D\varepsilon}{Dt} = -p(\operatorname{div} \mathbf{v}) + k\nabla^2 T + \rho h + \Phi \tag{10.12.16}$$

EXAMPLE 10.12.2 Show that, for a perfect gas, the energy equation (10.12.3) can be rewritten in the following forms:

$$\rho c_v \frac{DT}{Dt} = \frac{p}{\rho} \frac{D\rho}{Dt} + k\nabla^2 T + \rho h + \Phi \tag{10.12.17}$$

$$\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} + k\nabla^2 T + \rho h + \Phi \tag{10.12.18}$$

Solution For a perfect gas, we have

$$\varepsilon = c_v T \tag{10.12.19a}$$

$$p = \rho R T \tag{10.12.19b}$$

Substituting for ε from (10.12.19a) and for div $\mathbf{v}$ from (10.12.1) in (10.12.16), we obtain (10.12.17).

Next, using (10.4.19), expression (10.12.19b) yields

$$c_v T = c_p T - \frac{p}{\rho}$$

so that

$$c_v \frac{DT}{Dt} = c_p \frac{DT}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} + \frac{p}{\rho^2}\frac{D\rho}{Dt} \tag{10.12.20}$$

With the use of this expression in (10.12.17), we arrive at (10.12.18). ■

Note: In the absence of viscous effects, we have $\Phi \equiv 0$, and equations (10.12.17) and (10.12.18) reduce to equations (10.4.31) and (10.4.32), respectively.

10.13

INITIAL AND BOUNDARY CONDITIONS

Since the field functions for viscous fluid flows are the same as those for nonviscous fluid flows, initial and boundary conditions (10.5.1) to (10.5.5) hold for the viscous case also.

However, because of the presence of shear stresses in a viscous fluid, the condition to be satisfied on a rigid boundary surface S contacting the fluid is different from the condition (10.5.6) employed in the nonviscous case. For a viscous fluid flow, the condition (10.5.6) is usually replaced by the stronger condition:

$$\mathbf{v} = \mathbf{v}_S \tag{10.13.1}$$

on S, where $\mathbf{v}_S$ is the velocity with which the surface S moves. For a surface at rest, the condition becomes

$$\mathbf{v} = \mathbf{0} \qquad (10.13.1)'$$

on S.

Note that condition (10.5.6) is just a particular case of condition (10.13.1). Whereas condition (10.5.6) implies that the normal component of fluid velocity is the same as that of the solid boundary (at the point of contact), condition (10.3.1) implies such a restriction on the tangential component also. In other words, condition (10.13.1) implies that the fluid in contact with the solid surface must move with the surface. This amounts to saying that the fluid must adhere to the solid and therefore cannot slip over the surface of the solid. The condition (10.13.1) was first proposed by Stokes and is known as the *no-slip condition*. In order to satisfy this boundary condition, Prandtl (1905) made the hypothesis that within a thin layer of fluid adjacent to the boundary the relative fluid velocity increases rapidly from 0 at the solid boundary to the full value at its outer edge. This thin layer is called the *boundary layer* within which the viscosity effects are predominant. The condition (10.13.1) is employed as the standard boundary condition in common engineering problems. However, for high-altitude aerodynamical problems, this condition is known to be invalid, and one uses what are called *slip conditions*. We do not consider such conditions in this text.

EXAMPLE 10.13.1 Show that on a viscous fluid surface, the stress vector is given by (i) for incompressible fluid

$$\mathbf{s} = -p\mathbf{n} + 2\mu(\mathbf{n} \cdot \nabla)\mathbf{v} + \mu\mathbf{n} \times \mathbf{w} \tag{10.13.2a}$$

and (ii) for compressible fluid

$$\mathbf{s} = (-p + \lambda \operatorname{div} \mathbf{v})\mathbf{n} + 2\mu(\mathbf{n} \cdot \nabla)\mathbf{v} + \mu\mathbf{n} \times \mathbf{w} \tag{10.13.2b}$$

Solution For an incompressible viscous fluid, the material law is given by (10.12.8). Using this in Cauchy's law (7.4.8), we obtain

$$\mathbf{s} = \mathbf{T}\mathbf{n} = -p\mathbf{n} + \mu(\nabla\mathbf{v} + \nabla\mathbf{v}^T)\mathbf{n} = -p\mathbf{n} + 2\mu(\nabla\mathbf{v})\mathbf{n} + \mu(\nabla\mathbf{v}^T - \nabla\mathbf{v})\mathbf{n} \tag{10.13.3}$$

By use of identities (3.5.34) and (3.6.19), we find that

$$(\nabla\mathbf{v})\mathbf{n} = (\mathbf{n} \cdot \nabla)\mathbf{v}; \qquad (\nabla\mathbf{v}^T - \nabla\mathbf{v})\mathbf{n} = \mathbf{n} \times \operatorname{curl} \mathbf{v} \tag{10.13.4}$$

Using these in (10.13.3), we obtain (10.13.2a).

For a compressible viscous fluid, the material law is given by (10.12.4). Using this in Cauchy's law (7.4.8) we obtain the following counterpart of (10.13.3):

$$\mathbf{s} = \mathbf{T}\mathbf{n} = -p\mathbf{n} + \lambda(\operatorname{div} \mathbf{v})\mathbf{n} + 2\mu(\nabla\mathbf{v})\mathbf{n} + \mu(\nabla\mathbf{v}^T - \nabla\mathbf{v})\mathbf{n} \tag{10.13.5}$$

Using (10.13.4) in (10.13.5), we obtain (10.12.2b). ■

Note: Expressions (10.13.2a) and (10.13.2b) can be utilized to rewrite the boundary condition (10.5.2b) entirely in terms of $\mathbf{v}$ and p as follows

$$-p\mathbf{n} + 2\mu(\mathbf{n} \cdot \nabla)\mathbf{v} + \mu(\mathbf{n} \times \operatorname{curl} \mathbf{v}) = \mathbf{s}^* \tag{10.13.6a}$$

(incompressible case);

$$(-p + \lambda \operatorname{div} \mathbf{v})\mathbf{n} + 2\mu(\mathbf{n} \cdot \nabla)\mathbf{v} + \mu(\mathbf{n} \times \operatorname{curl} \mathbf{v}) = \mathbf{s}^* \tag{10.13.6b}$$

(compressible case), on S_τ for $t \geq 0$.

EXAMPLE 10.13.2 Show that at a fixed rigid solid surface contacting a viscous fluid, the stress vector is given by

$$\mathbf{s} = -p\mathbf{n} + \mu\mathbf{w} \times \mathbf{n} \tag{10.13.7a}$$

(incompressible case);

$$\mathbf{s} = [-p + (\lambda + 2\mu)\operatorname{div}\mathbf{v}]\mathbf{n} + \mu\mathbf{w} \times \mathbf{n} \tag{10.13.7b}$$

(compressible case). Hence, compute the normal and shear stresses exerted by the fluid on the solid surface.

Solution At a fixed rigid solid surface contacting a viscous fluid, we have $\mathbf{v} = \mathbf{0}$, by the no-slip boundary condition (10.13.1)′. Hence if C is an *arbitrary* simple closed curve chosen on this surface,

$$\int_C \mathbf{v} \times d\mathbf{x} = \mathbf{0} \tag{10.13.8}$$

By using a consequence of Stokes's theorem, given by (3.6.20), expression (10.13.8) becomes

$$\int_S [(\operatorname{div}\mathbf{v})\mathbf{n} - (\mathbf{n} \cdot \nabla)\mathbf{v} - \mathbf{n} \times \mathbf{w}]\, dS = \mathbf{0} \tag{10.13.9}$$

where S is the area on the solid surface, bounded by C. Since C is arbitrary, it follows that the integrand of the surface integral in (10.13.9) should vanish identically; hence,

$$(\mathbf{n} \cdot \nabla)\mathbf{v} = (\operatorname{div}\mathbf{v})\mathbf{n} + \mathbf{w} \times \mathbf{n} \tag{10.13.10}$$

on the solid surface considered.

Substituting for $(\mathbf{n} \cdot \nabla)\mathbf{v}$ from (10.13.10) in (10.13.2a) and noting that $\operatorname{div}\mathbf{v} = 0$ for an incompressible fluid, we get the result (10.13.7a). Similarly, (10.13.10) and (10.13.2b) yield the result (10.13.7b).

The normal stress exerted by the fluid on the solid surface is given by

$$\sigma = -\mathbf{s} \cdot \mathbf{n} \tag{10.13.11}$$

where $\mathbf{n}$ is directed *into* the fluid. For an incompressible fluid, (10.13.7a) and (10.13.11) yield $\sigma = p$. Since p is equal to the mean pressure $\bar{p}$ for an incompressible fluid, it follows that

$$\sigma = \bar{p} \tag{10.13.12}$$

Thus, in the case of an incompressible viscous fluid, the fluid elements exert just the mean stress on the surface, along the normal.

For a compressible fluid, we find by using (10.13.7b) in (10.13.11) that

$$\sigma = p - (\lambda + 2\mu)(\text{div}\,\mathbf{v}) \tag{10.13.13}$$

which on using (10.11.6) and (10.11.5) may be rewritten as

$$\sigma = \bar{p} - \tfrac{4}{3}\mu(\text{div}\,\mathbf{v}) \tag{10.13.14}$$

Thus, in the case of a compressible viscous fluid the normal stress exerted by the fluid on the surface is caused by the mean pressure as well as the rate of change in volume of fluid elements.

If $\mathbf{t}$ is a unit vector tangential to the surface, the magnitude of the shear stress exerted by the fluid on the solid surface is

$$\tau = |\mathbf{s} \cdot \mathbf{t}| \tag{10.13.15}$$

By using (10.13.7ab) in (10.13.15), we find that the following expression holds for both incompressible and compressible fluids:

$$\tau = \mu|[\mathbf{w}, \mathbf{n}, \mathbf{t}]| \tag{10.13.16}$$

Thus, τ depends only on the vorticity vector $\mathbf{w}$ for both incompressible and compressible viscous fluids.

Form (10.13.16) it also follows that, in irrotational motion, no shear stress is exerted on a fixed solid boundary. ■

EXAMPLE 10.13.3 Show that, at a fixed rigid solid surface contacting a viscous fluid, the vorticity vector lies tangential to the surface.

Solution Let C be an arbitrary simple closed curve on the surface and let S be the area of the surface enclosed by C. Then, by Stokes's theorem, we have

$$\int_C \mathbf{v} \cdot d\mathbf{x} = \int_S \mathbf{w} \cdot \mathbf{n}\, dS \tag{10.13.17}$$

But by the no-slip boundary condition $\mathbf{v} = \mathbf{0}$ on S. Hence the lefthand side of (10.13.17) vanishes identically. Since C is arbitrary, it follows that $\mathbf{w} \cdot \mathbf{n} = 0$ or that $\mathbf{w}$ is tangential to the surface, at every point. ■

10.14

NAVIER-STOKES EQUATION

We now deduce the equation of motion of a *viscous* fluid by eliminating the stresses from the governing equations.

Let us start with Stokes's law given by (10.12.4), namely,

$$\mathbf{T} = (-p + \lambda\,\text{div}\,\mathbf{v})\mathbf{I} + \mu(\nabla\mathbf{v} + \nabla\mathbf{v}^T) \tag{10.14.1}$$

Taking the divergence on both sides of this tensor equation and using the identities (3.5.35), (3.5.13) and (3.5.14), we obtain

$$\operatorname{div} \mathbf{T} = \nabla(-p + \lambda \operatorname{div} \mathbf{v}) + \mu(\nabla^2 \mathbf{v} + \nabla \operatorname{div} \mathbf{v}) \tag{10.14.2}$$

Substituting this expression into Cauchy's equation of motion (10.12.2) we arrive at the following equation of motion expressed in terms of p, ρ and $\mathbf{v}$:

$$\mu\nabla^2\mathbf{v} + (\lambda + \mu)\nabla(\operatorname{div} \mathbf{v}) - \nabla p + \rho\mathbf{b} = \rho\frac{D\mathbf{v}}{Dt} \tag{10.14.3}$$

This equation, essentially by Navier (1822) and Stokes (1845), is referred to as the *Navier–Stokes equation*. This equation holds for both compressible and incompressible viscous fluid flows; in the incompressible case, $\rho = \rho_0$ and $\operatorname{div} \mathbf{v} = 0$.

In the absence of viscosity, that is, if λ and μ are negligibly small, equation (10.14.3) reduces to Euler's equation of motion (10.6.2).

Using the identity (10.6.4), namely,

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{w} \times \mathbf{v} + \frac{1}{2}\nabla v^2 \tag{10.14.4}$$

equation (10.14.3) can be rewritten in the following alternative forms:

$$\mu\nabla^2\mathbf{v} + (\lambda + \mu)\nabla(\operatorname{div} \mathbf{v}) - \nabla p + \rho\mathbf{b} = \rho\left\{\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right\} \tag{10.14.5a}$$

$$\mu\nabla^2\mathbf{v} + (\lambda + \mu)\nabla(\operatorname{div} \mathbf{v}) - \nabla p + \rho\mathbf{b} = \rho\left\{\frac{\partial \mathbf{v}}{\partial t} + \mathbf{w} \times \mathbf{v} + \frac{1}{2}\nabla v^2\right\} \tag{10.14.5b}$$

For an incompressible viscous fluid, the Navier–Stokes equation (10.14.3) is often rewritten in the following form:

$$\nu\nabla^2\mathbf{v} - \frac{1}{\rho}\nabla p + \mathbf{b} = \frac{D\mathbf{v}}{Dt} \tag{10.14.6}$$

where

$$\nu = \mu/\rho. \tag{10.14.7}$$

The coefficient ν has dimension area/time and is called *kinematic viscosity*.

By using (10.14.4), the equation (10.14.6) can be rewritten in two explicit forms as follows:

$$\nu\nabla^2\mathbf{v} - \frac{1}{\rho}\nabla p + \mathbf{b} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \tag{10.14.8a}$$

$$\nu\nabla^2\mathbf{v} - \frac{1}{\rho}\nabla p + \mathbf{b} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{w} \times \mathbf{v} + \frac{1}{2}\nabla v^2 \tag{10.14.8b}$$

When the external force is conservative, that is, if $\mathbf{b} = -\nabla\chi$, and ρ is a constant we get the following useful form of equation (10.14.8b):

$$\nu\nabla^2\mathbf{v} - \nabla\left(\frac{p}{\rho} + \chi + \frac{1}{2}v^2\right) = \frac{\partial\mathbf{v}}{\partial t} + \mathbf{w}\times\mathbf{v} \tag{10.14.9}$$

In the case of a compressible viscous fluid for which Stokes's condition (10.11.8) holds, we have $\lambda = -(2/3)\mu$; see (10.11.5). Then, equation (10.14.3) involves only the shear viscosity μ.

Like Euler's equation of motion for a nonviscous fluid, the Navier–Stokes equation is the fundamental equation of motion for a viscous fluid. Whereas the Euler's equation is a first-order partial differential equation in spatial derivatives of velocity, the Navier–Stokes equation is of second order in these derivatives.

It may be noted that the Navier–Stokes equation (10.14.3) is strikingly analogous to Navier's equation (9.8.6) of classical elasticity. While (9.8.6) is a *linear* partial differential equation containing the displacement $\mathbf{u}$ as the only unknown, (10.14.3) is a *nonlinear* partial differential equation that contains the velocity $\mathbf{v}$, the density ρ and the pressure p as unknowns, the nonlinearity of the equation stemming from the presence of $(\mathbf{v}\cdot\nabla)\mathbf{v}$ in the acceleration. Thus, whereas (9.8.6) provides a complete set of field equations for the determination of $\mathbf{u}$, (10.14.3) does not provide a complete set of field equations for the determination of $\mathbf{v}$; it has to be supplemented by the equation of continuity in the incompressible case and, further, equations of state and an equation of heat conduction in the compressible case.

10.14.1 REYNOLDS NUMBER

Due to nonlinearity of the Navier–Stokes equation, no general method of solution is available. Exact solutions are known only in special cases. Approximate solutions are often obtained either by neglecting the *convective term* $(\mathbf{v}\cdot\nabla)\mathbf{v}$ in comparison with the *viscous term* $\nu\nabla^2\mathbf{v}$ or by neglecting the viscous term in comparison with the convective term. Thus, the ratio of the order of the convective term to that of the viscous term plays a crucial role in solving the Navier–Stokes equation approximately.

Suppose that, in a given flow, V is a reference speed and L is a reference length. Then, the magnitude of the convective term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ is of order $V(V/L) = V^2/L$ and the magnitude of the viscous term $\nu\nabla^2\mathbf{v}$ is of order $(\nu V/L^2)$. Thus,

$$\frac{|(\mathbf{v}\cdot\nabla)\mathbf{v}|}{|\nu(\nabla^2\mathbf{v})|} = \mathrm{O}\left(\frac{VL}{\nu}\right)$$

Since ν has dimension area/time, we note that (VL/ν) is a dimensionless quantity. This quantity is called the *Reynolds number*, after O. Reynolds, and we denote it by Re. Thus, the convective term $(\mathbf{v} \cdot \nabla)\mathbf{v}$ is negligible in comparison with the viscous term $\nu(\nabla^2\mathbf{v})$ when Re is very small, and the viscous term is negligible in comparison with the convective term when Re is very large. For small values of Re, the Navier-Stokes equation takes a linearized form; for example, equation (10.14.6) will reduce to equation (10.14.28) given later. For large values of Re, the effect of viscosity is significant only in boundary layers; at points outside a boundary layer the Navier-Stokes equation is approximated to Euler's equation valid for nonviscous fluids.

EXAMPLE 10.14.1 Find the pressure distribution such that the velocity field given by

$$v_1 = k(x_1^2 - x_2^2), \quad v_2 = 2kx_1x_2, \quad v_3 = 0, \quad (k = \text{constant}) \tag{10.14.10}$$

satisfies the Navier-Stokes equation for an incompressible fluid in the absence of body force.

Solution When written in the component form, the Navier-Stokes equation for an incompressible fluid given by (10.14.6) reads as follows, on using the identity (10.14.4) also taken in the component form:

$$\begin{aligned}
\frac{\partial v_1}{\partial t} + \left(v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2} + v_3\frac{\partial}{\partial x_3}\right)v_1 &= \nu\nabla^2 v_1 - \frac{1}{\rho}\frac{\partial p}{\partial x_1} + b_1 \\
\frac{\partial v_2}{\partial t} + \left(v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2} + v_3\frac{\partial}{\partial x_3}\right)v_2 &= \nu\nabla^2 v_2 - \frac{1}{\rho}\frac{\partial p}{\partial x_2} + b_2 \qquad (10.14.11)\\
\frac{\partial v_3}{\partial t} + \left(v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2} + v_3\frac{\partial}{\partial x_3}\right)v_3 &= \nu\nabla^2 v_3 - \frac{1}{\rho}\frac{\partial p}{\partial x_3} + b_3
\end{aligned}$$

Substituting the expressions for v_i from (10.14.10) in equations (10.14.11) and noting that $b_i = 0$, we obtain

$$\frac{\partial p}{\partial x_1} = -2k^2\rho x_1(x_1^2 + x_2^2) \tag{10.14.12}$$

$$\frac{\partial p}{\partial x_2} = -2k^2\rho x_2(x_1^2 + x_2^2) \tag{10.14.13}$$

$$\frac{\partial p}{\partial x_3} = 0 \tag{10.14.14}$$

Thus, the pressure-gradient in the x_3-direction should be 0. Accordingly, $p = p(x_1, x_2)$ so that

$$dp = \frac{\partial p}{\partial x_1} dx_1 + \frac{\partial p}{\partial x_2} dx_2$$

This yields, on using (10.14.12) and (10.14.13),

$$dp = -2k^2\rho(x_1^2 + x_2^2)(x_1\, dx_1 + x_2\, dx_2) = -k^2\rho d\{\tfrac{1}{2}(x_1^2 + x_2^2)^2\}$$

Hence

$$p = -\tfrac{1}{2}k^2\rho(x_1^2 + x_2^2)^2 + C \qquad (10.14.15)$$

where C is an arbitrary constant. If p^0 is the pressure at the origin, we get

$$p = p^0 - \tfrac{1}{2}k^2\rho(x_1^2 + x_2^2)^2 \qquad (10.14.16)$$

This is the required pressure distribution. ■

EXAMPLE 10.14.2 For steady flow of an incompressible viscous fluid under a conservative body force, prove the following:

(i)
$$\mathbf{v} \times \mathbf{w} = \nabla\left(\frac{p}{\rho} + \frac{1}{2}v^2 + \chi\right) + \nu \operatorname{curl} \mathbf{w} \qquad (10.14.17)$$

(ii)
$$(\mathbf{v} \cdot \nabla)\mathbf{w} - (\mathbf{w} \cdot \nabla)\mathbf{v} = \nu\nabla^2\mathbf{w} \qquad (10.14.18)$$

(iii) If

$$\mathbf{v} = \psi_{,2}\mathbf{e}_1 - \psi_{,1}\mathbf{e}_2 \qquad (10.14.19)$$

where $\psi = \psi(x_1, x_2)$, then

$$\begin{vmatrix} (\nabla^2\psi)_{,1} & (\nabla^2\psi)_{,2} \\ \psi_{,1} & \psi_{,2} \end{vmatrix} = \nu\nabla^4\psi \qquad (10.14.20)$$

Solution (i) For an incompressible viscous fluid moving under a conservative force, the Navier-Stokes equation is given by (10.14.9).

For steady flow $\partial\mathbf{v}/\partial t = \mathbf{0}$. Also, for an incompressible fluid,

$$\operatorname{curl} \mathbf{w} = \operatorname{curl} \operatorname{curl} \mathbf{v} = \operatorname{grad} \operatorname{div} \mathbf{v} - \nabla^2\mathbf{v} = -\nabla^2\mathbf{v} \qquad (10.14.21)$$

because $\operatorname{div} \mathbf{v} = 0$. Using these facts in equation (10.14.9), we obtain (10.14.17).

(ii) From (10.14.21), we get

$$\operatorname{curl} \operatorname{curl} \mathbf{w} = -\nabla^2\mathbf{w} \qquad (10.14.22)$$

Also, since $\operatorname{div} \mathbf{v} = 0$ and $\operatorname{div} \mathbf{w} = 0$, we find from identity (3.4.25) that

$$\operatorname{curl}(\mathbf{v} \times \mathbf{w}) = (\mathbf{w} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{w} \qquad (10.14.23)$$

Now, taking the curl on both sides of (10.14.17) and using (10.14.22), (10.14.23) and (3.4.16), we obtain (10.14.18).

(iii) When $\mathbf{v}$ is given by (10.14.19), we get

$$\mathbf{w} = \operatorname{curl} \mathbf{v} = -\nabla^2 \psi \mathbf{e}_3 \tag{10.14.24}$$

so that

$$(\mathbf{v} \cdot \nabla)\mathbf{w} = \left(\psi_{,2} \frac{\partial}{\partial x_1} - \psi_{,1} \frac{\partial}{\partial x_2} \right)(-\nabla^2 \psi)\mathbf{e}_3 \tag{10.14.25}$$

and

$$(\mathbf{w} \cdot \nabla)\mathbf{v} = \mathbf{0} \tag{10.14.26}$$

Using (10.14.24), (10.14.25) and (10.14.26) in (10.14.18), we obtain (10.14.20). ■

Note: The function ψ defined by (10.14.19) is known as *two-dimensional stream function.*

EXAMPLE 10.14.3 A fluid motion for which the Reynolds number is small (so that nonlinear terms in velocity are negligible) is known as a *creeping flow* or *Stokes's flow.* For a steady creeping flow of an incompressible viscous fluid under zero body force, show that p is a harmonic function. Deduce that ψ defined by (10.14.19) is a biharmonic function in this case.

Solution For creeping flow,

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \approx \frac{\partial \mathbf{v}}{\partial t} \tag{10.14.27}$$

Then the Navier–Stokes equation (10.14.6) becomes

$$\nu \nabla^2 \mathbf{v} - \frac{1}{\rho} \nabla p + \mathbf{b} = \frac{\partial \mathbf{v}}{\partial t} \tag{10.14.28}$$

For steady flow with zero body force, this equation reduces to

$$\nu \nabla^2 \mathbf{v} = \frac{1}{\rho} \nabla p \tag{10.14.29}$$

Taking the divergence of this equation and using the equation of continuity $\operatorname{div} \mathbf{v} = 0$, we get

$$\nabla^2 p = 0 \tag{10.14.30}$$

Thus, p is a harmonic function.

For ψ defined through the relation (10.14.19), we have

$$v_1 = \psi_{,2}, \qquad v_2 = -\psi_{,1}, \qquad v_3 = 0$$

Using these in equation (10.14.29), we get

$$p_{,1} = \rho\nu\nabla^2(\psi_{,2}); \qquad p_{,2} = -\rho\nu\nabla^2(\psi_{,1}) \tag{10.14.31}$$

From these, we obtain, respectively,

$$p_{,12} = \rho\nu\nabla^2(\psi_{,22}); \qquad p_{,21} = -\rho\nu\nabla^2(\psi_{,11}) \tag{10.14.32}$$

so that

$$\nu\rho\nabla^2(\psi_{,11} + \psi_{,22}) = -(p_{,21} - p_{,12}) = 0 \quad \text{or} \quad \nabla^4\psi = 0$$

Thus, ψ is a biharmonic function. ■

EXAMPLE 10.14.4 Consider a steady motion of an incompressible viscous fluid under a conservative body force. If

$$H_0 = \frac{1}{2}v^2 + \frac{p}{\rho} + \chi \tag{10.14.33}$$

prove the following.

(i) H_0 is constant along the field lines of the vector

$$\mathbf{f} = (\mathbf{v} \times \mathbf{w}) \times \operatorname{curl} \mathbf{w} \tag{10.14.34}$$

(ii)
$$\mathbf{v} \cdot \nabla H_0 = \nu(\nabla^2 H_0 - \mathbf{w}^2) \tag{10.14.35}$$

Solution For the motion considered, the relation (10.14.17) holds. Using (10.4.33), this relation can be rewritten as

$$\nabla H_0 = (\mathbf{v} \times \mathbf{w}) - \nu \operatorname{curl} \mathbf{w} \tag{10.14.36}$$

From (10.14.34) and (10.14.36) we readily see that $\mathbf{f} \cdot \nabla H_0 = 0$. Thus, ∇H_0 is orthogonal to $\mathbf{f}$ and hence to the field line of $\mathbf{f}$. But ∇H_0 is always orthogonal to the surfaces of constant H_0. Hence $\mathbf{f}$ must be tangential to a surface of constant H_0. That is, H_0 is constant along the field lines of $\mathbf{f}$.

From (10.14.36), we get

$$\nabla^2 H_0 = \operatorname{div}(\mathbf{v} \times \mathbf{w}) = \mathbf{w}^2 - \mathbf{v} \cdot \operatorname{curl} \mathbf{w} \tag{10.14.37}$$

and

$$\mathbf{v} \cdot \nabla H_0 = -\nu \mathbf{v} \cdot \operatorname{curl} \mathbf{w} \tag{10.14.38}$$

These relations together yield the relation (10.14.35). ■

EXAMPLE 10.14.5 For a nonsteady flow of an incompressible viscous fluid under conservative body force with $\operatorname{curl} \mathbf{w} = \nabla\xi$ for some scalar function ξ, show that the Navier–Stokes equation becomes

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{w} \times \mathbf{v} = -\nabla H^* \tag{10.14.39}$$

where

$$H^* = \frac{p}{\rho} + \frac{1}{2}v^2 + \chi + \nu\xi \tag{10.14.40}$$

Deduce the following.

(i) If the flow is of potential kind, then $H^* + (\partial\phi/\partial t) = f(t)$, where $f(t)$ is an arbitrary function of t.

(ii) If the flow is steady and $\mathbf{v} \times \mathbf{w} \neq \mathbf{0}$, then H^* is constant along stream lines and vortex lines.

(iii) If the flow is steady and $\mathbf{v} \times \mathbf{w} = \mathbf{0}$, then H^* is constant everywhere in the fluid.

Solution For an incompressible viscous fluid moving under a conservative body force, the Navier–Stokes equation is given by (10.14.9).

Since curl $\mathbf{w} = -\nabla^2\mathbf{v}$, the given condition, curl $\mathbf{w} = \nabla\xi$, yields $\nabla^2\mathbf{v} = -\nabla\xi$. Using this, equation (10.14.9) becomes (10.14.39), with H^* defined by (10.14.40).

We observe that equation (10.14.39) is strikingly similar to equation (10.8.1). Indeed, in the absence of viscous effects, function H^* defined by (10.14.40) reduces to Bernoulli's function H defined by (10.8.2). Following the steps that led to Bernoulli's equations (10.8.4), (10.8.7) and (10.8.8) from (10.8.1), we arrive at the desired results starting from the equation (10.14.39). ■

EXAMPLE 10.14.6 For an incompressible viscous fluid moving under a conservative body force, prove the following:

(i)
$$\nabla^2\left(\frac{p}{\rho} + \chi + \frac{1}{2}v^2\right) = \operatorname{div}(\mathbf{v} \times \mathbf{w}) \tag{10.14.41}$$

(ii)
$$\nabla^2\left(\frac{p}{\rho} + \chi\right) = \tfrac{1}{2}\mathbf{w}^2 - \mathbf{D} \cdot \mathbf{D} \tag{10.14.42}$$

Further, if the motion is irrotational, deduce that

(iii)
$$\nabla^2 v^2 = 2\mathbf{D} \cdot \mathbf{D} \geq 0 \tag{10.14.43}$$

Solution (i) For an incompressible viscous fluid moving under a conservative body force, the Navier-Stokes equation is given by (10.14.9). If we take divergence of this equation and note that div $\mathbf{v} = 0$, we get the relation (10.14.41).

(ii) Substituting for div($\mathbf{v} \times \mathbf{w}$) from the identity (6.3.21) (with div $\mathbf{v} = 0$) in the righthand side of the relation (10.14.41), we get the relation (10.14.42).

(iii) For an irrotational motion, $\mathbf{w} = \mathbf{0}$. In this case, subtracting (10.14.42) from (10.14.41) we get $\nabla^2 v^2 = 2\mathbf{D} \cdot \mathbf{D}$. Since $\mathbf{D} \cdot \mathbf{D} = d_{ij}d_{ij} \geq 0$, we obtain the inequality $\nabla^2 v^2 \geq 0$. Thus, (10.14.43) is proven. ■

EXAMPLE 10.14.7 For a flow of an incompressible viscous fluid under conservative body force, show that the vorticity equation is given by

$$\frac{D\mathbf{w}}{Dt} = (\mathbf{w} \cdot \nabla)\mathbf{v} + \nu\nabla^2\mathbf{w} \tag{10.14.44}$$

Deduce the following.

(i) If curl $\mathbf{w} = \nabla\xi$, equation (10.14.44) becomes

$$\frac{D\mathbf{w}}{Dt} = (\mathbf{w} \cdot \nabla)\mathbf{v} \tag{10.14.45}$$

(ii) If the motion is two-dimensional (where $v_3 \equiv 0$ and v_1 and v_2 are independent of x_3), equation (10.14.44) reduces to

$$\frac{Dw}{Dt} = \nu\nabla^2 w \tag{10.14.46}$$

where $w = w_3$.

(iii) If the motion is two-dimensional and in circles with centers on x_3 axis, equation (10.14.46) reduces to

$$\frac{\partial w}{\partial t} = \nu\nabla^2 w \tag{10.14.47}$$

Solution For the given flow, the Navier–Stokes equation is given by (10.14.6) with $\mathbf{b} = -\nabla\chi$. Taking curl on both sides of this equation, we get

$$\text{curl}\left(\frac{D\mathbf{v}}{Dt}\right) = \nu\nabla^2\mathbf{w} \tag{10.14.48}$$

Using this expression in Beltrami's vorticity equation (8.2.16), we obtain the equation (10.14.44).

(i) If $\text{curl}\,\mathbf{w} = \nabla\xi$, then the identity (3.4.27) yields $\nabla^2\mathbf{w} = \mathbf{0}$. Consequently, equation (10.14.44) reduces to equation (10.14.45).

(ii) If $v_3 \equiv 0$ and v_1 and v_2 are independent of x_3, we readily find

$$w_1 = w_2 = 0, \qquad w_3 = v_{2,1} - v_{1,2} \tag{10.14.49}$$

Consequently,

$$(\mathbf{w} \cdot \nabla)\mathbf{v} = \mathbf{0} \tag{10.14.50}$$

and equation (10.14.44) reduces to equation (10.14.46).

(iii) If the motion is two-dimensional and in circles with centers on the x_3 axis, we have $\mathbf{v} = v\mathbf{e}_\theta$, with $v = v(R, t)$, where R is the radial distance parallel to the x_1x_2 plane and $\mathbf{e}_\theta$ is the unit vector in the transverse direction (see Figure 10.14).

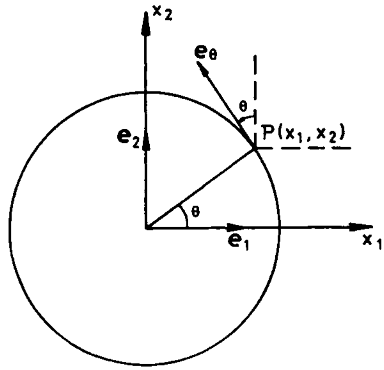

Figure 10.14. Example 10.14.7(iii).

From Figure 10.14, we find that

$$\mathbf{e}_\theta = (-\sin\theta)\mathbf{e}_1 + (\cos\theta)\mathbf{e}_2 = -\frac{x_2}{R}\mathbf{e}_1 + \frac{x_1}{R}\mathbf{e}_2$$

Thus,

$$\mathbf{v} = v\mathbf{e}_\theta = v[-x_2\mathbf{e}_1 + x_1\mathbf{e}_2]$$

so that

$$v_1 = -\frac{v}{R}x_2, \qquad v_2 = \frac{v}{R}x_1, \qquad v_3 = 0 \tag{10.14.51}$$

These give, on noting that $v = v(R, t)$ and $R_{,i} = x_i/R$, $i = 1, 2$,

$$v_{1,2} = -\frac{v}{R} - \frac{1}{R}\frac{\partial v}{\partial R}R_{,2}x_2 + \frac{v}{R^2}R_{,2}x_2 = -\frac{v}{R} - \frac{1}{R^2}\frac{\partial v}{\partial R}x_2^2 + \frac{v}{R^3}x_2^2 \tag{10.14.52a}$$

$$v_{2,1} = \frac{v}{R} + \frac{1}{R}\frac{\partial v}{\partial R}R_{,1}x_1 - \frac{v}{R^2}R_{,1}x_1 = \frac{v}{R} + \frac{1}{R^2}\frac{\partial v}{\partial R}x_1^2 - \frac{v}{R^3}x_1^2 \tag{10.14.52b}$$

Using these in (10.14.49), we obtain

$$w = \frac{v}{R} + \frac{\partial v}{\partial R} \tag{10.14.53}$$

(where $w = w_3$), which yields

$$w_{,i} = \left[\frac{\partial^2 v}{\partial R^2} - \frac{1}{R}\frac{\partial v}{\partial R} - \frac{v}{R^2}\right]\frac{x_i}{R}, \qquad i = 1, 2 \tag{10.14.54}$$

From (10.14.51), (10.14.53) and (10.14.54) we find that

$$(\mathbf{v}\cdot\nabla)w = v_1 w_{,1} + v_2 w_{,2} = 0$$

Consequently,

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} \tag{10.14.55}$$

Substituting this in (10.14.46), we obtain (10.14.47). ■

Note: Using (10.14.53) it can be easily shown that

$$\nabla^2 w = w_{,kk} = \frac{\partial^2 w}{\partial R^2} + \frac{1}{R}\frac{\partial w}{\partial R} \tag{10.14.56}$$

Then equation (10.14.47) takes the polar form

$$\frac{\partial w}{\partial t} = \nu\left[\frac{\partial^2 w}{\partial R^2} + \frac{1}{R}\frac{\partial w}{\partial R}\right] \tag{10.14.47$'$}$$

It can be verified that a solution of this partial differential equation is

$$w = \frac{A}{8\pi\nu t}\exp\left[-\frac{R^2}{4\nu t}\right] \tag{10.14.57}$$

where A is a nonnegative constant. This solution shows that w decays rapidly with time, a phenomenon called *diffusion of vorticity*.

EXAMPLE 10.14.8 Show that the rate of decrease in kinetic energy due to viscosity in a finite volume V of an incompressible fluid is given by

$$W = \mu\int_V \mathbf{w}^2\, dV - \mu\int_S \mathbf{n}\cdot(\mathbf{v}\times\mathbf{w})\, dS \tag{10.14.58}$$

where S is the boundary of V.

If S is a rigid solid surface at rest, deduce that

$$W = \mu\int_V \mathbf{w}^2\, dV = \int_V \Phi\, dV \tag{10.14.59}$$

Solution From (8.6.1), we recall that the kinetic energy of a continuum contained in a volume V is given by

$$K = \tfrac{1}{2}\int_V \rho(\mathbf{v}\cdot\mathbf{v})\, dV$$

Using (8.2.10), we obtain

$$\frac{DK}{Dt} = \frac{1}{2}\int_V \rho\frac{D}{Dt}(\mathbf{v}\cdot\mathbf{v})\, dV = \int_V \rho\mathbf{v}\cdot\frac{D\mathbf{v}}{Dt} \tag{10.14.60}$$

For an incompressible viscous fluid, expression (10.14.60) becomes, on using the Navier–Stokes equation (10.14.6),

$$\frac{DK}{Dt} = \mu \int_V \mathbf{v} \cdot \nabla^2 \mathbf{v}\, dV - \int_V \rho \mathbf{v} \cdot \nabla p\, dV + \int_V \rho \mathbf{v} \cdot \mathbf{b}\, dV \qquad (10.14.61)$$

The last two terms on the righthand side of this expression represents the rate of work done by the pressure and the body force, and the first term represents the contribution of viscosity to the rate of change of kinetic energy. Therefore, if W denotes the rate of *decrease* in kinetic energy due to viscosity, we have

$$W = -\mu \int_V \mathbf{v} \cdot \nabla^2 \mathbf{v}\, dV \qquad (10.14.62)$$

Since the fluid is incompressible, we note by using identities (3.4.27) and (3.4.24) that

$$\left.\begin{aligned} \nabla^2 \mathbf{v} &= -\operatorname{curl}\operatorname{curl}\mathbf{v} = -\operatorname{curl}\mathbf{w} \\ \operatorname{div}(\mathbf{v} \times \mathbf{w}) &= \mathbf{w}^2 - \mathbf{v} \cdot \operatorname{curl}\mathbf{w} \end{aligned}\right\} \qquad (10.14.63)$$

In view of these expressions, (10.14.62) becomes

$$W = \mu \int_V \mathbf{w}^2\, dV - \mu \int_V \operatorname{div}(\mathbf{v} \times \mathbf{w})\, dV \qquad (10.14.64)$$

This expression yields (10.14.58), on using the divergence theorem (3.6.1).

If S is a rigid solid surface at rest, then by the no-slip boundary condition we have $\mathbf{v} = \mathbf{0}$ on S. Consequently, the surface integral on the righthand side of (10.14.58) becomes identically 0, and we obtain

$$W = \mu \int_V \mathbf{w}^2\, dV \qquad (10.14.65)$$

which is the first part of (10.14.59). To obtain the other part, we recall expression (10.10.19). For an incompressible fluid, this expression becomes

$$\int_V \Phi\, dV = \mu \int_V \mathbf{w}^2\, dV + 2\mu \int_S \frac{D\mathbf{v}}{Dt} \cdot \mathbf{n}\, dS \qquad (10.14.66)$$

If S is a rigid surface at rest, the surface integral in the expression (10.14.66) vanishes and we get

$$\int_V \Phi\, dV = \mu \int_V \mathbf{w}^2 dV \qquad (10.14.67)$$

which is the second part of (10.14.59). ■

Expression (10.14.59) explicitly exhibits the remarkable relationship between the kinetic energy, the vorticity and the viscous dissipation of a fluid moving within a rigid enclosure.

EXAMPLE 10.14.9 For an incompressible viscous fluid moving under a conservative body force, show that the circulation I_c round a circuit c moving with the fluid is *not* constant in general. Deduce that I_c is constant if and only if curl $\mathbf{w} = \nabla\xi$, for some ξ.

Solution For the given flow, the Navier-Stokes equation is given by (10.14.6) with $\mathbf{b} = -\nabla\chi$.

From (6.6.4), we recall that the rate of change of circulation round a material circuit is given by

$$\frac{DI_c}{Dt} = \oint_c \frac{D\mathbf{v}}{Dt} \cdot d\mathbf{x} \tag{10.14.68}$$

Substituting for $D\mathbf{v}/Dt$ from (10.14.6) with $\mathbf{b} = -\nabla\chi$ in (10.14.68), we get

$$\frac{DI_c}{Dt} = \nu \oint_c \nabla^2\mathbf{v} \cdot d\mathbf{x} \tag{10.14.69}$$

Since the fluid is viscous, $\nu \neq 0$ and consequently $DI_c/Dt \neq 0$ when $\nabla^2\mathbf{v} \neq \mathbf{0}$. Thus, I_c is *not* constant in general.

Since $\nabla^2\mathbf{v} = -\text{curl}\,\mathbf{w}$ for an incompressible fluid, expression (10.14.69) can be rewritten as

$$\frac{DI_c}{Dt} = -\nu \oint_c \text{curl}\,\mathbf{w} \cdot d\mathbf{x} \tag{10.14.70}$$

Evidently, I_c = constant if and only if curl $\mathbf{w} = \nabla\xi$ for some scalar ξ. (Thus, Kelvin's circulation theorem is not generally valid for viscous fluids.) ■

10.15 SOME VISCOUS FLOW PROBLEMS

As remarked earlier, the presence of the nonlinear term in the Navier-Stokes equation makes the exact solution of the equation very difficult except for a very few cases. In this section we consider some simple examples of flows of an incompressible viscous fluid for which the Navier-Stokes equation admits exact solutions. The linearized Navier-Stokes equation is employed in the last example.

10.15.1 STEADY LAMINAR FLOW BETWEEN PARALLEL PLATES

First, we consider an incompressible viscous fluid bounded between two rigid infinite flat plates distance h apart. We suppose that the fluid flows steadily in a fixed direction parallel to the plates. Such a flow is called a

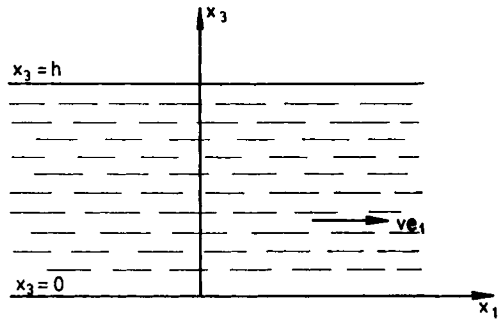

Figure 10.15. Steady laminar flow.

steady laminar flow. Assuming that there is no body force, the problem is to find the velocity field in the fluid.

We take the coordinate axes such that the x_1 axis is along the direction of flow and the x_3 axis is normal to the plates, with $x_3 = 0$ and $x_3 = h$ as the planes containing the plates; see Figure 10.15. It is convenient to refer to the plate in the plane $x_3 = 0$ as the lower plate and that in the plane $x_3 = h$ as the upper plate. Then the velocity is of the form $\mathbf{v} = v\mathbf{e}_1$, where $v = v(x_1, x_2, x_3)$, and the Navier-Stokes equation (10.14.6) gives the following equations for v and $p_{,i}$:

$$\nu\nabla^2 v - \frac{1}{\rho}p_{,1} = \frac{Dv}{Dt} \tag{10.15.1}$$

$$p_{,2} = 0, \tag{10.15.2a}$$

$$p_{,3} = 0 \tag{10.15.2b}$$

The equation of continuity (10.12.5) becomes

$$v_{,1} = 0 \tag{10.15.3}$$

If the fluid extends to infinity in the x_2 direction, then the variation of v in the x_2 direction may be neglected. Consequently, it follows that v varies only with x_3, and

$$\frac{Dv}{Dt} \equiv \frac{\partial v}{\partial t} + (\mathbf{v} \cdot \nabla)v = 0$$

Thus, under the assumptions made, inertia plays no role in the flow. From (10.15.2), it follows that $p = p(x_1)$, and equation (10.15.1) reduces to the ordinary linear differential equation

$$\mu\frac{d^2 v}{dx_3^2} = \frac{dp}{dx_1} \tag{10.15.4}$$

Thus the flow is induced entirely by the pressure gradient, $dp/dx_1 = p'(x_1)$ in the x_1 direction.

Integrating (10.15.4) successively twice, we obtain the following explicit expression for v:

$$\mu v = \tfrac{1}{2}x_3^2 p'(x_1) + x_2 f_1(x_1) + f_2(x_1) \tag{10.15.5}$$

where f_1 and f_2 are arbitrary functions of x_1. If we suppose that the lower plate ($x_3 = 0$) is stationary, then, by the no-slip boundary condition, we have $v = 0$ for $x_3 = 0$ and (10.15.5) gives $f_2(x_1) = 0$. If we compute $v_{,1}$ from (10.15.5) and use (10.15.3) we get

$$x_3 p''(x_1) = -2f_1'(x_1) \tag{10.15.6}$$

Since this equation holds for all x_1 and x_3 in $0 \le x_3 \le h$, we must have $p''(x_1) = 0$ and $f_1'(x_1) = 0$ so that

$$f_1(x_1) = C_1 \tag{10.15.7a}$$

$$p(x_1) = C_2 - Gx_1 \tag{10.15.7b}$$

where C_1, C_2 and G are arbitrary constants. All these results allow us to rewrite (10.15.5) in the form

$$\mu v = C_1 x_3 - \tfrac{1}{2}Gx_3^2 \tag{10.15.8}$$

This shows that the flow occurs only when at least one of the constants C_1 and G is nonzero.

Since $v_1 = v(x_3)$, $v_2 = v_3 = 0$, we find from the material law (10.12.8) that

$$\tau_{31} = \mu \frac{dv}{dx_3} \tag{10.15.9}$$

which, by using (10.15.8), reduces to

$$\tau_{31} = C_1 - Gx_3 \tag{10.15.10}$$

When the constants C_1 and G are known, (10.15.8) gives the velocity distribution and (10.15.10) gives the shear stress on plane elements parallel to the plates. Evidently, (10.15.8) represents the parabolic profile of the flow between the plates. Expression (10.15.10) shows that the shear stress τ_{31} varies linearly with height (increasing x_3 direction) with

$$C_1 = [\tau_{31}]_{x_3=0} \tag{10.15.11}$$

as the shear stress at the lower plate. Thus the constant C_1 involved in the solution (10.15.8) has a definite physical interpretation. It can be seen from (10.15.7b) that the other constant G in (10.15.8) also has physical significance; in fact,

$$G = -p'(x_1) \tag{10.15.12}$$

Thus, G represents the negative of the pressure gradient in the direction of the flow. Since G is a constant, the pressure gradient must be a constant for the flow. Further, since the flow takes place in the x_1 direction, the pressure does not increase in that direction; that is, $G \geq 0$.

Two particular cases of this problem are of interest.

Case i: Plane Couette Flow Suppose the flow is generated by the movement of the upper plate with a constant speed v_0 in the x_1 direction. Such a flow is called *plane Couette flow*. Then by the no-slip boundary condition, we have $v = v_0$ for $x_3 = h$, and (10.15.8) gives

$$C_1 = \frac{1}{h}\mu v_0 + \frac{1}{2}Gh \tag{10.15.13}$$

Putting this expression back into (10.15.8), we obtain

$$\mu v = \frac{1}{2}Gx_3(h - x_3) + \frac{\mu v_0}{h}x_3 \tag{10.15.14}$$

This determines the velocity distribution completely, provided the constant pressure-gradient $G = -p'(x_1)$ is known beforehand. For example, if this pressure-gradient is 0, then

$$v = \frac{v_0}{h}x_3 \tag{10.15.15}$$

This represents a *linear* distribution of velocity depicted in Figure 10.16a. In the figure, a few velocity vectors are indicated to make the flow easier to visualize. Such a diagram is called a *velocity profile*. The velocity profile for $p'(x_1) < 0$ is given in Figure 10.16b.

Expression (10.15.14) can be used to find the *mass flow rate per unit of width*; that is the mass of fluid that passes the plane x_1 = constant per unit of distance in the x_2 direction and per unit of time. This is given by

$$M = \int_0^h \rho v \, dx_3 = \frac{G}{2\nu}\int_0^h x_3(h - x_3)\, dx_3 + \frac{\rho v_0}{h}\int_0^h x_3 \, dx_3 = \frac{Gh^3}{12\nu} + \frac{\rho v_0 h}{2} \tag{10.15.16}$$

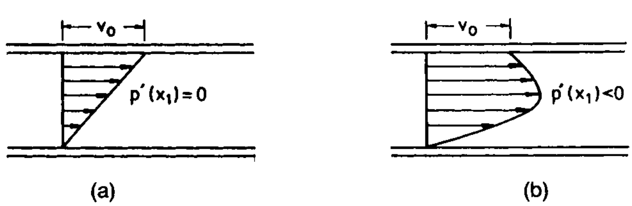

Figure 10.16. Velocity profiles for plane Coutte flow.

In the case of $p'(x_1) = 0$, we get

$$M = \frac{\rho h}{2} v_0 \tag{10.15.17}$$

Thus, in this case, M is independent of viscosity.

From (10.15.10) and (10.15.13), we get

$$\tau_{31} = \frac{\mu}{h} v_0 + G\left(\frac{h}{2} - x_3\right) \tag{10.15.18}$$

This expression gives the shear stress on a fluid element parallel to the plates (for Couette flow). The shear stress exerted on the plates is often called the *skin friction*. The skin friction on the lower plate is given by

$$[\tau_{31}]_{x_3=0} = \frac{\mu}{h} v_0 + \frac{Gh}{2} \tag{10.15.19}$$

and that on the upper plate is given by

$$[\tau_{31}]_{x_3=h} = \frac{\mu}{h} v_0 - \frac{Gh}{2} \tag{10.15.20}$$

When $p'(x_1) = 0$, (10.15.18) gives

$$\tau_{31} = \frac{\mu}{h} v_0 \tag{10.15.21}$$

Evidently, in this case, all fluid particles, including those in contact with the plates, experience the same shear stress. This stress is directly proportional to v_0 and inversely proportional to h and acts in the direction of the flow.

Case ii: Plane Poiseullie Flow Suppose the upper plate is also held stationary and the flow is generated only by the constant pressure gradient $p'(x_1) < 0$. Such a flow is called a *plane Poiseullie flow*. Then $v_0 = 0$ and $G > 0$, and (10.15.14) yields the following expression for velocity distribution:

$$\mu v = \tfrac{1}{2} G x_3 (h - x_3) \tag{10.15.22}$$

Evidently, the velocity distribution is now symmetrical about the midplane $x_3 = h/2$ and the velocity is maximum on this plane, with

$$v_{\max} = [v]_{x_3=h/2} = \frac{G}{8\mu} h^2 \tag{10.15.23}$$

The velocity profile is depicted in Figure 10.17.

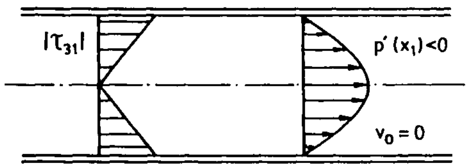

Figure 10.17. Plane Poiseullie flow.

Substituting for v from (10.15.22) in (10.15.9) we get the following expression for the shear stress on a fluid element parallel to the plates:

$$\tau_{31} = G\left(\frac{h}{2} - x_3\right) \tag{10.15.24}$$

We observe that $\tau_{31} = 0$ on the midplane $x_3 = h/2$ and that $\tau_{31} > 0$ for $0 \leq x_3 < h/2$ and $\tau_{31} < 0$ for $h/2 < x_3 \leq h$. That is, plane elements laying below the midplane experience a shear stress along the direction of the flow while those above experience a shear stress in the opposite direction, with the elements laying on the midplane remaining stress free.

From (10.15.24), we find that the skin friction on the lower plate is

$$[\tau_{31}]_{x_3=0} = \frac{Gh}{2} \tag{10.15.25}$$

and that on the upper plate is

$$[\tau_{31}]_{x_3=h} = -\frac{Gh}{2} \tag{10.15.26}$$

The magnitudes of the skin frictions on both the plates are equal to $Gh/2$. But, while the lower plate experiences a friction along the direction of the flow, the upper plate experiences a (backward) *drag*.

Expression (10.15.22) can be employed to find the mass flow rate per unit of width. This is given by

$$M = \int_0^h \rho v \, dx_3 = \frac{G}{2\nu} \int_0^h x_3(h - x_3)\, dx_3 = \frac{Gh^3}{12\nu} \tag{10.15.27}$$

Evidently, for given G and h, M varies inversely as ν. Expression (10.15.27) can be used to measure ν.

10.15.2. STEADY FLOW IN A STRAIGHT CONDUIT

We now consider the steady flow of an incompressible viscous fluid through a straight conduit of uniform cross section under a constant pressure gradient along the direction of flow. Such a flow is called the *Hagen-Poiseullie flow*. If we choose the axes such that the x_3 axis is along the direction of the conduit so that a right cross section of the conduit is bounded by a simple closed curve C parallel to the x_1x_2 plane, then the velocity at any point of the fluid is of the form $\mathbf{v} = v\mathbf{e}_3$, where $v = v(x_1, x_2, x_3)$, and the equation of continuity (10.12.5) yields $v_{,3} = 0$. Consequently, $v = v(x_1, x_2)$ and $(\mathbf{v} \cdot \nabla)\mathbf{v} = 0$. Since the flow is steady, we get $(D\mathbf{v}/Dt) = 0$. Under these conditions the Navier–Stokes equation (10.14.6) yields the following three component equations in the absence of body force:

$$p_{,1} = 0 \tag{10.15.28a}$$

$$p_{,2} = 0 \tag{10.15.28b}$$

$$p_{,3} = \mu(v_{,11} + v_{,22}) \tag{10.15.28c}$$

The first two of these equations show that $p = p(x_3)$. It is assumed that the pressure-gradient is a constant; hence,

$$\frac{dp}{dx_3} = -G \tag{10.15.29}$$

where G is a constant. Note that, for the fluid to flow in the x_3 direction, G has to be positive.

Equations (10.15.28c) and (10.15.29) yield the following governing equation for v:

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} = -\frac{G}{\mu} \tag{10.15.30}$$

If the conduit is stationary, the no-slip boundary condition yields

$$v = 0 \tag{10.15.31}$$

on C; see Figure 10.18.

We note that (10.15.30) is a two-dimensional Poisson equation. When solved under the boundary condition (10.15.31), this equation determines v uniquely. Once v is determined, the stresses developed in the fluid can be computed by use of the material law (10.12.8).

We now consider two particular cases for further analysis.

Case i: Circular Cross Section Suppose the conduit is of circular cross section. Then C is a circle whose equation can be taken as

$$x_1^2 + x_2^2 = a^2 \tag{10.15.32}$$

where a is the radius of the conduit.

Since $v = 0$ on C by (10.15.31), the equation (10.15.32) of C suggests that v is of the form

$$v = \alpha(x_1^2 + x_2^2 - a^2) \tag{10.15.33}$$

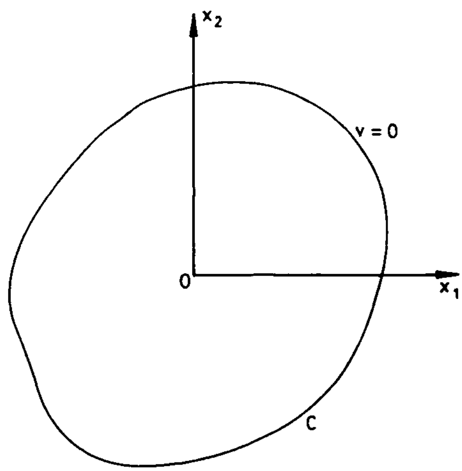

Figure 10.18. Cross section of a conduit.

where α is a constant. Substituting for v from (10.15.33) into equation (10.15.30), we find that $\alpha = -G/4\mu$. Thus,

$$v = \frac{G}{4\mu}(a^2 - x_1^2 - x_2^2) \tag{10.15.34}$$

is the expression for v that satisfies the governing equation (10.15.30) as well as the boundary condition (10.15.31). We note that the velocity distribution is in the form of a paraboloid of revolution; the velocity profile in a section parallel to the length of the pipe is depicted in Figure 10.19. We find that the velocity is maximum on the axis of the conduit, namely the x_3 axis, and

$$v_{\max} = \frac{G}{4\mu}a^2 \tag{10.15.35}$$

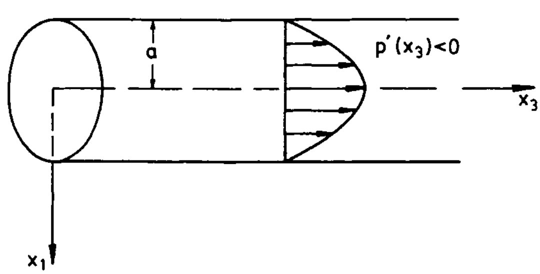

Figure 10.19. Velocity profile for case (i).

Expression (10.15.34) can be used to find the rate of mass of fluid that passes a cross section (x_3 = constant); this rate is given by

$$M = \int_A \rho v \, dA = 2\pi\rho \int_0^a vR \, dR \tag{10.15.36}$$

where A is the area of cross section, and $R = (x_1^2 + x_2^2)^{1/2}$. Using (10.15.34) in (10.15.36), we get

$$M = \frac{\rho \pi G a^4}{8\mu} \tag{10.15.37}$$

Consequently, the rate of volume of fluid that passes a cross section (x_3 = constant) is

$$\frac{M}{\rho} = \frac{\pi G a^4}{8\mu} \tag{10.15.38}$$

This is known as *Poiseullie's formula*. This formula is generally used to measure the viscosity of a fluid.

If we consider a cylindrical fluid surface element dS coaxial with the conduit and having radius R, then the components of the exterior unit normal to this element are

$$n_1 = \frac{x_1}{R}, \qquad n_2 = \frac{x_2}{R}, \qquad n_3 = 0 \tag{10.15.39}$$

The components of the stress vector on dS can be computed by use of Cauchy's law (7.4.9), the material law (10.12.8) and expressions (10.15.34) and (10.15.39). The stress components s_i thus obtained are

$$s_1 = -p\frac{x_1}{R} \tag{10.15.40a}$$

$$s_2 = -p\frac{x_2}{R} \tag{10.15.40b}$$

$$s_3 = -\frac{G}{2}R \tag{10.15.40c}$$

Together with (10.15.39), expressions (10.15.40ab) yield the normal stress on dS as $\sigma = \mathbf{s} \cdot \mathbf{n} = -p$, as expected and expression (10.16.40c) gives the shear stress on dS along the conduit. Evidently, this shear stress acts opposite to the direction of flow. It is to be noted that all the three stress components s_i are independent of viscosity. If dS is taken on the boundary surface, we get $-(G/2)a$ as the skin friction along the direction of flow.

Case ii: Elliptic Cross Section Suppose the conduit is of elliptic cross section. Then C is an ellipse whose equation can be taken as

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \tag{10.15.41}$$

In view of the boundary condition (10.15.31), v can be assumed in the form

$$v = \beta\left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1\right) \tag{10.15.42}$$

where β is a constant. Substituting for v from (10.15.42) into equation (10.15.30), we find that

$$\beta = -\frac{a^2b^2G}{2\mu(a^2 + b^2)} \tag{10.15.43}$$

Thus,

$$v = \frac{a^2b^2G}{2\mu(a^2 + b^2)}\left[1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2}\right] \tag{10.15.44}$$

is the expression for v that satisfies governing equation (10.15.30) as well as boundary condition (10.15.31). We note that the velocity distribution is paraboloidal in nature. Also, the velocity is maximum on the axis of the conduit, and $v_{\max} = -\beta$.

Using (10.15.44), we find that the rate of mass of fluid that passes a cross section is given by

$$M = \int_A \rho v\, dA = \frac{\rho a^2b^2G}{2\mu(a^2 + b^2)} \int_A \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dA \tag{10.15.45}$$

where A is the area of cross section. Setting

$$x_1 = a\xi \cos\theta, \qquad x_2 = b\xi \sin\theta \tag{10.15.46}$$

and noting that ξ varies from 0 to 1 and θ varies from 0 to 2π as the point (x_1, x_2) varies over A, we find from (10.15.45) that

$$M = \frac{\rho a^2b^2G}{2\mu(a^2 + b^2)} \int_0^{2\pi} \int_0^1 (\xi^2 - 1)\xi\, d\xi\, d\theta = \frac{\rho\pi a^3b^3G}{4\mu(a^2 + b^2)} \tag{10.15.47}$$

This is a generalization of Poiseullie's formula (10.15.38). If we set $b = a$ in (10.15.47), we recover (10.15.38).

Let us consider a fluid element dS on an elliptic surface coaxial with and similar to the boundary surface of the conduit. On this surface element,

$$\phi(x_1, x_2) \equiv \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = \text{constant} \tag{10.15.48}$$

The exterior unit normal to dS is therefore $\mathbf{n} = (\nabla\phi/|\nabla\phi|)$. From (10.15.48) and (10.15.46), we find that

$$\nabla\phi = \frac{2x_1}{a^2}\mathbf{e}_1 + \frac{2x_2}{b^2}\mathbf{e}_2 = \frac{2\xi}{ab}(b\cos\theta\,\mathbf{e}_1 + a\sin\theta\,\mathbf{e}_2)$$

$$|\nabla\phi| = \frac{2\xi}{ab}(b^2\cos^2\theta + a^2\sin^2\theta)^{1/2}$$

Hence the components of $\mathbf{n}$ are

$$n_1 = \frac{b}{N}\cos\theta, \qquad n_2 = \frac{a}{N}\sin\theta, \qquad n_3 = 0 \tag{10.15.49}$$

where

$$N = (b^2\cos^2\theta + a^2\sin^2\theta)^{1/2} \tag{10.15.50}$$

The components of the stress vector on dS can now be computed by using Cauchy's law (7.4.9), the material law (10.12.8) and expressions (10.15.44), (10.15.46) and (10.15.49). We find that

$$s_1 = \tau_{11}n_1 + \tau_{12}n_2 = -pn_1 = -p\frac{b}{N}\cos\theta \tag{10.15.51a}$$

$$s_2 = \tau_{21}n_1 + \tau_{22}n_2 = -pn_2 = -p\frac{a}{N}\sin\theta \tag{10.15.51b}$$

$$s_3 = \tau_{31}n_1 + \tau_{32}n_2 = \mu(v_{,1}n_1 + v_{,2}n_2) = -\frac{Gab\xi N}{a^2 + b^2} \tag{10.15.51c}$$

Together with (10.15.49), expressions (10.15.51a, b) yield the normal stress on dS as $\sigma = \mathbf{s}\cdot\mathbf{n} = -p$, as expected. Expression (10.15.51c) gives the shear stress on dS along the conduit. Evidently, this stress acts opposite to the direction of flow. It may be observed that all the stress components s_i are independent of the viscosity.

If dS is chosen on the boundary surface, for which $\xi = 1$, we get $s_3 = \tau$, where

$$\tau = -\frac{Gab}{a^2 + b^2}N \tag{10.15.52}$$

This gives the skin friction at a point on the conduit and this varies from one point to another. By using (10.15.50), it can be shown that the skin friction is maximum at the endpoints of the minor axis and minimum at the endpoints of the major axis of the boundary curve (10.15.41). If $a > b$, the maximum and minimum values of $|\tau|$ are

$$\max|\tau| = \frac{Ga^2b}{a^2 + b^2}; \qquad \min|\tau| = \frac{Gab^2}{a^2 + b^2} \tag{10.15.53}$$

When $b = a$, we have $N = a$ and $\xi = R/a$ and expressions (10.15.51) reduce to (10.15.40). Also, then, τ becomes equal to $-(G/2)a$ at all points of the boundary.

10.15.3 STEADY FLOW BETWEEN TWO COAXIAL ROTATING CYLINDERS

Here we consider the steady flow of an incompressible viscous fluid between two infinitely long coaxial circular cylinders due to the rotation of the cylinders with constant but different angular velocities about the common axis. Such a flow is called a *Couette flow*. Assuming that there is no body force, the problem is to find the velocity field.

We choose the axes such that the x_3 axis coincides with the axis of the cylinders and the x_1x_2 plane lies on a section of the tube formed by the cylindrical surfaces. Since the flow is caused by the rotation of the cylinders about the axis and the flow is steady, we assume that the fluid particles move in circular orbits centered on the axis and that the magnitude of velocity depends only on the distance of the particle from the axis. Accordingly, the velocity at any point of the fluid is taken in the form

$$\mathbf{v} = v(R)\mathbf{e}_\theta \tag{10.15.54}$$

where $R = (x_1^2 + x_2^2)^{1/2}$ is the distance of the point from the axis and $\mathbf{e}_\theta$ is the unit vector in the transverse direction (see Figure 10.20). If we denote the unit vector directed along the radial direction perpendicular to the axis by $\mathbf{e}_R$, then $\mathbf{e}_\theta = \mathbf{e}_3 \times \mathbf{e}_R$, so that

$$[\mathbf{e}_\theta]_i = \varepsilon_{ijk}[\mathbf{e}_3]_j[\mathbf{e}_R]_k = \varepsilon_{i3k}\frac{1}{R}x_k \tag{10.15.55a}$$

for $i, k = 1, 2$;

$$[\mathbf{e}_\theta]_3 = 0 \tag{10.15.55b}$$

Consequently, (10.15.54) gives

$$v_i = \varepsilon_{i3k}v(R)\frac{1}{R}x_k = \varepsilon_{i3k}\psi(R)x_k \tag{10.15.56a}$$

for $i, k = 1, 2$;

$$v_3 = 0 \tag{10.15.56b}$$

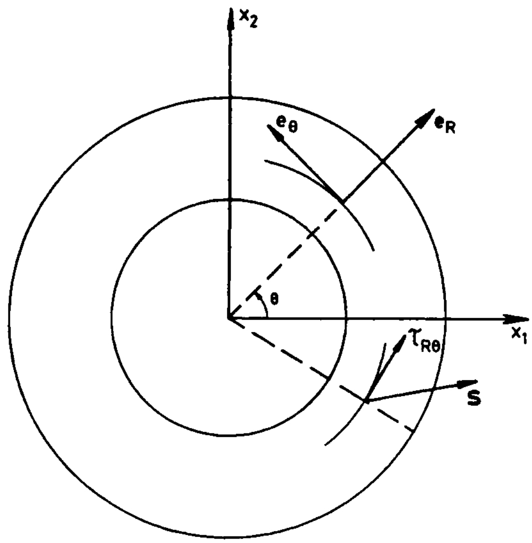

Figure 10.20. Cross section of a circular tube.

From (10.15.56a), we obtain the following relations valid for $i, j, k = 1, 2$:

$$v_{i,j} = \varepsilon_{i3j}\psi(R) + \varepsilon_{i3k}x_k x_j \frac{1}{R}\frac{d\psi}{dR} \tag{10.15.57a}$$

$$v_{i,jj} = \varepsilon_{i3k}\left\{\frac{3}{R}\frac{d\psi}{dR} + \frac{d^2\psi}{dR^2}\right\}x_k \tag{10.15.57b}$$

$$v_{i,j}v_j = \varepsilon_{i3j}\varepsilon_{j3k}x_k\,\psi^2 \tag{10.15.57c}$$

[The reader may note that explicit forms of (10.15.56a) and (10.15.57a) are contained in (10.14.51) and (10.14.52ab).]

From (10.15.57a) and (10.15.56b) we find that $\operatorname{div} \mathbf{v} = 0$. Thus, the equation of continuity is satisfied. Also, (10.15.55ab), (10.15.56b) and (10.15.57bc) give

$$\begin{aligned}(\nabla^2\mathbf{v})\cdot\mathbf{e}_\theta &= v_{i,jj}\left(\varepsilon_{i3k}\frac{1}{R}x_k\right) = \left[3\frac{d\psi}{dR} + R\frac{d^2\psi}{dR^2}\right]\\ &= \frac{d}{dR}\left[\frac{1}{R}\frac{d}{dR}(R^2\psi)\right] = \frac{d}{dR}\left[\frac{1}{R}\frac{d}{dR}(Rv)\right]\end{aligned} \tag{10.15.58a}$$

$$\{(\mathbf{v}\cdot\nabla)\mathbf{v}\}\cdot\mathbf{e}_\theta = v_{i,j}v_j\left(\varepsilon_{i3k}\frac{1}{R}x_k\right) = 0 \tag{10.15.58b}$$

Because of the axisymmetric and steady nature of the flow, we suppose that the pressure is a function of R only so that

$$\nabla p = \frac{dp}{dR}\mathbf{e}_R$$

from which it follows that

$$(\nabla p)\cdot\mathbf{e}_\theta = 0 \tag{10.15.59}$$

Taking the scalar product with $\mathbf{e}_\theta$ on both sides of the Navier–Stokes equation (10.14.6) with body force equal to 0 and using expressions (10.15.58) and (10.15.59) in the resulting equation, we obtain the following ordinary linear differential equation for the function $v(R)$:

$$\frac{d}{dR}\left[\frac{1}{R}\frac{d}{dR}(Rv)\right] = 0 \tag{10.15.60}$$

Integration of this equation yields

$$v(R) = AR + \frac{B}{R} \tag{10.15.61}$$

where A and B are arbitrary constants.

We verify that the component of the Navier–Stokes equation (10.14.6) along $\mathbf{e}_R$ yields, on using (10.15.57bc),

$$\frac{dp}{dR} = -\frac{\rho}{R}v^2 \tag{10.15.62}$$

and that the component along $\mathbf{e}_3$ is identically satisfied.

Thus, under the assumptions made, (10.15.61) gives the velocity field and (10.15.62) gives the pressure-gradient field when the constants A and B in (10.15.61) are determined by the boundary conditions.

Let a and b be the radii and ω_1 and ω_2 be the constant angular speeds of the inner and outer cylinders, respectively. Since the cylinders rotate rigidly about the x_3 axis, the velocities at points on these cylinders can be determined by (6.2.19). Thus, we find that $\omega_1 a\mathbf{e}_\theta$ and $\omega_2 b\mathbf{e}_\theta$ are the velocities at a point on the inner and outer cylinders, respectively. The no-slip boundary condition then yields $v(R) = \omega_1 a$ for $R = a$ and $v(R) = \omega_1 b$ for $R = b$. Using (10.15.61), these conditions yield the following two equations for the determination of the two constants A and B:

$$\begin{aligned} Aa + \frac{B}{a} &= \omega_1 a \\ Ab + \frac{B}{b} &= \omega_2 b \end{aligned} \tag{10.15.63}$$

Solving these equations we obtain

$$A = \frac{\omega_2 b^2 - \omega_1 a^2}{b^2 - a^2}, \qquad B = \frac{(\omega_1 - \omega_2)a^2 b^2}{b^2 - a^2} \tag{10.15.64}$$

Substituting these into (10.15.61), we obtain

$$v(R) = \frac{1}{b^2 - a^2}\left[(\omega_2 b^2 - \omega_1 a^2)R + (\omega_1 - \omega_2)\frac{a^2 b^2}{R}\right] \tag{10.15.65}$$

This determines the velocity field in the fluid. We immediately note that, if both the cylinders rotate with the same angular velocity $\omega\mathbf{e}_3$, then $v(R) = \omega R$. Hence, in this case, the fluid rotates like a rigid body along with boundary surfaces.

Substituting (10.15.65) in (10.15.62) we obtain the pressure-gradient at a point of the fluid.

Since the motion is essentially rotational, it is of interest to compute the vorticity at a point. By use of (10.15.56ab), we find that

$$w_i = [\text{curl}\,\mathbf{v}]_i = \varepsilon_{ijk} v_{k,j} = \begin{cases} 0, & \text{for } i = 1, 2 \\ \dfrac{1}{R}\dfrac{d}{dR}(Rv), & \text{for } i = 3 \end{cases}$$

Consequently, with the use of (10.15.65), we obtain

$$\mathbf{w} = \text{curl}\,\mathbf{v} = \frac{2(\omega_2 b^2 - \omega_1 a^2)}{b^2 - a^2}\mathbf{e}_3 \tag{10.15.66}$$

Evidently, the vorticity is constant and is directed along the axis. Further, the flow is irrotational if and only if the angular speeds of the cylinders obey the relation

$$\frac{\omega_2}{\omega_1} = \frac{a^2}{b^2} \tag{10.15.67}$$

The stresses produced at a point of the fluid can be computed by use of (10.15.56) in Stokes's law (10.12.8). Thus, we find

$$\tau_{ij} = -p\delta_{ij} + \mu(\varepsilon_{i3k}x_j + \varepsilon_{j3k}x_i)\frac{x_k}{R}\frac{d\psi}{dR}$$

for $i, j = 1, 2$; (10.15.68)

$$\tau_{31} = \tau_{32} = 0, \qquad \tau_{33} = -p$$

Cauchy's law (7.4.6) can be employed to find the stress vector $\mathbf{s}$ on a cylindrical surface R = constant in the fluid. The tangential component of this stress vector is given by

$$\mathbf{s} \cdot \mathbf{e}_\theta = s_i[\mathbf{e}_\theta]_i = \left(\tau_{ij}\frac{x_j}{R}\right)\left(\varepsilon_{i3k}\frac{1}{R}x_k\right) \tag{10.15.69}$$

Substituting for τ_{ij} from (10.15.68) in (10.15.69), we arrive at the following simple formula for the shear stress $\tau_{R\theta} \equiv \mathbf{s} \cdot \mathbf{e}_\theta$:

$$\tau_{R\theta} = \mu R \frac{d\psi}{dR} = \mu R \frac{d}{dR}\left[\frac{v}{R}\right] \tag{10.15.70}$$

By use of (10.15.65) in (10.15.70), we get

$$\tau_{R\theta} = \frac{2\mu a^2 b^2}{b^2 - a^2}\frac{\omega_2 - \omega_1}{R^2} \tag{10.15.71}$$

Evidently, the shear stress $\tau_{R\theta}$ is tensile or compressive accordingly as the outer cylinder rotates faster or slower than the inner cylinder. Also, the magnitude of this stress is maximum on the inner cylinder and minimum on the outer cylinder. The stress vanishes when both the cylinders rotate with the same speed.

Expression (10.15.71) can also be employed to compute the couple exerted by the fluid on the cylinders due to viscous drag. For example, the magnitude M of the moment of the couple per unit length in the axial direction on the inner cylinder is given by

$$M = \int_0^{2\pi} [\tau_{R\theta}R^2\, d\theta]_{R=a} = \frac{4\mu\pi a^2 b^2}{b^2 - a^2}|\omega_2 - \omega_1| \tag{10.15.72}$$

This result is used for measuring μ by a rotation viscometer.

Two interesting limiting cases are worth mentioning. For $a = 0$, the problem reduces to that of a flow inside a cylinder due to the rotation of the cylinder. In this case, we find from (10.15.65) that the motion is simply a rigid-body motion; the fluid rotates like a rigid body along with the cylinder. For $b \to \infty$ with a remaining finite and greater than 0, the problem reduces to that of a flow of an infinite fluid outside a cylinder due to the rotation of the cylinder. In this case, condition (10.15.67) is trivially satisfied and the flow is irrotational.

10.15.4 UNSTEADY FLOW NEAR A MOVING PLANE BOUNDARY

Here, we consider the unsteady motion of an incompressible viscous fluid occupying the half-space $x_3 > 0$ with a plane rigid plate at $x_3 = 0$ as its boundary. Suppose the fluid and the plate are initially at rest and that the flow is generated by the movement of the plate in a direction parallel to its own plane, say the x_1 direction; see Figure 10.21.

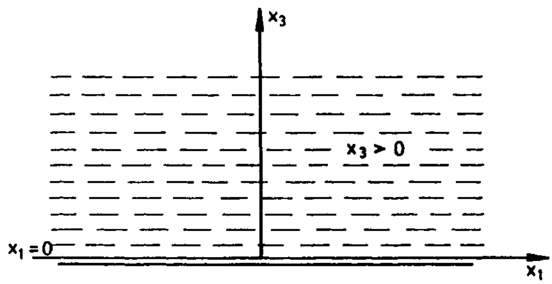

Figure 10.21. Unsteady flow near a plate.

When the plate moves in the x_1 direction it is natural to expect that the fluid also moves in the same direction so that the velocity of the fluid at any point is of the form $\mathbf{v} = v\mathbf{e}_1$. In the absence of body force, the Navier–Stokes equation (10.14.6) yields the following three component equations

$$\frac{Dv}{Dt} = -\frac{1}{\rho} p_{,1} + \nu \nabla^2 v \tag{10.15.73}$$

$$p_{,2} = 0 \qquad p_{,3} = 0 \tag{10.15.74}$$

The equation of continuity (10.12.5) becomes

$$\frac{\partial v}{\partial x_1} = 0 \tag{10.15.75}$$

Since the fluid extends to infinity in the x_2 direction, the variation of v in the x_2 direction may be neglected. Then, in view of (10.15.75), v varies only with x_3 and t. Consequently,

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + (\mathbf{v} \cdot \nabla)v = \frac{\partial v}{\partial t} \tag{10.15.76}$$

and equation (10.15.73) reduces to

$$\nu \frac{\partial^2 v}{\partial x_3^2} - \frac{\partial v}{\partial t} = \frac{1}{\rho} \frac{\partial p}{\partial x_1} \tag{10.15.77}$$

Equations (10.15.74) imply that p is independent of x_2 and x_3. Since v does not depend on x_1 by (10.15.75), the lefthand side of (10.15.77) is a function of x_3 and t, while the righthand side is a function of x_1 and t. Therefore, each side of (10.15.77) must be a function of t alone; that is,

$$\nu \frac{\partial^2 v}{\partial x_3^2} - \frac{\partial v}{\partial t} = f(t) \tag{10.15.78}$$

$$\frac{1}{\rho} \frac{\partial p}{\partial x_1} = f(t) \tag{10.15.79}$$

where $f(t)$ is an arbitrary function of t. It follows from (10.15.79) that

$$p = \rho x_1 f(t) + f_1(t) \tag{10.15.80}$$

where $f_1(t)$ is again an arbitrary function of t. Since the fluid extends to infinity in the x_1 direction, $p \to \infty$ as $x_1 \to \infty$ unless $f(t) \equiv 0$. But the infinite pressure is physically unrealistic. Therefore, we assume $f(t) = 0$ so that (10.15.78) and (10.15.80) become

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial x_3^2} \tag{10.15.81}$$

$$p = f_1(t) \tag{10.15.82}$$

Evidently, the velocity and pressure fields are now uncoupled; v can be determined independent of p. If desired, p can be obtained from a boundary condition prescribed at $x_3 = 0$.

Equation (10.15.81) is usually called the *diffusion equation*. To determine v we have to solve this equation under appropriate initial and boundary conditions. Since the fluid is assumed to be initially at rest, the initial condition is

$$v = 0 \tag{10.15.83}$$

in $x_3 > 0$ for $t = 0$.

As the plate is assumed to move in the x_1 direction, the no-slip boundary condition requires

$$v = V(t) \tag{10.15.84}$$

for $x_3 = 0$, $t > 0$, where $V(t)$ is the speed of the plate. This is a time-dependent boundary condition.

Since the flow is generated by the motion of the plate (at $x_3 = 0$), we may assume that the fluid particles far way from the plate remain unaffected by the movement of the plate. This leads to the condition

$$v \to 0 \tag{10.15.85}$$

as $x_3 \to \infty$ for $t > 0$. This is often called a *regularity condition* (or *boundary condition* at infinity).

We consider two particular cases of interest.

Case i: Oscillating Plate Suppose the plate oscillates with real constant amplitude V in its own plane with a given frequency ω so that

$$V(t) = V \cos \omega t \tag{10.15.86}$$

$t > 0$, and we are interested in the velocity field *strictly after* the flow is fully developed. Then the initial condition (10.15.83) is not required in our analysis. We now seek a solution of (10.15.81) in the form

$$v(x_3, t) = \mathrm{Re}[f(x_3)e^{i\omega t}] \tag{10.15.87}$$

where Re stands for the real part. Substituting this solution into (10.15.81) yields the following ordinary differential equation for $f(x_3)$:

$$\frac{d^2f}{dx_3^2} - \left(\frac{i\omega}{\nu}\right)f = 0 \tag{10.15.88}$$

The general solution of this equation subject to the regularity condition (10.15.85) is

$$f(x_3) = A\exp[-(1+i)\delta]x_3 \tag{10.15.89}$$

where A is an arbitrary constant and $\delta = \sqrt{\omega/2\nu} > 0$. In view of (10.15.84), (10.15.86) and (10.15.87), we find that $A = V$ and hence the final form of the solution for $v(x_3, t)$ is

$$v(x_3, t) = \text{Re}\{V\exp[-(1+i)\,\delta x_3 + i\omega t]\} = Ve^{-\delta x_3}\cos(\omega t - \delta x_3) \tag{10.15.90}$$

This shows that the fluid particles oscillate with the plate with the same frequency as that of the plate and that their amplitude, $V\exp(-\delta x_3)$, decreases exponentially with distance from the plate. In other words, in the fluid, shear waves that spread out from the oscillating plate propagate with exponentially decaying amplitude so that fluid oscillations are essentially confined to a layer adjacent to the plate. This layer is called the *Stokes's boundary layer* and the problem is called the *Stokes's problem*. The thickness of the layer is on the order $\delta^{-1} = (2\nu/\omega)^{1/2}$. Evidently, the thickness of the layer depends on the frequency of the plate: the layer becomes thicker as the frequency decreases and thinner as the frequency increases. In other words, a large amount of fluid oscillates with a slowly oscillating plate and only a small amount of fluid oscillates with a rapidly oscillating plate. The speed of oscillation is $(\omega/\delta) = \sqrt{2\nu\omega}$.

The use of the material law (10.12.8) and $\mathbf{v} = v\mathbf{e}_1$ gives the shear stress on a fluid element parallel to the plate as

$$\begin{aligned}\tau_{31} &= \mu v_{,3} = \mu\,\delta V e^{-\delta x_3}\{\sin(\omega t - \delta x_3) - \cos(\omega t - \delta x_3)\}\\ &= \sqrt{2}\mu\,\delta V e^{-\delta x_3}\sin\left(\omega t - \delta x_3 - \frac{\pi}{4}\right)\end{aligned} \tag{10.15.91}$$

This shear stress is maximum on the plate and decays exponentially with increasing distance from the plate, and

$$\max \tau_{31} = [\tau_{31}]_{x_3=0} = \sqrt{2}\,\alpha\mu V\sin\left(\omega t - \frac{\pi}{4}\right) \tag{10.15.92}$$

This is the skin friction acting on the plate.

Case ii: Impulsively Moved Plate This case arises when the plate is started impulsively from rest with a constant velocity $V\mathbf{e}_1$ and this velocity is maintained for all time $t > 0$.

To determine v in this case, we transform governing equation (10.15.81) to an ordinary differential equation by using the substitution

$$\eta = \frac{x_3}{2\sqrt{\nu t}}, \qquad v = VF(\eta), \qquad t > 0 \tag{10.15.93}$$

where η is called a *similarity variable*. Thus we find

$$\frac{\partial v}{\partial x_3} = \frac{\partial v}{\partial \eta} \cdot \frac{\partial \eta}{\partial x_3} = \frac{V}{2\sqrt{\nu t}} F'(\eta), \qquad \frac{\partial^2 v}{\partial x_3^2} = \frac{V}{4\nu t} F''(\eta)$$

$$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} = -\frac{V\nu x_3}{4(\nu t)^{3/2}} F'(\eta)$$

so that (10.15.81) reduces to

$$F''(\eta) + 2\eta F'(\eta) = 0 \tag{10.15.94}$$

The general solution of this first order ordinary linear differential equation for $F'(\eta)$ is

$$F'(\eta) = A\, e^{-\eta^2} \tag{10.15.95}$$

This gives $F(\eta)$ as

$$F(\eta) = A \int_0^\eta e^{-\eta^2}\, d\eta + B \tag{10.15.96}$$

where A and B are arbitrary constants.

From (10.15.84), (10.15.85) and (10.15.93), we note that $F(\eta)$ satisfies the following conditions:

$$F(\eta) = 1 \quad \text{for} \quad \eta = 0, \qquad F(\eta) \to 0 \text{ as } \eta \to \infty. \tag{10.15.97a,b}$$

Using these conditions in (10.15.96) and noting

$$\int_0^\infty \exp(-\eta^2)\, d\eta = \frac{\sqrt{\pi}}{2} \tag{10.15.98}$$

we find $B = 1$ and $A = -2/\sqrt{\pi}$. Thus the final form of v is given by

$$v = V\left[1 - \frac{2}{\sqrt{\pi}} \int_0^\eta \exp(-\eta^2)\, d\eta\right] \tag{10.15.99}$$

It can be verified that v decreases continuously from V to 0 as x_3 increases from 0 to infinity. As in the Stokes's problem, solution (10.15.99) exhibits a boundary layer nature; that is, the effect of the motion of the plate on the fluid flow decreases rapidly away from the plate. Here, the thickness of the boundary layer is on the order $\delta = \sqrt{\nu t}$, which grows with increasing time; the rate of growth is on the order $d\delta/dt \approx \sqrt{\nu/t}$, which decreases with increasing time. This problem is known as the *Rayleigh problem* and the associated boundary layer is called the *Rayleigh layer*.

Using the fact that $\mathbf{v} = v\mathbf{e}_1$, where v is given by (10.15.99), we find that the vorticity vector is directed along the x_2 axis with magnitude

$$|v_{,3}| = \frac{V}{\sqrt{\pi\nu t}}\exp\left(-\frac{x_3^2}{4\nu t}\right) \tag{10.15.100}$$

Evidently, the vorticity is 0 at $t = 0$ except at $x_3 = 0$, and it decays exponentially to 0 with increasing x_3. This implies that the vorticity is generated at the plate and diffuses outward within the Rayleigh layer.

By using Stokes's law (10.12.8), we find the shear stress on a fluid element parallel to the plate as given by

$$\tau_{31} = \mu v_{,3} = -\frac{\mu V}{\sqrt{\pi\nu t}}\exp\left(-\frac{x_3^2}{4\nu t}\right) \tag{10.15.101}$$

Evidently, the shear stress acts opposite to the direction of motion and its magnitude is maximum on the plate with

$$\max \tau_{31} = [\tau_{31}]_{x_3=0} = -\frac{\mu V}{\sqrt{\pi\nu t}} \tag{10.15.102}$$

This is the skin friction on the plate. Its magnitude varies inversely as $\sqrt{t}$.

10.15.5 SLOW AND STEADY FLOW PAST A RIGID SPHERE

As already indicated, for sufficiently small Reynolds numbers, the viscous forces exceed the inertia forces and the nonlinear terms in the Navier–Stokes equation may be neglected. In this case the governing equations for an incompressible fluid flow are given by (10.12.5) and (10.14.28). As an example, we consider here the steady flow past a rigid, fixed sphere in the absence of body force.

We assume that the fluid particles far away from the sphere are undisturbed and move with a uniform velocity V_∞ in the x_1 direction. Then, the velocity field satisfies the following conditions:

$$v_1 = V_\infty, \qquad v_2 = v_3 = 0 \qquad \text{as } r \to \infty \tag{10.15.103}$$

where r is the distance measured from the center of the sphere, which is taken as the origin (see Figure 10.22).

Since the sphere is rigid and stationary, the condition to be satisfied on the surface of the sphere is the no-slip boundary condition

$$\mathbf{v} = \mathbf{0} \quad \text{or} \quad v_i = 0 \qquad \text{for } r = a, \tag{10.15.104}$$

where a is the radius of the sphere.

Since the flow is steady and there is no body force, equations (10.14.29) and (10.14.30), which follow from equations (10.12.5) and (10.14.28), serve as the equations for determining p and v_i.

Noting that x_1/r^3 ($r \neq 0$) is a harmonic function, we take the pressure p, which is a harmonic function by (10.14.30), in the form

$$p = -\frac{A}{r^3}x_1 \tag{10.15.105}$$

where A is a constant. Substituting this value of p into (10.14.29) we obtain the following governing equations for v_i:

$$\begin{aligned} \mu\nabla^2 v_1 &= \frac{A}{r^5}(3x_1^2 - r^2) \\ \mu\nabla^2 v_2 &= \frac{3A}{r^5}x_1x_2 \\ \mu\nabla^2 v_3 &= \frac{3A}{r^5}x_1x_3 \end{aligned} \tag{10.15.106}$$

It can be verified that the solutions for v_i are

$$v_1 = V_\infty - \frac{x_1^2}{2\mu r^5}(Ar^2 - C) + \frac{1}{6\mu r^3}(Br^2 - C) \tag{10.15.107a}$$

$$v_2 = -\frac{x_1x_2}{2\mu r^5}(Ar^2 - D') \tag{10.15.107b}$$

$$v_3 = -\frac{x_1x_3}{2\mu r^5}(Ar^2 - D') \tag{10.15.107c}$$

where A, B, C, D' are constants. In view of conditions (10.15.103) and (10.15.104) we obtain

$$A = \tfrac{3}{2}\mu a V_\infty, \qquad B = -3A, \qquad C = D' = Aa^2$$

Substituting these into (10.15.105) and (10.15.107), we obtain the following solution for the pressure and velocity fields:

$$p = -\frac{3\mu a V_\infty}{2r^3} x_1 \tag{10.15.108}$$

$$v_1 = V_\infty - \frac{3aV_\infty}{4r^5}(r^2 - a^2)x_1^2 - \frac{aV_\infty}{4r^3}(3r^2 - a^2) \tag{10.15.109a}$$

$$v_2 = -\frac{3aV_\infty}{4r^5}(r^2 - a^2)x_1 x_2 \tag{10.15.109b}$$

$$v_3 = -\frac{3aV_\infty}{4r^5}(r^2 - a^2)x_1 x_3 \tag{10.15.109c}$$

This solution was first obtained by Stokes in 1851. The velocity distribution shows that the stream lines of the flow are symmetric about the equatorial plane normal to the direction of the flow, as depicted in Figure 10.22.

Since the fluid essentially moves in the x_1 direction, the resultant force on the surface of the sphere acts in the x_1 direction. This force, called the *drag* on the sphere, is

$$D = \int_S s_1 \, dS \tag{10.15.110}$$

where s_1 is, as usual, the x_1 component of the stress vector $\mathbf{s}$, and S is the surface of the sphere.

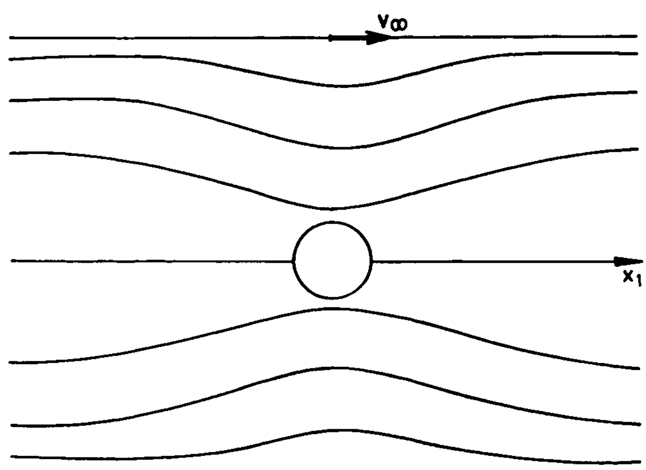

Figure 10.22. Slow and steady flow past a rigid sphere.

By using Cauchy's law (7.4.9) and noting that $n_i = x_i/a$ on S, expression (10.15.110) can be rewritten as

$$D = \frac{1}{a} \int_S (\tau_{11}x_1 + \tau_{12}x_2 + \tau_{13}x_3)\, dS \tag{10.15.111}$$

Computing τ_{1i} by using the material law (10.12.8) and results (10.15.108) and (10.15.109), substituting the resulting expressions onto the righthand side of (10.15.111) and subsequently evaluating the surface integral, we arrive at the following famous formula, known as *Stokes's formula*:

$$D = 6\mu\pi a V_\infty \tag{10.15.112}$$

Computations leading to (10.15.112) from (10.15.111) reveal that one-third of the righthand side of (10.15.112) is due to pressure and two-thirds is due to skin friction.

A nondimensional *drag coefficient* for the sphere, C_D, is defined by the relation

$$D = \tfrac{1}{2}\rho V_\infty^2 C_D(\pi a^2) \tag{10.15.113}$$

Using (10.15.112), we get

$$C_D = \frac{24}{Re} \tag{10.15.114}$$

where $Re = (2aV_\infty/\nu)(<1)$ is the Reynolds number based on the diameter of the sphere.

It has to be pointed out that Stokes's solution (10.15.109) is not valid at long distances from the sphere. In order to check this point, we estimate the orders of magnitude of the inertia term $\rho(\mathbf{v} \cdot \nabla)\mathbf{v}$ and the viscous term $\mu\nabla^2\mathbf{v}$. According to Stokes's solution, the inertia term is on the order $\rho(aV_\infty^2/r^2)$, whereas the viscous term is on the order $\mu a V_\infty/r^3$. Hence Stokes's solution holds only when

$$\rho\frac{aV_\infty^2}{r^2} \ll \frac{\mu a V_\infty}{r^3}$$

that is, $r \ll \nu/V_\infty$.

The difficulty associated with Stokes's solution was first resolved by Oseen in 1910. Using the hypothesis that v_1 is approximately equal to V_∞ at long distances from the sphere, he assumed $(\mathbf{v} \cdot \nabla)\mathbf{v} \approx V_\infty(\partial\mathbf{v}/\partial x_1)$ in the Navier–Stokes equation and replaced equation (10.14.29) by the following equation to study the problem:

$$\rho V_\infty \frac{\partial \mathbf{v}}{\partial x_1} = -\nabla p + \mu\nabla^2\mathbf{v} \tag{10.15.115}$$

We shall not attempt to find the solution of the problem on the basis of this equation, which is known as the *Oseen equation*. We just mention that the solution leads to the expression for the drag coefficient

$$C_D = \frac{24}{Re}\left(1 + \frac{3Re}{16}\right) \tag{10.15.116}$$

which agrees with experimental findings up to $Re = 2aV_\infty/\nu = 5$. Comparison of (10.15.114) and (10.15.116) shows that Stokes's formula (10.15.114) is a first approximation of Oseen's formula (10.15.116).

10.16

EXERCISES

1. If a fluid of uniform density is at rest under a constant gravitational field, show that the pressure varies linearly with depth.

2. For water, an approximate relationship between the pressure and density is $\rho = 1 + \alpha p$, where α is a constant. If ρ_a is the density at the surface of a lake, show that the density of water at depth h in the lake is given by $\rho = \rho_a e^{\alpha g h}$. Assume that p varies only with depth and gravity is the only body force.

3. At a height h above the earth's surface the density of air is ρ and pressure is p. At the surface, the corresponding values are ρ_0 and p_0. Assuming that the gravitational field is constant and $\rho = \rho_0 e^{-\alpha z}$, where α is a constant and z is the vertical distance above the earth's surface, show that

$$p = p_0 - \frac{\rho_0 g}{\alpha}(1 - e^{-\alpha z})$$

4. Assuming that the atmosphere around the earth is a perfect gas at rest in which the pressure-density relation at an altitude z is given by $(p/p_0) = (\rho/\rho_0)^\gamma$, where p_0 and ρ_0 are the pressure and density at a reference elevation z_0, determine expressions for T, p and ρ in terms of T_0, p_0, ρ_0, z and z_0. Here T and T_0 are temperatures at z and z_0, respectively.

5. A rigid sphere of radius a is fixed in an incompressible fluid at rest. The fluid extends to infinity in all directions and the pressure-density relation is $p = \alpha\rho$, where α is a constant. If every fluid particle is attracted towards the center O of the sphere by a force of magnitude β/r^2, where r is the distance from O, show that the pressure exerted on the surface of the sphere is proportional to $\exp(\beta/\alpha a)$.

6. Show that in an incompressible nonviscous fluid flow, the rate of work done by external forces is equal to the rate of change of kinetic energy and that the rate of change of internal energy is equal to the rate of heat input.

7. If $e = \varepsilon + (p/\rho)$ is the specific enthalpy, show that $de/dT = c_p$ for a perfect gas.

8. Show that the energy equation for a compressible nonviscous fluid flow can be put in the following forms:

(i) $$\rho\frac{D\varepsilon}{Dt} = -p \operatorname{div} \mathbf{v} - \operatorname{div} \mathbf{q} + \rho h$$

(ii)) $$\frac{D\varepsilon}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) + \frac{1}{\rho}\operatorname{div} \mathbf{q} - h = 0$$

(iii) $$\operatorname{div}(p\mathbf{v} + \mathbf{q}) - \rho(\mathbf{v} \cdot \mathbf{b}) - \rho h = \rho\frac{De}{Dt} = \frac{\partial}{\partial t}(\rho e) + \operatorname{div}(\rho e)$$

(iv) $$\rho\frac{D\bar{e}}{Dt} = \frac{\partial p}{\partial t} - \operatorname{div} \mathbf{q} + \rho h + \rho(\mathbf{b} \cdot \mathbf{v})$$

where $e = \varepsilon + (p/\rho)$ and $\bar{e} = e + (v^2/2)$.

9. Write down Euler's equation of motion in the suffix notation.

10. Show that the velocity field given by

$$v_1 = \alpha\frac{x_1^2 - x_2^2}{R^4}, \qquad v_2 = \frac{2\alpha x_1 x_2}{R^4}, \qquad v_3 = 0$$

where α is a nonzero constant and $R^2 = x_1^2 + x_2^2 \neq 0$, obeys Euler's equation of motion for an incompressible fluid. Determine the pressure distribution associated with this velocity field.

11. The velocity field in an incompressible nonviscous fluid flow is given by

$$v_1 = v_0 \cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a},$$

$$v_2 = v_0 \sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}, \qquad v_3 = 0$$

where v_0 and a are nonzero constants. Show that the pressure is given by

$$p = \frac{1}{4}v_0^2\left\{\cos\frac{\pi z}{a} - \cos\frac{\pi x}{a}\right\} + \text{constant}$$

12. For a steady irrotational flow of a nonviscous fluid of constant density under gravity, show that

$$p = p_0 - \rho g z - \tfrac{1}{2}\rho v^2$$

where p_0 is a constant.

13. For an irrotational steady flow of a nonviscous fluid under zero body force, show that $dp = -\rho v \, dv$.

14. A liquid column of uniform density moves in a vertical direction (against gravity) with a constant acceleration $\mathbf{a}$. Find the pressure at a point whose depth from the upper surface of the liquid is h.

15. For the liquid column considered in Example 10.6.2, show that the rise of the liquid above the vertex of the upper surface along the wall of the vessel is $\omega^2 a^2/2g$, where a is the radius of the vessel.

16. For a two-dimensional flow of an incompressible nonviscous fluid under zero body force, show that the vorticity is constant.

17. In the irrotational, two-dimensional flow of an incompressible nonviscous fluid, show that the velocity is constant in magnitude if and only if it is constant in direction as well.

18. The velocity field for a two-dimensional flow of an incompressible nonviscous fluid under zero body force is given by

$$v_1 = \psi_{,2}, \qquad v_2 = -\psi_{,1}, \qquad v_3 = 0$$

where $\psi = \psi(x_1, x_2, t)$. By using Euler's equation of motion show that ψ satisfies the equation

$$\frac{\partial}{\partial t}(\nabla^2 \psi) = \begin{vmatrix} \psi_{,1} & \psi_{,2} \\ \nabla^2 \psi_{,1} & \nabla^2 \psi_{,2} \end{vmatrix}$$

19. Show that in a compressible nonviscous fluid flow, the rate of change of circulation round a circuit c is given by

$$\frac{DI_c}{Dt} = -\int_S \left\{ \nabla\left(\frac{1}{\rho}\right) \times \nabla p - \operatorname{curl} \mathbf{b} \right\} \cdot \mathbf{n}\, dS$$

where S is a surface for which c is the rim.

20. Show that for an elastic fluid moving under conservative body force, Cauchy's vorticity equation (6.6.8) can be rewritten as

$$\frac{\mathbf{w}}{\rho} = \frac{\mathbf{w}_0}{\rho_0} \cdot \nabla^0 \mathbf{x}$$

21. For an elastic fluid moving under conservative force, show that

$$\frac{D}{Dt}\int_V \mathbf{w}\, dV = \int_S (\mathbf{w} \cdot \mathbf{n})\mathbf{v}\, dS$$

where V is a material volume and S is its boundary surface.

22. Show that the equation (10.7.8) can be rewritten as

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{w} \times \mathbf{v} = -\nabla(P + \chi + \tfrac{1}{2}v^2)$$

Deduce that $\nabla^2(P + \chi + \frac{1}{2}v^2) = w^2 + \mathbf{v} \cdot \nabla^2 \mathbf{v} - \mathbf{v} \cdot \nabla(\operatorname{div} \mathbf{v}) - (\partial/\partial t)(\operatorname{div} \mathbf{v})$.

23. Show that in the absence of body force the only possible steady flow of an elastic fluid for which $\mathbf{v} = v(x_1)\mathbf{e}_1$, $\rho = \rho(x_1)$, $P = P(x_1)$ is the one for which v, ρ and P are all constants, independent of x_1.

24. For a nonviscous perfect gas moving in the x_1 direction, show that

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x_1^2}\{\rho(v^2 + RT)\}$$

25. For a steady, irrotational flow of an adiabatic isentropic gas in the absence of body force, show that the speed of sound is given by

$$c_s^2 = c_0^2 - \frac{\gamma - 1}{2} v^2$$

where c_0 is the speed at a reference condition.

26. Show that for a perfect gas $c_s = \sqrt{\gamma RT}$.

27. Write equation (10.7.24) in the unabridged form.

28. Show that the Bernoulli's equation (10.8.7) reduces to

(i) $p \log(p/\rho) + (1/2)v^2 + \chi =$ constant, for a perfect gas.
(ii) $\gamma p/(\gamma - 1)\rho + (1/2)v^2 + \chi =$ constant, for an isentropic perfect gas.

29. A gas, in which p and ρ are related by an adiabatic relation $p = \beta\rho^n$, flows in a steady state in a horizontal conduit. Using Bernoulli's equation show that the maximum speed is given by

$$v_{\max} = c_0\left(\frac{2}{n - 1}\right)^{1/2}$$

where c_0 is the speed of sound at a reference condition.

30. A nonviscous incompressible fluid flows steadily around the outside of a fixed vertical cylinder of radius a so that the fluid particles move with speed a/R in horizontal circles concentric with the boundary of the cylinder, where R is the radial distance from the axis of the cylinder. Show that the flow is irrotational. If the surface of the fluid is open to the atmosphere, show, by using the Bernoulli's equation, that

$$2gz = 1 - (a^2/R^2)$$

for appropriate choice of the coordinate axes.

31. Find the stress matrix associated with the following velocity field occurring in a viscous fluid flow:

(i) $v_1 = 0, \quad v_2 = 0, \quad v_3 = x_2$

(ii) $v_1 = 0, \quad v_2 = x_2^2 - x_3^2, \quad v_3 = -2x_2x_3$

(iii) $v_1 = v_1(x_1, x_2), \quad v_2 = v_2(x_1, x_2), \quad v_3 \equiv 0$

32. For a certain motion of an incompressible viscous fluid, the velocity is of the form $\mathbf{v} = v(x_1, t)\mathbf{e}_1$. Verify that the motion is necessarily irrotational and nonisochoric. Find the stress components developed in the fluid. Deduce that the stress vector on planes parallel to and perpendicular to the direction of flow is a pure pressure.

33. For a viscous fluid, show that

$$\mathbf{T}^{(d)} = (\bar{p} - p)\mathbf{I} + \kappa(\mathrm{tr}\,\mathbf{D})\mathbf{I} + 2\mu\mathbf{D}^{(d)}$$

34. Show that for a compressible viscous fluid,

$$\bar{p} = p + \frac{\kappa}{\rho}\frac{D\rho}{Dt}$$

35. Show that

$$\Phi = \kappa I_D^2 + 4\mu II_{\mathbf{D}^{(d)}} = \mathrm{tr}(\mathbf{T}^{(v)}\mathbf{D})$$

36. The stress matrix at a given point of a viscous fluid with zero bulk viscosity is

$$[\mathbf{T}] = \begin{bmatrix} 0 & 2 & 0 \\ 2 & -4 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

Find the viscous stress matrix $[\mathbf{T}^{(v)}]$.

37. For a compressible viscous fluid with zero bulk viscosity contained in a volume V, show that the total surface force on the boundary S of V is given by

$$\int_S \mathbf{s}\,dS = \int_V (\mathrm{div}\,\mathbf{T}^{(v)} - \nabla p)\,dV = \int_V [2\mu\,\mathrm{div}\,\mathbf{D}^{(d)} - \nabla p]\,dV = \int_V [\mu\nabla^2\mathbf{v} - \nabla p]\,dV$$

38. Show that the energy equation (10.12.3) can be expressed in the form

$$\rho\frac{D\varepsilon}{Dt} = \frac{p}{\rho}\frac{D\rho}{Dt} + k\nabla^2 T + \rho h + \Phi$$

39. For an irrotational flow of a viscous fluid adjacent to a plane rigid wall, determine how the tangential velocity varies with the normal direction from the wall.

40. Show that the magnitude of normal stress on a stationary boundary with which a compressible viscous fluid is in contact is $\bar{p} + (4/3)\nu(D\rho/Dt)$.

41. For an irrotational flow of an incompressible viscous fluid in a region bounded by a rigid boundary, show that the velocity is uniquely determined by prescribing the normal component of velocity on the boundary.

42. Write down the Navier–Stokes equation (10.14.3) in the suffix notation.

43. For an irrotational flow of an incompressible fluid, show that the Navier–Stokes equation reduces to Euler's equation.

44. For an incompressible viscous fluid moving under conservative body force, show that

$$\frac{DI_c}{Dt} = \nu\nabla^2 I_c$$

45. For an incompressible viscous fluid moving under conservative body force, show that the vorticity obeys the equation

$$\frac{\partial \mathbf{w}}{\partial t} = \text{curl}(\mathbf{v} \times \mathbf{w}) + \nu\nabla^2\mathbf{w}$$

46. Generalize expression (10.14.20) to the nonsteady case, other conditions being the same.

47. Assuming that μ/p is a constant, show that, for an irrotational motion of a compressible fluid with zero bulk viscosity under a conservative body force, the Navier-Stokes equation has an integral in the form

$$\int \frac{dp}{\rho} + \frac{1}{2}v^2 + \chi - \frac{\partial \phi}{\partial t} + \frac{4}{3}\nu\nabla^2\phi = f(t)$$

48. In an incompressible fluid moving under zero body force, the velocity components are of the form

$$v_1 = -\tfrac{1}{2}\alpha x_1 - f(R)x_2$$

$$v_2 = -\tfrac{1}{2}\alpha x_2 + f(R)x_2$$

$$v_3 = \alpha x_3$$

where $R = (x_1^2 + x_2^2)^{1/2}$ and α is a positive constant. Show that the pressure distribution is of the form

$$p = p_0 - \tfrac{1}{8}\rho\alpha^2(R^2 + 4x_3^2) + \rho \int R\{f(R)\}^2\, dR$$

where p_0 is a constant.

49. For the plane Poiseullie flow, show that the vorticity is a harmonic function.

50. An incompressible viscous fluid bounded between two inclined parallel, rigid plates flows down under gravity. If the upper plate is moving in the direction of flow with speed $v_0(t)$ and the lower plate is held stationary, obtain the velocity distribution in the fluid.

51. An incompressible viscous fluid flows steadily down an inclined plane under gravity. The fluid layer is of uniform thickness and the upper layer is exposed to atmospheric pressure. Obtain the velocity distribution in the fluid.

52. For the Hagen–Poiseullie flow in a circular conduit show that the vorticity is perpendicular to the conduit and that its magnitude varies directly with the radial distance from the axis.

53. For the steady flow of an incompressible viscous fluid through a straight triangular tube bounded by the planes $x = a$, $y = \pm(1/\sqrt{3})x$ under a constant pressure gradient and zero body force, show that the velocity distribution is given by

$$v = \frac{G}{4\mu a}(x - a)(3y^2 - x^2)$$

Deduce that the mass flow rate in a cross section is

$$M = \frac{Ga^4}{60\sqrt{3}\,\nu}$$

54. An incompressible viscous fluid moves steadily in a horizontal elliptic pipe of length L placed along the x axis due to a pressure difference P maintained across the ends. If $y^2 + 4z^2 = 4$ is the equation of the boundary of a cross section, show that the total volume of fluid delivered through the pipe per unit of time is $2\pi P/5\mu L$.

55. An incompressible viscous fluid flows steadily through a region bounded by a solid cylinder of radius a and a coaxial tube of radius b, $b > a$. The cylinder and the tube are stationary, there is no body force and the motion is due to a constant pressure gradient parallel to the axis. Show that the velocity distribution is given by

$$v = \frac{G}{4\mu}[(a^2 - R^2) + (b^2 - a^2)\{\log(R/a)/\log(b/a)\}]$$

where R is, as usual, the radial distance from the axis. Deduce that the shear stress and rate of mass flow are, respectively,

$$\tau = \frac{g}{4}\left[\frac{b^2 - a^2}{R\log(b/a)} - 2R\right], \qquad M = \frac{\pi G}{8\nu}\left[b^4 - a^4 - \frac{(b^2 - a^2)^2}{\log(b/a)}\right]$$

56. An incompressible viscous fluid flows steadily through a region bounded by a solid cylinder of radius a and a coaxial tube of radius b, $b > a$. The motion is due to the movement of the cylinder with speed v_0 along its axis. The tube is stationary; there is no body force; and there is no pressure gradient. Show that the velocity distribution is given by $v = v_0 \log(b/R)/\log(b/a)$ and that the shear stress along the axis is $\tau = -\mu v_0/\{R\log(b/a)\}$, where R is, as usual, the radial distance from the axis.

57. Carry out the computations involved in obtaining (10.15.112) from (10.15.110).