# Algebraic Combinatorics: HW6

various.

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**Problem 1** (Trees with prescribed degrees and Cayley's formula).

(a) Given positive integers $d_1, d_2, \ldots, d_n$ such that $\sum d_i = 2n-2$, show that the number of labelled trees on $[n]$ such that vertex $i$ has degree $d_i$ for each $i$ is

$$\frac{(n-2)!}{\prod (d_i - 1)!}.$$

(b) Prove Cayley's formula from (a).

(c) What is the number of all trees on $n$ vertices with exactly $n-l$ leaves? (Hint: You may use (a) and leave your answer in terms of *Stirning's number of the second kind.*)

*Proof.* (a) The proof via Prüfer codes is a trivial arrangement argument. Proceeding by induction on $n$, the base case $n=1$ is also trivial. So, assume that the statement holds for $n-1$, that is, there are

$$\frac{(n-3)!}{\prod_{i=1}^{n-1}(d_i-1)!}$$

ways to create a labelled tree with $\sum d_i = 2n-4$.

For $n$ vertices, notice that there must be at least one vertex with degree 1 since the sum of degrees is $2n-2$. Assign $d_n$ to be the vertex with degree 1. Then, we have $n-1$ remaining vertices ways to connect the $n$th vertex to. If we connect to the $k$th vertex, then we are interested in the number of labelled trees with degrees $d_1, d_2, \ldots d_k - 1, \ldots, d_{n-1}$, which is $\frac{(n-3)!}{\prod_{i=1}^{n-1}(d_i-1)!} \cdot (d_k - 1)$. So, the total number of valid trees on $[n]$ vertices is

$$\begin{aligned}
\sum_{k=1}^{n-1} \frac{(n-3)!}{\prod_{i=1}^{n-1}(d_i-1)!} \cdot (d_k - 1) &= \frac{(n-3)!}{\prod_{i=1}^{n-1}(d_i-1)!} \cdot \sum_{k=1}^{n-1}(d_k - 1) \\
&= \frac{(n-3)!}{\prod_{i=1}^{n-1}(d_i-1)!} \cdot (2n-3-(n-1)) \\
&= \frac{(n-2)!}{\prod_{i=1}^{n-1}(d_i-1)!} \cdot \frac{1}{0!} \\
&= \frac{(n-2)!}{\prod_{i=1}^{n}(d_i-1)!},
\end{aligned}$$

as desired. The proof by induction is complete.

(b) Finding the total number of labelled trees on $[n]$ is the same to summing $\frac{(n-2)!}{\prod(d_i-1)!}$ over all possible degrees $d_1, d_2, \ldots, d_n$ such that $\sum d_i = 2n-2$. Note that $\frac{(n-2)!}{\prod(d_i-1)!}$ is the number of ways to order $n-2$ numbers in a row, where $x$ appears $d_x - 1$ times, so our sum is counting the number of ways to list $n-2$ integers in an ordered row, where each number is between 1 and $n$, which is precisely $n^{n-2}$.

This proof effectively travels through Prüfer codes.

(c) Let vertices $l+1$ through $n$ be the leaves, so $d_{l+1} = d_{l+2} = \cdots = d_n = 1$. The sum of the remaining degrees is $2n-2-(n-l) = n+l-2$. By (a), the number of trees is then

$$\frac{(n-2)!}{\prod_{i=1}^{l}(d_i-1)!}$$

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**Problem 2** (Counting Spanning trees of $K_{m,n}$)**.** Find the value of $\tau(K_{m,n})$ using:

(i) Matrix-Tree Theorem.

(ii) Combinatorial argument, say, that of Prüfer or Joyal.

(iii) Let $L$ be the Laplacian of $K_{m,n}$.

(a) Find a simple upper bound on rank$(L - mI)$.

*Proof.* 1. Recall that the Matrix-Tree Theorem states that $\tau(G) = |L_0|$. We let vertices $v_1$ through $v_m$ be within the first partite and $v_{m+1}$ through $v_{m+n}$ be within the second partite. Then, the Laplacian of $K_{m,n}$ is

$$L = \begin{pmatrix} mI_n & -J \\ -J & nI_m \end{pmatrix},$$

where $J$ is the all-ones matrix. Then, we have

$$L_0 = \begin{pmatrix} mI & -J \\ -J & nI \end{pmatrix}_{\{v_1,v_2,\ldots,v_{m+n-1}\}} = \begin{pmatrix} mI & -J \\ -J & nI \end{pmatrix}_{\{v_1,v_2,\ldots,v_m\}}.$$

The determinant of this matrix is

$$mI \cdot nI - J^2 = mIn - nJ^2 = mIn - nJn = n(m-n),$$

so $\tau(K_{m,n}) = n(m-n)$.

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**Problem 4.** Starting at a point $x_0$ we walk along the edges of a connected graph $G$ according to the following rules:

- We never use the same edge twice in the same direction.
- Whenever we arrive at a point $x \neq x_0$ not previously visited, we mark the edge along which we entered $x$. We use the marked edge to leave $x$ only if we must, that is, if we have used all the other edges before.

> Show that we get stuck at $x_0$, and that, by then, every edge has been traversed in both direction.

*Proof.* Set the vertex at which we get stuck to be $v$. Suppose, for the sake of contradiction, then $v \neq x_0$. Then, if we consider the number of times we have traversed an edge to arrive *at* $v$, this must be one more than the number of times we have traversed an edge to leave *from* $v$. Thus, there must be at least one edge that we have not traversed in both directions, so we may traverse that edge and leave. Therefore, we must be stuck at $x_0$.

We now show that, when we get stuck at $x_0$, every other edge must be traversed in both directions, where we duplicate each edge and direct it, to create the enter-exit effect. Then, consider the "in-tree" rooted at $x_0$. Suppose, for the sake of contradiction, that we get stuck at $x_0$ when there is still some untraversed edges. At least one of those eddges but be an edge of the in-tree rooted at $v$, and let the point closest to $x_0$ (in the in-tree) on an untraversed edge of the in-tree be $y$. Since we travel to and from $y$ an equal number of times, there is an "exit" from $y$ we don't use, so $y$ is closer to $x_0$ than the initial vertex on the untraversed edge, meaning that $y = v$ (by the choice of edge). But, if there is an in-edge to $x_0$, then we must certainly also have an unused out-edge, contradicting our stuckness. So, all edges are traversed in our manually constructed directed graph, meaning all edges are traversed twice in the original graph. ■

> **Problem 5** (Universal cycles for $S_n$). (i) Let $n \geq 3$. Show that there does not exist a sequence $a_1, a_2, \ldots, a_{n!}$ such that all the $n!$ contiguous blocks $a_i, a_{i+1}, \ldots, a_{i+n-1}$ (subscripts taken modulo $n!$) are all the $n!$ permutations of $S_n$.
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> (ii) Show that for all $n \geq 1$, there exist a sequence $a_1, a_2, \ldots, a_{n!}$ such that all the $n!$ contiguous blocks $a_i, a_{i+1}, \ldots, a_{i+n-1}$ consists of the first $n-1$ terms $b_1, b_2, \ldots, b_{n-1}$ of all permutations $b_1, b_2, \ldots, b_n$ of $[n]$.
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> Such sequences are called **universal cycles** for $S_n$ (for example, for $n = 3$, 123213 is a universal cycle.)
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> (iii) For $n = 3$, find the number of universal cycles beginning with 123.