第七周作业

1. 耦合谐振子的哈密顿量为

$$H=\frac{1}{2m}(\hat{p}_1^2+\hat{p}_2^2)+\frac{1}{2}m\omega^2(\hat{x}_1^2+\hat{x}_2^2)+\lambda\hat{x}_1\hat{x}_2,$$

其中

$$\hat{p}_1=-i\hbar\frac{\partial}{\partial x_1},\qquad \hat{p}_2=-i\hbar\frac{\partial}{\partial x_2}.$$

x_1,p_1和x_2,p_2分属于不同的自由度。设$\lambda<m\omega^2$，试求这耦合谐振子的能级。提示：对于耦合谐振子，可以用坐标变换的办法将问题化成两个独立的一维谐振子问题。

解 令

$$\hat{x}_1=\frac{1}{\sqrt{2}}(\hat{y}_1+\hat{y}_2),\qquad \hat{x}_2=\frac{1}{\sqrt{2}}(\hat{y}_1-\hat{y}_2),$$

即

$$\hat{y}_1=\frac{1}{\sqrt{2}}(\hat{x}_1+\hat{x}_2),\qquad \hat{y}_2=\frac{1}{\sqrt{2}}(\hat{x}_1-\hat{x}_2),$$

容易证明

$$\hat{x}_1^2+\hat{x}_2^2=\hat{y}_1^2+\hat{y}_2^2,$$

$$\hat{x}_1\hat{x}_2=\frac{1}{2}(\hat{y}_1^2-\hat{y}_2^2),$$

$$\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}=\frac{1}{2}\left(\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}\right)^2+\frac{1}{2}\left(\frac{\partial}{\partial y_1}-\frac{\partial}{\partial y_2}\right)^2$$

$$=\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2},$$

因此哈密顿量可以表示成

$$H=-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}\right)+\frac{1}{2}m\omega^2(\hat{y}_1^2+\hat{y}_2^2)+\frac{\lambda}{2}(\hat{y}_1^2-\hat{y}_2^2)$$

$$=-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}\right)+\frac{1}{2}m\omega_1^2\hat{y}_1^2+\frac{1}{2}m\omega_2^2\hat{y}_2^2,$$

其中

$$\omega_1^2=\omega^2+\frac{\lambda}{m},\qquad \omega_2^2=\omega^2-\frac{\lambda}{m}.$$

此哈密顿量正是两个独立谐振子能量算符之和，因此能量本征值和本征函数为

$$E_{N_1N_2}=\left(N_1+\frac{1}{2}\right)\hbar\omega_1+\left(N_2+\frac{1}{2}\right)\hbar\omega_2,$$

$$\psi_{N_1N_2}(y_1,y_2)=\psi_{N_1}(y_1)\psi_{N_2}(y_2),\qquad N_1,N_2=0,1,2,\cdots$$

2. 在上题中，没有耦合项$\lambda\hat{x}_1\hat{x}_2$时，自由振子本征态记为$\psi_{n_1n_2}(x_1,x_2)=$

$\psi_{n_1}(x_1)\psi_{n_2}(x_2)$，其中$n_1, n_2 = 0,1,2,\cdots$，$\psi_n(x)$为一维谐振子的能量本征函数。耦合振子本征态记为$\psi_{N_1N_2}(y_1, y_2)$，其中$N_1, N_2 = 0,1,2,\cdots$，$y_1, y_2$为变换后的坐标。试对于$\psi_{N_1N_2}$态计算$\hat{n}_1, \hat{n}_2$的平均值。

解　引入升降算符和粒子数算符

$$\hat{n}_1 = \hat{a}_1^\dagger \hat{a}_1, \qquad \hat{n}_2 = \hat{a}_2^\dagger \hat{a}_2,$$

$$\hat{a}_1 = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_1 + \frac{i}{m\omega}\hat{p}_1\right),\quad \hat{a}_2 = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_2 + \frac{i}{m\omega}\hat{p}_2\right),$$

$$\hat{N}_1 = \hat{b}_1^\dagger \hat{b}_1, \qquad \hat{N}_2 = \hat{b}_2^\dagger \hat{b}_2,$$

$$\hat{b}_1 = \sqrt{\frac{m\omega_1}{2\hbar}}\left(\hat{y}_1 + \frac{i}{m\omega_1}\hat{p}(y_1)\right), \qquad \hat{p}(y_1) = -i\hbar\frac{\partial}{\partial y_1},$$

$$\hat{b}_2 = \sqrt{\frac{m\omega_2}{2\hbar}}\left(\hat{y}_2 + \frac{i}{m\omega_2}\hat{p}(y_2)\right), \qquad \hat{p}(y_2) = -i\hbar\frac{\partial}{\partial y_2}.$$

容易求得

$$\begin{aligned}
\hat{n}_1 &= \frac{m\omega}{2\hbar}\hat{x}_1^2 + \frac{1}{2m\omega\hbar}\hat{p}_1^2 + \frac{i}{2\hbar}[\hat{x}_1, \hat{p}_1] \\
&= \frac{m\omega}{2\hbar}\hat{x}_1^2 + \frac{1}{2m\omega\hbar}\hat{p}_1^2 - \frac{1}{2} \\
&= \frac{m\omega}{4\hbar}(\hat{y}_1 + \hat{y}_2)^2 - \frac{\hbar}{4m\omega}\left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\right)^2 - \frac{1}{2} \\
&= \frac{m\omega}{4\hbar}(\hat{y}_1^2 + \hat{y}_2^2 + 2\hat{y}_1\hat{y}_2) + \frac{1}{4m\omega\hbar}[\hat{p}^2(y_1) + \hat{p}^2(y_2) + 2\hat{p}(y_1)\hat{p}(y_2)] - \frac{1}{2},
\end{aligned}$$

$$\begin{aligned}
\hat{n}_2 &= \frac{m\omega}{2\hbar}\hat{x}_2^2 + \frac{1}{2m\omega\hbar}\hat{p}_2^2 - \frac{1}{2} \\
&= \frac{m\omega}{4\hbar}(\hat{y}_1^2 + \hat{y}_2^2 - 2\hat{y}_1\hat{y}_2) + \frac{1}{4m\omega\hbar}[\hat{p}^2(y_1) + \hat{p}^2(y_2) - 2\hat{p}(y_1)\hat{p}(y_2)] - \frac{1}{2}.
\end{aligned}$$

在$\psi_{N_1N_2}$态中计算平均值时，

$$\langle \hat{y}_1^2 \rangle = \frac{\hbar}{2m\omega_1}\langle \hat{b}_1^{\dagger 2} + \hat{b}_1^2 + 2\hat{N}_1 + 1 \rangle = \frac{\hbar}{m\omega_1}\left(N_1 + \frac{1}{2}\right),$$

$$\langle \hat{y}_2^2 \rangle = \frac{\hbar}{m\omega_2}\left(N_2 + \frac{1}{2}\right),$$

$$\begin{aligned}
\langle \hat{y}_1\hat{y}_2 \rangle &= \frac{\hbar}{2m\sqrt{\omega_1\omega_2}}\langle (\hat{b}_1^\dagger + \hat{b}_1)(\hat{b}_2^\dagger + \hat{b}_2) \rangle \\
&= \frac{\hbar}{2m\sqrt{\omega_1\omega_2}}\langle \hat{b}_1^\dagger\hat{b}_2^\dagger + \hat{b}_1\hat{b}_2 + \hat{b}_1^\dagger\hat{b}_2 + \hat{b}_1\hat{b}_2^\dagger \rangle = 0,
\end{aligned}$$

$$\langle \hat{p}^2(y_1) \rangle = \frac{m\omega_1\hbar}{2}\langle 2\hat{N}_1 + 1 - \hat{b}_1^{\dagger 2} - \hat{b}_1^2 \rangle = m\omega_1\hbar\left(N_1 + \frac{1}{2}\right),$$

$$\langle \hat{p}^2(y_2) \rangle = m\omega_2\hbar\left(N_2 + \frac{1}{2}\right),$$

$$\begin{aligned}\langle \hat{p}(y_1)\hat{p}(y_2) \rangle &= -\frac{m\hbar\sqrt{\omega_1\omega_2}}{2}\langle (\hat{b}_1^\dagger - \hat{b}_1)(\hat{b}_2^\dagger - \hat{b}_2) \rangle \\ &= -\frac{m\hbar\sqrt{\omega_1\omega_2}}{2}\langle \hat{b}_1^\dagger\hat{b}_2^\dagger + \hat{b}_1\hat{b}_2 - \hat{b}_1^\dagger\hat{b}_2 - \hat{b}_1\hat{b}_2^\dagger \rangle = 0.\end{aligned}$$

因此

$$\begin{aligned}\langle \hat{n}_1 \rangle &= \frac{m\omega}{4\hbar}\left[\frac{\hbar}{m\omega_1}\left(N_1 + \frac{1}{2}\right) + \frac{\hbar}{m\omega_2}\left(N_2 + \frac{1}{2}\right)\right] \\ &\quad + \frac{1}{4m\omega\hbar}\left[m\omega_1\hbar\left(N_1 + \frac{1}{2}\right) + m\omega_2\hbar\left(N_2 + \frac{1}{2}\right)\right] - \frac{1}{2} \\ &= \frac{1}{4}\left[\left(N_1 + \frac{1}{2}\right)\left(\frac{\omega}{\omega_1} + \frac{\omega_1}{\omega}\right) + \left(N_2 + \frac{1}{2}\right)\left(\frac{\omega}{\omega_2} + \frac{\omega_2}{\omega}\right)\right] - \frac{1}{2}.\end{aligned}$$

类似地，可以求出

$$\langle \hat{n}_2 \rangle = \langle \hat{n}_1 \rangle.$$

如耦合强度$\lambda \to 0$，则$\omega_1 = \omega_2 = \omega$，这时

$$\langle \hat{n}_1 \rangle = \langle \hat{n}_2 \rangle \to \frac{1}{2}(N_1 + N_2).$$