Algebraic Combinatorics - HW7

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Problem 1 (Lattice Paths avoiding a certain set of points)**.** Let P be a fixed lattice (ballot) path from $(0,0)$ to (m,n). Let T be a set of interior points on P (that is, some subset of points on P other than $(0,0)$ and (m,n)). Let $f_{m,n}(T)$ be the number of lattice paths from $(0,0)$ to (m,n) that avoid all of T.

(i) Use a Corollary of Gessel-Viennot Lemma to find an expression for $f_{m,n}(T)$.

(ii) Use Inclusion-Exclusion to find an expression for $f_{m,n}(T)$.

Proof. (i) We begin with the formula for the total number of lattice paths from $(0,0)$ to (m,n): $\binom{m+n}{m}$. First, consider the case where T has only one point with coordinates (a_1,b_1). The number of lattice paths from $(0,0)$ to (a_1,b_1) is $\binom{a_1+b_1}{a_1}$, and the number of lattice paths from (a_1,b_1) to (m,n) is $\binom{(m-a_1)+(n-b_1)}{m-a}$. Multiplying these two together, we get $\binom{a_1+b_1}{a_1} * \binom{(m-a_1)+(n-b_1)}{m-a}$ as the number of lattice paths we must avoid, so the total number of lattice paths from $(0,0)$ to (a_1,b_1).

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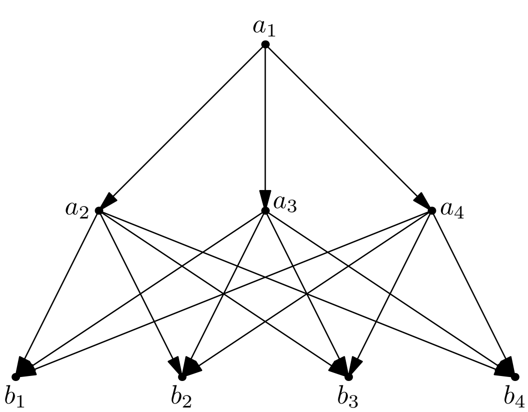
Problem 2 (Linear dependency and Gessel-Vienot)**.** Let $A_{n\times n}$ be a matrix with linearly dependent rows. Show by using Gessel-Viennot Lemma that $|A| = 0$.

Proof. Let $A_{n\times n}$ represent the edge matrix of a weighted DAG from $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Let $e_{ij} = [A_{ij}]$. Then, since the rows are linearly independent, the first row can be written as a linera combination of the other rows, say

$$e_{1j} = \sum_{2\leq i\leq n} c_i e_{ij}.$$

We can construct our graph with a_1 above all the other a_i, with the example for $n = 4$ shown below:



We can set the edge from a_1 to a_k to have weight c_k, whence we have the linear combination relationship in A. For example,

$$e_{11} = e_{21}c_2 + e_{31}c_3 + e_{41}c_4.$$

By the Gessel-Viennot Lemma, the determinant of A is the sum of the weights of all path systems from A to B. However, notice that the path system from a_1 to $b_{\sigma(a_1)}$ will have an odd number of vertices, while the path system from all other a_i to $b_{\sigma(a_i)}$ will have an even number of vertices, while $|A| + |B| = 2n$ is even. Therefore, there are no path systems, so

$$\det A = \sum_{\mathscr{P}} \operatorname{sign}(\mathscr{P})\omega(\mathscr{P}) = 0.$$

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> **Problem 3** (GCD matrix)**.** Let $S = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{N}$. let the **GCD matrix** M of S have entries $m_{ij} = \gcd(a_i, a_j)$. Prove that if S is closed under taking divisors, then
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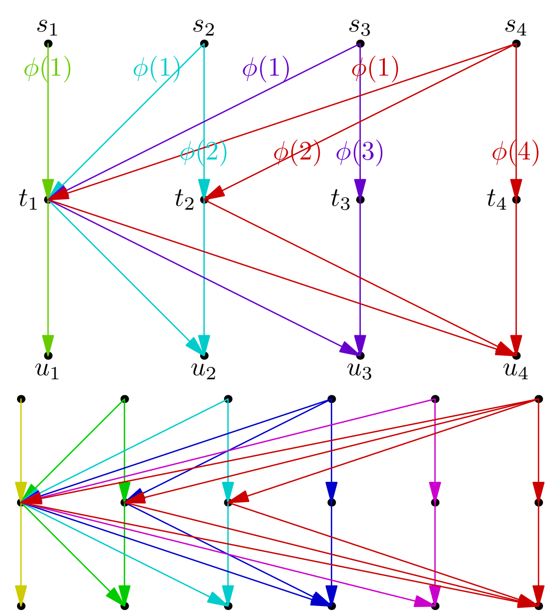
> $$|M| = \prod_{i=1}^{n} \varphi(a_i).$$
>
> (*Hint:* Form a certain digraph using three copies of S, and then put certain edges and weights on them.)

Proof. We form a digraph between S, T, and U, where T and U are two copies of S, placing edges in a way that allows us to make use of the Totient Function's divisor sum. For a given $1 \leq i \leq n$, let d be a divisor of i. Then, we place an edge from s_i to t_d of weight $\phi(d)$ and an edge from t_d to u_i of weight 1. We give the examples of $n = 4$ and $n = 6$, where the colorings are relevant for our path counting argument:



Consider the edge matrix A of this graph, where a_{ij} is the number of paths from s_i to u_j. By our setup,

$$a_{ij} = \sum_{d|i,j} \phi(d) = \sum_{d|(i,j)} \phi(d) = (i,j),$$

so $A = M$. By the Gessel-Viennot Lemma, $|M| = |A|$, which is the sum of the weights of all path systems from S to U. Notice that $\sigma(s_1)$ is necessarily u_1, whence $\sigma(s_p) = u_p$ for all prime p. We can continue inductively on the number of divisors to conclude that $\sigma(s_k) = u_k$ for all $1 \leq k \leq n$, so we only have one path system $\mathscr{P}$, and $\omega(\mathscr{P}) = \prod_{i=1}^{n} \varphi(i)$, as desired. ■

Problem 4 (Determinant of a matrix of Stirling Numbers)**.** For $m \geq 0$, $n \geq 1$, prove the following identity. Here $S_{n,k}$ is the Stirling number of the 2^{nd} kind.

$$\det \begin{pmatrix} S_{m+1,1} & S_{m+1,2} & \cdots & S_{m+1,n} \\ S_{m+2,1} & S_{m+2,2} & \cdots & S_{m+2,n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{m+n,1} & S_{m+n,2} & \cdots & S_{m+n,n} \end{pmatrix} = (n!)^m$$

Proof. The Stirling number of the second kind $S_{n,k}$ is the number of ways to partition a set of n elements into k partitions. We with to construct a DAG from A to B (where $|A| = |B| = n$) such that the number of paths from a_i to b_j is $S_{m+i,j}$. ■