

# Algebraic Combinatorics HW 3

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**Problem 1** (Symmetric polynomial and unimodality).  $f(x) = p_0 + p_1x + p_2x^2 + \cdots + p_nx^n$  is *symmetric* if for all  $i$ ,

$$p_i = p_{n-i}$$

It is *unimodal* if for some fixed  $j$ ,

$$p_0 \leq p_1 \leq \cdots \leq p_{j-1} \leq p_j \geq p_{j+1} \geq \cdots \geq p_{n-1} \geq p_n$$

Let  $F(q)$ ,  $G(q)$  be symmetric and unimodal polynomials with non-negative real coefficients. Show that  $F(q)G(q)$  is also symmetric (easy) and unimodal (less easy).

*Proof.*

□

**Problem 2** (Log-concavity of Binomial coefficients). A sequence  $a_1, a_2, a_3, \dots, a_n$  is *logarithmically concave* if

$$a_i^2 \geq a_{i-1}a_{i+1} ; \forall i$$

- (i) Show that if a sequence of positive terms is log-concave, then it is also unimodal.
- (ii) It is easy to see algebraically that the sequence  $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}$  is log-concave. Give a combinatorial proof of this fact.

*Proof.*

□

**Problem 3** (Uniqueness in Sperner's Thm). Show that equality in Sperner's Theorem for  $B_n$  is achieved only by the middle (middle two) rank(s) if  $n$  is even (odd). (*Hint*: If not, then move the example closer to the middle rank(s))

*Proof.*

□

**Problem 4** (A generalization of Sperner's Thm). Let  $P$  be a rank-symmetric, rank-unimodal poset. Show that if  $P$  has a symmetric chain decomposition, then it has *strong Sperner property*, that is, for any  $j \geq 1$ , the largest size of a union of  $j$  antichains is equal to the size of the largest  $j$  levels of  $P$ . (*Remark*:  $j = 1$  corresponds to Sperner's Theorem when  $P = B_n$ )

*Proof.*

□