Algebraic Combinatorics: HW6

various.

April 1, 2024

Problem 1 (Trees with prescribed degrees and Cayley's formula).

(a) Given positive integers $d_1, d_2, \ldots, d_n$ such that $\sum d_i = 2n-2$, show that the number of labelled trees on $[n]$ such that vertex i has degree d_i for each i is

$$\frac{(n-2)!}{\prod (d_i-1)!}.$$

(b) Prove Cayley's formula from (a).

(c) What is the number of all trees on n vertices with exactly $n-l$ leaves? (Hint: You may use (a) and leave your answer in terms of *Stirning's number of the second kind.*)

Proof. (a) The proof via Prüfer codes is a trivial arrangement argument. Proceeding by induction on n, the base case $n=1$ is also trivial. So, assume that the statement holds for $n-1$, that is, there are

$$\frac{(n-3)!}{\prod_{i=1}^{n-1}(d_i-1)!}$$

ways to create a labelled tree with $\sum d_i = 2n-4$.

For n vertices, notice that there must be at least one vertex with degree 1 since the sum of degrees is $2n-2$. Assign d_n to be the vertex with degree 1. Then, we have $n-1$ remaining vertices ways to connect the nth vertex to. If we connect to the kth vertex, then we are interested in the number of labelled trees with degrees $d_1, d_2, \ldots d_k - 1, \ldots, d_{n-1}$, which is $\frac{(n-3)!}{\prod_{i=1}^{n-1}(d_i-1)!} \cdot (d_k - 1)$. So, the total number of valid trees on $[n]$ vertices is

$$\begin{aligned}
\sum_{k=1}^{n-1} \frac{(n-3)!}{\prod_{i=1}^{n-1}(d_i-1)!} \cdot (d_k-1) &= \frac{(n-3)!}{\prod_{i=1}^{n-1}(d_i-1)!} \cdot \sum_{k=1}^{n-1}(d_k-1) \\
&= \frac{(n-3)!}{\prod_{i=1}^{n-1}(d_i-1)!} \cdot (2n-3-(n-1)) \\
&= \frac{(n-2)!}{\prod_{i=1}^{n-1}(d_i-1)!} \cdot \frac{1}{0!} \\
&= \frac{(n-2)!}{\prod_{i=1}^{n}(d_i-1)!},
\end{aligned}$$

as desired. The proof by induction is complete.

(b) Finding the total number of labelled trees on $[n]$ is the same to summing $\frac{(n-2)!}{\prod(d_i-1)!}$ over all possible degrees $d_1, d_2, \ldots, d_n$ such that $\sum d_i = 2n-2$. Note that $\frac{(n-2)!}{\prod(d_i-1)!}$ is the number of ways to order $n-2$ numbers in a row, where x appears $d_x - 1$ times, so our sum is counting the number of ways to list $n-2$ integers in an ordered row, where each number is between 1 and n, which is precisely n^{n-2}.

This proof effectively travels through Prüfer codes.

(c) Let vertices $l+1$ through n be the leaves, so $d_{l+1} = d_{l+2} = \cdots = d_n = 1$. The sum of the remaining degrees is $2n-2-(n-l) = n+l-2$. By (a), the number of trees is then

$$\frac{(n-2)!}{\prod_{i=1}^{l}(d_i-1)!}$$

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Problem 2 (Counting Spanning trees of $K_{m,n}$)**.** Find the value of $\tau(K_{m,n})$ using:

1. Matrix-Tree Theorem.
2. Combinatorial argument, say, that of Prüfer or Joyal.
3. Let L be the Laplacian of $K_{m,n}$.
   (a) Find a simple upper bound on $\operatorname{rank}(L - mI)$.

Proof. 1. Recall that the Matrix-Tree Theorem states that $\tau(G) = |L_0|$. We let vertices v_1 through v_m be within the first partite and v_{m+1} through v_{m+n} be within the second partite. Then, the Laplacian of $K_{m,n}$ is

$$L = \begin{pmatrix} mI_n & -J \\ -J & nI_m \end{pmatrix},$$

where J is the all-ones matrix. Then, we have

$$L_0 = \begin{pmatrix} mI & -J \\ -J & nI \end{pmatrix}_{\{v_1, v_2, \ldots, v_{m+n-1}\}} = \begin{pmatrix} mI & -J \\ -J & nI \end{pmatrix}_{\{v_1, v_2, \ldots, v_m\}}.$$

The determinant of this matrix is

$$mI \cdot nI - J^2 = mIn - nJ^2 = mIn - nJn = n(m-n),$$

so $\tau(K_{m,n}) = n(m-n)$.

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Problem 3. Starting at a point x_0 we walk along the edges of a connected graph G according to the following rules:

- We never use the same edge twice in the same direction.
- Whenever we arrive at a point $x \neq x_0$ not previously visited, we mark the edge along which we entered x. We use the marked edge to leave x only if we must, that is, if we have used all the other edges before.

> Show that we get stuck at x_0, and that, by then, every edge has been traversed in both direction.

Proof. Set the vertex at which we get stuck to be v. Suppose, for the sake of contradiction, then $v \neq x_0$. Then, if we consider the number of times we have traversed an edge to arrive *at* v, this must be one more than the number of times we have traversed an edge to leave *from* v. Thus, there must be at least one edge that we have not traversed in both directions, so we may traverse that edge and leave. Therefore, we must be stuck at x_0.

We now show that, when we get stuck at x_0, every other edge must be traversed in both directions. ■