Algebraic Combinatorics - HW8

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4/16/2024

Problem 1 (Saving electricity through Linear Algebra and Graph Theory). (i) Consider the vector space $V = \mathbb{F}_2^n$ (over $\mathbb{F}_2$). Let $A_{n\times n}$ be a symmetric matrix over $\mathbb{F}_2$. Consider the diagonal of A, as a column vector $\overrightarrow{d}$. Prove that

$$\vec{d} \in Col(A)$$

(*Hint*: First show that $\vec{v}^T A\vec{v} = \vec{d}^T\vec{v}$ for all $\vec{v} \in V$. Then show the required result by contradiction.)

(ii) Given a graph G, show (using (i)) that $V(G)$ can be partitioned into V_1 and V_2 such that $G[V_1]$ is an even graph (that is, all degrees are even) and for axll vertices $v \in V_2$, $|N(v) \cap V_1|$ is odd.

(iii) Assume that there is a bulb and a button at each vertex of a graph G. The connections are made such that pushing the button at a vertex once, will change the status of the bulb and its neighbors. Initially all bulbs are on. Show (using (ii)) that one can push some buttons and turn all the bulbs off. Does this remind you of a game that you may have played as a kid?

Proof. By matrix multiplication,

$$v^T Av = \sum_{a_{ij}\in A} a_{ij}v_iv_j = \sum a_{ii}v_i^2 + 0 = d^T v.$$

Note that $(Ax)^T y = (Ay)^T x$ by dot product properties.

Thus, the image of A is the orthogonal complement of the kernel of $A^T = A$.

If $Av = 0$, then $d^T v = 0$, so d is orthogonal to the Kernel of A, whence $d \in \mathrm{Col}(A)$. ∎

Problem 2 (Cycle Space and Bond Space in Graphs + some applications). Given an undirected connected graph G with n vertices and m edges, let each subset of the edge set $E(G)$ be represented by its characteristic binary vector. Let

$$\begin{aligned} \mathcal{C}(G) &= \{\text{all even subgraphs of } G\} \\ \mathcal{B}(G) &= \{\text{all minimal edge-cuts of } G\} \\ M(G) &= \text{incidence matrix of } G \end{aligned}$$

(i) Show that both $\mathcal{C}(G)$ and $\mathcal{B}(G)$ are subspaces of $\mathbb{F}_2^m$ (over $\mathbb{F}_2$).

(ii) Show that the *stars* at any $n-1$ vertices of G are linearly independent and forms a basis in $\mathcal{B}(G)$, thus

$$\mathcal{B}(G) = \mathrm{Row}(M(G)) \quad \text{and} \quad \dim(\mathcal{B}(G)) = n - 1.$$

(iii) Given any spanning tree T of G, show that the *fundamental cycles* (as described in class) are linearly independent in $\mathcal{C}(G)$. Then show that

$$\mathcal{C}(G) = (\mathcal{B}(G))^{\perp} \quad \mathcal{C}(G) = \mathrm{Nul}(M(G)) \quad \dim\big(\mathcal{C}(G)\big) = m - n + 1$$

(iv) Use the above to show that:

(a) A graph is bipartite $\Leftrightarrow$ every circuit is of even length. (*Hint*: Show that $\overrightarrow{j} = (1, 1, \cdots, 1) \in \mathcal{B}(G)$)

(b) Show that for any graph G, $E(G)$ can be partitioned into an even graph and an edge-cut.

(c) Use (b) to give a new proof of $\#1(ii)$ (and hence $\#1(iii)$)

Proof. Clearly, over F_2, scalar multiplication is no issue. We must show that the "addition" of two graphs in $\mathcal{C}$ or $\mathcal{B}$, that is, their symmetric difference, remains in $\mathcal{C}$ or $\mathcal{B}$, respectively. $\mathcal{C}$ is clear; given $C_1, C_2 \in \mathcal{C}$, consider the degree of a vertez in $C_1 + C_2$. Clearly, the sum of two even degrees is even, and is unchanged over F_2. For $\mathcal{B}$, consider two minimal edge cuts B_1 and B_2, yielding partitions v_1, v_1' and v_2, v_2', respectively. B_1 consists of all edges between v_1 and v_1' and likewise for B_2, v_2, and v_2'.

The only edges that remain are those in $B_1 \setminus B_2$ or $B_2 \setminus B_1$. These connect vertices in $v_1 \setminus v_2$ to $v_1' \setminus v_2'$ or $v_2 \setminus v_1$ to $v_2' \setminus v_1'$. FINISH THIS ARGUMENT.

Let s_v be the star at vertex v. All s_i are evidently in $\mathcal{B}$, and if we have a minimal edge-cut b between $v_{i_1}, \ldots, v_{i_k}$ and $v_{j_1}, \ldots, v_{j_l}$, then note that $b = \sum_{c=1}^{k} s_{v_{i_c}}$, remembering that we are over $\mathbb{F}_2$. (Another consequence of working over this field is that the sum is identical to $\sum_{c=1}^{l} s_{v_{j_c}}$.) We now need only write $s_1 + s_2 + \cdots + s_{n-1} = s_n$, which we can do by edge-counting: for an edge between v_n and v_m, s_m includes this edge, and no other s_i does. Thus, the edge's coordinate remains nonzero in the partial sum of the s_i. Similarly, for an edge between v_m and v_o with $m, o \neq n$, s_m and s_o both include this edge, so it is not included in the partial sum. Thus, the only remaining edges in the partial sum are those connected to v_n, whence the sum is s_n, so we can write the nth star as a sum of the others. A quick examination of the $n - 1$ other stars shows that they are linearly independent, by consider the coordinate representing an edge between v_i and v_n: it is nonzero only in s_i, so if $\sum_i a_i s_i = 0$, all $a_i = 0$.

The star for vertex k represents the transpose of the kth row of $M(G)$, so $\mathcal{B}(G) = \mathrm{Row}(M(G))$. The dimension of $\mathcal{B}$ is the number of linearly independent stars (elements in the basis), which is $n - 1$.

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