Algebraic Combinatorics HW 3

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Problem 1 (Symmetric polynomial and unimodality)**.** $f(x) = p_0 + p_1x + p_2x^2 + \cdots + p_nx^n$ is *symmetric* if for all i,

$$p_i = p_{n-i}$$

It is *unimodal* if for some fixed j,

$$p_0 \leq p_1 \leq \cdots \leq p_{j-1} \leq p_j \geq p_{j+1} \geq \cdots \geq p_{n-1} \geq p_n$$

Let $F(q)$, $G(q)$ be symmetric and unimodal polynomials with non-negative real coefficients. Show that $F(q)G(q)$ is also symmetric (easy) and unimodal (less easy).

Proof. □

Problem 2 (Log-concavity of Binomial coefficients)**.** A sequence $a_1, a_2, a_3, \cdots, a_n$ is *logarithmically concave* if

$$a_i^2 \geq a_{i-1}a_{i+1} \ ; \ \forall i$$

(i) Show that if a sequence of positive terms is log-concave, then it is also unimodal.

(ii) It is easy to see algebraically that the sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \cdots, \binom{n}{n}$ is log-concave. Give a combinatorial proof of this fact.

Proof. □

Problem 3 (Uniqueness in Sperner's Thm)**.** Show that equality in Sperner's Theorem for B_n is achieved only by the middle (middle two) rank(s) if n is even (odd). (*Hint*: If not, then move the example closer to the middle rank(s))

Proof. □

Problem 4 (A generalization of Sperner's Thm)**.** Let P be a rank-symmetric, rank-unimodal poset. Show that if P has a symmetric chain decomposition, then it has *strong Sperner property*, that is, for any $j \geq 1$, the largest size of a union of j antichains is equal to the size of the largest j levels of P. (*Remark*: $j = 1$ corresponds to Sperner's Theorem when $P = B_n$)

Proof. □