Counting Eulerian Circuits.

- A closed walk without repeating edges is a circuit.
- A circuit without repeating vertices is a cycle.
- A circuit using every edge (exactly once) is an Eulerian circuit.
- A cycle using every vertex (exactly once) is a Hamiltonian cycle (i.e, a spanning cycle).

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Thm [Euler 1736; Good]

A connected digraph G is

$$\boxed{\text{Eulerian} \iff G \text{ is `balanced'}} \quad (d^+(v) = d^-(v), \text{ for all } v).$$

Pf: easy; say use induction.

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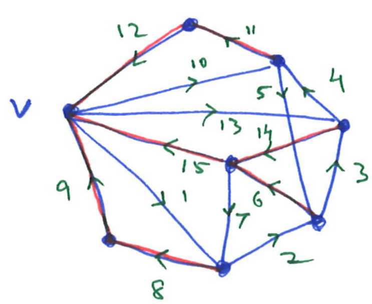
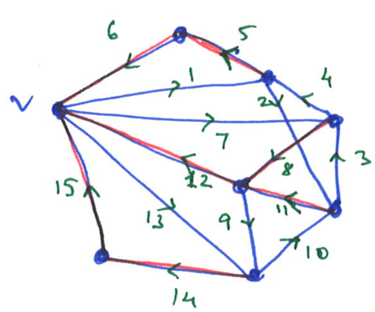
$\boxed{Q}$. How do we count # Eulerian tours in G?

Consider the following two Eulerian tours in the same G.





If one considers the last edge leaving every $u (\neq v)$ we get an 'in-tree' rooted at v.

Thm [BEST Thm ; Ehrenfest + de-Bruijn 1951
Tutte- Smith . 1941]

let $\boxed{G}$ be a connected balanced digraph. (actually, an orientation of the (undirected) graph G)

let $\boxed{e} \in E(G)$ with v as its tail.

let $\boxed{\tau^-(v)}$ = # sp. in-trees rooted at v.

$\tau^+(v)$ = # sp. out-trees . . . .

let $\boxed{\epsilon(G,e)}$ = # Eulerian tours starting at e ; then.

$$\boxed{\epsilon(G,e) = \tau^-(v) \cdot \prod_{u \in V(G)} (d^+(u) - 1)!} \quad ; \quad \forall e.$$

Pf: let $\mathcal{E} = \underset{\overset{\shortparallel}{e}}{e_1}, e_2, \ldots, e_m$ be an Eulerian tour in G

For each $u \ (\neq v)$, let $e(u)$ = last exit-edge from u.

Claim 1: $S = \{e(u) : u \ (\neq v)\}$ forms a sp. in-tree rooted at v.

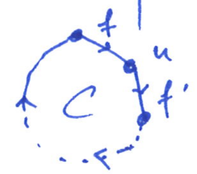
Pf: • S has $n-1$ edges

• For any two edges $f_1, f_2 \in S$;
it cannot be that $f_1 = e(u) = f_2$.

• $\nexists f \in S : f = e(v)$

• S does not create any (directed) cycle C.

If not, let f be the last edge in the tour $\mathcal{E}$ belonging to C. But then, there must be an edge $f' \in C$ such that the final vertex of f = initial vertex of f' = u $(\neq v)$



But there is no way to exit u via f', since f occurs after f' in $\mathcal{E}$; a contradiction.

Claim 2: Converse of Claim 1 is true.

Given G, and a sp. in-tree T rooted at v, we create an Eulerian tour $\mathcal{E}$ as follows:

" Start with any out-edge of v.
Continue to choose any edge possible to continue the tour; except we never choose an edge of T unless we have to".

Pf: We must show that we never get stuck until all edges are used. Moreover, the set of last exits of $\mathcal{E}$ from u $(\neq v)$ coincides with the edges of T.

The only way to get stuck is to end up at v.
Now let us assume that there were still some unused edges. At least one of these edges must be a last-exit edge; that is, an edge of T.
Let f be the unique edge of T which is unused, with u = initial vertex of f being closest to v (in T). Let $y = \text{final}(f)$.
Suppose $y \neq v$. Since we enter y as often as we leave it, we don't use the last exit of y.
This means y is closer to v in T than u.
$\therefore y = v$. But this means we can leave v; a contradiction.
So we have shown that every Eulerian tour $\mathcal{E}$ starting at e has an associated "last exit" in-tree, with root v.
On the other hand, given an in-tree T with root v; we can obtain all Eulerian tours $\mathcal{E}$ starting at e such that $T = T(\mathcal{E})$.

$$\therefore \epsilon(G, e) = \tau^{-}(v) \cdot \prod_{u \in V(G)} (d^{+}(u) - 1)!$$

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Obs: $\boxed{\tau^-(u_1) = \tau^-(u_2)}$, $\forall\, u_1, u_2 \in V(G)$.

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Q. How do we find $\tau^-(v)$?

Thm: [Directed Matrix-Tree Thm ; Tutte]

Let G be a connected digraph on $\{v_1, v_2, \ldots, v_n\}$

Let $L^+(G)$ be defined as

$$L^+_{ij} = \begin{cases} d^+(v_i) & ; \quad i = j \\ -m_{ij} & ; \quad i \neq j \text{ and there are } m_{ij} \text{ edges } (v_i, v_j) \end{cases}$$

Note $\boxed{L^+(G) = D^+ - A'}$; $A'_{ij} = \#$ edges (v_i, v_j)

Let $\boxed{L^+_0(G)}$ be the matrix obtained by deleting the last row and column; then

$$\boxed{\tau^-(v_n) = |L^+_0|}$$

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Obs: $\boxed{\text{DMTT} \Rightarrow \text{MTT}}$ (why ?)

Given undirected G, create $\hat{G}$ by duplicating every edge and giving them opposite direction.

Show $L(G) = L^+(\hat{G})$.

Also sp. trees in $G \Leftrightarrow$ sp in-trees of $\hat{G}$.

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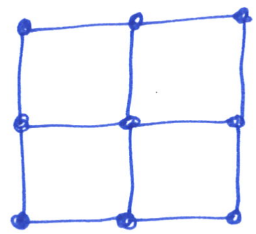
ex: [Efficient mail carrier]

City blocks and intersections are represented by the following graph G.



He (she) must walk along each block twice and end where he started from. i.e we must find # Eulerian walks in $\hat{G}$.

This graph G has 192 spanning trees.

$\therefore$ # eulerian walks beginning with a fixed edge $= 192 \cdot (1!)^4 (2!)^4 (3!)^1$ $= 18432$.

$\therefore$ # different eulerian tours $= 18432 \times 24 = \boxed{442368}$

Assuming mail is delivered 250 times a year The carrier has ~ 1769 years before he would have to repeat a route !.

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ex: [Counting de Bruijn sequences]

A binary de Bruijn sequence of degree n is a binary sequence $A = a_1 a_2 a_3 \ldots a_{2^n}$ such that every binary sequence $b_1 b_2 \cdots b_n$ of length n occurs exactly once as a contiguous block of A, that is $a_i a_{i+1} \cdots a_{i+n-1}$ where indices are taken mod 2^n.

- $n=2$: 0011 $\qquad \# = 1$
- $n=3$: 00010111 00011101 $\qquad \# = 2$

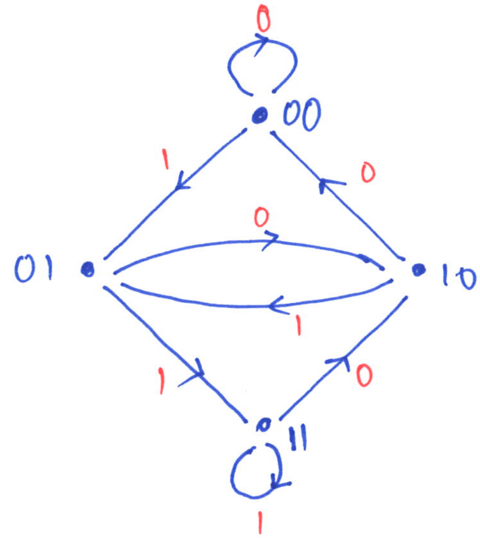
$\vdots \qquad \vdots$

- $n=8$: $\vdots$ $\qquad \# = 1329227995784915872903807060280344576$

$\vdots$

To enumerate de Bruijn sequences, we will establish a bijection between them and Eulerian tours in a directed graph D_n (de Bruijn graph of degree n).

for $\underline{n=3}$:



Eulorian tour:

$11 \xrightarrow{0} 10 \xrightarrow{0} 00 \xrightarrow{0} 00 \xrightarrow{1} 01 \xrightarrow{0} 10 \xrightarrow{1} 01 \xrightarrow{1} 11 \xrightarrow{1} 11$

$\Updownarrow$

0001011 ; a de Bruijn sequence .

Note: in any D_n , $d^+ = d^- = 2$

$\therefore$ # de Bruijn seq. of degree n (upto cyclic symmetry) $=$ # sp. in-trees of D_n $= \tau^-(v) = |L_0^+| = \boxed{\frac{1}{2^{n-1}} \lambda_1 \cdot \lambda_2 \cdots \lambda_{2^n-1}}$

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Thm: E. values of $L^+(D_n)$ are
- 0 (multiplicity 1)
- 2 (" $2^{n-1}-1$)

$\therefore |L_0^+| = \frac{1}{2^{n-1}} \cdot 2^{2^{n-1}-1} = \boxed{2^{2^{n-1}-n}}$

Pf: Let $A'(D_n)$ be the directed adjacency matrix of D_n.

Let $D^+(D_n)$ be the diagonal matrix of outdegrees of D_n.

then, $L^+ = D^+ - A'$

$= 2I - A'$.

$\therefore$ e.values of L^+ are $2 -$ e.values of A'

Now $A_{uv} = \begin{cases} 1 & \text{if } (u,v) \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$

Also note $A^{n-1} = J = 2^{n-1} \times 2^{n-1}$ matrix of all 1's (why?).

we know e.values of J are $\underbrace{2^{n-1}}_{\text{once}}, \underbrace{0, 0, \cdots, 0}_{2^{n-1}-1 \text{ times}}$.

$\therefore$ e.values of A are $\underbrace{2\omega}, 0, 0, \cdots, 0$

$\omega = (n-1)^{th}$ root of unity

but $tr(A) = 2 \Rightarrow \omega = 1$.

$\therefore$ e. values of L^+ are $\underbrace{0}_{\text{once}}, \underbrace{2, 2, \cdots, 2}_{2^{n-1}-1 \text{ times}}$.

$\therefore |L_0^+| = \frac{1}{2^{n-1}}\left[2^{2^{n-1}-1}\right] = 2^{2^{n-1}-n}$.

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Remark: • If we do not care about cyclic symmetry.

$\#$ de-Bruijn seq of deg $n = 2^{2^{n-1}}$ (call it N).

$\#$ binary seq. of length ~~2~~ $2^n = 2^{2^n} = N^2$.

Q. Is there an explicit bijection.

$\psi : B_n \times B_n \longrightarrow A_n$ (B_n = set of all de-Bruijn seq of deg n

A_n = set of all bin. seq of length 2^n).

Yes! found by Bidkhori-Kishore (2009).

• P. Diaconis & R. Graham's book has many entertaining applications of these ideas to Magic !!

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