SSY281 Model Predictive Control

Assignment 5

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Question 1

As stated in the lecture notes (page 103) the function $V(x(k)) = x(k)^T S x(k)$ may be a lynapunov function if it fulfulls the criteria for a lynaponuv function and thus proving stability. The solution to finding S is done by solving the riccati equation.

$$V(x^+) - V(x) = (Ax)^T S(Ax) - x^T Sx = x^T(A^T SA - S) = -x^T Qx$$

Choosing Q to identity matrix and solving in matlab with both *idare()* and *dlyap()* to be:

$$S = \begin{bmatrix} 1.2895 & 0.5921 \\ 0.5921 & 2.6316 \end{bmatrix} \tag{1}$$

Question 2

To prove that it is a Lyapunov function we plug Q, R and the solution to the LQ problem into the equation and show that $V(f(x)) - V(x) = -\alpha_3(x)$ where α_3 is positive definite, that is the "energy" is decaying along the solution.

$$V(u, x_0) = \sum_{i=0}^{\infty} \left(x(i)^T Q x(i) + u(i)^T R u(i)\right) = \left\{ Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ R = 1 \right\} = \sum_{i=0}^{\infty} \left(x(i)^T x(i) + u(i)^T u(i)\right)$$

$$V(x_0) = \left\{ \begin{matrix} \text{express as solution} \\ \text{to the LQ problem} \end{matrix} \right\} = \sum_{i=0}^{\infty} \left(x(i)^T x(i) + (-Kx(i))^T(-Kx(i))\right) = \sum_{i=0}^{\infty} \left(x(i)^T x(i) + x(i)^T K^T K x(i)\right)$$

For $V(x_0)$ to be lyapunov then $\underbrace{V(f(x))}_{V(x^+)} - V(x) = -\alpha_3(x)$ must be satisfied where α_3 pos def.

$x^+ = x_{k+1}$

$x = x_k$

Hence: $V(x^+) - V(x) = \sum_{i=k+1}^{\infty} \left(x(i)^T x(i) + x(i)^T K^T K x(i)\right) - \sum_{i=k}^{\infty} \left(x(i)^T x(i) + x(i)^T K^T K x(i)\right) =$

$= -x(k)^T x(k) - x(k)^T K^T K x(k) =$

$= -\underbrace{\left(x_0^T x_0 + x_0^T K^T K x_0\right)}_{\text{pos. def.}} \Rightarrow$ It is a lyapunov function!

Figure 1: Proof of V(u,x0) being a Lyapunov function

Question 3

(a)

The shortest N that stabalizes the system was found to be $N = 5$ using the developed RH controller in assignment 2.

(b)

If we look at how K is formed when $N = 1$ we can see that Q has no effect on K and hence doesn't affect stability.

$$P(N) = P(1) = P_f$$
$$K(0) = (R + B^T P_f B)^{-1} B^T P_f A$$

(c)

As stated in the lecture notes (page 107), if we choose the terminal cost to be the value function of the unconstrained LQ problem $V_f(x) = x^T P x$ where P is the solution to the riccati equation we can find a stabilizing solution, hence we just solve the discrete algebraic ricatti equations to obtain the unique stabilizing solution P. Since P doesn't change it doesn't matter what N we choose. The solution was found with the *idare()* command in matlab and P_f is shown below.

$$P_f = \begin{bmatrix} 13.6586 & 7.9930 \\ 7.9930 & 39.4371 \end{bmatrix} \tag{2}$$

(c)

Another R which stabilizes the system was found to be $R = 0.1337$ when $P_f = Q$.

Question 4

As described in question (2b), Q has no impact when $N = 1$ and the unconstrained LQ controller gains are calculated using:

$$K = (R + B^T P_f B)^{-1} B^T P_f A \tag{3}$$

If we plug in the given A, B, P_f we can see that:

$$B^T P_f B = \begin{bmatrix} 0 & 0 & ... & 1 \end{bmatrix} P_f \begin{bmatrix} 0 & 0 & ... & 1 \end{bmatrix}^T = p_n$$
$$B^T P_f A = \begin{bmatrix} 0 & 0 & ... & 1 \end{bmatrix} P_f A = \begin{bmatrix} 0 & 0 & ... & p_n \end{bmatrix} A = 0_{1\times n}$$
$$K = 0_{1\times n}$$

If $K = 0_{1\times n}$ it won't affect the system and since the original system as unstable it will remain unstable. Hence R and Q cannot affect the stability if the system.

Question 5

(a)

The controller gain when $N = 1$ is calculated by $K = -(R+B^T P_f B)^{-1} B^T P_f A$ and the eigenvalues of the closed loop system becomes the solution to $|A+BK-\lambda I| = 0$. The eigenvalues was found to be:

$$\lambda_{1,2} = \frac{6\,R + 1 \pm \sqrt{12\,R^2 - 12\,R + 1}}{2\,(R+1)} \tag{4}$$

For the system to be stable the poles have to remain inside the unit circle and hence:

$$|\lambda_{1,2}| = \left| \frac{6\,R + 1 \pm \sqrt{12\,R^2 - 12\,R + 1}}{2\,(R+1)} \right| < 1 \tag{5}$$

The solution can be found when splitting the solution into two parts, one where the eigenvalues are real and another when they are complex conjugate pairs. The derivation for finding the solutions to the complex conjugate is shown in fig 2 and both solutions were calculated using using *wolfram alpha* and the result is shown below.

$$\text{Real solution: } \begin{cases} 0 < R < \dfrac{3-\sqrt{6}}{6} \end{cases}$$
$$\text{Complex conj. sol.: } \begin{cases} 0 < R < 0.2 \end{cases}$$
$$\text{Complete solution: } \mathrm{Re}_{sol} \cup \ \mathrm{Im}_{\ sol} \Rightarrow R \in (0, 0.2)$$

$$K(k) = \left(R + B^T P(k+1) B\right)^{-1} B^T P(k+1) A$$

$$\lambda = \mathrm{eig}(A + Bk)$$ (+ because how we did in prev. assignment, that is we return $-k$)

$$\Rightarrow \det\left(A + B\left(R + B^T \underbrace{P(k+1)}_{P_f} B\right)^{-1} B^T \underbrace{P(k+1)}_{P_f} A - \lambda I\right) = 0$$

$$\Rightarrow \frac{6R - \lambda - 6R\lambda + R\lambda^2 + \lambda^2}{R+1} = 0$$

Solve matlab $\Rightarrow$ $$|\lambda_{1,2}| = \left|\frac{6R + 1 \pm \sqrt{12R^2 - 12R + 1}}{2(R+1)}\right| \underbrace{<}_{\text{stable}} 1$$

assume eigenvalues to be partly complex $\Rightarrow$ $$|\lambda_{1,2}| = \left|\underbrace{\frac{6R+1}{2(R+1)}}_{a} \pm i \underbrace{\frac{\sqrt{-(12R^2 - 12R + 1)}}{2(R+1)}}_{b}\right| < 1$$

$$\sqrt{12R^2 - 12R + 1} = \sqrt{-1(-12R^2 + 12R - 1)} = i\sqrt{-(12R^2 - 12R + 1)}$$

complex conjugate pairs $\Rightarrow \lambda_2 = \lambda_1^* \Rightarrow \underbrace{\lambda_1}_{<1} \cdot \underbrace{\lambda_2}_{<1} < 1$

$$\Rightarrow \lambda_1 \cdot \lambda_2 = (a + jb)(a - jb) = a^2 + b^2 = \frac{(6R+1)^2}{(2R+2)^2} + \frac{-(12R^2 - 12R + 1)}{(2R+2)^2} < 1$$

$$\Leftrightarrow 36R^2 + 12R \cancel{+1} - 12R^2 + 12R \cancel{-1} < 4R^2 + 8R + 4$$

$$\Leftrightarrow 20R^2 + 16R - 4 < 0 \Leftrightarrow R^2 + \frac{16}{20}R - \frac{4}{20} < 0$$

$$\Leftrightarrow R^2 + \frac{4}{5}R - \frac{1}{5} < 0 \quad PQ \Rightarrow R = \begin{cases} 0{,}2 \\ \underbrace{-1{,}8}_{R > 0} \end{cases} \Rightarrow 0 < R < 0{,}2$$

Figure 2: Complex conjugate solution

(b)

We can find the solution by doing as we did in (5a) but the controller gains are acquired a little bit different since $N = 2$.

$$P(1) = Q + A^T P_f A - A^T P_f B(R + B^T P_f B)^{-1} B^T P_f A$$
$$K(0) = -(R + B^T P(1)B)^{-1} B^T P(1)A$$

The same procedure of splitting the eigenvalues into two different cases, real and complex conjugate pairs and using wolfram alpha the bounds on R was once again found:

$$\text{Real solution: } \begin{cases} 0 < R < 0.38 \\ 2.62 < R \leq 3.12 \end{cases}$$
$$\text{Complex conj. sol.: } \begin{cases} 3.12 < R < 4.49 \end{cases}$$
$$\text{Complete solution: } \mathrm{Re}_{sol} \cup \ \mathrm{Im}_{\ sol} \Rightarrow R \in (0, 0.38) \cup (2.62, 4.49)$$

Question 6

If we design the closed loop system with an infinite-time LQ controller and introduce $V(x) = x^T P x$ to be a possible lyapunov function and then check the conditions for it beeing a lyapunov function we can prove stability. The equations are shown below and the full derivation is shown in fig 3

$$P = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA$$
$$K = -(R + B^T PB)^{-1} B^T PA$$
$$A_{cl} = A + BK$$
$$V(x) = x^T Px$$
$$V(x^+) - V(x) = (A_{cl}x)^T P(A_{cl}x) - x^T Px = ... = -x^T (KRK + Q)x$$

Since $V(x) = x^T Sx$ is non negative (quadratic), continous, zero at zero, strictly increasing and unbounded if fulfills the first two criterion. The third criteria is proved by $V(x^+) = -x^T (KRK + Q)x$ since KRX is quadratic and both R, Q are positive definite $KRK + Q \geq 0$. Thus the criteria for a lyapunov function is fulfilled and the closed loop system is stable.

Q6

$$P = Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A$$

1 $$k = -(R + B^T P B)^{-1} B^T P A$$

$$A_{cl} = A + Bk = A - B(R + B^T P B)^{-1} B^T P A$$

To check stability we introduce $V(x) = x^T P x$ as the lyapunov func and check conditions

$$V(x^+) - V(x) = (A_{cl} x)^T P (A_{cl} x) - x^T P x = x^T (A_{cl}^T P A_{cl} - P) x =$$

$$= x^T (A_{cl}^T P A_{cl} - (Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A)) x =$$

$$= x^T (A_{cl}^T P A_{cl} - (Q + A^T P A + A^T P B k)) x =$$

$$= x^T ((A + Bk)^T P (A + Bk) - (Q + A^T P A + A^T P B k)) x = \dots =$$

$$= x^T (k^T (B^T P A + B^T P B k) - Q) x = \left\{ B^T P A \overset{1}{=} -(R + B^T P B) k \right\} =$$

$$= x^T (k^T (-(R + B^T P B) k + B^T P B k) - Q) x =$$

$$= x^T (-k^T R k - Q) x$$

$$= -x^T (\underbrace{kRk}_{\geq 0} + \underbrace{Q}_{\geq 0}) x$$

Since kRx is quadratic and R & Q pos. def

$kRk + Q \geq 0$.

Thus the criteria is fulfilled and the closed loop system is stable

Figure 3: Proof of V(x) beeing a lynapunov function