# HW 8 - Cycle Space and Bond Space (Due Tuesday 4/16)

**1**. [Saving electricity through Linear Algebra and Graph Theory]

($i$) Consider the vector space $V = \mathbb{F}_2^n$ (over $\mathbb{F}_2$). Let $A_{n\times n}$ be a symmetric matrix over $\mathbb{F}_2$. Consider the diagonal of $A$, as a column vector $\overrightarrow{d}$. Prove that

$$\overrightarrow{d} \in Col(A)$$

(*Hint*: First show that $\overrightarrow{v}^T A \overrightarrow{v} = \overrightarrow{d}^T \overrightarrow{v}$ for all $\overrightarrow{v} \in V$. Then show the required result by contradiction.)

($ii$) Given a graph $G$, show (using ($i$)) that $V(G)$ can be partitioned into $V_1$ and $V_2$ such that $G[V_1]$ is an even graph (that is, all degrees are even) and for all vertices $v \in V_2$, $|N(v) \cap V_1|$ is odd.

($iii$) Assume that there is a bulb and a button at each vertex of a graph $G$. The connections are made such that pushing the button at a vertex once, will change the status of the bulb and its neighbors. Initially all bulbs are on. Show (using ($ii$)) that one can push some buttons and turn all the bulbs off. Does this remind you of a game that you may have played as a kid ?

(*Remark*: This can also be done by a purely Graph-theoretic argument, avoiding Linear Algebra. )

**2**. [Cycle Space and Bond Space in Graphs + some applications]

Given an undirected connected graph $G$ with $n$ vertices and $m$ edges, let each subset of the edge set $E(G)$ be represented by its characteristic binary vector. Let

$\mathcal{C}(G) = \{$all even subgraphs of $G\}$ ; $\mathcal{B}(G) = \{$all minimal edge-cuts of $G\}$ ; $M(G) =$ incidence matrix of $G$

($i$) Show that both $\mathcal{C}(G)$ and $\mathcal{B}(G)$ are subspaces of $\mathbb{F}_2^m$ (over $\mathbb{F}_2$).

($ii$) Show that the *stars* at any $n-1$ vertices of $G$ are linearly independent and forms a basis in $\mathcal{B}(G)$, thus

$$\mathcal{B}(G) = Row\big(M(G)\big) \ ; \ dim\big(\mathcal{B}(G)\big) = n-1$$

($iii$) Given any spanning tree $T$ of $G$, show that the *fundamental cycles* (as described in class) are linearly independent in $\mathcal{C}(G)$. Then show that

$$\mathcal{C}(G) = \big(\mathcal{B}(G)\big)^{\perp} \ , \ \mathcal{C}(G) = Nul\big(M(G)\big) \ ; \ dim\big(\mathcal{C}(G)\big) = m-n+1$$

($iv$) Use the above to show that:

- ($a$) A graph is bipartite $\iff$ every circuit is of even length. $\big($*Hint*: Show that $\overrightarrow{j} = (1, 1, \cdots, 1) \in \mathcal{B}(G)\big)$
- ($b$) Show that for any graph $G$, $E(G)$ can be partitioned into an even graph and an edge-cut.
- ($c$) Use ($b$) to give a new proof of #1($ii$) (and hence #1($iii$))

**3**. [Totally Unimodular Matrices and an extension of the Matrix-Tree Theorem; **Not for credit**]

A matrix is called *totally unimodular* if every square submatrix of it has determinant either 0 or $\pm 1$.

Let $G$ be a loopless connected graph with $n$ vertices, $m$ edges and incidence matrix $M(G)$. The proof of Matrix-Tree Theorem (shown in class) says that $M(G)$ is totally unimodular. Cauchy-Binet formula then says that

$$\tau(G) = \left|M_0 M_0^T\right|$$

Note that the rows of $M_0$ are precisely the incidence vectors of some $n-1$ stars of $G$, which we know from #2($ii$), forms a basis of $\mathcal{B}(G)$. In fact, a lot more is true:

Let $T$ be a spanning tree of $G$. Let $B$ and $C$ be the basis matrices for $\mathcal{B}(G)$ and $\mathcal{C}(G)$ w.r.to $T$, then one can similarly show that $B$ and $C$ are also totally unimodular and that

$$\tau(G) = \left|BB^T\right| = \left|CC^T\right|$$