第十二周作业答案

1. 在一个一维的问题中，一个粒子的哈密顿量算符为

$$H = \frac{1}{2m}P^2 + V(x),$$

算符X和P满足关系式$[X,P]=i\hbar$。H的本征矢用$|\varphi_n\rangle$表示，即有$H|\varphi_n\rangle = E_n|\varphi_n\rangle$，其中$n$是离散指标。

a. 试证

$$\langle\varphi_n|P|\varphi_{n'}\rangle = \alpha\langle\varphi_n|X|\varphi_{n'}\rangle,$$

式中的α是一个只依赖于E_n与$E_{n'}$之差的系数。试求α（提示：考虑对易子$[X,H]$）。

b. 利用封闭性关系式，从上面的结果导出：

$$\sum_{n'}(E_n - E_{n'})^2|\langle\varphi_n|X|\varphi_{n'}\rangle|^2 = \frac{\hbar^2}{m^2}\langle\varphi_n|P^2|\varphi_n\rangle.$$

证明：

a.

$$\begin{aligned}
[X,H] &= \frac{1}{2m}[X,P^2] + [X,V] \\
&= \frac{1}{2m}([X,P]P + P[X,P]) \\
&= \frac{i\hbar}{m}P, \\
\langle\varphi_n|P|\varphi_{n'}\rangle &= -i\frac{m}{\hbar}\langle\varphi_n|[X,H]|\varphi_{n'}\rangle \\
&= -i\frac{m}{\hbar}\langle\varphi_n|XH|\varphi_{n'}\rangle + i\frac{m}{\hbar}\langle\varphi_n|HX|\varphi_{n'}\rangle \\
&= i\frac{m}{\hbar}(E_n^* - E_{n'})\langle\varphi_n|X|\varphi_{n'}\rangle \\
&= i\frac{m}{\hbar}(E_n - E_{n'})\langle\varphi_n|X|\varphi_{n'}\rangle, \\
\alpha &= i\frac{m}{\hbar}(E_n - E_{n'}).
\end{aligned}$$

这里厄米算符H的本征值为实数。

b. 利用封闭性关系$I = \sum|\varphi_n\rangle\langle\varphi_n|$,

$$\begin{aligned}
\frac{\hbar^2}{m^2}\langle\varphi_n|P^2|\varphi_n\rangle &= \frac{\hbar^2}{m^2}\sum_{n'}\langle\varphi_n|P|\varphi_{n'}\rangle\langle\varphi_{n'}|P|\varphi_n\rangle \\
&= -\sum_{n'}(E_n - E_{n'})\langle\varphi_n|X|\varphi_{n'}\rangle(E_{n'} - E_n)\langle\varphi_{n'}|X|\varphi_n\rangle \\
&= \sum_{n'}(E_n - E_{n'})^2|\langle\varphi_n|X|\varphi_{n'}\rangle|^2.
\end{aligned}$$