Homework 2

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Task(1) Find a linear model to solve the regression problem.

The MSE loss can be written as:

$$L(\beta) = \|Y - X\beta\|_2^2$$

The gradient can be written as:

$$\begin{aligned} \frac{\partial L(\beta)}{\partial \beta} &= \frac{\partial \|Y - X\beta\|_2^2}{\partial \beta} \\ &= -2X^\top (Y - X\beta) \end{aligned}$$

Using gradient descent to calculate β:

$$\beta_{t+1} = \beta_t - \eta \cdot \left(-2X^\top (Y - X\beta)\right)$$

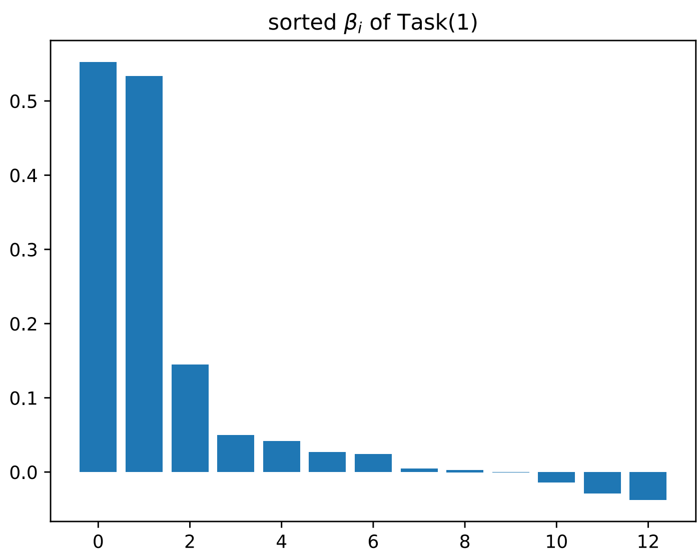
Where η is learning rate.

$$Error_{\text{test}} = \sum_{i=1}^{m} (y_1 - f(X_i))^2$$

After 100000 iterations with $\eta = 0.001$, the $Error_{\text{test}}$ defined as above is 190.52631008965514.
The stdout is:

```
Task(1):
beta = [ 5.33052946e-01  4.12641248e-02  4.92870780e-02  1.44349983e-01
  2.40886173e-02  2.75480488e-03  4.20735754e-03 -3.03859472e-04
 -2.86934505e-02 -3.71851433e-02 -1.35701831e-02  2.66545668e-02
  5.51986584e-01]
loss = 190.52631008965514
```

And the visualization of the weight vector β is:



It can be seen from the figure above that 2 of elements in β plays a giant positive role. While other β_i are relatively small.

Task(2) Using Ridge Regression to solve the regression problem.

The loss can be written as:

$$L(\beta) = \|Y - X\beta\|_2^2 + \lambda\|\beta\|_2^2$$

The gradient can be written as:

$$\begin{aligned}\frac{\partial L(\beta)}{\partial \beta} &= \frac{\partial \|Y - X\beta\|_2^2 + \lambda\|\beta\|_2^2}{\partial \beta} \\ &= -2X^\top (Y - X\beta) + 2\lambda\beta\end{aligned}$$

Let

$$\frac{\partial L(\beta)}{\partial \beta} = 0$$

We have

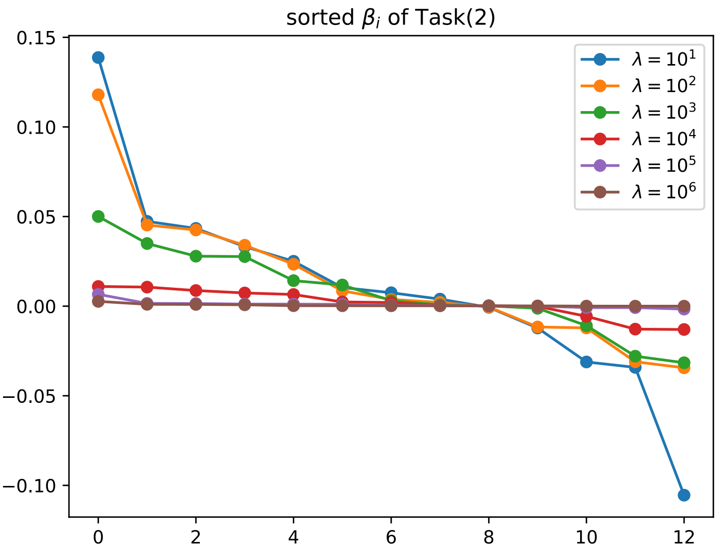
$$\beta = \left(X^\top X + \lambda I\right)^{-1} X^\top Y$$

Try out different λ and find the minimul loss. The stdout is

```
Task(2):
lambda = 10                   loss = 193.1719239298978
lambda = 100                  loss = 190.64552450846662
lambda = 1000                 loss = 186.57235083360067
lambda = 10000                loss = 173.33544686440175
lambda = 100000               loss = 168.967456043671
lambda = 1000000              loss = 178.7360808699323

best_beta = [ 8.21943627e-05  1.54299477e-03  1.00523867e-03  1.16273744e-03
  3.83008956e-04  6.61266799e-03  1.45027551e-03  4.79238031e-04
 -1.62703231e-03 -8.11543685e-04 -7.97680714e-04  1.13904484e-03
 -1.46033335e-05]
best_loss = 168.967456043671
```

When $\lambda = 10^5$, the loss is minimul.
And the visualization of the weight vector β is:



It can be seen from the figure above that as λ goes higher, β_i goes smaller.
And all β_i is a lot less than the result in Task(1). This shows that using Ridge Regression can prevent β_i becoming too large. And the loss is also smaller than the loss in Task(1), which shows that Ridge Regression can prevent over-fitting.
However, when λ goes too high, the loss becomes larger because if β_i is too small, it cannot represent enough feature to predict the value of Y.

Task(3) Using RBF kernel regression to solve the regression problem.

The loss can be written as:

$$L(c) = \|Y - Kc\|^2 + \lambda c^\top K c$$

where

$$K(x_i, x_j) = \phi(x_i)^\top \phi(x_j)$$

$$\begin{aligned} K &= \begin{bmatrix} \phi(x_1)^\top\phi(x_1) & \cdots & \phi(x_1)^\top\phi(x_n) \\ \vdots & \ddots & \vdots \\ \phi(x_m)^\top\phi(x_1) & \cdots & \phi(x_m)^\top\phi(x_n) \end{bmatrix} \\ &= \begin{bmatrix} \phi(x_1)^\top \\ \vdots \\ \phi(x_n)^\top \end{bmatrix} \begin{bmatrix} \phi(x_1) & \cdots & \phi(x_n) \end{bmatrix} \end{aligned}$$

Let

$$\frac{\partial L(c)}{\partial c} = 0$$

We have

$$c = (K + \lambda I)^{-1} Y$$

Case we have

$$\begin{aligned} Y_{\text{train}}^{\text{pred}} &= Kc \\ &= \begin{bmatrix} \phi(x_1)^\top \\ \vdots \\ \phi(x_n)^\top \end{bmatrix} \begin{bmatrix} \phi(x_1) & \cdots & \phi(x_n) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\ &= \begin{bmatrix} \phi(x_1)^\top \\ \vdots \\ \phi(x_n)^\top \end{bmatrix} \sum_{i=1}^{n} c_i \phi(x_i) \end{aligned}$$

in the training, we have

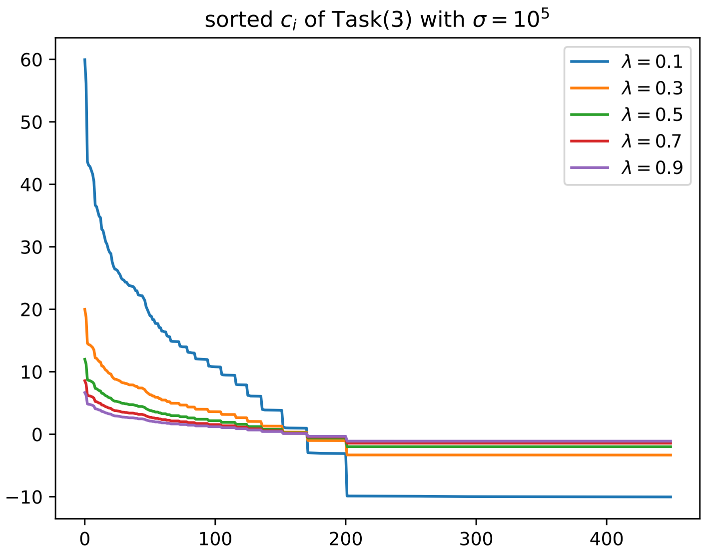
$$\begin{aligned} Y_{\text{test}}^{\text{pred}} &= \begin{bmatrix} \phi(x_1)^\top \\ \vdots \\ \phi(x_m)^\top \end{bmatrix} \sum_{i=1}^{n} c_i \phi(x_i) \\ &= \begin{bmatrix} \phi(x_1)^\top \\ \vdots \\ \phi(x_m)^\top \end{bmatrix} \begin{bmatrix} \phi(x_1) & \cdots & \phi(x_n) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\ &= \begin{bmatrix} \phi(x_1)^\top\phi(x_1) & \cdots & \phi(x_1)^\top\phi(x_n) \\ \vdots & \ddots & \vdots \\ \phi(x_m)^\top\phi(x_1) & \cdots & \phi(x_m)^\top\phi(x_n) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \end{aligned}$$

in testing.
Set $\sigma = 10^5$ and try out different λ to find the minimul loss. The stdout is

```
Task(3):
lambda = 0.1                    loss = 155.29346939385917
lambda = 0.3                    loss = 155.1889666982272
lambda = 0.5                    loss = 155.1868258245468
lambda = 0.7                    loss = 155.19959272106766
lambda = 0.9                    loss = 155.21735678971228

best_c = [...]
best_loss = 155.1868258245468
```

Case c is too long, I won't put it in this report. c can be obtained via executing the code.
We can see that the best $\lambda = 0.5$.
And the visualization of the weight vector c is:



It can be seen from the figure above that as λ goes higher, c_i goes smaller.
And the loss is much smaller than other linar model. Because kernal function of higher dimension have a better capacity to represent the feature.

Task(4) Using Spline Regression to solve the regression problem.

```
Task(4):
task(4) 涉及高维的spline regression， 课堂上没有教。这一个task可以不用做。
```

Task(5) Using Lasso Regression to solve the regression problem.

The loss can be written as:

$$L(\beta) = \frac{1}{2}\|Y - X\beta\|_2^2 + \lambda\|\beta\|_1$$

The gradient can be written as:

$$\begin{aligned}\frac{\partial L(\beta)}{\partial \beta} &= \frac{\partial \frac{1}{2}\|Y - X\beta\|_2^2 + \lambda\|\beta\|_1}{\partial \beta} \\ &= \lambda \cdot \operatorname{sign}(\beta) - X^\top (Y - X\beta)\end{aligned}$$

Using gradient descent to calculate β:

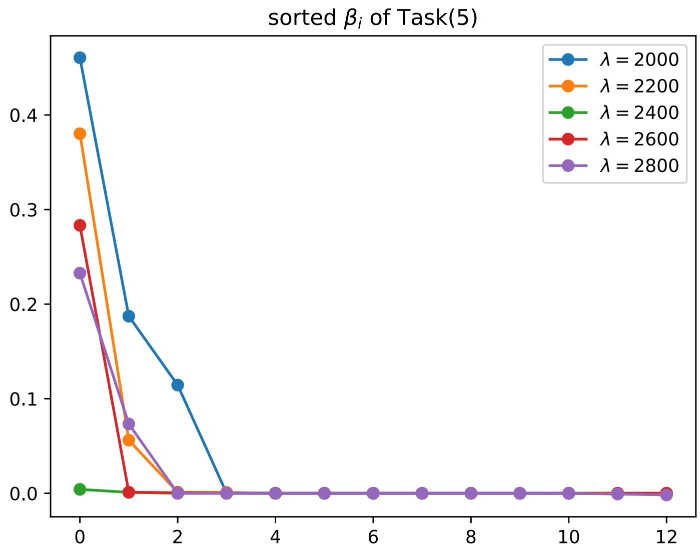
$$\beta_{t+1} = \beta_t - \eta \cdot \left(\lambda \cdot \operatorname{sign}(\beta) - X^\top (Y - X\beta)\right)$$

Where η is learning rate.
Set the iteration num be 100000 and $\eta = 10^{-4}$. Try out different λ and the stdout is:

```
Task(5):
lambda = 2000                  loss = 156.82043402173144
lambda = 2200                  loss = 179.0831038785226
lambda = 2400                  loss = 174.8449522295677
lambda = 2600                  loss = 175.93794339250366
lambda = 2800                  loss = 158.93595080304917

best_beta = [ 4.60576688e-01  3.43504798e-06 -3.74674312e-06  6.88011475e-06
  1.14431192e-01 -1.60789531e-06 -3.12163388e-06  1.61820114e-04
  8.20035892e-06  6.95914888e-06 -4.12246454e-06 -3.27022231e-06
  1.87220832e-01]
best_loss = 156.82043402173144
```

We can see that the best $\lambda = 2000$.
And the visualization of the weight vector β is:



It can be seen from the figure above that most of the β_i is zero compared with the result in Task(2). This shows that compared with ridge regression, lasso regression produce sparse solutions, making some of the unimportant feature coefficients zero to simplify the model and improving generalization performance.