Verified Analysis of Random Binary Tree Structures

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Isabelle

Contributions

- Quicksort

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- Random Binary Search Trees

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- Treaps

Discrete distributions in Isabelle/HOL

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- Isomorphic to the set of functions $f : \alpha \to \mathbb{R}$ with $f(x) \geq 0$ and $\sum_{x::\alpha} f(x) = 1$
- Giry monad allows composing PMFs: $\mathbf{do}\ \{x \leftarrow A;\ y \leftarrow B\ x;\ \mathbf{return}\ (f\ x\ y)\}$

Quicksort

Deterministic quicksort with first element as pivot:

$$\mathsf{qs} :: \alpha\ \mathsf{list} \to \alpha\ \mathsf{list}$$

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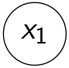
- $E[\mathsf{snd}(\mathsf{rqs\ xs})] = 2(n+1)H_n - 4n \sim 2n \ln n$
- $\mathsf{avqs} = \mathsf{rqs}$

Random Binary Search Trees

What happens when we insert distinct elements $x_1, \ldots, x_n$ into an empty BST?



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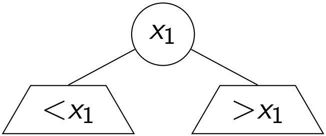

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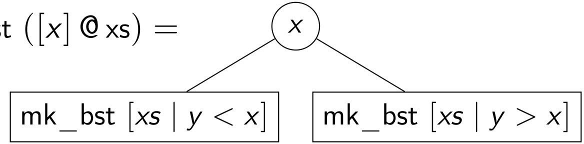

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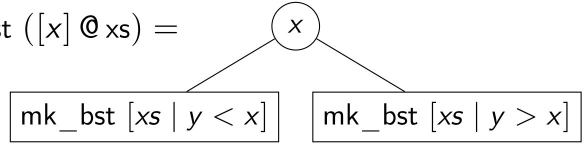

Let us now add elements from a set A in random order:

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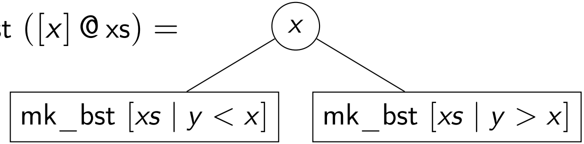

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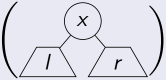

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$\Longrightarrow$ Hence average access time is $\sim 2\ln n$.

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The actual behaviour is $\approx 2.988 \log_2 n$ [Reed 2003].

Treaps

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A Nice Solution: Treaps [Aragon & Seidel 1989]

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Randomised Treap

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$\text{rinss}\ []\ t = \textbf{return}\ t$
$\text{rinss}\ ([x]\ @\ xs)\ t = \textbf{do}\ \{t' \leftarrow \text{rins}\ x\ t;\ \text{rinss}\ xs\ t'\}$

Proof of the main result on treaps:

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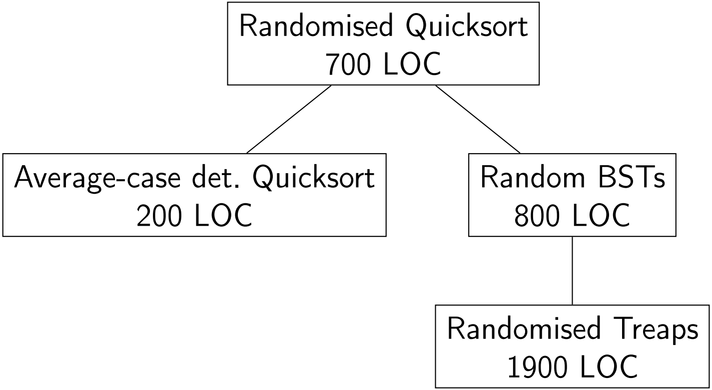
- The 'Node' constructor is a measurable function from $\mathcal{T}(M) \otimes M \otimes \mathcal{T}(M)$ to $\mathcal{T}(M)$
- Other tree operations (projections, primitive recursion) are similarly measurable


Randomised Quicksort
700 LOC
Average-case det. Quicksort
200 LOC
Random BSTs
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1900 LOC

Bonus: Executability

value random_bst $\{1, 2, 3 :: \mathrm{int}\}$

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pmf_of_alist

$$[(\langle\langle\langle\langle\rangle, 1, \langle\rangle\rangle, 2, \langle\rangle\rangle, 3, \langle\rangle\rangle, 1/6),$$
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int tree pmf

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\begin{array}{l}
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\qquad\ (\langle\langle\langle\rangle, 1, \langle\langle\rangle, 2, \langle\rangle\rangle\rangle, 3, \langle\rangle\rangle, 1/6), \\
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\qquad\ (\langle\langle\langle\rangle, 1, \langle\rangle\rangle, 2, \langle\langle\rangle, 3, \langle\rangle\rangle\rangle, 1/3), \\
\qquad\ (\langle\langle\rangle, 1, \langle\langle\rangle, 2, \langle\langle\rangle, 3, \langle\rangle\rangle\rangle\rangle, 1/6)] \\
\qquad\qquad \text{int tree pmf}
\end{array}
$$

value measure_pmf.expectation $(\text{random\_bst } \{1..6 :: \text{int}\})$ height

Bonus: Executability

$$
\begin{array}{l}
\textbf{value}\ \text{random\_bst}\ \{1, 2, 3 :: \text{int}\} \\
\quad \text{pmf\_of\_alist} \\
\qquad [(\langle\langle\langle\langle\rangle, 1, \langle\rangle\rangle, 2, \langle\rangle\rangle, 3, \langle\rangle\rangle, 1/6), \\
\qquad\ (\langle\langle\langle\rangle, 1, \langle\langle\rangle, 2, \langle\rangle\rangle\rangle, 3, \langle\rangle\rangle, 1/6), \\
\qquad\ (\langle\langle\rangle, 1, \langle\langle\langle\rangle, 2, \langle\rangle\rangle, 3, \langle\rangle\rangle\rangle, 1/6), \\
\qquad\ (\langle\langle\langle\rangle, 1, \langle\rangle\rangle, 2, \langle\langle\rangle, 3, \langle\rangle\rangle\rangle, 1/3), \\
\qquad\ (\langle\langle\rangle, 1, \langle\langle\rangle, 2, \langle\langle\rangle, 3, \langle\rangle\rangle\rangle\rangle, 1/6)] \\
\qquad\quad \text{int tree pmf}
\end{array}
$$

$$
\begin{array}{l}
\textbf{value}\ \text{measure\_pmf.expectation}\ (\text{random\_bst}\ \{1..6 :: \text{int}\})\ \text{height} \\
\quad 65\ /\ 15 :: \text{real}
\end{array}
$$

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- Skip Lists (already done, [Haslbeck & E. 2018])