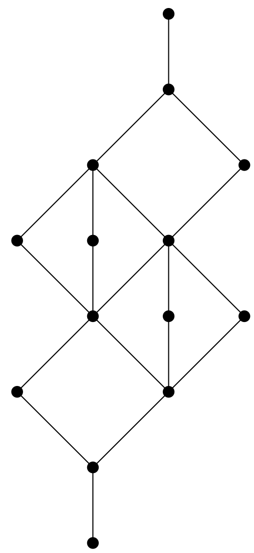
HW 4 - Group Actions on B_n (Due Thursday 2/29)

0. [**Warm-up**; **Not-for-credit**] Draw the Hasse diagram of the poset of nonisomorphic simple graphs with 5 vertices (with subgraph containment ordering). What is the size of the largest antichain ? How many antichains have this size ?

1. [Some (counter)-examples]

(i) Give an example of a finite graded poset P with the Sperner property, together with a group G acting on P, such that P/G is *not* Sperner (From our lectures we know that P cannot be B_n)

(ii) Consider the poset P whose Hasse diagram is given by



Find a subgroup G of S_7 such that $P \cong B_7/G$ or else prove that such a group does not exist.

2. [Binary Necklace Poset]

A $(0,1)$-*necklace* of *length* n and *weight* i is a circular arrangement of i 1's and $n-i$ 0's. For instance, the $(0,1)$-necklaces of length 6 and weight 3 are (writing a circular arrangement linearly) $000111, 001011, 010011$ and 010101. Cyclic shifts of a linear word represent the same necklace.

Let N_n denote the set of all $(0,1)$-necklaces of length n. Define a partial order $\leq$ on N_n by letting $u \leq v$ if we can obtain v from u by changing some 0's to 1's. Clearly N_n is graded of rank n, where the rank of a necklace being its weight.

(i) (easy) Show that N_n is rank-symmetric, rank-unimodal and Sperner.

(ii) (difficult; not-for-credit) Show that N_n has a symmetric chain decomposition.

(iii) (unsolved; not-for-credit) Show that every quotient poset B_n/G has a symmetric chain decompositions.

3. [Transitive Group Action]

Suppose X is a finite set with n elements. Let G be a group of permutations on X. Thus G acts on 2^X. We say that G acts *transitively* on the j-element subsets if for every two j-element subsets S and T, there is a $\pi \in G$ for which $\pi \cdot S = T$. Show that if G acts transitively on j-element subsets for some $j \leq \frac{n}{2}$, then G acts transitively on i-element subsets for all $0 \leq i \leq j$.

4. [On Switching-reconstructability; **for Grad students**]

(i) Let $\mathcal{G}_n$ be the set of all simple graphs on $[n]$, so $|\mathcal{G}_n| = 2^{\binom{n}{2}}$. Given $G \in \mathcal{G}_n$, let G_i be the graph obtained by *switching* at vertex i, that is, deleting all edges incident to i, and adding every edge from i that is not in G. Define a linear transformation

$$\phi : \mathbb{R}\mathcal{G}_n \to \mathbb{R}\mathcal{G}_n \text{ by } \phi(G) = G_1 + G_2 + \cdots + G_n$$

Show that ϕ is invertible iff $n \not\equiv 0 \pmod 4$

(ii) The graph G is *switching-reconstructible* if it can be uniquely reconstructed from the (multi)set of *unlabelled* vertex switches G_i. Show that G is switching-reconstructible if $n \not\equiv 0 \pmod 4$.

(iii) (unsolved; not-for-credit) Show that G is switching-reconstructible if $n \neq 4$

(iv) Show that the number of edges can be determined from the multiset of unlabelled G_i's if $n \neq 4$. Find two graphs with 4 vertices and a different number of edges, but with the same unlabelled $G_i's$

(v) Define G to be *weakly switching-reconstructible* if it can be uniquely reconstructed from the multiset of *labelled* vertex switches G_i. That is, we are given each G_i as a labelled graph, but we are not told the vertex i that was switched. Show that G is weakly switching-reconstructible if $n \neq 4$, but that G need not be weakly switching-reconstructible if $n = 4$.