

Algebraic Combinatorics: HW6

various.

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Problem 1 (Trees with prescribed degrees and Cayley's formula).

- (a) Given positive integers d_1, d_2, \dots, d_n such that $\sum d_i = 2n - 2$, show that the number of labelled trees on $[n]$ such that vertex i has degree d_i for each i is

$$\frac{(n-2)!}{\prod (d_i - 1)!}.$$

- (b) Prove Cayley's formula from (a).
 (c) What is the number of all trees on n vertices with exactly $n - l$ leaves? (Hint: You may use (a) and leave your answer in terms of *Stirling's number of the second kind*.)

Proof. (a) The proof via Prüfer codes is a trivial arrangement argument. Proceeding by induction on n , the base case $n = 1$ is also trivial. So, assume that the statement holds for $n - 1$, that is, there are

$$\frac{(n-3)!}{\prod_{i=1}^{n-1} (d_i - 1)!}$$

ways to create a labelled tree with $\sum d_i = 2n - 4$.

For n vertices, notice that there must be at least one vertex with degree 1 since the sum of degrees is $2n - 2$. Assign d_n to be the vertex with degree 1. Then, we have $n - 1$ remaining vertices ways to connect the n th vertex to. If we connect to the k th vertex, then we are interested in the number of labelled trees with degrees $d_1, d_2, \dots, d_k - 1, \dots, d_{n-1}$, which is $\frac{(n-3)!}{\prod_{i=1}^{n-1} (d_i - 1)!} \cdot (d_k - 1)$. So, the total number of valid trees on $[n]$ vertices is

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{(n-3)!}{\prod_{i=1}^{n-1} (d_i - 1)!} \cdot (d_k - 1) &= \frac{(n-3)!}{\prod_{i=1}^{n-1} (d_i - 1)!} \cdot \sum_{k=1}^{n-1} (d_k - 1) \\ &= \frac{(n-3)!}{\prod_{i=1}^{n-1} (d_i - 1)!} \cdot (2n - 3 - (n - 1)) \\ &= \frac{(n-2)!}{\prod_{i=1}^{n-1} (d_i - 1)!} \cdot \frac{1}{0!} \\ &= \frac{(n-2)!}{\prod_{i=1}^n (d_i - 1)!}, \end{aligned}$$

as desired. The proof by induction is complete.

- (b) Finding the total number of labelled trees on $[n]$ is the same to summing $\frac{(n-2)!}{\prod (d_i-1)!}$ over all possible degrees d_1, d_2, \dots, d_n such that $\sum d_i = 2n - 2$. Note that $\frac{(n-2)!}{\prod (d_i-1)!}$ is the number of ways to order $n - 2$ numbers in a row, where x appears $d_x - 1$ times, so our sum is counting the number of ways to list $n - 2$ integers in an ordered row, where each number is between 1 and n , which is precisely n^{n-2} .

This proof effectively travels through Prüfer codes.

- (c) Let vertices $l + 1$ through n be the leaves, so $d_{l+1} = d_{l+2} = \dots = d_n = 1$. The sum of the remaining degrees is $2n - 2 - (n - l) = n + l - 2$. By (a), the number of trees is then

$$\frac{(n-2)!}{\prod_{i=1}^l (d_i - 1)!}$$

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Problem 2 (Counting Spanning trees of $K_{m,n}$). Find the value of $\tau(K_{m,n})$ using:

- (i) Matrix-Tree Theorem.
- (ii) Combinatorial argument, say, that of Prüfer or Joyal.
- (iii) Let L be the Laplacian of $K_{m,n}$.
 - (a) Find a simple upper bound on $\text{rank}(L - mI)$.
 - (b) Deduce a lower bound on the multiplicity of eigenvalue L equal to m .
 - (c) Assume $m \neq n$ and do the same for n .
 - (d) Find the remaining eigenvalues of L .
 - (e) Use (a)-(d) to compute $\tau(K_{m,n})$.

Proof. (i) Recall that the Matrix-Tree Theorem states that $\tau(G) = |L_0|$. We let vertices v_1 through v_m be within the first partite and v_{m+1} through v_{m+n} be within the second partite. Then, the Laplacian of $K_{m,n}$ is

$$L = \begin{pmatrix} nI_m & -J_{m \times n} \\ -J_{n \times m} & mI_n \end{pmatrix},$$

Then, we have

$$L_0 = \begin{pmatrix} nI_m & -J_{m \times (n-1)} \\ -J_{(n-1) \times m} & mI_{n-1} \end{pmatrix},$$

and we may use row operations and $\tau(K_{m,n}) = |L_0| = n^{m-1}m^{n-1}$.

- (ii) First, we note that one of the partites of $K_{m,n}$ must be larger than the other. [EXPLANATION OF THIS] Therefore, WLOG, let $n > m$. We divide the vertices of $K_{m,n}$ into two sets, A and B , where $|A| = m$ and $|B| = n$. Then, we delete the smallest leaf in B and append its neighbor to the Prüfer code. We repeat this process until $|B| = m$. Then, we delete the smallest leaf in A and append its neighbor to the Prüfer code, followed by deleting the smallest leaf in B and appending its neighbor, etc.

From the first part of the process, we have n^{m-n} ways to choose the neighbors of the leaves in B , and from the second part of the process, we have $m^{m-1}n^{m-1}$ ways to choose the neighbors of the leaves in A . Whence,

$$\tau(K_{m,n}) = m - nm^{m-1}n^{m-1} = n^{m-1}m^{n-1}$$

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Problem 4. Starting at a point x_0 we walk along the edges of a connected graph G according to the following rules:

- We never use the same edge twice in the same direction.
- Whenever we arrive at a point $x \neq x_0$ not previously visited, we mark the edge along which we entered x . We use the marked edge to leave x only if we must, that is, if we have used all the other edges before.

Show that we get stuck at x_0 , and that, by then, every edge has been traversed in both direction.

Proof. Set the vertex at which we get stuck to be v . Suppose, for the sake of contradiction, then $v \neq x_0$. Then, if we consider the number of times we have traversed an edge to arrive *at* v , this must be one more than the number of times we have traversed an edge to leave *from* v . Thus, there must be at least one edge that we have not traversed in both directions, so we may traverse that edge and leave. Therefore, we must be stuck at x_0 .

We now show that, when we get stuck at x_0 , every other edge must be traversed in both directions, where we duplicate each edge and direct it, to create the enter-exit effect. Then, consider the “in-tree” rooted at x_0 . Suppose, for the sake of contradiction, that we get stuck at x_0 when there is still some untraversed edges. At least one of those edges but be an edge of the in-tree rooted at v , and let the point closest to x_0 (in the in-tree) on an untraversed edge of the in-tree be y . Since we travel to and from y an equal number of times, there is an “exit” from y we don’t use, so y is closer to x_0 than the initial vertex on the untraversed edge, meaning that $y = v$ (by the choice of edge). But, if there is an in-edge to x_0 , then we must certainly also have an unused out-edge, contradicting our stuckness. So, all edges are traversed in our manually constructed directed graph, meaning all edges are traversed twice in the original graph. ■

Problem 5 (Universal cycles for S_n). (i) Let $n \geq 3$. Show that there does not exist a sequence $a_1, a_2, \dots, a_{n!}$ such that all the $n!$ contiguous blocks $a_i, a_{i+1}, \dots, a_{i+n-1}$ (subscripts taken modulo $n!$) are all the $n!$ permutations of S_n .

- (ii) Show that for all $n \geq 1$, there exist a sequence $a_1, a_2, \dots, a_{n!}$ such that all the $n!$ contiguous blocks $a_i, a_{i+1}, \dots, a_{i+n-2}$ consists of the first $n-1$ terms b_1, b_2, \dots, b_{n-1} of all permutations b_1, b_2, \dots, b_n of $[n]$.

Such sequences are called **universal cycles** for S_n (for example, for $n = 3$, 123213 is a universal cycle.)

- (iii) For $n = 3$, find the number of universal cycles beginning with 123.

Proof. (i) There are $(n-1)!$ sequences that begin with a given i (integer $1 \leq i \leq n$), so the i s must be evenly spaced throughout the sequence, which means that the sequence is just $(n-1)!$ copies of a_1, a_2, \dots, a_n , which will certainly not contain all $n!$ permutations of S_n .

For the example of $n = 3$, the requirement of 2 blocks beginning with 1, 2, and 3 force a structure of $ABCABC$, which, because of the repeat, doesn't contain all $n!$ permutations.

(ii)

(iii) The beginning 123 gives blocks of 12 and 23, so the next term can be a block of 32 or 31, giving cases of:

- (a) 1232: we've accounted for 23 already, so the next number must be 1. In 12321, we already have a 12, so the next number must be 3, yielding the universal cycle 123213.
- (b) 1231: we've accounted for 12 already, so the next number must be 3. In 12313, we already have a 31, so the last number must be a 2, yielding the universal cycle 123132.

We confirm that all permutations are present, so the two universal cycles are

$$\boxed{12313, 123132}.$$

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