Algebraic Combinatorics - HW7

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> **Problem 1** (Lattice Paths avoiding a certain set of points)**.** Let P be a fixed lattice (ballot) path from $(0,0)$ to (m,n). Let T be a set of interior points on P (that is, some subset of points on P other than $(0,0)$ and (m,n)). Let $f_{m,n}(T)$ be the number of lattice paths from $(0,0)$ to (m,n) that avoid all of T.
>
> (i) Use a Corollary of Gessel-Viennot Lemma to find an expression for $f_{m,n}(T)$.
>
> (ii) Use Inclusion-Exclusion to find an expression for $f_{m,n}(T)$.

Proof. (i) Let A be an ordered list of points $P_i = (a_i, b_i)$ on T, with last element $(0,0)$. Let B be an ordered list of the same P_i, except with last element (m,n) instead of $(0,0)$. Construct the directed acyclic graph G, the grid of points from $(0,0)$ to (m,n), with edges going up or right with weight 1. Our path matrix M is a $|A| \times |B|$ matrix, that is, an $(|T|+1) \times (|T|+1)$ matrix. The number of paths from P_i to P_j is naturally $\binom{a_j - a_i + b_j - b_i}{a_j - a_i}$ (note this is zero when P_i is after P_j). Thus, Gessel-Viennot Lemma gives us that the sum of signed vertex-disjoint paths from A to B is $\det M$. However, since the permutations are signed, the only path system that survives is that the identity, which counts the number of paths from $(0,0)$ to (m,n) avoiding T. Therefore,

$$f_{(m,n)}(T) = \det(\left[\binom{a_j - a_i + b_j - b_i}{a_j - a_i}\right])$$

(ii) We use PIE to count the number of ballot paths that intersect at least one vertex in T. Let $t_1, t_2, \ldots, t_k$ be the vertices of T, where $t_i = (x_i, y_i)$. Let B be the set of all "bad" ballot paths from $(0,0)$ to (m,n), and consider the subsets $B_i \subset B$ where B_i is the set of all ballot paths that intersect t_i. Then, by PIE,

$$|B| = \sum_{1 \le i \le k} |B_i| - \sum_{1 \le i < j \le k} |B_i \cap B_j| + \cdots,$$

that is,

$$\sum_{i=1}^{k} \left((-1)^{i+1} \sum_{1 \le c_1 < \cdots < c_i \le k} |B_{c_1} \cap B_{c_2} \cap \cdots \cap B_{c_i}| \right)$$

Notice that $|B_{c_1} \cap \cdots \cap B_{c_i}|$ is the number of ballot paths through $t_{c_1}, \ldots, t_{c_i}$, which is

$$\prod_{v=0}^{i} \binom{x_{c_{v+1}} - x_{c_v} + y_{c_{v+1}} - y_{c_v}}{x_{c_{v+1}} - x_{c_v}},$$

where $x_{c_0} = y_{c_0} = 0$ and $x_{c_{i+1}} = m$, $y_{c_{i+1}} = n$, whence

$$|B| = \sum_{i=1}^{k} \left((-1)^{i+1} \sum_{1 \le c_1 < \cdots < c_i \le k} \prod_{v=0}^{i} \binom{x_{c_{v+1}} - x_{c_v} + y_{c_{v+1}} - y_{c_v}}{x_{c_{v+1}} - x_{c_v}} \right).$$

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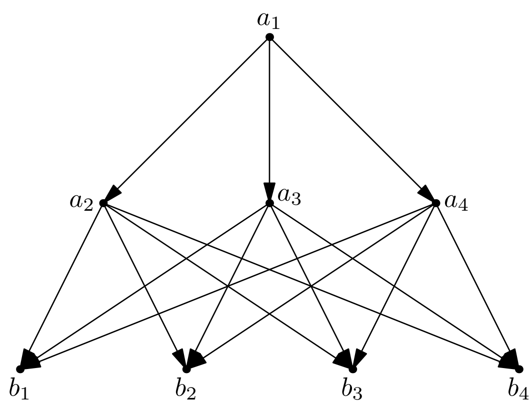
Problem 2 (Linear dependency and Gessel-Vienot)**.** Let $A_{n\times n}$ be a matrix with linearly dependent rows. Show by using Gessel-Viennot Lemma that $|A| = 0$.

Proof. Let $A_{n\times n}$ represent the edge matrix of a weighted DAG from $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Let $e_{ij} = [A_{ij}]$. Then, since the rows are linearly independent, the first row can be written as a linear combination of the other rows, say

$$e_{1j} = \sum_{2\le i\le n} c_i e_{ij}.$$

We can construct our graph with a_1 above all the other a_i, with the example for $n = 4$ shown below:



We can set the edge from a_1 to a_k to have weight c_k, whence we have the linear combination relationship in A. For example,

$$e_{11} = e_{21}c_2 + e_{31}c_3 + e_{41}c_4.$$

By the Gessel-Viennot Lemma, the determinant of A is the sum of the weights of all path systems from A to B. However, notice that the path system from a_1 to $b_{\sigma(a_1)}$ will have an odd number of vertices, while the path system from all other a_i to $b_{\sigma(a_i)}$ will have an even number of vertices, while $|A| + |B| = 2n$ is even. Therefore, there are no path systems, so

$$\det A = \sum_{\mathscr{P}} \operatorname{sign}(\mathscr{P})\omega(\mathscr{P}) = 0.$$

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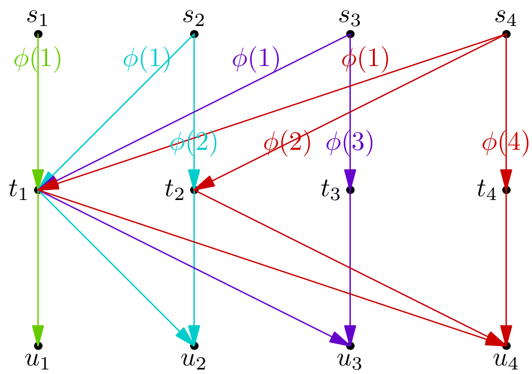
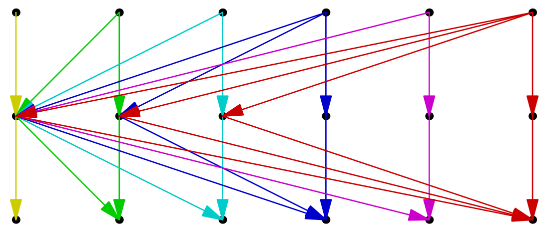
Problem 3 (GCD matrix)**.** Let $S = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{N}$. let the **GCD matrix** M of S have entries $m_{ij} = \gcd(a_i, a_j)$. Prove that if S is closed under taking divisors, then

$$|M| = \prod_{i=1}^{n} \varphi(a_i).$$

(*Hint:* Form a certain digraph using three copies of S, and then put certain edges and weights on them.)

Proof. We form a digraph between S, T, and U, where T and U are two copies of S, placing edges in a way that allows us to make use of the Totient Function's divisor sum. For a given $1 \le i \le n$, let d be a divisor of i. Then, we place an edge from s_i to t_d of weight $\phi(d)$ and an edge from t_d to u_i of weight 1. We give the examples of $n = 4$ and $n = 6$, where the colorings are relevant for our path counting argument:





Consider the edge matrix A of this graph, where a_{ij} is the number of paths from s_i to u_j. By our setup,

$$a_{ij} = \sum_{d|i,j} \phi(d) = \sum_{d|(i,j)} \phi(d) = (i,j),$$

so $A = M$. By the Gessel-Viennot Lemma, $|M| = |A|$, which is the sum of the weights of all path systems from S to U. Notice that $\sigma(s_1)$ is necessarily u_1, whence $\sigma(s_p) = u_p$ for all prime p. We can continue inductively on the number of divisors to conclude that $\sigma(s_k) = u_k$ for all $1 \leq k \leq n$, so we only have one path system $\mathscr{P}$, and $\omega(\mathscr{P}) = \prod_{i=1}^{n} \varphi(i)$, as desired. ■

Problem 4 (Determinant of a matrix of Stirling Numbers)**.** For $m \geq 0$, $n \geq 1$, prove the following identity. Here $S_{n,k}$ is the Stirling number of the 2^{nd} kind.

$$\det \begin{pmatrix} S_{m+1,1} & S_{m+1,2} & \cdots & S_{m+1,n} \\ S_{m+2,1} & S_{m+2,2} & \cdots & S_{m+2,n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{m+n,1} & S_{m+n,2} & \cdots & S_{m+n,n} \end{pmatrix} = (n!)^m$$

Proof. Construct a digraph as shown in the hint, with vertices x_n through x_1, followed by $m+1$ vertices on the horizontal axis and y_1 through y_n on the vertical axis.

With a digraph as shown below, we use the Gessel-Viennot lemma on the path systems from the x_i to y_i.

We construct the path matrix by considering the weighted sum of path systems from the x_i to the y_i. From the weighting, the only surviving path is the identity. the number of ways to get from x_i to y_i is $n!$: we must start by taking i up-right steps, since the paths must be vertex disjoint. We then take m steps to the right, each of which has width i. Thus, the weight of the surviving path system is $(n!)^m$.

The number of paths from x_i to y_j is $S_{m+i,j}$ as well: a path from x_i to y_j involves taking j steps up-right and $m+i-j$ steps right. We are effectively splitting the $m+i$ steps to the right across the j levels, where a setp up from the y_k to y_{k+1} level counts as a step to the right for the kth level. Thus, we are partioning $m+i$ steps into j subsets, and there are $S_{m+i,j}$ ways to do this.

Thus, Gessel-Viennot gives the desired result. ■