Algebraic Combinatorics: HW5

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> **Problem 1** (Pairing $2n$ people)**.** The Lemma to prove Burnside's Lemma states that when a group G acts on X, the number of permutations that map x to y is the same for all y in the orbit of x. Use this to count the ways to pair up $2n$ people. (*Remark*: This can also be shown by a very easy combinatorial argument.)

Proof. Consider two rows of n people, and represent this ordering by an arrangement of the numbers from 1 to $2n$, where the $2i-1$ and $2i$th people are in a pair. Construct the set X (size $2n!$), and let our group $G = S_n \oplus (S_2)^n$ (swapping all n pairs, and shuffling them amongst each other). Then, notice that there is only one permutation that maps one pairing arrangement to another, so there will be orbits of size $|G|$, for a total number of possible distinct pairings of

$$\frac{|X|}{|G|} = \frac{2n!}{n!2^n}.$$

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> **Problem 2.** Partitions of an integer n are obtained from compositions of n by ignoring the order of the parts. Use Burnside's Lemma and the symetric group S_3 to count the partitions of 9 into three parts. List all such partitions explicitly.

Proof. S_3 fixes various partitions as follows:

- (123) and (132) fix only the $3+3+3$ partition. There is 1 such partition.
- (12), (13), and (23) fix one ordering of the $1|1|7$, $2|2|5$, $4|4|1$, and $3|3|3$ partitions. For example, (23) fixes $7|1|1, 5|2|2, 1|4|4, 3|3|3$.
- The identity fixes all $\binom{9-3+2}{2} = \binom{8}{2} = 28$ partitions.

Thus, by Burnside's Lemma, the total number of partitions is

$$\frac{1+1+4+4+4+28}{6} = \boxed{7}.$$

By simple enumeration, the partitions are as follows:

- $1|1|7$

- 1|2|6
- 1|3|5
- 1|4|4
- 2|2|5
- 2|3|4
- 3|3|3

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Problem 3 (Burnside's $\Rightarrow$ Fermat's Theorem)**.** Let p, n, l be positive integers with p prime. Use induction (on l) and Burnside's Lemma to prove that

$$p^l \bigg| \left(n^{p^l} - n^{p^{l-1}}\right)$$

Proof. We first show the base case $p|n^p - n$.

Consider the number of ways to color a p-necklace with n colors, where we are only concerned about rotational symmetries. From p's primality, the fixed points of any non-identity rotation are the a constant-color necklaces. Thus, the number of distinct colorings is

$$\frac{1}{p}(n^p + (p-1)n),$$

so $p|n^p + (p-1)n$. Therefore, $p|n^p - n$.

Now, assume that the statement holds for all integers up to $l-1$. Consider the number of ways to color a p^l-necklace with n colors, where we are concerned with rotational symmetries, that is, the cyclic group $G \cong \mathbb{Z}_{p^l}$. G has a generator π, the unit rotation.

Consider the number of fixed points of π^i, where $0 \leq i < p^l$. If $i \neq 0$ and $p|i$, then π^i fixes $n^{p^{l-1}}$ necklaces, since our necklace has $p^l/p = p^{l-1}$ free beads. There are $p^{l-1} - 1$ such i.

Otherwise, we fix the n constant color necklaces, or when $i = 0$, fix all n^{p^l} necklaces.

Thus, by Burnside's Lemma, the number of colorings is

$$\frac{1}{p^l}((p^{l-1}-1)n^{p^{l-1}} + (p^l - p^{l-1})(n) + (n^{p^l})) = \frac{1}{p^l}(n^{p^l} - n^{p^{l-1}} + p^{l-1}(n^{p^{l-1}} - n) + p^l n).$$

Notice that

$$\begin{aligned} n^{p^{l-1}} - n &= (n^{p^{l-1}} - n^{p^{l-2}}) + (n^{p^{l-2}} - n^{p^{l-3}}) + \cdots + (n^p - n) \\ &\equiv 0 \pmod{p}, \end{aligned}$$

so $p^l | p^{l-1}(n^{p^{l-1}} - n)$, whence

$$\frac{1}{p^l}(n^{p^l} - n^{p^{l-1}})$$

is an integer, so $p^l | n^{p^l} - n^{p^{l-1}}$, as desired. Therefore, by strong induction, our proof is complete.

And that's the end of the proof. I hope you enjoyed it. It was a pleasure to do. Goodbye, goodbye, goodbye. Goodbye, goodbye, goodbye. Goodbye, goodbye, goodbye. Goodbye, goodbye, goodbye. I will continnue this conclusion by induction. The base case is "goodbye." The inductive step is "goodbye." The conclusion is "goodbye." The proof is complete. Goodbye. ■

Problem 4 (Crowns with missing jewels; Extra-Credit)**.** A crown with n places for diamonds is missing k of them. How many distinguishable ways can this happen? In other words, how many convex k-gons can be formed from the vertices of a regular n-gon, where two k-gons are considered distinguishable when they do not arise from each other by rotations.

Proof. We apply Polyá's Enumeration Theorem. Let G be the cyclic group of order n. We want the coefficient of the z^k term in

$$\frac{1}{n}\sum_{d|n}\phi(d)z_d^{n/d},$$

and $z_d = (1 + z^d)$ (z is like the 'generator'). There is a nonzero coefficient of z^k in $(1 + z^d)^{n/d}$ when $k|d$, and this coefficient is $\binom{n/d}{k/d}$. Thus, our desired coefficient is the sum

$$\boxed{\frac{1}{n}\sum_{d|\gcd(k,n)}\phi(d)\binom{n/d}{k/d}}.$$

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