Algebraic Combinatorics - HW7

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Problem 1 (Lattice Paths avoiding a certain set of points)**.** Let P be a fixed lattice (ballot) path from $(0,0)$ to (m,n). Let T be a set of interior points on P (that is, some subset of points on P other than $(0,0)$ and (m,n)). Let $f_{m,n}(T)$ be the number of lattice paths from $(0,0)$ to (m,n) that avoid all of T.

(i) Use a Corollary of Gessel-Viennot Lemma to find an expression for $f_{m,n}(T)$.

(ii) Use Inclusion-Exclusion to find an expression for $f_{m,n}(T)$.

Proof. (i) hum.

(ii) We use PIE to count the number of ballot paths that intersect at least one vertex in T. Let $t_1, t_2, \dots, t_k$ be the vertices of T, where $t_i = (x_i, y_i)$. Let B be the set of all "bad" ballot paths from $(0,0)$ to (m,n), and consider the subsets $B_i \subset B$ where B_i is the set of all ballot paths that intersect t_i. Then, by PIE,

$$|B| = \sum_{1\le i\le k} |B_i| - \sum_{1\le i<j\le k} |B_i \cap B_j| + \cdots,$$

that is,

$$\sum_{i=1}^{k}\left((-1)^{i+1} \sum_{1\le c_1<\cdots<c_i\le k} |B_{c_1}\cap B_{c_2}\cap\cdots\cap B_{c_i}|\right)$$

Notice that $|B_{c_1}\cap\cdots\cap B_{c_i}|$ is the number of ballot paths through $t_{c_1},\dots,t_{c_i}$, which is

$$\prod_{v=0}^{i}\binom{x_{c_{v+1}} - x_{c_v} + y_{c_{v+1}} - y_{c_v}}{x_{c_{v+1}} - x_{c_v}},$$

where $x_{c_0} = y_{c_0} = 0$ and $x_{c_{i+1}} = m$, $y_{c_{i+1}} = n$, whence

$$|B| = \sum_{i=1}^{k}\left((-1)^{i+1} \sum_{1\le c_1<\cdots<c_i\le k} \prod_{v=0}^{i}\binom{x_{c_{v+1}} - x_{c_v} + y_{c_{v+1}} - y_{c_v}}{x_{c_{v+1}} - x_{c_v}}\right).$$

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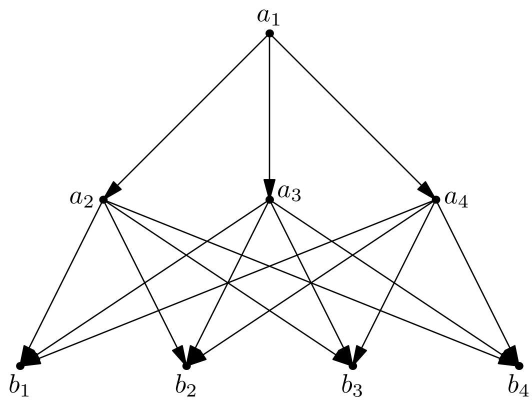
Problem 2 (Linear dependency and Gessel-Vienot)**.** Let $A_{n\times n}$ be a matrix with linearly dependent rows. Show by using Gessel-Viennot Lemma that $|A| = 0$.

Proof. Let $A_{n\times n}$ represent the edge matrix of a weighted DAG from $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_n\}$. Let $e_{ij} = [A_{ij}]$. Then, since the rows are linearly independent, the first row can be written as a linear combination of the other rows, say

$$e_{1j} = \sum_{2\le i\le n} c_i e_{ij}.$$

We can construct our graph with a_1 above all the other a_i, with the example for $n = 4$ shown below:



We can set the edge from a_1 to a_k to have weight c_k, whence we have the linear combination relationship in A. For example,

$$e_{11} = e_{21}c_2 + e_{31}c_3 + e_{41}c_4.$$

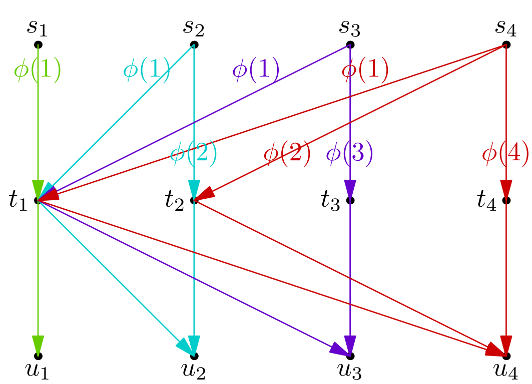
By the Gessel-Viennot Lemma, the determinant of A is the sum of the weights of all path systems from A to B. However, notice that the path system from a_1 to $b_{\sigma(a_1)}$ will have an odd number of vertices, while the path system from all other a_i to $b_{\sigma(a_i)}$ will have an even number of vertices, while $|A| + |B| = 2n$ is even. Therefore, there are no path systems, so

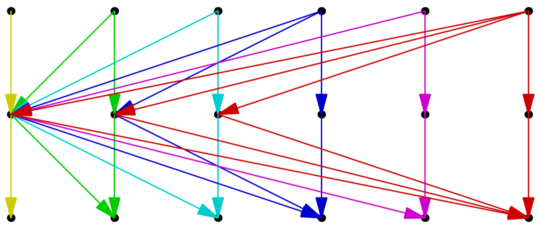
$$\det A = \sum_{\mathscr{P}} \operatorname{sign}(\mathscr{P})\omega(\mathscr{P}) = 0.$$

■

> **Problem 3** (GCD matrix)**.** Let $S = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{N}$. let the **GCD matrix** M of S have entries $m_{ij} = \gcd(a_i, a_j)$. Prove that if S is closed under taking divisors, then
>
> $$|M| = \prod_{i=1}^{n} \varphi(a_i).$$
>
> (*Hint:* Form a certain digraph using three copies of S, and then put certain edges and weights on them.)

Proof. We form a digraph between S, T, and U, where T and U are two copies of S, placing edges in a way that allows us to make use of the Totient Function's divisor sum. For a given $1 \leq i \leq n$, let d be a divisor of i. Then, we place an edge from s_i to t_d of weight $\phi(d)$ and an edge from t_d to u_i of weight 1. We give the examples of $n = 4$ and $n = 6$, where the colorings are relevant for our path counting argument:





Consider the edge matrix A of this graph, where a_{ij} is the number of paths from s_i to u_j. By our setup,

$$a_{ij} = \sum_{d|i,j} \phi(d) = \sum_{d|(i,j)} \phi(d) = (i,j),$$

so $A = M$. By the Gessel-Viennot Lemma, $|M| = |A|$, which is the sum of the weights of all path systems from S to U. Notice that $\sigma(s_1)$ is necessarily u_1, whence $\sigma(s_p) = u_p$ for all prime p. We can continue inductively on the number of divisors to conclude that $\sigma(s_k) = u_k$ for all $1 \le k \le n$, so we only have one path system $\mathscr{P}$, and $\omega(\mathscr{P}) = \prod_{i=1}^{n} \varphi(i)$, as desired. ■

Problem 4 (Determinant of a matrix of Stirling Numbers)**.** For $m \ge 0$, $n \ge 1$, prove the following identity. Here $S_{n,k}$ is the Stirling number of the 2$^{\text{nd}}$ kind.

$$\det \begin{pmatrix} S_{m+1,1} & S_{m+1,2} & \cdots & S_{m+1,n} \\ S_{m+2,1} & S_{m+2,2} & \cdots & S_{m+2,n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{m+n,1} & S_{m+n,2} & \cdots & S_{m+n,n} \end{pmatrix} = (n!)^m$$

Proof. The Stirling number of the second kind $S_{n,k}$ is the number of ways to partition a set of n elements into k partitions. We wish to construct a DAG from A to B (where $|A| = |B| = n$) such that the number of paths from a_i to b_j is $S_{m+i,j}$. ■