Algebraic Combinatorics HW 1

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Problem 1 (Closed Walks in K_n)**.** Find a combinatorial proof of the fact that # closed walks of length l in K_n from some vertex to itself is

$$\frac{1}{n}\left((n-1)^l + (n-1)(-1)^l\right)$$

Proof. By symmetry, we can simply count the number of closed l-walks from v_1. We write each walk as an ordered list of vertices, so that we must find the number of walks

$$v_{i_1}, v_{i_2}, \cdots, v_{i_l}, v_{i_{l+1}}$$

with all $v_{i_k} \neq v_{i_{k+1}}$ and $v_{i_1} = v_{i_{l+1}} = v_1$.

We start by counting the number of walks if we allow $v_{i_l} = v_{i_{l+1}}$, and then subtract the overcounting. After the first, each of the $l-1$ subsequent vertices have $n-1$ options, for $(n-1)^{l-1}$ total sequences. We msut now subtract the number of sequences where $v_{i_{l-1}} = v_1$.

Conceptually, we are counting the number of closed l walks by counting the number of open $l-1$ walks, which is the number of total $l-1$ walks minus the number of closed $l-1$-walks.[1]

We now subtract the number of sequences where $v_{i_l} = v_1$, Repeating the same reasoning, we can overcount by counting $(n-1)^{l-2}$ sequences, and then subtracting the number of sequences where $v_{i_{l-1}} = v_{i_l} = v_1$. We continue this process, finishing with $(n-1)^1$ closed 2-walks, making the number of closed l-walks from a vertex to itself is

$$\begin{aligned}
&(n-1)^{l-1} - (n-1)^{l-2} + (n-1)^{l-3} + \cdots + (-1)^l(n-1) \\
&= \sum_{k=1}^{l-1} (n-1)^k (-1)^{l-1+k} \\
&= (-1)^{l-1} \sum_{k=1}^{l-1} (1-n)^k \\
&= (-1)^{l-1} \frac{(1-n)\left((1-n)^{l-1} - 1\right)}{1-n-1} \\
&= (-1)^l \frac{(1-n)^l - (1-n)}{n} \\
&= \frac{1}{n}\left((n-1)^l + (n-1)(-1)^l\right),
\end{aligned}$$

as desired. $\square$

[1] This suggests that we could complete via an inductive proof, using

$$(n-1)^l - \frac{1}{n}\left((n-1)^{l-1} + (n-1)(-1)^{l-1}\right) = \frac{1}{n}\left((n-1)^l + (n-1)(-1)^l\right).$$

Problem 2 (Eigenvalues of some bipartite graphs)**.**

(i) Let $G[A, B]$ be a bipartite graph with partite sets A, B. Show by a walk-counting argument that the non-zero eigenvalues of G come in pairs $\pm\lambda$.
(Eigenvalues of K_{rs}) Consider the complete bipartite graph $K_{r,s}$ (that is, having partite sets of size r and s)

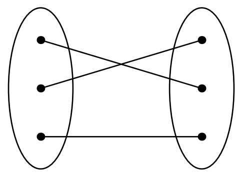
(ii) Use purely combinatorial reasoning to compute the number of closed walks of length l in $K_{r,s}$.

(iii) Deduce the eigenvalues of $K_{r,s}$.
(Eigenvalues of $K_{n,n} - nK_2$) Let H_n be the graph $K_{n,n}$ with a perfect matching removed.

(iv) Show that the eigenvalues of H_n are

$$\pm 1(n-1 \text{ times}), \pm(n-1)(\text{once each}).$$

Proof.



A B

(i) Every step on a walk takes us between partite sets A and B. Thus, there are no $2l+1$-walks, meaning that

$$\sum (\lambda_i)^{2l+1} = 0,$$

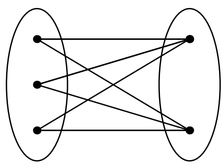
so

$$\sum (-\lambda_i)^{2l+1} = \sum (\lambda_i)^{2l+1} = 0.$$

As

$$\sum (\lambda_i)^{2l} = \sum (-\lambda_i)^{2l},$$

$\sum \lambda_i^k$ and $\sum (-\lambda_i)^k$ agree for all positive integers k, so the $-\lambda_i$ are simply a permutation of the λ_i, meaning that all nonzero eigenvalues come in $\pm\lambda$ pairs.



r (A) s (B)

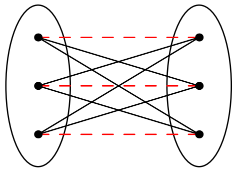
(ii) Call the partite with r elements A and the partite with s elements B. If l is odd, there are zero walks. So, we assume l is even. If we begin our l-walk in A, we know go from A to B $\frac{l}{2}$ times and B to A $\frac{l}{2}$ times. Each time we go from A to B, we have s options. Each time we go from B to A, we have r options, except the last step, at which point we must return to our original vertex, for which we have r choices. There are thus $s^{l/2}r^{l/2-1}r = (rs)^{l/2}$ l-walks beginning in A, and an identical argument gives $(rs)^{l/2}$ l-walks beginning in B. The number of l-walks is thus

$$\begin{cases} 0 & l \text{ odd} \\ 2(rs)^{\frac{l}{2}} & l \text{ even} \end{cases}.$$

(iii) $\sum \lambda_i^l$ and $(rs)^l + (-rs)^l + (n-2)\cdot(0)^l$ agree for all positive l, so the eigenvalues of $K_{r,s}$ are

$$\pm rs, 0 \ (r+s-2 \text{ times}).$$

We now consider the $K_{n,n} - nK_2$ graph, providing $n = 3$ as an example:



(iv) We aim to find the number of l-walks (for even l) on $K_{n,n} - nK_2$. If we write the partites as $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ such that the a_i and b_i are not connected, but a_i and b_j for $i \neq j$ are connected, then our walk-counting problem becomes analogous to the K_n problem. Any valid will alternate between A and B, but have *no index repeated twice in a row.* That is, a_1b_2 is a valid step, while a_1b_1 is not. We can use this to establish an bijection between l-walks starting in A and l-walks on K_n, meaning that the *total* number of l-walks on $K_{n,n} - nK_2$ is

$$2n\left(\frac{1}{n}\left((n-1)^l + (n-1)(-1)^l\right)\right) = 2(n-1)^l + 2(n-1)(-1)^l.$$

For odd l, the number of walks is 0, and so $\sum \lambda_i^l$ agrees for all positive l, as

$$(n-1)^l + (1-n)^l + (n-1)((-1)^l + (1)^l),$$

meaning that our eigenvalues are

$$\pm 1 (n-1 \text{ times}), \pm(n-1) (\text{once each}).$$

□

Problem 3 (On the largest eigenvalue of $A(G)$; **Extra credit**).

(i) Let G be a graph with max degree $\Delta(G)$. Let λ_1 be the largest eigenvalue of $A(G)$. Show that $\lambda_1 \leq \Delta(G)$.

(ii) Let G be a simple graph with m edges. Show that $\lambda_1 \leq \sqrt{2m}$.

Proof. (i)

□

Problem 4.

(i) Start with n coins heads up. Choose a coin at random and turn it over. Do this a total of m times. What is the probability that all coins will have heads up?

(ii) Same as (i), except now compute the probability that all coins have tails up.

(iii) Same as (i), but now we turn over two coins at a time.

Solution.

(i.) Flipping m coins can be modeled as an m-walk on the graph of the n-hypercube, Q_n. Thus, the probability we seek is

$$\frac{\#\text{closed } m-\text{walks}}{\#m-\text{walks}} \text{ on } \mathbb{Z}_2^n.$$

There are n^m such walks, both because each vertex has degree n/each step flips one of n coordinates and that we choose amongst n coins each flip.

In order to find the number of closed m-walks on Q_n, we consider the eigenvalues of $A(Q_n)$, which are of the form $n-2i$ as i ranges from 0 to n, where $\lambda_i = n-2i$ has multiplicity $\binom{n}{i}$. Thus, the number of closed m-walks on Q_n is

$$\sum_{i=0}^{n} \binom{n}{i} (n-2i)^m,$$

so the desired probability is

$$\frac{\sum_{i=0}^{n} \binom{n}{i} (n-2i)^m}{n^m}.$$

(ii.) Starting with all heads and ending with all tails is akin to walking from v_1 to v_{2^n}. So, we seek the ratio

$$\frac{((A(Q_n))^m)_{1,2^n}}{n^m},$$

and applying Corollary 2.5 from the textbook gives a probability of

$$\frac{\sum_{i=0}^{n} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{n-k}{i-j} (n-2i)^l}{2^n n^m}.$$

(iii.) The adjacency matrix $A(Q_n)^2$ represents the number of ways to flip two coins, but it allows us to flip the same coin twice in a row, result in nonzero diagonals. Thus, if we subtract some scalar multiple of the identity from $A(Q_n)^2$, yielding a matrix with zeroes along the diagonal, we have the adjacency matrix representing the two-flip graph. Each a_{ii}

$\square$

Problem 5. Let G_n be the graph with vertex set $\mathbb{Z}_2^n$ with the edge set defined as: u and v are adjacent iff they differ in exactly two coordinates (that is, $\omega(u+v)=2$). What are the eigenvalues of G_n?

Proof. Recall from lecture that the eigenvalues of $A(Q_n)$ are of the form $n-2i$ as i ranges from 0 to n, where $\lambda_i = n-2i$ has multiplicity $\binom{n}{i}$. Consider $A(Q_n)^2$. We aim to show that

$$A(Q_n)^2 = nI_n + 2A(G_n)$$

via a walk counting argument. $A(Q_n)^2_{ij}$ counts the number of 2-walks from i to j (vertices on the hypercube). When $i=j$, this is precisely $\deg i = n$. When $i \neq j$, for there to be a 2-walk from i to j, we must have $\omega(i+j)=2$. Each step flips one coordinate of i, and since there must be exactly two coordinates to flip, we have $2! = 2$ such walks. Thus, $A(Q_n)^2_{ij} = 2$ when $\omega(i+j)=2$. $2A(G_n)$ is the matrix with 2s when $\omega(i+j)=2$ and zeroes elsewhere, and nI_n is the matrix with ns along the diagonal. Therefore,

$$A(Q_n)^2 = nI_n + 2A(G_n),$$

so for any given eigenvalue λ_i of Q_n, since $A(Q_n)$ is diagonalizable, λ_i^2 is an eigenvalue of $A(Q_n)^2$. Thus,

$$\begin{aligned}
\det(A(Q_n)^2 - \lambda_i^2 I_n) &= 0 \\
\det(nI_n + 2A(G_n) - \lambda_i^2 I_n) &= 0 \\
\det(2A(G_n) - (\lambda_i^2 - n)I_n) &= 0 \\
\det\left(A(G_n) - \frac{\lambda_i^2 - n}{2} I_n\right) &= 0,
\end{aligned}$$

so $\frac{\lambda_i^2 - n}{2} = \frac{(n-2i)^2 - n}{2}$ is an eigenvalue of G_n, as i ranges from 0 to n. $\square$