HW 6 - Enumeration of Spanning trees and Eulerian circuits (Due Tuesday 04/02)

1. [Trees with prescribed degrees and Cayley's formula]

(a) Given positive integers $d_1, d_2, ..., d_n$ such that $\sum d_i = 2n - 2$, show that the number of (labelled) tress on $[n]$ such that vertex i has degree d_i for each i is

$$\frac{(n-2)!}{\prod(d_i - 1)!}$$

(*Remark*: Note that for any graph on n vertices with m edges, we always have $\sum d_i = 2m$, thus for a tree, $\sum d_i = 2n - 2$. One can conversely show (say, by induction) that if $\sum d_i = 2n - 2$, there must exist at least one tree with this degree sequence.

Note, the problem can easily be done using Prüfer code. Can you use induction to show the same ?)

(b) Prove Cayley's formula from (a).

(c) What is the number of all trees on n vertices with exactly $n - l$ leaves ? (*Hint*: You may use (a), and leave your answer in a terms of *Stirling's number of the second kind.*)

2. [Counting Spanning trees of $K_{m,n}$] Find the value of $\tau(K_{m,n})$ using:

(i) Matrix-Tree Theorem.

(ii) Combinatorial argument, say, that of Prüfer or Joyal.

(iii) Let L be the Laplacian of $K_{m,n}$.

- (a) Find a simple upper bound on $rank(L - mI)$.
- (b) Deduce a lower bound on the multiplicity of eigenvalue of L equal to m.
- (c) Assume $m \neq n$ and do the same for n.
- (d) Find the remaining eigenvalues of L.
- (e) Use (a) – (d) to compute $\tau(K_{m,n})$

3. [**Not for credit**] Let $n \geq 5$ and G_n be a graph with vertex set $\mathbb{Z}_n$ with edges $\{i, i+1\}$ and $\{i, i+2\}$ for all $i \in \mathbb{Z}_n$. Show that $\tau(G_n) = nF_n^2$, where F_n is the n^{th} Fibonacci number (where $F_1 = 1, F_2 = 1, F_{n+2} = F_{n+1} + F_n$).

4. [Labyrinth Problem] Starting at a point x_0 we walk along the edges of a connected graph G according to the following rules:

- We never use the same edge twice in the same direction.
- Whenever we arrive at a point $x \neq x_0$ not previously visited, we mark the edge along which we entered x. We use the marked edge to leave x only if we must, that is, if we have used all the other edges before.

Show that we get stuck at x_0 and that, by then, every edge has been traversed in both directions.

5. [Universal cycles for S_n]

(i) Let $n \geq 3$. Show that there does not exist a sequence $a_1, a_2, \cdots, a_{n!}$ such that all the $n!$ contiguous blocks $a_i, a_{1+1}, \cdots a_{i+n-1}$ (subscripts taken modulo $n!$) are all the $n!$ permutations of S_n.

(ii) Show that for all $n \geq 1$, there exist a sequence $a_1, a_2, \cdots, a_{n!}$ such that all the $n!$ contiguous blocks $a_i, a_{1+1}, \cdots a_{i+n-2}$ consists of the first $n - 1$ terms $b_1, b_2, \cdots, b_{n-1}$ of all permutations $b_1, b_2, \cdots, b_n$ of $[n]$. Such sequences are called *universal cycles* for S_n(For example, for $n = 3$, 123213 is such a universal cycle)

(iii) For $n = 3$, find the number of universal cycles beginning with 123.

(iv) (unsolved, not for credit) Find U_n, the number of universal cycles for S_n beginning with $123 \cdots n$. It is known that

$$U_4 = 2^7 \cdot 3$$

$$U_5 = 2^{33} \cdot 3^8 \cdot 5^3$$

$$U_6 = 2^{190} \cdot 3^{49} \cdot 5^{33}$$

$$U_7 = 2^{1217} \cdot 3^{123} \cdot 5^{119} \cdot 7^5 \cdot 11^{28} \cdot 43^{35} \cdot 73^{20} \cdot 79^{21} \cdot 109^{35}$$

(*Remark*: Perhaps some divisibilities can be explained by more powerful machinery of Representation Theory of S_n)