## Algebraic Combinatorics - HW7

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**Problem 1** (Lattice Paths avoiding a certain set of points). Let P be a fixed lattice (ballot) path from (0,0) to (m,n). Let T be a set of interior points on P (that is, some subset of points on P other than (0,0) and (m,n)). Let  $f_{m,n}(T)$  be the number of lattice paths from (0,0) to (m,n) that avoid all of T.

- (i) Use a Corollary of Gessel-Viennot Lemma to find an expression for  $f_{m,n}(T)$ .
- (ii) Use Inclusion-Exclusion to find an expression for  $f_{m,n}(T)$ .

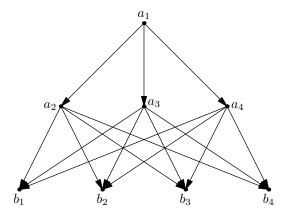
Proof. (i) We begin with the formula for the total number of lattice paths from (0,0) to (m,n):  $\binom{m+n}{m}$ . First, consider the case where T has only one point with coordinates  $(a_1,b_1)$ . The number of lattice paths from (0,0) to  $(a_1,b_1)$  is  $\binom{a_1+b_1}{a_1}$ , and the number of lattice paths from  $(a_1,b_1)$  to (m,n) is  $\binom{(m-a_1)+(n-b_1)}{m-a}$ . Multiplying these two together, we get  $\binom{a_1+b_1}{a_1} * \binom{(m-a_1)+(n-b_1)}{m-a}$  as the number of lattice paths we must avoid, so the total number of lattice paths from (0,0) to  $(a_1,b_1)$ .

**Problem 2** (Linear dependency and Gessel-Vienot). Let  $A_{n\times n}$  be a matrix with linearly dependent rows. Show by using Gessel-Viennot Lemma that |A|=0.

*Proof.* Let  $A_{n\times n}$  represent the edge matrix of a weighted DAG from  $A=\{a_1,a_2,\ldots,a_n\}$  and  $B=\{b_1,b_2,\ldots,b_n\}$ . Let  $e_{ij}=[A_{ij}]$ . Then, since the rows are linearly independent, the first row can be written as a linear combination of the other rows, say

$$e_{1j} = \sum_{2 \le i \le n} c_i e_{ij}.$$

We can construct our graph with  $a_1$  above all the other  $a_i$ , with the example for n=4 shown below:



We can set the edge from  $a_1$  to  $a_k$  to have weight  $c_k$ , whence we have the linear combination relationship in A. For example,

$$e_{11} = e_{21}c_2 + e_{31}c_3 + e_{41}c_4.$$

By the Gessel-Viennot Lemma, the determinant of A is the sum of the weights of all path systems from A to B. However, notice that the path system from  $a_1$  to  $b_{\sigma(a_1)}$  will have an odd number of vertices, while the path system from all other  $a_i$  to  $b_{\sigma(a_i)}$  will have an even number of vertices, while |A| + |B| = 2n is even. Therefore, there are no path systems, so

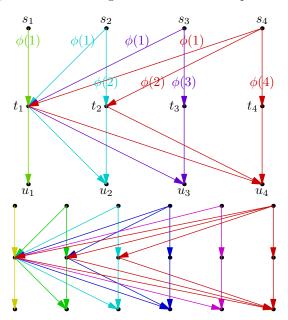
$$\det A = \sum_{\mathscr{P}} \operatorname{sign}(\mathscr{P}) \omega(\mathscr{P}) = 0.$$

**Problem 3** (GCD matrix). Let  $S = \{a_1, a_2, \dots, a_n\} \subset \mathbb{N}$ . let the GCD matrix M of S have entries  $m_{ij} = \gcd(a_i, a_j)$ . Prove that if S is closed under taking divisors, then

$$|M| = \prod_{i=1}^{n} \varphi(a_i).$$

(*Hint:* Form a certain digraph using three copies of S, and then put certain edges and weights on them.)

*Proof.* We form a digraph between S, T, and U, where T and U are two copies of S, placing edges in a way that allows us to make use of the Totient Function's divisor sum. For a given  $1 \le i \le n$ , let d be a divisor of i. Then, we place an edge from  $s_i$  to  $t_d$  of weight  $\phi(d)$  and an edge from  $t_d$  to  $u_i$  of weight 1. We give the examples of n = 4 and n = 6, where the colorings are relevant for our path counting argument:



Consider the edge matrix A of this graph, where  $a_{ij}$  is the number of paths from  $s_i$  to  $u_j$ . By our setup,

$$a_{ij} = \sum_{d|i,j} \phi(d) = \sum_{d|(i,j)} \phi(d) = (i,j),$$

so A=M. By the Gessel-Viennot Lemma, |M|=|A|, which is the sum of the weights of all path systems from S to U. Notice that  $\sigma(s_1)$  is necessarily  $u_1$ , whence  $\sigma(s_p)=u_p$  for all prime p. We can continue inductively on the number of divisors to conclude that  $\sigma(s_k)=u_k$  for all  $1 \leq k \leq n$ , so we only have one path system  $\mathscr{P}$ , and  $\omega(\mathscr{P})=\prod_{i=1}^n \varphi(i)$ , as desired.

**Problem 4** (Determinant of a matrix of Stirling Numbers). For  $m \ge 0$ ,  $n \ge 1$ , prove the following identity. Here  $S_{n,k}$  is the Stirling number of the  $2^{\text{nd}}$  kind.

$$\det \begin{pmatrix} S_{m+1,1} & S_{m+1,2} & \cdots & S_{m+1,n} \\ S_{m+2,1} & S_{m+2,2} & \cdots & S_{m+2,n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{m+n,1} & S_{m+n,2} & \cdots & S_{m+n,n} \end{pmatrix} = (n!)^m$$

*Proof.* The Stirling number of the second kind  $S_{n,k}$  is the number of ways to partition a set of n elements into k partitions. We wish to construct a DAG from A to B (where |A| = |B| = n) such that the number of paths from  $a_i$  to  $b_j$  is  $S_{m+i,j}$ .