HW4

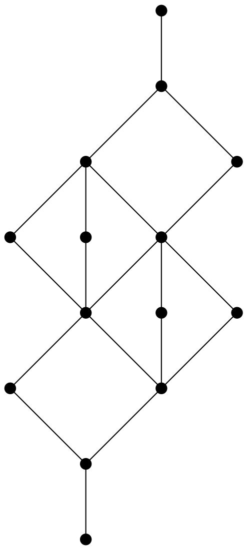
People

Date

Problem 1 (Some (counter)-example)**.**

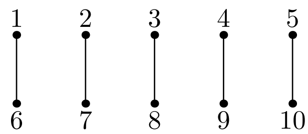
(i) Give an example of a finite graded poset P with the Sperner property, together with a group G acting on P, such that P/G is *not* Sperner.

(ii) Consider the poset P whose Hasse diagram is given by

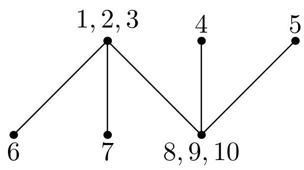


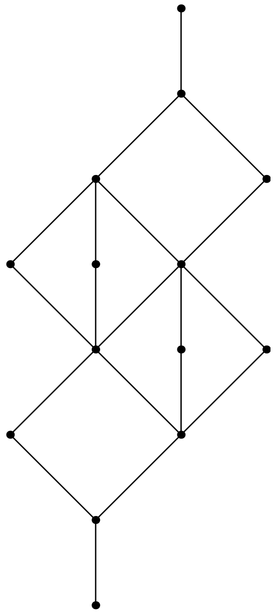
Find a subgroup G of S_7 such that $P \cong B_7/G$ or else prove that such a group does not exist.

Proof. (i) We draw a Hasse diagram for P:



We see that P is Sperner by inspection; its largest antichain is of length four, and each rank has four elements. Let $G = ((1,2,3),(8,9,10))$ be the group generated by the permutations $(1,2,3)$ and $(8,9,10)$. By drawing the Hasse diagram of P/G, we see that is clearly not Sperner:



(ii) 

□

Problem 2 (Binary Necklace Poset)**.** A $(0,1)$-*necklace* of *length* n and *weight* i is a circular arrangement of i 1's and $n-i$ 0's. For instance, the $(0,1)$-necklaces of length 6 and weight 3 are (writing a circular arrangement linearly) $000111, 001011, 010011$ and 010101. Cyclic shifts of a linear word represent the same necklace.

(i) (easy) Show that N_n is rank-symmetric, rank-unimodal and Sperner.

Proof. Notice that $N_n \cong B_n/G$, where G is the group generated by the shift-by-one permutation $(1,2)(2,3),\dots,(n-1,n)$. This action also naturally preserves order, because order is strictly adding beads. Thus, by Theorem 5.8, N_n is rank-symmetric, rank-unimodal, and Sperner. □

Problem 3. Suppose X is a finite set with n elements. Let G be a group of permutations on X. Thus G acts on 2^X. We say that G acts *transitively* on the j-element subsets if for every two j-element subsets S and T, there is a $\pi \in G$ for which $\pi \cdot S = T$. Show that if G acts transitively on j-element subsets for some $j \le \frac{n}{2}$, then G acts transitively on i-element subsets for all $0 \le i \le j$.

Proof. We proceed by induction on i, with G acting transitively on i-element subsets. Consider any two $i-1$-element subsets S_1 and T_1. Since $i \le \frac{n}{2}$, we are guaranteed at least one element $a \in X$ not in S_1 or T_1. Define the function ϕ from the $i-1$ to the i-subsets of X as the addition of a. Then, $\phi(S_1)$ and $\phi(T_1)$ are i-subsets, and by the inductive hypothesis, there is some $\pi \in G$ for which $\pi \cdot \phi(S_1) = \phi(T_1)$. Note that

$$\pi(\phi(S_1)) = \pi(S_1 \cup \{a\}) = T_1 \cup \{a\},$$

and so since this is a permutation, we must have $\pi \cdot S_1 = T_1$. Thus, G acts transitively on $i-1$-element subsets, and the induction is complete. □