Algebraic Combinatorics - HW7

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Problem 1 (Lattice Paths avoiding a certain set of points)**.** Let P be a fixed lattice (ballot) path from $(0,0)$ to (m,n). Let T be a set of interior points on P (that is, some subset of points on P other than $(0,0)$ and (m,n)). Let $f_{m,n}(T)$ be the number of lattice paths from $(0,0)$ to (m,n) that avoid all of T.

(i) Use a Corollary of Gessel-Viennot Lemma to find an expression for $f_{m,n}(T)$.

(ii) Use Inclusion-Exclusion to find an expression for $f_{m,n}(T)$.

Problem 2 (Linear dependency and Gessel-Vienot)**.** Let $A_{n\times n}$ be a matrix with linearly dependent rows. Show by using Gessel-Viennot Lemma that $|A| = 0$.

Problem 3 (GCD matrix)**.** Let $S = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{N}$. let the **GCD matrix** M of S have entries $m_{ij} = \gcd(a_i, a_j)$. Prove that if S is closed under taking divisors, then

$$|M| = \prod_{i=1}^{n} \varphi(a_i).$$

(*Hint:* Form a certain digraph using three copies of S, and then put certain edges and weights on them.)

Problem 4 (Determinant of a marix of Stirling Numbers)**.** For $m \geq 0$, $n \geq 1$, prove the following identity. Here $S_{n,k}$ is the Stirling number of the 2^{nd} kind.

$$\det \begin{pmatrix} S_{m+1,1} & S_{m+1,2} & \cdots & S_{m+1,n} \\ S_{m+2,1} & S_{m+2,2} & \cdots & S_{m+2,n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{m+n,1} & S_{m+n,2} & \cdots & S_{m+n,n} \end{pmatrix} = (n!)^m$$