

# Algebraic Combinatorics HW 1

EVAN LIM

2-3-2024

**Problem 1** (Closed Walks in  $K_n$ ). Find a combinatorial proof of the fact that  $\#$  closed walks of length  $l$  in  $K_n$  from some vertex to itself is

$$\frac{1}{n} \left( (n-1)^l + (n-1)(-1)^l \right)$$

*Proof.* We'll count the number of closed  $l$ -walks from  $v_1$  to  $v_1$ , which, by symmetry, is the number of closed  $l$ -walks from any vertex to itself. We write each walk as an ordered list of vertices, so that we must find the number of walks

$$v_1, v_{i_1}, v_{i_2}, \dots, v_{i_{l-1}}, v_1,$$

where  $v_{i_1} \neq v_1$ ,  $v_{i_{l-1}} \neq v_1$ , and  $v_{i_k} \neq v_{i_{k+1}}$  for all  $1 \leq k \leq l-2$ . We count starting with  $(n-1)^l$ , or the number of  $l-1$ -walks starting with  $v_1$ : each subsequent vertex is distinct from the preceding one, for  $n-1$  options. However, we overcount by the number of sequences where  $v_{i_{l-1}} = v_1$ , which is the number of closed  $l-1$ -walks<sup>1</sup>. The idea is that we take any  $l-1$ -walk, and make our next step  $v_1$ , but we need to subtract the closed  $l-1$  walks, which would make our final step impossible. Repeating the same reasoning, we can find the number of closed  $l-1$ -walks by taking  $(n-1)^{l-1}$  and then subtracting the number of closed  $l-2$ -walks. In this way, and finishing with  $n-1$  2-walks, the number of closed  $l$ -walks from a vertex to itself is

$$\begin{aligned} & (n-1)^{l-1} - (n-1)^{l-2} + (n-1)^{l-3} + \dots + (-1)^l (n-1) \\ &= \sum_{k=1}^{l-1} (n-1)^k (-1)^{l-1+k} \\ &= (-1)^{l-1} \sum_{k=1}^{l-1} (1-n)^k \\ &= (-1)^{l-1} \frac{(1-n)((1-n)^{l-1} - 1)}{1-n-1} \\ &= (-1)^l \frac{(1-n)^l - (1-n)}{n} \\ &= \frac{1}{n} \left( (n-1)^l + (n-1)(-1)^l \right), \end{aligned}$$

as desired. □

---

<sup>1</sup>This suggests that we could complete via an inductive proof, using

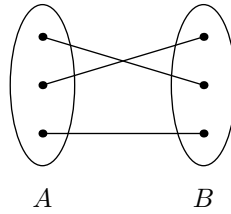
$$(n-1)^l - \frac{1}{n} \left( (n-1)^{l-1} + (n-1)(-1)^{l-1} \right) = \frac{1}{n} \left( (n-1)^l + (n-1)(-1)^l \right).$$

**Problem 2** (Eigenvalues of some bipartite graphs).

- (i) Let  $G[A, B]$  be a bipartite graph with partite sets  $A, B$ . Show by a walk-counting argument that the non-zero eigenvalues of  $G$  come in pairs  $\pm\lambda$ .  
 (Eigenvalues of  $K_{rs}$ ) Consider the complete bipartite graph  $K_{r,s}$  (that is, having partite sets of size  $r$  and  $s$ )
- (ii) Use purely combinatorial reasoning to compute the number of closed walks of length  $l$  in  $K_{r,s}$ .
- (iii) Deduce the eigenvalues of  $K_{r,s}$ .  
 (Eigenvalues of  $K_{n,n} - nK_2$ ) Let  $H_n$  be the graph  $K_{n,n}$  with a perfect matching removed.
- (iv) Show that the eigenvalues of  $H_n$  are

$$\pm 1(n-1 \text{ times}), \pm(n-1)(\text{once each}).$$

*Proof.*



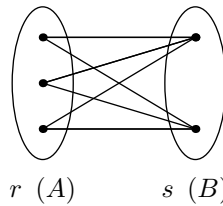
- (i) Every step on a walk takes us between partite sets  $A$  and  $B$ . Thus, there are no  $2l+1$ -walks, meaning that

$$\sum (\lambda_i)^{2l+1} = 0,$$

so

$$\sum (-\lambda_i)^{2l+1} = -\sum (\lambda_i)^{2l+1} = 0 = \sum (\lambda_i)^{2l+1}.$$

As  $\sum (\lambda_i)^{2l} = \sum (-\lambda_i)^{2l}$ ,  $\sum \lambda_i^k$  and  $\sum (-\lambda_i)^k$  agree for all positive integers  $k$ , so the  $-\lambda_i$  are simply a permutation of the  $\lambda_i$ , meaning that all nonzero eigenvalues come in  $\pm\lambda$  pairs.



- (ii) Call the partite with  $r$  elements  $A$  and the partite with  $s$  elements  $B$ . If  $l$  is odd, there are zero walks. So, we assume  $l$  is even. If we begin our  $l$ -walk in  $A$ , we know go from  $A$  to  $B$   $l/2$  times and  $B$  to  $A$   $l/2$  times. Each time we go from  $A$  to  $B$ , we have  $s$  options. Each time we go from  $B$  to  $A$ , we have  $r$  options, except the last step, at which point we must return to our original vertex, for which we have  $r$  choices. There are thus  $s^{l/2} r^{l/2} = (rs)^{l/2}$   $l$ -walks beginning in  $A$ , and an identical argument gives  $(rs)^{l/2}$   $l$ -walks beginning in  $B$ . There are thus

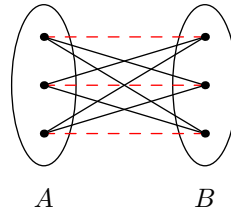
$$\begin{cases} 0 & l \equiv 1 \pmod{2} \\ 2(rs)^{l/2} & l \equiv 0 \pmod{2} \end{cases}$$

$l$ -walks.

- (iii)  $\sum \lambda_i^l$  and  $(rs)^l + (-rs)^l + (n-2) \cdot (0)^l$  agree for all positive  $l$ , so the eigenvalues of  $K_{r,s}$  are

$$\pm rs, 0 \text{ (} r+s-2 \text{ times)}.$$

We now consider the  $K_{n,n} - nK_2$  graph, providing  $n = 3$  as an example:



- (iv) We aim to find the number of  $l$ -walks (for even  $l$ ) on  $K_{n,n} - nK_2$ . If we write the partites as  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  such that the  $a_i$  and  $b_i$  are not connected, but  $a_i$  and  $b_j$  for  $i \neq j$  are connected, then our walk-counting problem becomes analogous to the  $K_n$  problem. Any valid walk will alternate between  $A$  and  $B$ , but have *no index repeated twice in a row*. That is,  $a_1 b_2$  is a valid step, while  $a_1 b_1$  is not. We can use this to establish a bijection between  $l$ -walks starting in  $A$  and  $l$ -walks on  $K_n$ , meaning that the *total* number of  $l$ -walks on  $K_{n,n} - nK_2$  is

$$2n \left( \frac{1}{n} \left( (n-1)^l + (n-1)(-1)^l \right) \right) = 2(n-1)^l + 2(n-1)(-1)^l.$$

For odd  $l$ , the number of walks is clearly 0, and so  $\sum \lambda_i^l$  agrees for all positive  $l$  with

$$(n-1)^l + (1-n)^l + (n-1)((-1)^l + (1)^l),$$

meaning that our eigenvalues are

$$\pm 1(n-1 \text{ times}), \pm(n-1)(\text{once each}).$$

□

**Problem 3** (On the largest eigenvalue of  $A(G)$ ; Extra credit).

- (i) Let  $G$  be a graph with max degree  $\Delta(G)$ . Let  $\lambda_1$  be the largest eigenvalue of  $A(G)$ . Show that  $\lambda_1 \leq \Delta(G)$ .
- (ii) Let  $G$  be a simple graph with  $m$  edges. Show that  $\lambda_1 \leq \sqrt{2m}$ .

*Proof.* (i)

□

**Problem 4.**

- (i) Start with  $n$  coins heads up. Choose a coin at random and turn it over. Do this a total of  $m$  times. What is the probability that all coins will have heads up?
- (ii) Same as (i), except now compute the probability that all coins have tails up.
- (iii) Same as (i), but now we turn over two coins at a time.

*Solution.* (i) By writing the chain of coins as a string of ones and zeros (where a zero corresponds to a tail), each flip changes exactly one digit. We can thus view the process of  $m$  flips as an  $m$ -walk on an  $n$ -dimensional hypercube, making our probability

$$\frac{\# \text{ closed } m\text{-walks from a given vertex to itself}}{\# m\text{-walks starting at a given vertex}},$$

which is also

$$\frac{\# \text{ closed } m\text{-walks}}{\# m\text{-walks}} \text{ on } \mathbb{Z}_2^n.$$

We find the number of closed  $m$ -walks on  $\mathbb{Z}_2^n$  by finding the graph's eigenvalues. We view  $\mathbb{Z}_2^n$  as the graph direct sum of  $\mathbb{Z}_2^{n-1}$  and  $\mathbb{Z}_2$ , that is, a bipartite graph with partites of size  $2^{n-1}$ . From Q2(ii), for even  $m$ , we have

$$2(2^{n-1}2^{n-1})^m = (2^{2n-2})^m + (-2^{2n-2})^m$$

closed  $m$ -walks. For odd  $m$ , we have  $0 = (2^{2n-2})^m + (-2^{2n-2})^m$  closed  $m$ -walks. Therefore,  $(2^{2n-2})^m + (-2^{2n-2})^m$  and  $\sum \lambda_i^m$  agree for all positive  $m$ , so the eigenvalues are precisely  $\pm 2^{2n-2}$  and some number of 0s. Thus, the number of closed  $m$ -walks is  $(2^{2n-2})^m + (-2^{2n-2})^m$ . There are naturally  $n^m$  total possible  $m$ -walks, by the combinatorial argument that we choose between  $n$  coins to flip  $m$  times (or that each edge has). Therefore, the probability that a given  $m$ -walk on this hypercube is closed, which is also the probability we end our  $m$ -flips with all heads, is

$$\boxed{\frac{(4^{n-1})^m + (-4^{n-1})^m}{n^m}}.$$

- (ii) We want to find the number of  $m$ -walks from vertex  $\underbrace{000 \cdots 000}_{n \text{ zeroes}}$  to vertex  $\underbrace{111 \cdots 111}_{n \text{ ones}}$ , which is  $(A(G)^m)_{0,2^n}$ . Note that since the eigenvalues we previously found are  $\pm 2^{2n-2}$  and  $2^n - 2$  zeroes, we can diagonalize

$$A(G) = S \text{diag}(2^{2n-2}, -2^{2n-2}, \underbrace{0, 0, 0, \dots, 0}_{2^n-2}) S^{-1},$$

with some permutation of these diagonal elements. Then,

$$\begin{aligned} (A(G))^{k+2} &= S \text{diag}(2^{(2n-2)(k+2)}, (-2^{2n-2})^{k+2}, 0, \dots) S^{-1} \\ &= S \text{diag}((2^{2n-2})^2 2^{(2n-2)k}, (2^{2n-2})^2 (-2^{2n-2})^k, 0, \dots) S^{-1} \\ &= 2^{4n-4} S \text{diag}(2^{(2n-2)k}, (-2^{2n-2})^k, 0, \dots) S^{-1} \\ &= 2^{4n-4} (A(G))^k, \end{aligned}$$

so if we know the number of valid  $m$ -walks, we know the number of valid  $m + 2c$  walks. That is, if we know the number of ways to go from all heads to all tails in  $m$  moves, we know the number of ways to do so in all  $m + 2c$  moves. The fewest number of moves in which we could flip from all heads to all tails is  $n$ , which is simply flipping each coin. There are also naturally  $n!$  ways to do our flipping in  $n$  moves, so if  $m$  is the same parity as  $n$ , the number of valid  $m$ -walks is

$$2^{(4n-4)\frac{m-n}{2}} n! = 4^{(n-2)(m-n)} n!.$$

Notice that, by the coin-flipping mechanism, when the  $m$  and  $n$  are not the same parity, we have no possibilities (we will be in the wrong partite of the graph).

$$\frac{4^{(n-2)(m-n)} n!}{n^m}$$

FINISH EXPLANATION

□

**Problem 5.** Let  $G_n$  be the graph with vertex set  $\mathbb{Z}_2^n$  with the edge set defined as:  $u$  and  $v$  are adjacent iff they differ in exactly two coordinates (that is,  $\omega(u+v) = 2$ ). What are the eigenvalues of  $G_n$ ?