第十一周作业答案

1. 我们用$|\varphi_n\rangle$表示厄米算符H的本征态（譬如，H可以是任何物理体系的哈密顿算符），假设全体$|\varphi_n\rangle$构成一个离散的正交归一基。算符$U(m,n)$定义是

$$U(m,n)=|\varphi_m\rangle\langle\varphi_n|,$$

   a. 计算$U(m,n)$的伴随算符$U^\dagger(m,n)$，
   b. 计算对易子$[H,U(m,n)]$，
   c. 证明：

$$U(m,n)U^\dagger(p,q)=\delta_{n,q}U(m,p),$$

   d. 计算算符$U(m,n)$的迹$\mathrm{Tr}\{U(m,n)\}$,
   e. 设A是一个算符，它的矩阵元是$A_{mn}=\langle\varphi_m|A|\varphi_n\rangle$；试证：

$$A=\sum_{m,n}A_{mn}U(m,n),$$

   f. 试证：$A_{pq}=\mathrm{Tr}\{AU^\dagger(p,q)\}$。

解：

a.

$$U^\dagger(m,n)=(|\varphi_m\rangle\langle\varphi_n|)^\dagger=|\varphi_n\rangle\langle\varphi_m|.$$

b.

因为$H|\varphi_n\rangle=\lambda_n|\varphi_n\rangle$，

所以$<\varphi_n|H^\dagger=\lambda_n^*<\varphi_n|$，也就是$<\varphi_n|H=\lambda_n^*<\varphi_n|$

$$\begin{aligned}[H,U(m,n)]&=H|\varphi_m\rangle\langle\varphi_n|-|\varphi_m\rangle\langle\varphi_n|H\\&=\lambda_m|\varphi_m\rangle\langle\varphi_n|-\lambda_n^*|\varphi_m\rangle\langle\varphi_n|\\&=(\lambda_m-\lambda_n)U(m,n),\end{aligned}$$

其中$\lambda_{m,n}$是$|\varphi_{m,n}\rangle$态对应的本征值，且厄米算符的本征值为实数。

c.

$$\begin{aligned}U(m,n)U^\dagger(p,q)&=|\varphi_m\rangle\langle\varphi_n|\left(|\varphi_p\rangle\langle\varphi_q|\right)^\dagger\\&=|\varphi_m\rangle\langle\varphi_n|\varphi_q\rangle\langle\varphi_p|\\&=\delta_{n,q}|\varphi_m\rangle\langle\varphi_p|=\delta_{n,q}U(m,p).\end{aligned}$$

d.

$$\mathrm{Tr}\{U(m,n)\}=\sum_n\langle\varphi_n|U(n,n)|\varphi_n\rangle=\sum_n 1=N.$$

N是$\{|\varphi_n\rangle\}$这一离散正交归一基的维度（总数）

e. 利用态空间中恒等算符的表达式

$$A=\sum_m|\varphi_m\rangle\langle\varphi_m|A\sum_n|\varphi_n\rangle\langle\varphi_n|$$

$$= \sum_{m,n} |\varphi_m\rangle\langle\varphi_m|A|\varphi_n\rangle\langle\varphi_n|$$

$$= \sum_{m,n} A_{mn}|\varphi_m\rangle\langle\varphi_n|$$

$$= \sum_{m,n} A_{mn}U(m,n).$$

f.

$$\mathrm{Tr}\{AU^\dagger(p,q)\} = \sum_n \langle\varphi_n|AU^\dagger(p,q)|\varphi_n\rangle$$

$$= \sum_n \langle\varphi_n|A|\varphi_q\rangle\langle\varphi_p|\varphi_n\rangle$$

$$= \sum_n \langle\varphi_n|A|\varphi_q\rangle\delta_{np} = A_{pq}.$$

2. 在一个二维矢量空间中，考虑这样一个算符，它在正交归一基$\{|1\rangle,|2\rangle\}$中的矩阵为：

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

   a. σ_y是厄米算符吗？试计算它的本征值和本征矢（要给出它们在基$\{|1\rangle,|2\rangle\}$中的已归一化的展开式）。
   b. 计算在这些本征矢上的投影算符的矩阵，然后证明它们满足正交归一关系式和封闭性关系式。
   c. 同样是上面这些问题，但矩阵为三维空间的矩阵

$$L_y = \frac{\hbar}{2i}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}.$$

解：

a.

$$\sigma_y^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y,$$

即σ_y是厄米算符。

用特征方程求解算符的本征值和本征矢：

$$\sigma_y|\psi\rangle = \lambda|\psi\rangle,$$

$$\sum_m \langle n|\sigma_y|m\rangle\langle m|\psi\rangle = \lambda\langle n|\psi\rangle,$$

$$\sum_m \langle n|\sigma_y|m\rangle c_m = \lambda c_n, \qquad c_n = \langle n|\psi\rangle,$$

$$\mathrm{Det}(\sigma_y - \lambda I) = \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = 0,$$
$$\lambda^2 - 1 = 0, \qquad \lambda_\pm = \pm 1,$$
$$ic_{1\pm} - \lambda_\pm c_{2\pm} = 0,$$
$$|\psi_+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle),$$
$$|\psi_-\rangle = \frac{1}{\sqrt{2}}(|1\rangle - i|2\rangle).$$

b. 投影算符$P_{\psi_\pm} = |\psi_\pm\rangle\langle\psi_\pm|$的矩阵元为

$$\langle m|P_{\psi_\pm}|n\rangle = \langle m|\psi_\pm\rangle\langle\psi_\pm|n\rangle = c_{m\pm}c_{n\pm}^*,$$

矩阵表示为

$$P_{\psi_+} = |\psi_+\rangle\langle\psi_+| = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix},$$

$$P_{\psi_-} = |\psi_-\rangle\langle\psi_-| = \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix}.$$

$\{|\psi_+\rangle, |\psi_-\rangle\}$正交归一性：

$$\langle\psi_\alpha|\psi_\alpha\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\alpha\frac{i}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \alpha\frac{i}{\sqrt{2}} \end{pmatrix} = 1, \ \ \alpha = \pm,$$

$$\langle\psi_+|\psi_-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} = \langle\psi_-|\psi_+\rangle^* = 0,$$

$$\langle\psi_\alpha|\psi_\beta\rangle = \delta_{\alpha\beta}.$$

封闭性关系

$$P_{\{|\psi_+\rangle,|\psi_-\rangle\}} = \sum_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| = P_{\psi_+} + P_{\psi_-}$$

$$=\begin{pmatrix}\frac{1}{2} & -\frac{i}{2}\\ \frac{i}{2} & \frac{1}{2}\end{pmatrix}+\begin{pmatrix}\frac{1}{2} & \frac{i}{2}\\ -\frac{i}{2} & \frac{1}{2}\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=I.$$

即$\{|\psi_+\rangle,|\psi_-\rangle\}$满足正交归一关系式和封闭性关系式。

c. 对于$L_y=\frac{\hbar}{2i}\begin{pmatrix}0 & \sqrt{2} & 0\\ -\sqrt{2} & 0 & \sqrt{2}\\ 0 & -\sqrt{2} & 0\end{pmatrix}$,

$$L_y^\dagger=-\frac{\hbar}{2i}\begin{pmatrix}0 & -\sqrt{2} & 0\\ \sqrt{2} & 0 & -\sqrt{2}\\ 0 & \sqrt{2} & 0\end{pmatrix}=L_y,$$

求解本征值

$$\mathrm{Det}\left(L_y-\lambda I\right)=\frac{\hbar}{2i}\begin{vmatrix}-2i\lambda/\hbar & \sqrt{2} & 0\\ -\sqrt{2} & -2i\lambda/\hbar & \sqrt{2}\\ 0 & -\sqrt{2} & -2i\lambda/\hbar\end{vmatrix}=0,$$

$$-\frac{2i\lambda}{\hbar}\left(-\frac{4\lambda^2}{\hbar^2}+2\right)-\frac{4i\lambda}{\hbar}=\frac{2i\lambda}{\hbar}\left(\frac{4\lambda^2}{\hbar^2}-4\right)=0,$$

$$\lambda_1=\hbar,\qquad |\psi_1\rangle=\frac{1}{2}\left(|1\rangle+i\sqrt{2}|2\rangle-|3\rangle\right),$$

$$\lambda_2=0,\qquad |\psi_2\rangle=\frac{1}{\sqrt{2}}(|1\rangle+|3\rangle),$$

$$\lambda_3=-\hbar,\qquad |\psi_3\rangle=\frac{1}{2}\left(|1\rangle-i\sqrt{2}|2\rangle-|3\rangle\right),$$

投影算符为

$$P_{\psi_1}=|\psi_1\rangle\langle\psi_1|=\begin{pmatrix}\frac{1}{4} & -i\frac{\sqrt{2}}{4} & -\frac{1}{4}\\ i\frac{\sqrt{2}}{4} & \frac{1}{2} & -i\frac{\sqrt{2}}{4}\\ -\frac{1}{4} & i\frac{\sqrt{2}}{4} & \frac{1}{4}\end{pmatrix},$$

$$P_{\psi_2}=|\psi_2\rangle\langle\psi_2|=\begin{pmatrix}\frac{1}{2} & 0 & \frac{1}{2}\\ 0 & 0 & 0\\ \frac{1}{2} & 0 & \frac{1}{2}\end{pmatrix},$$

$$P_{\psi_3} = |\psi_3\rangle\langle\psi_3| = \begin{pmatrix} \frac{1}{4} & i\frac{\sqrt{2}}{4} & -\frac{1}{4} \\ -i\frac{\sqrt{2}}{4} & \frac{1}{2} & i\frac{\sqrt{2}}{4} \\ -\frac{1}{4} & -i\frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix},$$

证明$\{|\psi_i\rangle\}$满足正交归一关系式和封闭性关系式：

$$\langle\psi_i|\psi_i\rangle = \sum_n \langle\psi_i|n\rangle\langle n|\psi_i\rangle = \sum_n |\langle n|\psi_i\rangle|^2 = 1, \qquad i = 1,2,3$$

$$\langle\psi_1|\psi_2\rangle = \begin{pmatrix} \frac{1}{2} & -\frac{i}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \langle\psi_2|\psi_1\rangle^* = 0,$$

$$\langle\psi_3|\psi_2\rangle = \begin{pmatrix} \frac{1}{2} & \frac{i}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \langle\psi_2|\psi_3\rangle^* = 0,$$

$$\langle\psi_1|\psi_3\rangle = \begin{pmatrix} \frac{1}{2} & -\frac{i}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -\frac{i}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix} = \langle\psi_3|\psi_1\rangle^* = 0,$$

$$\langle\psi_i|\psi_j\rangle = \delta_{ij},$$

$$P_{\{|\psi_i\rangle\}} = \sum_i |\psi_i\rangle\langle\psi_i| = P_{\psi_1} + P_{\psi_2} + P_{\psi_3}$$

$$= \begin{pmatrix} \frac{1}{4} & -i\frac{\sqrt{2}}{4} & -\frac{1}{4} \\ i\frac{\sqrt{2}}{4} & \frac{1}{2} & -i\frac{\sqrt{2}}{4} \\ -\frac{1}{4} & i\frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{4} & i\frac{\sqrt{2}}{4} & -\frac{1}{4} \\ -i\frac{\sqrt{2}}{4} & \frac{1}{2} & i\frac{\sqrt{2}}{4} \\ -\frac{1}{4} & -i\frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix}$$

$$= I.$$

3．矩阵σ_x的定义为：

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

试证：

$$e^{i\alpha\sigma_x} = I\cos\alpha + i\sigma_x \sin\alpha,$$

其中I是2×2单位矩阵。

证明

$$\sigma_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I,$$

$$\sigma_x^{2n+1} = \sigma_x,$$

$$\sigma_x^{2n} = I,$$

$$\begin{aligned} e^{i\alpha\sigma_x} &= \sum_n \frac{i^n}{n!}\alpha^n \sigma_x^n \\ &= \sum_n \frac{i^{2n}}{(2n)!}\alpha^{2n}\sigma_x^{2n} + \sum_n \frac{i^{2n+1}}{(2n+1)!}\alpha^{2n+1}\sigma_x^{2n+1} \\ &= \sum_n \frac{i^{2n}}{(2n)!}\alpha^{2n} I + \sum_n \frac{i^{2n+1}}{(2n+1)!}\alpha^{2n+1}\sigma_x \\ &= I\cos\alpha + i\sigma_x \sin\alpha. \end{aligned}$$