

# Algebraic Combinatorics: HW6

various.

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## Problem 1 (Trees with prescribed degrees and Cayley's formula).

- (a) Given positive integers  $d_1, d_2, \dots, d_n$  such that  $\sum d_i = 2n - 2$ , show that the number of labelled trees on  $[n]$  such that vertex  $i$  has degree  $d_i$  for each  $i$  is

$$\frac{(n-2)!}{\prod (d_i - 1)!}.$$

- (b) Prove Cayley's formula from (a).  
 (c) What is the number of all trees on  $n$  vertices with exactly  $n - l$  leaves? (Hint: You may use (a) and leave your answer in terms of *Stirling's number of the second kind*.)

*Proof.* (a) The proof via Prüfer codes is a trivial arrangement argument. Proceeding by induction on  $n$ , the base case  $n = 1$  is also trivial. So, assume that the statement holds for  $n - 1$ , that is, there are

$$\frac{(n-3)!}{\prod_{i=1}^{n-1} (d_i - 1)!}$$

ways to create a labelled tree with  $\sum d_i = 2n - 4$ .

For  $n$  vertices, notice that there must be at least one vertex with degree 1 since the sum of degrees is  $2n - 2$ . Assign  $d_n$  to be the vertex with degree 1. Then, we have  $n - 1$  remaining vertices ways to connect the  $n$ th vertex to. If we connect to the  $k$ th vertex, then we are interested in the number of labelled trees with degrees  $d_1, d_2, \dots, d_k - 1, \dots, d_{n-1}$ , which is  $\frac{(n-3)!}{\prod_{i=1}^{n-1} (d_i - 1)!} \cdot (d_k - 1)$ . So, the total number of valid trees on  $[n]$  vertices is

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{(n-3)!}{\prod_{i=1}^{n-1} (d_i - 1)!} \cdot (d_k - 1) &= \frac{(n-3)!}{\prod_{i=1}^{n-1} (d_i - 1)!} \cdot \sum_{k=1}^{n-1} (d_k - 1) \\ &= \frac{(n-3)!}{\prod_{i=1}^{n-1} (d_i - 1)!} \cdot (2n - 3 - (n - 1)) \\ &= \frac{(n-2)!}{\prod_{i=1}^{n-1} (d_i - 1)!} \cdot \frac{1}{0!} \\ &= \frac{(n-2)!}{\prod_{i=1}^n (d_i - 1)!}, \end{aligned}$$

as desired. The proof by induction is complete.

- (b) Finding the total number of labelled trees on  $[n]$  is the same to summing  $\frac{(n-2)!}{\prod (d_i-1)!}$  over all possible degrees  $d_1, d_2, \dots, d_n$  such that  $\sum d_i = 2n - 2$ . Note that  $\frac{(n-2)!}{\prod (d_i-1)!}$  is the number of ways to order  $n - 2$  numbers in a row, where  $x$  appears  $d_x - 1$  times, so our sum is counting the number of ways to list  $n - 2$  integers in an ordered row, where each number is between 1 and  $n$ , which is precisely  $n^{n-2}$ .

This proof effectively travels through Prüfer codes.

- (c) Let vertices  $l + 1$  through  $n$  be the leaves, so  $d_{l+1} = d_{l+2} = \dots = d_n = 1$ . The sum of the remaining degrees is  $2n - 2 - (n - l) = n + l - 2$ . By (a), the number of trees is then

$$\frac{(n-2)!}{\prod_{i=1}^l (d_i - 1)!}$$

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**Problem 2** (Counting Spanning trees of  $K_{m,n}$ ). Find the value of  $\tau(K_{m,n})$  using:

- (i) Matrix-Tree Theorem.
- (ii) Combinatorial argument, say, that of Prüfer or Joyal.
- (iii) Let  $L$  be the Laplacian of  $K_{m,n}$ .
  - (a) Find a simple upper bound on  $\text{rank}(L - mI)$ .

*Proof.* 1. Recall that the Matrix-Tree Theorem states that  $\tau(G) = |L_0|$ . We let vertices  $v_1$  through  $v_m$  be within the first partite and  $v_{m+1}$  through  $v_{m+n}$  be within the second partite. Then, the Laplacian of  $K_{m,n}$  is

$$L = \begin{pmatrix} mI_n & -J \\ -J & nI_m \end{pmatrix},$$

where  $J$  is the all-ones matrix. Then, we have

$$L_0 = \begin{pmatrix} mI & -J \\ -J & nI \end{pmatrix}_{\{v_1, v_2, \dots, v_{m+n-1}\}} = \begin{pmatrix} mI & -J \\ -J & nI \end{pmatrix}_{\{v_1, v_2, \dots, v_m\}}.$$

The determinant of this matrix is

$$mI \cdot nI - J^2 = mIn - nJ^2 = mIn - nJn = n(m - n),$$

so  $\tau(K_{m,n}) = n(m - n)$ .

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**Problem 4.** Starting at a point  $x_0$  we walk along the edges of a connected graph  $G$  according to the following rules:

- We never use the same edge twice in the same direction.
- Whenever we arrive at a point  $x \neq x_0$  not previously visited, we mark the edge along which we entered  $x$ . We use the marked edge to leave  $x$  only if we must, that is, if we have used all the other edges before.

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Show that we get stuck at  $x_0$ , and that, by then, every edge has been traversed in both direction.

*Proof.* Set the vertex at which we get stuck to be  $v$ . Suppose, for the sake of contradiction, then  $v \neq x_0$ . Then, if we consider the number of times we have traversed an edge to arrive *at*  $v$ , this must be one more than the number of times we have traversed an edge to leave *from*  $v$ . Thus, there must be at least one edge that we have not traversed in both directions, so we may traverse that edge and leave. Therefore, we must be stuck at  $x_0$ .

We now show that, when we get stuck at  $x_0$ , every other edge must be traversed in both directions, where we duplicate each edge and direct it, to create the enter-exit effect. Then, consider the “in-tree” rooted at  $x_0$ . Suppose, for the sake of contradiction, that we get stuck at  $x_0$  when there is still some untraversed edges. At least one of those edges but be an edge of the in-tree rooted at  $v$ , and let the point closest to  $x_0$  (in the in-tree) on an untraversed edge of the in-tree be  $y$ . Since we travel to and from  $y$  an equal number of times, there is an “exit” from  $y$  we don’t use, so  $y$  is closer to  $x_0$  than the initial vertex on the untraversed edge, meaning that  $y = v$  (by the choice of edge). But, if there is an in-edge to  $x_0$ , then we must certainly also have an unused out-edge, contradicting our stuckness. So, all edges are traversed in our manually constructed directed graph, meaning all edges are traversed twice in the original graph. ■

- Problem 5 (Universal cycles for  $S_n$ ).** (i) Let  $n \geq 3$ . Show that there does not exist a sequence  $a_1, a_2, \dots, a_{n!}$  such that all the  $n!$  contiguous blocks  $a_i, a_{i+1}, \dots, a_{i+n-1}$  (subscripts taken modulo  $n!$ ) are all the  $n!$  permutations of  $S_n$ .
- (ii) Show that for all  $n \geq 1$ , there exist a sequence  $a_1, a_2, \dots, a_{n!}$  such that all the  $n!$  contiguous blocks  $a_i, a_{i+1}, \dots, a_{i+n-2}$  consists of the first  $n-1$  terms  $b_1, b_2, \dots, b_{n-1}$  of all permutations  $b_1, b_2, \dots, b_n$  of  $[n]$ .  
Such sequences are called **universal cycles** for  $S_n$  (for example, for  $n = 3$ , 123213 is a universal cycle.)
- (iii) For  $n = 3$ , find the number of universal cycles beginning with 123.

*Proof.* (i) There are  $(n-1)!$  sequences that begin with a given  $i$  (integer  $1 \leq i \leq n$ ), so the is must be evenly spaced throughout the sequence, which means that the sequence is just  $(n-1)!$  copies of  $a_1, a_2, \dots, a_n$ , which will certainly not contain all  $n!$  permutations of  $S_n$ .

For the example of  $n = 3$ , the requirement of 2 blocks beginning with 1, 2, and 3 force a structure of  $ABCABC$ , which, because of the repeat, doesn’t contain all  $n!$  permutations.

(ii)

(iii) The beginning 123 gives blocks of 12 and 23, so the next term can be a block of 32 or 31, giving cases of:

- (a) 1232: we’ve accounted for 23 already, so the next number must be 1. In 12321, we already have a 12, so the next number must be 3, yielding the universal cycle 123213.
- (b) 1231: we’ve accounted for 12 already, so the next number must be 3. In 12313, we already have a 31, so the last number must be a 2, yielding the universal cycle 123132.

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We confirm that all permutations are present, so the two universal cycles are

$12313, 123132$ .

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