

Algebraic Combinatorics HW 3

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Problem 1 (Symmetric polynomial and unimodality; Extra-Credit). $f(x) = p_0 + p_1x + p_2x^2 + \cdots + p_nx^n$ is *symmetric* if for all i ,

$$p_i = p_{n-i}$$

It is *unimodal* if for some fixed j ,

$$p_0 \leq p_1 \leq \cdots \leq p_{j-1} \leq p_j \geq p_{j+1} \geq \cdots \geq p_{n-1} \geq p_n$$

Let $F(q)$, $G(q)$ be symmetric and unimodal polynomials with non-negative real coefficients. Show that $F(q)G(q)$ is also symmetric (easy) and unimodal (less easy).

Proof. Let F and G be n -degree polynomials with $F(q) = \sum f_i q^i$ and $G(q) = \sum g_i q^i$. Let $H(q) = F(q)G(q) = \sum h_i q^i$ be the $2n$ -degree polynomial that is the product of F and G . Then,

$$h_i = \sum_{a+b=i} f_a g_b = \sum_{a+b=i} f_{n-a} g_{n-b} = \sum_{c+d=2n-i} f_c g_d = h_{2n-i},$$

so $F(q)G(q)$ is symmetric.

Since F is unimodal, we can assign it center term f_j . F 's symmetry also allows us to write it as a sum of “chunks” $x^{j-c} + x^{j-c+1} + \cdots + x^{j+c-1} + x^{j+c}$ centered at f_j . We can do similarly for G , writing it as a sum of “chunks” $x^{l-c} + x^{l-c+1} + \cdots + x^{l+c}$ with maximal coefficient q_l . Then, the product of F and G is the sum of products of said chunks, and since

$$\begin{aligned} & (x^{j-c} + x^{j-c+1} + \cdots + x^{j+c-1} + x^{j+c})(x^{l-d} + x^{l-d+1} + \cdots + x^{l+d}) \\ &= x^{j+l-c-d}(1 + x + \cdots + x^{2c})(1 + x + \cdots + x^{2d}) \end{aligned}$$

is unimodal, $F(q)G(q)$ is a sum of unimodal polynomials and thus unimodal. \square

Problem 2 (Log-concavity of Binomial coefficients). A sequence $a_1, a_2, a_3, \cdots, a_n$ is *logarithmically concave* if

$$a_i^2 \geq a_{i-1}a_{i+1} ; \forall i$$

- (i) Show that if a sequence of positive terms is log-concave, then it is also unimodal.
- (ii) It is easy to see algebraically that the sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \cdots, \binom{n}{n}$ is log-concave. Give a combinatorial proof of this fact.

Proof. (i) We rewrite the given relation as

$$\frac{a_i}{a_{i-1}} \geq \frac{a_{i+1}}{a_i}.$$

This means that the ratio of consecutive terms is non-increasing. Thus, if the sequence is not either nonincreasing or nondecreasing (in which case it is naturally unimodal), it is initially increasing and must decrease at some point. This means that the sequence is unimodal.

(ii) Consider $\binom{n}{k}$ for $0 \leq k \leq n$. We need to show

$$\binom{n}{k}^2 \geq \binom{n}{k-1} \binom{n}{k+1}$$

Let $|A| = k$ and $|B| = n - k$.

There are $\binom{n}{k}^2$ ways to independently choose k elements for A and $n - k$ elements for B then form separate pairs with these elements (Using $\binom{n}{k} = \binom{n}{n-k}$). $\binom{n}{k-1} \binom{n}{k+1}$ shows the amount of ways to choose $k - 1$ elements from a set of n elements and $k + 1$ elements from a different set of elements. The pairs formed from these choices will have one element in common, and this common element can be chosen in n ways. Now, notice that A and B together have n elements, so the common element in the pairs from the right side can be any of the n elements in A or B . Since choosing from A and B allows for more pairs, $\binom{n}{k}^2 \geq \binom{n}{k-1} \binom{n}{k+1}$, i.e. $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$ is a log concave sequence. \square

Problem 3 (Uniqueness in Sperner's Thm). Show that equality in Sperner's Theorem for B_n is achieved only by the middle (middle two) rank(s) if n is even (odd). (*Hint:* If not, then move the example closer to the middle rank(s))

Proof. Let $l > \frac{n+1}{2}$ correspond to a minimum above-middle rank l . Consider the section of the graph of elements with ranks l (A) and $l - 1$ (δA). The shadow of A is evidently δA , and we aim to show that there are more elements in δA than in A . Considering the connections between A and δA gives us a bipartite graph, where each vertex of A is connected to l vertices in δA (l elements to remove) and each vertex of δA is connected to $n - l + 1$ vertices in A ($n - (l - 1)$ vertices not in any given element of δA). For l as chosen, notice that

$$\frac{l}{n - l + 1} > 1,$$

so there are more elements in δA than A , and so equality cannot be achieved in both A and δA (the middle-rank). This argument also shows that the ranks above middle strictly increase in number as we approach the middle rank.

A converse argument, with $l < \frac{n+1}{2}$ and A and δA constructed as before, gives the same bipartite graph, but $\frac{l}{n-l+1} < 1$, so as we move down, the ranks decrease in size.

Thus, because of this strict approach to equality, equality is only achieved when $\frac{n-1}{2} \leq l \leq \frac{n+1}{2}$, which is the middle (middle two) rank(s) if n is even (odd), as desired. \square

Problem 4 (A generalization of Sperner's Thm). Let P be a rank-symmetric, rank-unimodal poset. Show that if P has a symmetric chain decomposition, then it has *strong Sperner property*, that is, for any $j \geq 1$, the largest size of a union of j antichains is equal to the size of the largest j levels of P . (*Remark:* $j = 1$ corresponds to Sperner's Theorem when $P = B_n$)

Proof. Consider a collection A of j antichains A_1, A_2, \dots, A_j . The number of intersections between any chain C_i in the symmetric decomposition $P = C_1 \cup C_2 \cup \dots \cup C_p$ and A is at most the minimum of $|C_i|$ and j , as no chain can intersect an antichain more than once. Also, since the C_i are disjoint by definition,

$$|F| \leq \sum_i |C_i \cap F| = \sum_i \min\{|C_i|, j\},$$

and we can rewrite this sum based on the length of the chains in the symmetric chain decomposition.

We find the number of chains C_i with $|C_i| = l$. By the chain's symmetry, the chain runs from the $\frac{n-(l-1)}{2}$ th rank to the $\frac{n+l-1}{2}$ th rank ($|C_i| = (l-1) + 1 = l$). The number of such chains is the number of chains that reach rank $\frac{n-l+1}{2}$ minus the number of chains that reach rank $\frac{n-l-1}{2}$, which is also $p_{\frac{n-l+1}{2}} - p_{\frac{n-l-1}{2}}$. For ease of calculation, we can define a permutation¹ $f(x)$ on $0, 1, \dots, n+1$ to take care of the parity issues with $n-l-1$ while arranging them in decreasing order. That is, $f(x)$ gives the rank that is $\lfloor \frac{x}{2} \rfloor$ ranks away from the middle rank(s). Explicitly, for odd n , $f(x) = \frac{n+(-1)^x(2\lfloor \frac{x}{2} \rfloor + 1)}{2}$; dual definition for even n . Thus, we can simplify our expression for the number of chains with l vertices to $p_{f(l-1)} - p_{f(l)}$. When l is an invalid parity, we have $f(l-1) = f(l)$, so we can write

$$\begin{aligned} \sum_i \min\{|C_i|, j\} &= \sum_{l=1}^{j-1} l |\{i : |C_i| = l\}| + \sum_{l=j}^{n+1} j |\{i : |C_i| = l\}| \\ &= \sum_{l=1}^{j-1} l (p_{f(l-1)} - p_{f(l)}) + j \sum_{l=j}^{n+1} p_{f(l-1)} - p_{f(l)} \\ &= \sum_{l=1}^j p_{f(l-1)} + 0 - p_{n+1} \\ &= \sum_{l=0}^{j-1} p_{f(l)}, \end{aligned}$$

which is precisely the sum of the j largest p_i . □

¹this step was due to a hint from another person who was at camp with me.