第四周作业

考虑高斯波包$\phi(k)=Ae^{-(k-k_0)^2d^2}$所描述的一维自由粒子，求（要求写出具体求解步骤）：

(1) 归一化后的$\psi(x,t)$与波包的概率分布；

(2) 任取必要的常数，作图画出 3 个不同时间点的波包概率分布；

(3) 波包坐标的平均值表达式，并结合(2)中的图进行分析；

(4) 位置坐标方差表达式，并结合(2)中的图进行分析。

解(1)

$$
\begin{aligned}
\psi(x,t)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{ikx-i\frac{\hbar k^2}{2m}t}\mathrm{d}k\\
&=\frac{1}{\sqrt{2\pi}}e^{ik_0x-i\omega(k_0)t}\int_{-\infty}^{\infty}\phi(k)e^{i(k-k_0)(x-vt)-i\frac{\beta}{2}(k-k_0)^2t}\mathrm{d}k\,,
\end{aligned}
$$

$$\omega(k)=\omega(k_0)+v(k-k_0)+\frac{1}{2}\beta(k-k_0)^2,$$

$$v=\left(\frac{\mathrm{d}\omega}{\mathrm{d}k}\right)_{k_0}=\frac{\hbar k_0}{m},\qquad \beta=\left(\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}\right)_{k_0}=\frac{\hbar}{m},$$

$$
\begin{aligned}
\psi(x,t)&=\frac{1}{\sqrt{2\pi}}e^{ik_0x-i\omega(k_0)t}\int_{-\infty}^{\infty}Ae^{i(k-k_0)(x-vt)-(k-k_0)^2\left(\frac{i}{2}\beta t+d^2\right)}\mathrm{d}k\\
&=\frac{1}{\sqrt{2\pi}}e^{ik_0x-i\omega(k_0)t}\int_{-\infty}^{\infty}Ae^{-\left(k-k_0-i\frac{x-vt}{i\beta t+2d^2}\right)^2\left(\frac{i\beta t}{2}+d^2\right)-\frac{(x-vt)^2}{2(i\beta t+2d^2)}}\mathrm{d}k\\
&=\frac{A}{\sqrt{\pi(2d^2+i\beta t)}}e^{ik_0x-i\frac{\hbar k_0^2}{2m}t-\frac{(x-vt)^2}{2(2d^2+i\beta t)}}\int_{-\infty}^{\infty}e^{-\xi^2}\mathrm{d}\xi\\
&=\frac{A}{\sqrt{2d^2+i\beta t}}e^{ik_0x-i\frac{\hbar k_0^2}{2m}t-\frac{(x-vt)^2}{2(2d^2+i\beta t)}},\\
&=\frac{A}{\sqrt{2d^2(1+i\Delta)}}e^{ik_0x-i\frac{\hbar k_0^2}{2m}t-\frac{(x-vt)^2}{4d^2(1+i\Delta)}},\qquad \Delta=\frac{\hbar}{2md^2}t.
\end{aligned}
$$

其中$\int_{-\infty}^{\infty}e^{-\xi^2}\mathrm{d}\xi=\sqrt{\pi}$。再利用归一化条件，

$$\int_{-\infty}^{\infty}|\psi(x,t)|^2\mathrm{d}x=\frac{A^2}{2d^2\sqrt{1+\Delta^2}}\int_{-\infty}^{\infty}e^{-\frac{(x-vt)^2}{2d^2(1+\Delta^2)}}\mathrm{d}x=\frac{A^2\sqrt{\pi}}{\sqrt{2d^2}}=1,$$

$$A=\left(\frac{2d^2}{\pi}\right)^{1/4},$$

$$\psi(x,t)=\frac{1}{(2\pi d^2)^{1/4}\sqrt{1+i\Delta}}e^{ik_0x-i\frac{\hbar k_0^2}{2m}t-\frac{(x-vt)^2}{4d^2(1+i\Delta)}}.$$

波包的概率分布为

$$|\psi(x,t)|^2=\frac{1}{d\sqrt{2\pi(1+\Delta^2)}}e^{-\frac{(x-vt)^2}{2d^2(1+\Delta^2)}}.$$

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(3) 波包坐标平均值为

$$\begin{aligned}
\langle x\rangle&=\int_{-\infty}^{\infty}|\psi(x,t)|^2x\mathrm{d}x=\int_{-\infty}^{\infty}\frac{x}{d\sqrt{2\pi(1+\Delta^2)}}e^{-\frac{(x-vt)^2}{2d^2(1+\Delta^2)}}\mathrm{d}x\\
&=\int_{-\infty}^{\infty}\frac{x-vt}{d\sqrt{2\pi(1+\Delta^2)}}e^{-\frac{(x-vt)^2}{2d^2(1+\Delta^2)}}\mathrm{d}x+vt\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-vt)^2}{2d^2(1+\Delta^2)}}}{d\sqrt{2\pi(1+\Delta^2)}}\mathrm{d}x\\
&=\int_{0}^{\infty}\frac{e^{-\frac{\xi}{2d^2(1+\Delta^2)}}}{2d\sqrt{2\pi(1+\Delta^2)}}\mathrm{d}\xi+\int_{\infty}^{0}\frac{e^{-\frac{\xi}{2d^2(1+\Delta^2)}}}{2d\sqrt{2\pi(1+\Delta^2)}}\mathrm{d}\xi+vt\\
&=vt.
\end{aligned}$$

(4) 位置坐标方差为

$$\begin{aligned}
\sigma_x^2&=\int_{-\infty}^{\infty}(x-\langle x\rangle)^2|\psi(x,t)|^2\mathrm{d}x\\
&=\int_{-\infty}^{\infty}\frac{(x-vt)^2}{d\sqrt{2\pi(1+\Delta^2)}}e^{-\frac{(x-vt)^2}{2d^2(1+\Delta^2)}}\mathrm{d}x\\
&=-\frac{d\sqrt{1+\Delta^2}(x-vt)}{\sqrt{2\pi}}e^{-\frac{(x-vt)^2}{2d^2(1+\Delta^2)}}\Bigg|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\frac{d\sqrt{1+\Delta^2}}{\sqrt{2\pi}}e^{-\frac{(x-vt)^2}{2d^2(1+\Delta^2)}}\mathrm{d}x\\
&=d^2(1+\Delta^2).
\end{aligned}$$