Algebraic Combinatorics: HW6

various.

April 1, 2024

Problem 1 (Trees with prescribed degrees and Cayley's formula)**.**

(a) Given positive integers $d_1, d_2, \ldots, d_n$ such that $\sum d_i = 2n-2$, show that the number of labelled trees on $[n]$ such that vertex i has degree d_i for each i is

$$\frac{(n-2)!}{\prod(d_i-1)!}.$$

(b) Prove Cayley's formula from (a).

(c) What is the number of all trees on n vertices with exactly $n-l$ leaves? (Hint: You may use (a) and leave your answer in terms of *Stirning's number of the second kind.*)

Proof. (a) The proof via Prüfer codes is a trivial arrangement argument. Proceeding by induction on n, the base case $n=1$ is also trivial. So, assume that the statement holds for $n-1$, that is, there are

$$\frac{(n-3)!}{\prod_{i=1}^{n-1}(d_i-1)!}$$

ways to create a labelled tree with $\sum d_i = 2n-4$.

For n vertices, notice that there must be at least one vertex with degree 1 since the sum of degrees is $2n-2$. Assign d_n to be the vertex with degree 1. Then, we have $n-1$ remaining vertices ways to connect the nth vertex to. If we connect to the kth vertex, then we are interested in the number of labelled trees with degrees $d_1, d_2, \ldots d_k - 1, \ldots, d_{n-1}$, which is $\frac{(n-3)!}{\prod_{i=1}^{n-1}(d_i-1)!} \cdot (d_k-1)$. So, the total number of valid trees on $[n]$ vertices is

$$\begin{aligned}
\sum_{k=1}^{n-1} \frac{(n-3)!}{\prod_{i=1}^{n-1}(d_i-1)!} \cdot (d_k-1) &= \frac{(n-3)!}{\prod_{i=1}^{n-1}(d_i-1)!} \cdot \sum_{k=1}^{n-1}(d_k-1) \\
&= \frac{(n-3)!}{\prod_{i=1}^{n-1}(d_i-1)!} \cdot (2n-3-(n-1)) \\
&= \frac{(n-2)!}{\prod_{i=1}^{n-1}(d_i-1)!} \cdot \frac{1}{0!} \\
&= \frac{(n-2)!}{\prod_{i=1}^{n}(d_i-1)!},
\end{aligned}$$

as desired. The proof by induction is complete.

(b) Finding the total number of labelled trees on $[n]$ is the same to summing $\frac{(n-2)!}{\prod(d_i-1)!}$ over all possible degrees $d_1, d_2, \ldots, d_n$ such that $\sum d_i = 2n-2$. Note that $\frac{(n-2)!}{\prod(d_i-1)!}$ is the number of ways to order $n-2$ numbers in a row, where x appears $d_x - 1$ times, so our sum is counting the number of ways to list $n-2$ integers in an ordered row, where each number is between 1 and n, which is precisely n^{n-2}.

This proof effectively travels through Prüfer codes.

(c) Let vertices $l+1$ through n be the leaves, so $d_{l+1} = d_{l+2} = \cdots = d_n = 1$. The sum of the remaining degrees is $2n-2-(n-l) = n+l-2$. By (a), the number of trees is then

$$\frac{(n-2)!}{\prod_{i=1}^{l}(d_i-1)!}$$

■

Problem 2 (Counting Spanning trees of $K_{m,n}$)**.** Find the value of $\tau(K_{m,n})$ using:

1. Matrix-Tree Theorem.
2. Combinatorial argument, say,

Problem 3. Starting at a point x_0 we walk along the edges of a connected graph G according to the following rules:

- We never use the same edge twice in the same direction.
- Whenever we arrive at a point $x \neq x_0$ not previously visited, we mark the edge along which we entered x. We use the marked edge to leave x only if we must, that is, if we have used all the other edges before.

Show that we get stuck at x_0, and that, by then, every edge has been traversed in both direction.

Proof. We first show that we must get stuck at x_0. For the purposes of getting stuck, we don't particularly ■