第六周作业

1. 证明一维谐振子的升降算符满足

(1) $\hat{a}_-^\dagger = \hat{a}_+$，

(2) $[\hat{a}_-, \hat{a}_+] = 1$，

(3) $[\hat{a}_+, \hat{a}_+\hat{a}_-] = -\hat{a}_+$，

$[\hat{a}_-, \hat{a}_+\hat{a}_-] = \hat{a}_-$。

证(1)

$$\begin{aligned}
\hat{a}_+ &= \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x}), \\
\hat{a}_- &= \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega\hat{x}), \\
\hat{a}_-^\dagger &= \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p}^\dagger + m\omega\hat{x}^\dagger) \\
&= \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x}) = \hat{a}_+.
\end{aligned}$$

(2)

$$\begin{aligned}
[\hat{a}_-, \hat{a}_+] &= \left[\frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega\hat{x}), \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x})\right] \\
&= \frac{1}{2\hbar m\omega}[i\hat{p} + m\omega\hat{x}, -i\hat{p} + m\omega\hat{x}] \\
&= \frac{1}{2\hbar m\omega}(im\omega[\hat{p}, \hat{x}] - im\omega[\hat{x}, \hat{p}]) \\
&= \frac{i}{2\hbar}(-i\hbar - i\hbar) = 1.
\end{aligned}$$

(3)

$$\begin{aligned}
[\hat{a}_+, \hat{a}_+\hat{a}_-] &= [\hat{a}_+, \hat{a}_+]\hat{a}_- + \hat{a}_+[\hat{a}_+, \hat{a}_-] \\
&= \hat{a}_+[\hat{a}_+, \hat{a}_-], \\
[\hat{a}_+, \hat{a}_-] &= -1, \\
[\hat{a}_+, \hat{a}_+\hat{a}_-] &= -\hat{a}_+, \\
[\hat{a}_-, \hat{a}_+\hat{a}_-] &= [\hat{a}_-, \hat{a}_+]\hat{a}_- + \hat{a}_+[\hat{a}_-, \hat{a}_-] \\
&= [\hat{a}_-, \hat{a}_+]\hat{a}_-, \\
[\hat{a}_-, \hat{a}_+] &= 1, \\
[\hat{a}_-, \hat{a}_+\hat{a}_-] &= \hat{a}_-.
\end{aligned}$$

2. 对于谐振子的能量本征态ψ_n，

(1) 计算$\hat{x}$、$\hat{p}$的平均值，

(2) 计算$\hat{x}^2$、$\hat{p}^2$的平均值，

(3) 计算$\Delta x = (\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2)^{\frac{1}{2}}$、$\Delta p = (\langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2)^{\frac{1}{2}}$。

解(1)

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-),$$

$$\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}_+ - \hat{a}_-),$$

$$\hat{a}_-\psi_n = \sqrt{n}\psi_{n-1},$$

$$\hat{a}_+\psi_n = \sqrt{n+1}\psi_{n+1},$$

$$\hat{x}\psi_n = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+\psi_n + \hat{a}_-\psi_n)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\psi_{n+1} + \sqrt{n}\psi_{n-1}\right),$$

$$\hat{p}\psi_n = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}_+\psi_n - \hat{a}_-\psi_n)$$

$$= i\sqrt{\frac{m\omega\hbar}{2}}\left(\sqrt{n+1}\psi_{n+1} - \sqrt{n}\psi_{n-1}\right),$$

由本征态的正交归一化条件$\langle\psi_n, \psi_{n'}\rangle = \delta_{nn'}$，

$$\langle\hat{x}\rangle = \langle\psi_n, \hat{x}\psi_n\rangle = 0,$$

$$\langle\hat{p}\rangle = \langle\psi_n, \hat{p}\psi_n\rangle = 0,$$

这个结论也可以利用波函数$\psi_n(x)$的宇称性而得出。

(2)
$$\hat{x}^2 = \frac{\hbar}{2m\omega}(\hat{a}_+ + \hat{a}_-)^2$$

$$= \frac{\hbar}{2m\omega}(\hat{a}_+^2 + \hat{a}_-^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+)$$

$$= \frac{\hbar}{2m\omega}\left(\hat{a}_+^2 + \hat{a}_-^2 + 2\hat{N} + 1\right),$$

$$\hat{p}^2 = -\frac{m\omega\hbar}{2}(\hat{a}_+ - \hat{a}_-)^2$$

$$= -\frac{m\omega\hbar}{2}(\hat{a}_+^2 + \hat{a}_-^2 - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+)$$

$$= \frac{m\omega\hbar}{2}\left(2\hat{N} + 1 - \hat{a}_+^2 - \hat{a}_-^2\right),$$

由于

$$\hat{a}_+^2\psi_n = \sqrt{(n+1)(n+2)}\psi_{n+2},$$

$$\hat{a}_-^2\psi_n = \sqrt{n(n-1)}\psi_{n-2},$$

根据正交条件

$$\langle\psi_n, \hat{a}_+^2\psi_n\rangle = 0, \qquad \langle\psi_n, \hat{a}_-^2\psi_n\rangle = 0,$$

因此

$$\langle\hat{x}^2\rangle = \langle\psi_n, \hat{x}^2\psi_n\rangle = \frac{\hbar}{2m\omega}\left\langle\psi_n, \left(2\hat{N}+1\right)\psi_n\right\rangle = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right),$$

$$\langle \hat{p}^2 \rangle = \langle \psi_n, \hat{p}^2 \psi_n \rangle = \frac{m\omega\hbar}{2} \left\langle \psi_n, \left(2\widehat{N}+1\right)\psi_n \right\rangle = m\omega\hbar\left(n+\frac{1}{2}\right).$$

(3) $$\Delta x = (\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2)^{\frac{1}{2}} = \sqrt{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)},$$

$$\Delta p = (\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2)^{\frac{1}{2}} = \sqrt{m\omega\hbar\left(n+\frac{1}{2}\right)},$$

$$\Delta x \cdot \Delta p = \hbar\left(n+\frac{1}{2}\right).$$

对于基态，$n=0, \Delta x \cdot \Delta p = \hbar/2$，刚好是测不准关系所规定的下限。