Algebraic Combinatorics HW 2

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Problem 1.1 (Random Walks on $\mathbb{Z}$)**.** Consider a random walk on $\mathbb{Z}$ where we start at 0 and move from i to $i+1$ or $i-1$ with equal probability.

(i) Prove that we eventually return to 0 with probability 1.

(ii) Compute a_i explicitly and conclude that

$$\sum_{n=0}^{\infty} \frac{a_n}{2^n} = 1$$

What is the sequence a_n called in Math literature?

(iii) Prove that each number n is visited at least once with probability 1.

(iv) Let H_n denote the expected # steps needed to reach n for the first time. What is wrong with the following argument?

We claim that $H_n = cn$ for some constant c. This is true for $n = 0$. So let $n > 0$. On the average, we need H_1 steps to reach 1, and then H_{n-1} steps to reach n starting from 1. Hence

$$H_n = H_1 + H_{n-1} = c + c(n-1) = cn; H_1 = c$$

Proof. (i) Denote the probability that, starting at a, we eventually reach b, by $P(a \to b)$. Then, notice that our second step either brings us to ± 2 or to 0. Thus,

$$P(0 \to 0) = \frac{1}{2} + \frac{1}{2}P(2 \to 0) = \frac{1}{2} + \frac{1}{2}P(2 \to 1)P(1 \to 0) = \frac{1}{2} + \frac{1}{2}P(1 \to 0)^2.$$

We find $P(1 \to 0)$ using a similar method as in class:

$$P(1 \to 0) = \frac{1}{2} + \frac{1}{2}(P(2 \to 0)) = \frac{1}{2} + \frac{1}{2}P(1 \to 0)^2,$$

so $P(1 \to 0)^2 - 2P(1 \to 0) + 1 = 0$, so $P(1 \to 0) = 1$. Thus,

$$P(0 \to 0) = \frac{1}{2} + \frac{1}{2} = 1.$$

(ii) We compute a_i, the number of i-walks from 0 to 0. Clearly, if i is odd, $a_{2i+1} = 0$. If i is even, then we simply choose $\frac{i}{2}$ left steps out of i total steps, so $a_{2i} = \binom{2i}{i}$. Thus,

$$a_n = \binom{n}{n/2} \frac{1 + (-1)^n}{2}.$$

This sequence is OEIS A126869, alternating zero and B-type Catalan numbers/ middle binomial coefficients. The probability of returning to the origin after n steps is $\frac{a_n}{2^n}$, and since we know we eventually return to the origin, it follows that

$$\sum_{n=0}^{\infty} \frac{a_n}{2^n} = 1.$$

(iii) As shown in part (i), $P(1 \to 0) = 1$, and by symmetry, $P(0 \to 1) = 1$. Thus, at some point, we reach 1. Then, since $P(0 \to 1) = P(n \to n+1)$, we will reach 2 at some point. Noting that $P(n \to n-1) = P(n \to n+1) = 1$, we see that we can reach any n at some point with probability 1.

(iv) The flaw in this argument is that it fails to calculate H_1. Suppose, for the sake of contradiction, that H_1 is finite. Then, $H_1 = \frac{1}{2} + \frac{1}{2}(H_2 + 1)$, so

$$2H_1 = 2 + 2H_1 + 1,$$

which is moderately problematic, and a contradiction. Thus, we cannot say $H_1 = c$. $\square$

Problem 1.2 (Some examples of Hitting times)**.**

(i) Find the hitting time between any two vertices of K_n.

(ii) Find the hitting time between the endpoints of P_n (a path on n vertices).

(iii) Find the hitting time between an endpoint of P_n and a vertex at distance k from it.

(iv) Find the hitting time between two vertices of C_n (cycle of n vertices) at distance k.

(v) Find the hitting time between two 'antipodal' vertices of Q_3.

Proof. (i) We exploit unfair labor market conditions. That is, we exploit symmetry in the graph: the hitting time between any two distinct vertices of K_n is the same. Let this hitting time be x, then

$$x = \frac{1}{n-1} \cdot 1 + \frac{n-2}{n-1}(x+1),$$

whence

$$x = (n-1)^2.$$

(ii) Label our path with vertices $\{1, 2, 3, \dots, n\}$. Denote by c_i the expected number of steps to go from vertex i to $i+1$. We build a recursion relation for the c_i:

$$c_i = \frac{1}{2} \cdot 1 + \frac{1}{2}(1 + c_{i-1} + c_i)$$

gives

$$c_i = 2 + c_{i-1}.$$

Clearly, $c_1 = 1$, so $c_i = 2i - 1$, and the hitting time between the endpoints of P_n, or the hitting time from 1 to n, is

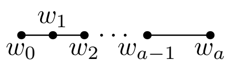
$$c_1 + c_2 + \cdots + c_{n-1} = (n-1)^2.$$

(iii) This is simply the sum

$$\sum_{i=1}^{k} c_i = k^2.$$

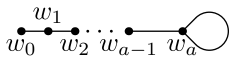
(iv) WLOG we're considering vertices of distance k from v_1. For odd n, we can take a step and have distance from a vertex unchanged (v_3 and v_4 are equidistant from v_1 on C_5). This observation motivates us to consider the graph with vertices w_i that represent distance between our vertex and v_1.

For even $n = 2a$, we have



and so wish to find the hitting time between w_k and w_0. This is an analogous setup to the previous part, so our hitting time is k^2.

For odd $n = 2a + 1$,



which slightly scuppers attempts to make a direct corollary of part (ii). That said, we can define c_i to be the expected steps between w_i and w_{i-1}. Notice that (for $i < a$)

$$c_i = \frac{1}{2} \cdot 1 + \frac{1}{2}(1 + c_{i+1} + c_i) \Rightarrow c_i = 2 + c_{i+1},$$

and as

$$c_a = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (1 + c_a) \Rightarrow c_a = 2,$$

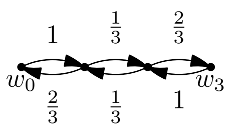
$c_i = 2 + 2a - 2i$. We compute

$$\sum_{i=1}^{k} c_i = (2 + 2a)(k) - k(k+1) = k(n-k).$$

The hitting time is thus:

$$\begin{cases} k^2 & n \text{ even} \\ k(n-k) & n \text{ odd} \end{cases}.$$

(v) Rather than working with the transition matrix for the hypercube, we represent the hypercube as a random walk along a path where each node represents the hamming distance of the current vertex from the starting vertex. We are guaranteed to move to a vertex with hamming distance of 1. From there we have a 1/3 chance of moving back but a 2/3 chance of moving to a node with hamming distance of 2.



From these probabilities, we can construct the following transition matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Letting v be the antipodal (fourth) vertex:

$$I - M[v] = \begin{pmatrix} 1 & -1 & 0 \\ -1/3 & 1 & -2/3 \\ 0 & -2/3 & 1 \end{pmatrix}$$

therefore

$$(I - M[v])^{-2} = \begin{pmatrix} 16 & 81/2 & 30 \\ 27/2 & 36 & 27 \\ 10 & 27 & 21 \end{pmatrix}$$

and thus

$$(I - M[v])^{-2} \begin{pmatrix} 0 \\ 0 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 10 \\ 9 \\ 7 \end{pmatrix}$$

whence

$$H(v_1, v) = 10$$

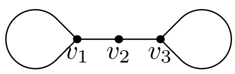
□

Problem 1.3. (i) Show that the following may hold for some graphs G (including regular graphs)

$$H(u, v) \neq H(v, u), \text{ for some } u, v \in V(G).$$

(ii) If u and v have the same degree, then the probability that a random walk starting at u visits v before returning to u is equal to the probability that a random walk starting at v vists u before returning to v. What can be said if the degrees of u and v are different?

Proof. (i) Suppose we have a graph in which vertices v_1 and v_3 each have one loop and are both connected to v_2:



The transition matrix for this graph is

$$M = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

and the hitting times towards v_2 are

$$H(v_1, v_2) = H(v_3, v_2) = 2.$$

These, however, do not match the hitting times approaching v_1:

$$H(v_2, v_1) = 4 \neq H(v_2, v_3) = 2.$$

Therefore, we have found a graph for which $H(u, v) \neq H(v, u)$ for some $u, v \in V(G)$.

(ii) If the degrees of u and v are different, we'd expect the two probabilities to be in the ratio $\deg u$ to $\deg v$: Consider the expected number of steps until we reach v from u, which is $\frac{2e}{\deg u}$. The expected number of steps until we reach u after visiting v is $H(v, u) + H(u, v)$. This seems to suggest that the probability of reaching v before returning to u is inversely proportional to $\deg u$, and likewise for the other probability. So, if the degrees of u and v are different, the probabilities' ratio is the degrees' ratio.

□