Algebraic Combinatorics HW 2

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Problem 1 (Random Walks on $\mathbb{Z}$)**.** Consider a random walk on $\mathbb{Z}$ where we start at 0 and move from i to $i+1$ or $i-1$ with equal probability.

(i) Prove that we eventually return to 0 with probability 1.

(ii) Compute a_i explicitly and conclude that

$$\sum_{n=0}^{\infty} \frac{a_n}{2_n} = 1$$

What is the sequence a_n called in Math literature?

(iii) Prove that each number n is visited at least once with probability 1.

(iv) Let H_n denote the expected # steps needed to reach n for the first time. What is wrong with the following argument? We claim that $H_n = cn$ for some constant c. This is true for $n = 0$. So let $n > 0$. On the average, we need H_1 steps to reach 1, and then H_{n-1} steps to reach n starting from 1. Hence

$$H_n = 1 + H_{n-1} = c + c(n-1) = cn; H_1 = c$$

Proof. (i) Let a_i be the number of ways to return to 0 after i steps.

□

Problem 2 (Some examples of Hitting times)**.** (i) Find the hitting time between any two vertices of K_n.

(ii) Find the hitting time between the endpoints of P_n (a path on n vertices).

(iii) Find the hitting time between an endpoint of P_n and a vertex at distance k from it.

(iv) Find the hitting time between two vertices of C_n (cycle of n vertices) at distance k.

(v) Find the hitting time between two 'antipodal' vertices of Q_3.

Problem 3. (i) Show that the following may hold for some graphs G (including regular graphs)

$$H(u, v) \neq H(v, u), \text{ for some } u, v \in V(G).$$

(ii) If u and v have the same degree, then the probability that a random walk starting at u vists v before returning to u is equal to the probability that a random walk starting at v vists u before returning to u. What can be said if the degrees of u and v are different?