HW4

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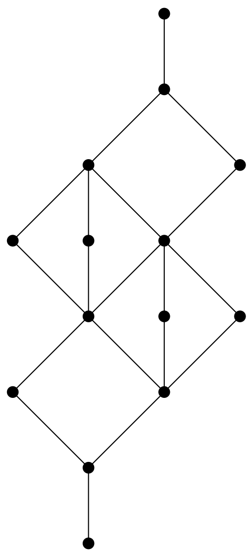
Problem 1 (Some (counter)-example).

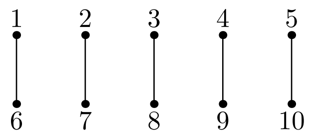
(i) Give an example of a finite graded poset P with the Sperner property, together with a group G acting on P, such that P/G is *not* Sperner.

(ii) Consider the poset P whose Hasse diagram is given by

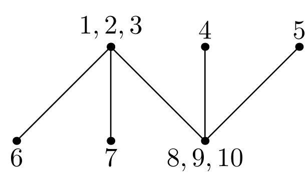


Find a subgroup G of S_7 such that $P \cong B_7/G$ or else prove that such a group does not exist.

Proof (Dallin and Evan): (i) We draw a Hasse diagram for P:



We see that P is Sperner by inspection; its largest antichain is of length four, and each rank has four elements. Let $G = ((1,2,3),(8,9,10))$ be the group generated by the permutations $(1,2,3)$ and $(8,9,10)$. By drawing the Hasse diagram of P/G, we see that is clearly not Sperner:



(ii) We claim that B_n/G where G is a group generated by permutations is isomorphic to B_m, where $m \leq n$. This result is since we effectively end up with the set of all subsets of $\{O_1, O_2, \ldots\}$ where the O_i are orbits in $\{1, 2, \ldots, n\}$ under the action of G. This will be isomorphic to B_m under the mapping $O_m \to m$. However, the given poset has ranks of $1, 1, 2, 3, 3, 2, 1, 1$, which are not the ranks of any boolean algebra.

□

Problem 2 (Binary Necklace Poset)**.** A $(0, 1)$-*necklace* of *length* n and *weight* i is a circular arrangement of i 1's and $n - i$ 0's. For instance, the $(0, 1)$-necklaces of length 6 and weight 3 are (writing a circular arrangement linearly) $000111, 001011, 010011$ and 010101. Cyclic shifts of a linear word represent the same necklace.

(i) (easy) Show that N_n is rank-symmetric, rank-unimodal and Sperner.

Proof (Evan): Notice that $N_n \cong B_n/G$, where G is the group generated by the shift-by-one permutation $(1, 2)(2, 3), \ldots, (n - 1, n)$. This action also naturally preserves order, because order is strictly adding beads. Thus, by Theorem 5.8, N_n is rank-symmetric, rank-unimodal, and Sperner. □

Problem 3. Suppose X is a finite set with n elements. Let G be a group of permutations on X. Thus G acts on 2^X. We say that G acts *transitively* on the j-element subsets if for every two j-element subsets S and T, there is a $\pi \in G$ for which $\pi \cdot S = T$. Show that if G acts transitively on j-element subsets for some $j \leq \frac{n}{2}$, then G acts transitively on i-element subsets for all $0 \leq i \leq j$.

Proof (Dallin and Evan): We proceed by induction on i, with G acting transitively on i-element subsets. Consider any two $i-1$-element subsets S_1 and T_1. Since $i \leq \frac{n}{2}$, we are guaranteed at least one element $a \in X$ not in S_1 or T_1. Define the function ϕ from the $i - 1$ to the i-subsets of X as the addition of a. Then, $\phi(S_1)$ and $\phi(T_1)$ are i-subsets, and by the inductive hypothesis, there is some $\pi \in G$ for which $\pi \cdot \phi(S_1) = \phi(T_1)$. Note that

$$\pi(\phi(S_1)) = \pi(S_1 \cup \{a\}) = T_1 \cup \{a\},$$

and since π is a permutation, we must have $\pi \cdot S_1 = T_1$. Thus, G acts transitively on $i - 1$-element subsets, and the induction is complete. □