Algebraic Combinatorics - HW8

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Problem 1 (Saving electricity through Linear Algebra and Graph Theory). (i) Consider the vector space $V = \mathbb{F}_2^n$ (over $\mathbb{F}_2$). Let $A_{n\times n}$ be a symmetric matrix over $\mathbb{F}_2$. Consider the diagonal of A, as a column vector $\overrightarrow{d}$. Prove that

$$\vec{d} \in Col(A)$$

(*Hint*: First show that $\vec{v}^T A\vec{v} = \vec{d}^T\vec{v}$ for all $\vec{v} \in V$. Then show the required result by contradiction.)

(ii) Given a graph G, show (using (i)) that $V(G)$ can be partitioned into V_1 and V_2 such that $G[V_1]$ is an even graph (that is, all degrees are even) and for all vertices $v \in V_2$, $|N(v) \cap V_1|$ is odd.

(iii) Assume that there is a bulb and a button at each vertex of a graph G. The connections are made such that pushing the button at a vertex once, will change the status of the bulb and its neighbors. Initially all bulbs are on. Show (using (ii)) that one can push some buttons and turn all the bulbs off. Does this remind you of a game that you may have played as a kid?

Proof. By matrix multiplication,

$$v^T Av = \sum_{a_{ij}\in A} a_{ij}v_iv_j = \sum a_{ii}v_i^2 + 0 = d^T v.$$

Note that $(Ax)^T y = (Ay)^T x$ by dot product properties.

Thus, the image of A is the orthogonal complement of the kernel of $A^T = A$.

If $Av = 0$, then $d^T v = 0$, so d is orthogonal to the Kernel of A, whence $d \in \mathrm{Col}(A)$.

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Problem 2 (Cycle Space and Bond Space in Graphs + some applications). Given an undirected connected graph G with n vertices and m edges, let each subset of the edge set $E(G)$ be represented by its characteristic binary vector. Let

$$\mathcal{C}(G) = \{\text{all even subgraphs of } G\}$$
$$\mathcal{B}(G) = \{\text{all minimal edge-cuts of } G\}$$
$$M(G) = \text{incidence matrix of } G$$

(i) Show that both $\mathcal{C}(G)$ and $\mathcal{B}(G)$ are subspaces of $\mathbb{F}_2^m$ (over $\mathbb{F}_2$).

(ii) Show that the *stars* at any $n-1$ vertices of G are linearly independent and forms a basis in $\mathcal{B}(G)$, thus

$$\mathcal{B}(G) = \mathrm{Row}(M(G)) \quad \text{and} \quad \dim(\mathcal{B}(G)) = n-1.$$

(iii) Given any spanning tree T of G, show that the *fundamental cycles* (as described in class) are linearly independent in $\mathcal{C}(G)$. Then show that

$$\mathcal{C}(G) = (\mathcal{B}(G))^{\perp} \quad \mathcal{C}(G) = \mathrm{Nul}(M(G)) \quad \dim\big(\mathcal{C}(G)\big) = m - n + 1$$

(iv) Use the above to show that:

(a) A graph is bipartite $\Leftrightarrow$ every circuit is of even length. (*Hint*: Show that $\overrightarrow{j} = (1, 1, \cdots, 1) \in \mathcal{B}(G)$)

(b) Show that for any graph G, $E(G)$ can be partitioned into an even graph and an edge-cut.

(c) Use (b) to give a new proof of $\#1(ii)$ (and hence $\#1(iii)$)

Proof. Clearly, over F_2, scalar multiplication is no issue. We must show that the "addition" of two graphs in $\mathcal{C}$ or $\mathcal{B}$, that is, their symmetric difference, remains in $\mathcal{C}$ or $\mathcal{B}$, respectively. $\mathcal{C}$ is clear; given $C_1, C_2 \in \mathcal{C}$, consider the degree of a vertez in $C_1 + C_2$. Clearly, the sum of two even degrees is even, and is unchanged over F_2. For $\mathcal{B}$, consider two minimal edge cuts B_1 and B_2, yielding partitions v_1, v_1' and v_2, v_2', respectively. B_1 consists of all edges between v_1 and v_1' and likewise for B_2, v_2, and v_2'.

The only edges that remain are those in $B_1 \setminus B_2$ or $B_2 \setminus B_1$. These connect vertices in $v_1 \setminus v_2$ to $v_1' \setminus v_2'$ or $v_2 \setminus v_1$ to $v_2' \setminus v_1'$. ■