Algebraic Combinatorics - HW8

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4/16/2024

Problem 1 (Saving electricity through Linear Algebra and Graph Theory)**.** (i) Consider the vector space $V = \mathbb{F}_2^n$ (over $\mathbb{F}_2$). Let $A_{n\times n}$ be a symmetric matrix over $\mathbb{F}_2$. Consider the diagonal of A, as a column vector $\vec{d}$. Prove that

$$\vec{d} \in Col(A)$$

(*Hint*: First show that $\vec{v}^T A\vec{v} = \vec{d}^T\vec{v}$ for all $\vec{v} \in V$. Then show the required result by contradiction.)

(ii) Given a graph G, show (using (i)) that $V(G)$ can be partitioned into V_1 and V_2 such that $G[V_1]$ is an even graph (that is, all degrees are even) and for all vertices $v \in V_2$, $|N(v) \cap V_1|$ is odd.

(iii) Assume that there is a bulb and a button at each vertex of a graph G. The connections are made such that pushing the button at a vertex once, will change the status of the bulb and its neighbors. Initially all bulbs are on. Show (using (ii)) that one can push some buttons and turn all the bulbs off. Does this remind you of a game that you may have played as a kid?

Proof. ∎

Problem 2 (Cycle Space and Bond Space in Graphs + some applications)**.** Given an undirected connected graph G with n vertices and m edges, let each subset of the edge set $E(G)$ be represented by its characteristic binary vector. Let

$$\begin{aligned} \mathcal{C}(G) &= \{\text{all even subgraphs of } G\} \\ \mathcal{B}(G) &= \{\text{all minimal edge-cuts of } G\} \\ M(G) &= \text{incidence matrix of } G \end{aligned}$$

(i) Show that both $\mathcal{C}(G)$ and $\mathcal{B}(G)$ are subspaces of $\mathbb{F}_2^m$ (over $\mathbb{F}_2$).

(ii) Show that the *stars* at any $n-1$ vertices of G are linearly independent and forms a basis in $\mathcal{B}(G)$, thus

$$\mathcal{B}(G) = \text{Row}(M(G)) \quad \text{and} \quad \dim(\mathcal{B}(G)) = n-1.$$

(iii) Given any spanning tree T of G, show that the *fundamental cycles* (as described in class) are linearly independent in $\mathcal{C}(G)$. Then show that

$$\mathcal{C}(G) = (\mathcal{B}(G))^{\perp} \quad \mathcal{C}(G) = \text{Nul}(M(G)) \quad \dim\big(\mathcal{C}(G)\big) = m-n+1$$

(iv) Use the above to show that:

(a) A graph is bipartite $\Leftrightarrow$ every circuit is of even length. (*Hint*: Show that $\overrightarrow{j} = (1, 1, \cdots, 1) \in \mathcal{B}(G)$)

(b) Show that for any graph G, $E(G)$ can be partitioned into an even graph and an edge-cut.

(c) Use (b) to give a new proof of $\#1(ii)$ (and hence $\#1(iii)$)