Algebraic Combinatorics: HW6

Dallin, Evan

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Problem 1 (Trees with prescribed degrees and Cayley's formula).

(a) Given positive integers $d_1, d_2, \dots, d_n$ such that $\sum d_i = 2n-2$, show that the number of labelled trees on $[n]$ such that vertex i has degree d_i for each i is

$$\frac{(n-2)!}{\prod (d_i - 1)!}.$$

(b) Prove Cayley's formula from (a).

(c) What is the number of all trees on n vertices with exactly $n - l$ leaves? (Hint: You may use (a) and leave your answer in terms of *Stirling's number of the second kind*.)

Proof. (a) The proof via Prüfer codes is a trivial arrangement argument. Proceeding by induction on n, the base case $n = 1$ is also trivial. So, assume that the statement holds for $n - 1$, that is, there are

$$\frac{(n-3)!}{\prod_{i=1}^{n-1} (d_i - 1)!}$$

ways to create a labelled tree with $\sum d_i = 2n - 4$.

For n vertices, notice that there must be at least one vertex with degree 1 since the sum of degrees is $2n - 2$. Assign d_n to be the vertex with degree 1. Then, we have $n - 1$ remaining vertices ways to connect the nth vertex to. If we connect to the kth vertex, then we are interested in the number of labelled trees with degrees $d_1, d_2, \dots d_k - 1, \dots, d_{n-1}$, which is $\frac{(n-3)!}{\prod_{i=1}^{n-1}(d_i-1)!} \cdot (d_k - 1)$. So, the total number of valid trees on $[n]$ vertices is

$$\begin{aligned}
\sum_{k=1}^{n-1} \frac{(n-3)!}{\prod_{i=1}^{n-1} (d_i - 1)!} \cdot (d_k - 1) &= \frac{(n-3)!}{\prod_{i=1}^{n-1} (d_i - 1)!} \cdot \sum_{k=1}^{n-1} (d_k - 1) \\
&= \frac{(n-3)!}{\prod_{i=1}^{n-1} (d_i - 1)!} \cdot (2n - 3 - (n-1)) \\
&= \frac{(n-2)!}{\prod_{i=1}^{n-1} (d_i - 1)!} \cdot \frac{1}{0!} \\
&= \frac{(n-2)!}{\prod_{i=1}^{n} (d_i - 1)!},
\end{aligned}$$

as desired. The proof by induction is complete.

(b) Finding the total number of labelled trees on $[n]$ is the same to summing $\frac{(n-2)!}{\prod(d_i-1)!}$ over all possible degrees $d_1, d_2, \ldots, d_n$ such that $\sum d_i = 2n-2$. Note that $\frac{(n-2)!}{\prod(d_i-1)!}$ is the number of ways to order $n-2$ numbers in a row, where x appears $d_x - 1$ times, so our sum is counting the number of ways to list $n-2$ integers in an ordered row, where each number is between 1 and n, which is precisely n^{n-2}.

This proof effectively travels through Prüfer codes.

(c) There are $\binom{n}{l}$ ways to choose the l non-leaves. We count the number of trees for any given combination of leaves.

WLOG, let vertices $l+1$ through n be the leaves, so $d_{l+1} = d_{l+2} = \cdots = d_n = 1$. The sum $\sum_{i=1}^{l} d_i - 1$ is then $2n-2-(n-l)-l = n-2$. By (a), the number of trees with this degree condition is then

$$\sum_{\sum_{i=1}^{l} d_i - 1 = n-2} \frac{(n-2)!}{\prod_{i=1}^{l}(d_i-1)!},$$

and setting the $d_i' = d_i - 1$ gives, where $1 \leq i \leq l$,

$$\sum_{\sum d_i' = n-2} \frac{(n-2)!}{\prod d_i'!},$$

which is precisely the number of ways to partition $n-2$ objects into l labelled partitions, also l times the number of ways to partition $n-2$ into l partitions, which is the stirling number $\left\{ {n-2 \atop l} \right\}$.

Thus, there are

$$\frac{n!}{(n-l)!} \left\{ {n-2 \atop l} \right\}$$

trees on $[n]$ with $n-l$ leaves.

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Problem 2 (Counting Spanning trees of $K_{m,n}$)**.** Find the value of $\tau(K_{m,n})$ using:

(i) Matrix-Tree Theorem.

(ii) Combinatorial argument, say, that of Prüfer or Joyal.

(iii) Let L be the Laplacian of $K_{m,n}$.

   (a) Find a simple upper bound on $\operatorname{rank}(L - mI)$.

   (b) Deduce a lower bound on the multiplicity of eigenvalue L equal to m.

   (c) Assume $m \neq n$ and do the same for n.

   (d) Find the remaining eigenvalues of L.

   (e) Use (a)-(d) to compute $\tau(K_{m,n})$.

Proof. (i) Recall that the Matrix-Tree Theorem states that $\tau(G) = |L_0|$. We let vertices v_1 through v_m be within the first partite and v_{m+1} through v_{m+n} be within the second partite.

Then, the Laplacian of $K_{m,n}$ is

$$L = \begin{pmatrix} nI_m & -J_{m\times n} \\ -J_{n\times m} & mI_n \end{pmatrix},$$

Then, we have

$$L_0 = \begin{pmatrix} nI_m & -J_{m\times(n-1)} \\ -J_{(n-1)\times m} & mI_{n-1} \end{pmatrix}.$$

Now all we must do is evaluate $|L_0|$. Adding all of the rows to the first row, we get

$$|L_0| = \begin{pmatrix} 1 & \underbrace{\cdots}_{m \text{ ones}} & & 1 & 0 & \cdots & 0 \\ 0 & n & 0 & 0 & -1 & \cdots & -1 \\ 0 & 0 & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & n & -1 & \cdots & -1 \\ 0 & \cdots & & 0 & m & 0 & 0 \\ 0 & \cdots & & 0 & 0 & \ddots & 0 \\ 0 & \cdots & & 0 & 0 & 0 & m \end{pmatrix},$$

so

$$|L_0| = \det(\operatorname{diag}\{1, \underbrace{m, \ldots, m}_{n-1}, \underbrace{n, \ldots, n}_{m-1}\}) = m^{n-1}n^{m-1}.$$

(ii) First, WLOG, we let $n \geq m$. We divide the vertices of $K_{m,n}$ into two sets, A and B, where $|A| = m$ and $|B| = n$. We first must verify that a leaf exists in B. This can be shown by contradiction: If the degree of every vertex in B were to be > 1, the total number of edges in our tree would be $\geq 2n$. This would be greater than the total number of edges, $m + n - 1$, a contradiction.

We proceed by deleting the smallest leaf in B and append its neighbor to the Prüfer code. We repeat this process until $|B| = m$. Then, we delete the smallest leaf in A and append its neighbor to the Prüfer code, followed by deleting the smallest leaf in B and appending its neighbor, etc.

From the first part of the process, we have n^{m-n} ways to choose the neighbors of the leaves in B, and from the second part of the process, we have $m^{m-1}n^{m-1}$ ways to choose the neighbors of the leaves in A. Whence,

$$\tau(K_{m,n}) = m - nm^{m-1}n^{m-1} = n^{m-1}m^{n-1}.$$

(iii) (a) $L - mI$ is a matrix

$$\begin{pmatrix} (n-m)I_m & -J_{m\times n} \\ -J_{n\times m} & 0 \end{pmatrix},$$

and since all of the rows in the lower blocks are the same (m -1s and n 0s), the rank is at most $m + 1$.

(b) m's multiplicity is at least $(m + n) - \operatorname{rank}(L - mI) \geq m + n - m - 1 = n - 1$.

(c) $L - nI$ is the matrix

$$\begin{pmatrix} 0 & -J_{m\times n} \\ -J_{n\times m} & (m-n)I_n \end{pmatrix},$$

and again we can collapse the top m rows into one, for a rank of $n+1$. n's multiplicity is then at least $m-1$.

(d) We have two more eigenvalues of multiplicity one (or one eigenvalue of multiplicity two) to find. Clearly, 0 is an eigenvalue with multiplicity one. Then, informed by the answer, we notice that $m+n$ is also an eigenvalue with multiplicity one.

(e) By Corollary (i) from lecture, we have

$$\tau(K_{m,n}) = \frac{1}{m+n}(m+n)(n^{m-1})(m^{n-1}) = m^{n-1}n^{m-1}.$$

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Problem 4. Starting at a point x_0 we walk along the edges of a connected graph G according to the following rules:

- We never use the same edge twice in the same direction.
- Whenever we arrive at a point $x \neq x_0$ not previously visited, we mark the edge along which we entered x. We use the marked edge to leave x only if we must, that is, if we have used all the other edges before.

Show that we get stuck at x_0, and that, by then, every edge has been traversed in both direction.

Proof. Set the vertex at which we get stuck to be v. Suppose, for the sake of contradiction, then $v \neq x_0$. Then, if we consider the number of times we have traversed an edge to arrive *at* v, this must be one more than the number of times we have traversed an edge to leave *from* v. Thus, there must be at least one edge that we have not traversed in both directions, so we may traverse that edge and leave. Therefore, we must be stuck at x_0.

We now show that, when we get stuck at x_0, every other edge must be traversed in both directions, where we duplicate each edge and direct it, to create the enter-exit effect. Then, consider the "in-tree" rooted at x_0. Suppose, for the sake of contradiction, that we get stuck at x_0 when there is still some untraversed edges. At least one of those eddges but be an edge of the in-tree rooted at v, and let the point closest to x_0 (in the in-tree) on an untraversed edge of the in-tree be y. Since we travel to and from y an equal number of times, there is an "exit" from y we don't use, so y is closer to x_0 than the initial vertex on the untraversed edge, meaning that $y = v$ (by the choice of edge). But, if there is an in-edge to x_0, then we must certainly also have an unused out-edge, contradicting our stuckness. So, all edges are traversed in our manually constructed directed graph, meaning all edges are traversed twice in the original graph. ■

Problem 5 (Universal cycles for S_n). (i) Let $n \geq 3$. Show that there does not exist a sequence $a_1, a_2, \ldots, a_{n!}$ such that all the $n!$ contiguous blocks $a_i, a_{i+1}, \ldots, a_{i+n-1}$ (subscripts taken modulo $n!$) are all the $n!$ permutations of S_n.

(ii) Show that for all $n \geq 1$, there exist a sequence $a_1, a_2, \ldots, a_{n!}$ such that all the $n!$ contiguous blocks $a_i, a_{i+1}, \ldots, a_{i+n-2}$ consists of the first $n-1$ terms $b_1, b_2, \ldots, b_{n-1}$ of all permutations

$b_1, b_2, \ldots, b_n$ of $[n]$.

Such sequences are called **universal cycles** for S_n (for example, for $n = 3$, 123213 is a universal cycle.)

(iii) For $n = 3$, find the number of universal cycles beginning with 123.

Proof. (i) There are $(n-1)!$ sequences that begin with a given i (integer $1 \leq i \leq n$), so the is must be evenly spaced throughout the sequence, which means that the sequence is just $(n-1)!$ copis of $a_1, a_2, \ldots, a_n$, which will certainly not contain all $n!$ permutations of S_n.

For the example of $n = 3$, the requirement of 2 blocks beginning wth $1, 2$, and 3 force a structure of $ABCABC$, which, because of the repeat, doesn't contain all $n!$ permutations.

(ii) Consider a way to create a digraph of the elements of S_n (written as n-strings), where we have a directed edge from ρ_1 to ρ_2 if the middle $n-2$ elements of ρ_1 are the same as the first $n-2$ elements of ρ_2. In this sense, a step is like a left shift of the n-string, where the first character is deleted and we append some character to the end, i.e. from 123 to 234 or 231. We wish to show that this graph is Eulerian.

Firstly, by the setup, it is clear that the graph is connected (we may perform a "shifting" walk until we get from one permutation to another). Then, notice that $\delta^-(v) = \delta^+(v) = 2$ for all v, by the string setup, so the digraph is balanced and thus Eulerian.

(iii) The beginning 123 gives blocks of 12 and 23, so the next term can be a block of 32 or 31, giving cases of:

(a) 1232: we've accounted for 23 already, so the next number must be 1. In 12321, we already have a 12, so the next number must be 3, yielding the universal cycle 123213.

(b) 1231: we've accounted for 12 already, so the next number must be 3. In 12313, we already have a 31, so the last number must be a 2, yielding the universal cycle 123132.

We confirm that all permutations are present, so the two universal cycles are

$$\boxed{12313, 123132}.$$

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