HW 10 - Lin. Alg. Method in Combinatorics (Due Tue 05/07)

(All problems are Extra-Credit)

1. [Oddtown/Eventown revisited]

(i) [Eventown] Prove the upper bound of $2^{\lfloor n/2 \rfloor}$ in Eventown. Show that it is achievable for every n.

(ii) [Reverse Oddtown]: Let us switch the role of *odd* and *even* in the Oddtown Rules. Prove that, still no more than n clubs can be formed. Show that the maximum is n when n is odd and $n-1$ when n is even.

(iii) [Maximal vs. Maximum families in Oddtown and Eventown]

(a) Show that in Eventown, if there are fewer than $2^{\lfloor n/2 \rfloor}$ clubs, then there is always a room for a new club without violating the rules; that is, every maximal family is also maximum.

(b) This is not true for Oddtown, that is, show that, given $0 \le t \le \frac{n-1}{2}$, there is a way of forming a maximal family of $n - 2t$ clubs.

(*Remark*: Other variations: Bipartite Oddtown, Skew Oddtown, mod-p^k-town, mod-s-town, etc.)

2. [Spherical 2-distance Sets] Define the $(n-1)$-dimensional unit sphere as

$$S^{n-1} = \left\{ \overrightarrow{x} \in \mathbb{R}^n : \|\overrightarrow{x}\| = 1 \right\}$$

A *spherical* 2-*distance set* is a 2-distance subset of S^{n-1}. Let $m_s(n)$ be the maximum size of a spherical 2-distance set. Show that

$$\frac{n(n+1)}{2} \le m_s(n) \le \frac{n(n+3)}{2}$$

Let $A_1, \cdots, A_m \subset [n]$. Assume that their pairwise symmetric differences have only two sizes. Prove that

$$m \le \frac{n(n+3)}{2}$$

Improve the upper bound to the following and show that it is tight

$$m \le 1 + \frac{n(n+1)}{2}$$

3. [(Uniform) Ray-Chaudhuri − Wilson 1975] Let $L \subseteq \{0, 1, \cdots, n-1\}$ such that $|L| = s$ and $A_1, \cdots, A_m$ be a k-uniform, L-intersecting family of subsets of $[n]$, then

$$m \le \binom{n}{s}$$

(*Hint*: Start as in the proof of non-uniform RW taking the same polynomials $f_1, \cdots, f_m$ of degree at most s. Then for each $I \subseteq [n]$ with $|I| \le s-1$, associate polynomials $g_I(x) = \left(\left(\sum_{j=1}^{n} x_j\right) - k \right) \prod_{i \in I} x_i$ of degree s. Show that f_is and g_Is together are linearly independent.)

4. [Hadwiger-Nelson problem: Coloring Unit Distance Graphs; **Not for credit**] Color the points in $\mathbb{R}^n$ with minimum number of colors such that every two points unit distance apart, gets different colors. Equivalently, one can consider an infinite graph with vertices as points in $\mathbb{R}^n$ and two points are adjacent iff they are at distance 1. Let $c(n)$ be the chromatic number of such a *unit-distance graph*. Finding the exact value of $c(n)$ is open even for the plane ($n = 2$). What we know so far is

$$5 \le c(2) \le 7$$

The lower bound of 4 is given by the *Moser spindle*. The better lower bound of 5 is an unit-distance graph on 1581 vertices and was found in 2018 by computer scientist and biologist **Aubrey de Grey**. The proof was computer assisted. A smaller graph on 509 vertices was found by the crowd-sourced online collaboration **Polymath Project**. In general,

$$(1.1)^n \le c(n) \le \left(2\sqrt{2} + o(1)\right)^n$$

(*Remark*: The lower bound construction is given by **Frankl-Wilson**'s Omitted intersection Theorem.)