Algebraic Combinatorics HW 3

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Problem 1 (Symmetric polynomial and unimodality; Extra-Credit)**.** $f(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n$ is *symmetric* if for all i,

$$p_i = p_{n-i}$$

It is *unimodal* if for some fixed j,

$$p_0 \leq p_1 \leq \cdots \leq p_{j-1} \leq p_j \geq p_{j+1} \geq \cdots \geq p_{n-1} \geq p_n$$

Let $F(q)$, $G(q)$ be symmetric and unimodal polynomials with non-negative real coefficients. Show that $F(q)G(q)$ is also symmetric (easy) and unimodal (less easy).

Proof. □

Problem 2 (Log-concavity of Binomial coefficients)**.** A sequence $a_1, a_2, a_3, \cdots, a_n$ is *logarithmically concave* if

$$a_i^2 \geq a_{i-1}a_{i+1} \ ; \ \forall i$$

(i) Show that if a sequence of positive terms is log-concave, then it is also unimodal.

(ii) It is easy to see algebraically that the sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \cdots, \binom{n}{n}$ is log-concave. Give a combinatorial proof of this fact.

Proof.

1. We rewrite the given relation as

$$\frac{a_i}{a_{i-1}} \geq \frac{a_{i+1}}{a_i}.$$

This means that the ratio of consecutive terms is non-increasing. Thus, even if the sequence is initially increasing, it must eventually decrease. This means that the sequence is unimodal.

2. Consider $\binom{n}{k}$ for $0 \leq k \leq n$. We need to show

$$\binom{n}{k}^2 \geq \binom{n}{k-1}\binom{n}{k+1}$$

Let $|A| = k, B = n - k$.
Notice the $\binom{n}{k}^2$ is the amount of ways to choose k elements from A and k elements from B independently then form separate pairs with these elements (Using $\binom{n}{k} = \binom{n}{n-k}$). $\binom{n}{k-1}\binom{n}{k+1}$ shows the amount of ways to choose $k-1$ elements from a set of n elements and $k+1$ elements from a different set of elements. The pairs formed from these choices will have one element in common, and this common element can

be chosen in n ways. Now, notice that A and B together have n elements, so the common element in the pairs from the right side can be any of the n elements in A or B. Since choosing from A and B allows for more pairs, $\binom{n}{k}^2 \geq \binom{n}{k-1}\binom{n}{k+1}$, i.e. $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ is a log concave sequence.

□

Problem 3 (Uniqueness in Sperner's Thm)**.** Show that equality in Sperner's Theorem for B_n is achieved only by the middle (middle two) rank(s) if n is even (odd). (*Hint*: If not, then move the example closer to the middle rank(s))

Proof. Let $l \geq \frac{n+1}{2}$ correspond to a rank above B_n above the middle of the poset. We call this rank A. Let δA correspond to the $l-1$ rank. The shadow of elements in A consists of l elements in δA. Thus, we can represent the map from A to δA as a bipartite graph. There are l connections from A to δA, □

Problem 4 (A generalization of Sperner's Thm)**.** Let P be a rank-symmetric, rank-unimodal poset. Show that if P has a symmetric chain decomposition, then it has *strong Sperner property*, that is, for any $j \geq 1$, the largest size of a union of j antichains is equal to the size of the largest j levels of P. (*Remark*: $j = 1$ corresponds to Sperner's Theorem when $P = B_n$)

Proof. We choose j antichains we call $A_1, A_2, \ldots, A_j$. Each of these antichains will intersect every other chain at most once. Thus, we have

$$\# \text{ of intersections} = \min\{k, |A_j|\}$$

□