Optimization Methods **Fall 2019**

Homework 1

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Notice

- The submission email is: **njuoptfall2019@163.com**.
- Please use the provided LaTeX file as a template. If you are not familiar with LaTeX, you can also use Word to generate a **PDF** file.

Problem 1: Norms

A function $f : \mathbb{R}^n \leftarrow \mathbb{R}$ with $\mathbf{dom} f = \mathbb{R}^n$ is called a *norm* if

- f is nonnegative: $f(x) \geq 0$ for all $x \in \mathbb{R}^n$
- f is definite: $f(x) = 0$ only if $x = 0$
- f is homogeneous: $f(tx) = |t|f(x)$, for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$
- f satisfies the triangle inequality: $f(x + y) \leq f(x) + f(y)$, for all $x, y \in v$

We use the notation $f(x) = \|x\|$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. The associated dual norm, denoted $\|\cdot\|_*$, is defined as

$$\|z\|_* = \sup\{z^T x | \|x\| \leq 1\}$$

a) Prove that $\|\cdot\|_*$ is a valid norm.

b) Prove that the dual of the Euclidean norm (ℓ_2-norm) is the Euclidean norm, *i.e.*, prove that

$$\|z\|_{2*} = \sup\{z^T x | \|x\|_2 \leq 1\} = \|z\|_2$$

.

(*Hint:* Use CauchySchwarz inequality.)

Solution.

a) $\because \|z\|_* = \sup\{z^T x | \|x\| \leq 1\} = \sup\{|z^T x| | \|x\| \leq 1\}\}$

$\because |z^T x| \geq 0$

$\therefore \|z\|_* \geq |z^T x| \geq 0$

$\because \forall x \in \mathbb{R}^n, \text{if } z = 0,\ \text{then } z^T x = 0$

$\therefore \|z\|_* = 0$ only if $z = 0$

$\because \|tz\|_* = \sup\{tz^T x | \|x\| \leq 1\} = \sup\{|t||z^T x| | \|x\| \leq 1\} = |t| \sup\{z^T x | \|x\| \leq 1\} = |t|\|z\|_*$

$\therefore \|tz\|_* = |t|\|z\|_*$

$\because \|z_1 + z_2\|_* = \sup\{z_1^T x + z_2^T x | \|x\| \leq 1\} \leq \sup\{z_1^T x | \|x\| \leq 1\} + \sup\{z_2^T x | \|x\| \leq 1\} = \|z_1\|_* + \|z_2\|_*$

$\therefore \|z_1 + z_2\|_* \leq \|z_1\|_* + \|z_2\|_*$

$\therefore \|\cdot\|$ is a valid norm.

b) $\because$ From Cauchy-Schwarz inequality, we have $|z^T x| \leq \sqrt{\sum_{i=1}^{n}(z^T)^2 * \sum_{i=1}^{n}(x)^2} = \|z\|_{2*}\|x\|_{2*}$

$\because$ For nonzero z, the value of x that maximize $z^T x$ over $\|x\|_2 \leq 1$ is $\frac{z}{\|z\|_2}$

$\therefore$ Let $x = \frac{z}{\|z\|_2}$, We get $\sup\{|z^T x| | \|x\|_{2*} \leq 1\} = \|z\|_2 \|\frac{z}{\|z\|_2}\|_2 = \|z\|_2$

$\therefore \|z\|_{2*} = \|z\|_2$

$\square$

Problem 2: Affine and Convex Sets

Affine sets C_a and convex C_c sets are the sets satisfying the constraints below:

$$\begin{aligned} \theta x_1 + (1-\theta)x_2 \in C_a \\ \text{s.t. } x_1, x_2 \in C_a \end{aligned} \tag{1}$$

$$\begin{aligned} \theta x_1 + (1-\theta)x_2 \in C_c \\ \text{s.t. } x_1, x_2 \in C_c, 0 \geq \theta \leq 1 \end{aligned} \tag{2}$$

a) Is the set $\{\alpha \in \mathbb{R}^k | p(0) = 1, |p(t)| \leq 1 \text{ for } \alpha \leq t \leq \beta\}$, where $p(t) = \alpha_1 + \alpha_2 t + \cdots + \alpha_k t^{k-1}$, affine?

b) Determine if each set below is convex.

1) $\{(x, y) \in \mathbf{R}^2_{++} | x/y \leq 1\}$.
2) $\{(x, y) \in \mathbf{R}^2_{++} | x/y \geq 1\}$.
3) $\{(x, y) \in \mathbf{R}^2_{+} | xy \leq 1\}$.
4) $\{(x, y) \in \mathbf{R}^2_{+} | xy \geq 1\}$.
5) $\{(x, y) \in \mathbf{R}^2 | y = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}\}$.

Solution.

a) Assume $C = \{\alpha \in \mathbb{R}^k | p(0) = 1, |p(t)| \leq 1 \text{ for } \alpha \leq t \leq \beta\}$, where $p(t) = \alpha_1 + \alpha_2 t + \cdots + \alpha_k t^{k-1}$

Let $x1,\ x2 \in C$. For $\forall \theta$, we have:

$$\begin{aligned}
&|[\theta x_1 + (1-\theta)x_2] + [\theta x_1 + (1-\theta)x_2]t + \cdots + [\theta x_1 + (1-\theta)x_2]t^{k-1}| \\
&= |(\theta x_1 + \theta x_1 t + \cdots + \theta x_1 t^{k-1}) + [(1-\theta)x_2 + (1-\theta)x_2 t + \cdots + (1-\theta)x_2 t^{k-1}]| \\
&\leq |(\theta x_1 + \theta x_1 t + \cdots + \theta x_1 t^{k-1})| + |[(1-\theta)x_2 + (1-\theta)x_2 t + \cdots + (1-\theta)x_2 t^{k-1}]| \\
&= |\theta + (1-\theta)| \\
&= 1 \\
&\therefore \forall(\theta x_1 + (1-\theta)x_2) \in C \\
&\therefore \text{Set } \{\alpha \in \mathbb{R}^k | p(0) = 1, |p(t)| \leq 1 \text{ for } \alpha \leq t \leq \beta\},\ \text{where } p(t) = \alpha_1 + \alpha_2 t + \cdots + \alpha_k t^{k-1}, \text{ is affine.}
\end{aligned}$$

b)

1) Assume $(x_1, y_1), (x_2, y_2) \in \{(x, y) \in \mathbf{R}^2_{++} | x/y \leq 1\}$

$$\begin{aligned}
&\because \frac{x}{y} \leq 1 \\
&\therefore x \leq y \\
&\therefore \text{For } \forall \theta \text{ that satisfies } 0 \leq \theta \leq 1,\ \text{we have:} \\
&\quad \theta(x_1, y_1) + (1-\theta)(x_2, y_2) \\
&= (\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2) \\
&= \frac{\theta x_1 + (1-\theta)x_2}{\theta y_1 + (1-\theta)y_2} \\
&\leq \frac{\theta y_1 + (1-\theta)y_2}{\theta y_1 + (1-\theta)y_2} = 1 \\
&\therefore \{(x, y) \in \mathbf{R}^2_{++} | x/y \leq 1\} \text{ is convex.}
\end{aligned}$$

2) In a similar way with 1), Assume $(x_1, y_1), (x_2, y_2) \in \{(x, y) \in \mathbf{R}^2_{++} | x/y \geq 1\}$

$$\begin{aligned}
&\because \text{For } \forall \theta \text{ that satisfies } 0 \leq \theta \leq 1,\ \text{we have:} \\
&\quad \theta(x_1, y_1) + (1-\theta)(x_2, y_2) = \frac{\theta x_1 + (1-\theta)x_2}{\theta y_1 + (1-\theta)y_2} \geq \frac{\theta y_1 + (1-\theta)y_2}{\theta y_1 + (1-\theta)y_2} = 1 \\
&\therefore \{(x, y) \in \mathbf{R}^2_{++} | x/y \geq 1\} \text{ is convex.}
\end{aligned}$$

3) Let $(2,\frac{1}{2}),(\frac{1}{2},2) \in \{(x,y) \in \mathbf{R}_+^2 | xy \le 1\}$

$\because$ For $\forall\theta$ that satisfies $0 \le \theta \le 1$, assume $(x,y) = \theta(2,\frac{1}{2}) + (1-\theta)(\frac{1}{2},2) = (\frac{3}{2}\theta + \frac{1}{2}, -\frac{3}{2}\theta + 2)$

$\therefore (\frac{3}{2}\theta + \frac{1}{2})(-\frac{3}{2}\theta + 2) = -\frac{9}{4}\theta^2 + \frac{9}{4}\theta + 1 = -\frac{9}{4}(\theta - \frac{1}{2})^2 + \frac{25}{16}$

$\therefore \exists\theta = \frac{1}{2} \in [0,1], xy = \frac{25}{16} \notin \{(x,y) \in \mathbf{R}_+^2 | xy \le 1\}$

$\therefore \{(x,y) \in \mathbf{R}_+^2 | xy \le 1\}$ isn't convex.

4) Let $(1,1),(-1,-1) \in \{(x,y) \in \mathbf{R}_+^2 | xy \ge 1\}$

$\because (For)\forall\theta$ that satisfies $0 \le \theta \le 1$, assume $(x,y) = \theta(1,1) + (1-\theta)(-1,-1) = (2\theta - 1, 2\theta - 1)$

$\therefore$ Let $\theta = \frac{1}{2} \in [0,1]$, we have $xy = (2 \times \frac{1}{2} - 1)(2 \times \frac{1}{2} - 1) = 0 \notin \{(x,y) \in \mathbf{R}_+^2 | xy \ge 1\}$

$\therefore \{(x,y) \in \mathbf{R}_+^2 | xy \ge 1\}$ isn't convex.

5) Assume $(x_1,y_1),(x_2,y_2) \in \{(x,y) \in \mathbf{R}^2 | y = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}\}$

$$
\begin{aligned}
\therefore \quad & \theta y_1 + (1-\theta)y_2 \\
&= \theta\frac{e^{x_1} - e^{-x_1}}{e^{x_1} + e^{-x_1}} + (1-\theta)\frac{e^{x_2} - e^{-x_2}}{e^{x_2} + e^{-x_2}} \\
&= \theta\frac{e^{x_1}(1-\frac{1}{e})}{e^{x_1}(1+\frac{1}{e})} + (1-\theta)\frac{e^{x_2}(1-\frac{1}{e})}{e^{x_2}(1+\frac{1}{e})} \\
&= (\theta + 1 - \theta)\frac{1-\frac{1}{e}}{1+\frac{1}{e}} \\
&= \frac{1-\frac{1}{e}}{1+\frac{1}{e}}
\end{aligned}
$$

$$
\begin{aligned}
\because \quad & \tanh(\theta x_1 + (1-\theta)x_2) \\
&= \frac{e^{\theta x_1 + (1-\theta)x_2} - e^{-(\theta x_1 + (1-\theta)x_2)}}{e^{\theta x_1 + (1-\theta)x_2} + e^{-(\theta x_1 + (1-\theta)x_2)}} \\
&= \frac{e^{\theta x_1 + (1-\theta)x_2}(1-\frac{1}{e})}{e^{\theta x_1 + (1-\theta)x_2}(1+\frac{1}{e})} \\
&= \frac{1-\frac{1}{e}}{1+\frac{1}{e}}
\end{aligned}
$$

$\therefore$ For $\forall\theta$ that satisfies $0 \le \theta \le 1, \theta y_1 + (1-\theta)y_2 = \tanh(\theta x_1 + (1-\theta)x_2)$

$\therefore \{(x,y) \in \mathbf{R}^2 | y = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}\}$ is convex.

□

Problem 3: Examples

a) Let $C \subseteq \mathbb{R}^n$ be the solution set of a quadratic inequality,

$$C = \{x \in \mathbb{R}^n | x^\top Ax + b^\top x + c \le 0\}, \tag{3}$$

with $A \in \mathbb{S}^n, b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

1) Show that C is convex if $A \succeq 0$.
2) Is the following statement true? The intersection of C and the hyperplane defined by $g^\top x + h = 0$ is convex if $A + \lambda gg^\top \succeq 0$ for some $\lambda \in \mathbb{R}$.

b) The polar of $C \subseteq \mathbb{R}^n$ is defined as the set

$$C^\circ = \{y \in \mathbb{R}^n | y^\top x \le 1 \text{ for all } x \in C\}$$

.

1) Show that C° is affine.
2) What is a polar of a polyhedra?
3) What is the polar of the unit ball for a norm $||\cdot||$?
4) Show that if C is closed and convex, with $0 \in C$, then $(C^\circ)^\circ = C$

Solution.

$a)$ 1) Assume $x_1, x_2 \in C, \theta \in [0,1]$

$\because \theta x_1^T A x_1 + \theta b^T x_1 + \theta c \leq 0,\ (1-\theta)x_2^T A x_2 + (1-\theta)b^T x_2 + (1-\theta)c \leq 0$

$\therefore \quad (\theta x_1 + (1-\theta)x_2)^T A(\theta x_1 + (1-\theta)x_2) + b^T(\theta x_1 + (1-\theta)x_2) + c$

$= \theta^2 x_1^T A x_1 + \theta(1-\theta)x_1^T A x_2 + \theta(1-\theta)x_2^T A x_1 + (1-\theta)^2 x_2^T A x_2 + \theta b^T x_1 + (1-\theta)b^T x_2 + \theta c + (1-\theta)c$

$\leq \theta^2 x_1^T A x_1 + \theta(1-\theta)x_1^T A x_2 + \theta(1-\theta)x_2^T A x_1 + (1-\theta)^2 x_2^T A x_2 - \theta x_1^T A x_1 - (1-\theta)x_2^T A x_2$

$= -\theta(1-\theta)x_1^T A x_1 + \theta(1-\theta)x_1^T A x_2 + \theta(1-\theta)x_2^T A x_1 - \theta(1-\theta)x_2^T A x_2$

$= \theta(1-\theta)(x_1^T A x_2 + x_2^T A x_1 - x_1^T A x_1 - x_2^T A x_2)$

$= -\theta(1-\theta)(x_1^T - x_2^T)A(x_1 - x_2)$

$\because A \succeq 0$

$\therefore (x_1^T - x_2^T)A(x_1 - x_2) \geq 0$

$\because \theta \in [0,1]$

$\therefore (\theta x_1 + (1-\theta)x_2)^T A(\theta x_1 + (1-\theta)x_2) + b^T(\theta x_1 + (1-\theta)x_2) + c \leq 0$

$\therefore C$ is convex if $A \succeq 0$

2) (Red color means I have referred to solutions here.)

Assume $x_1, x_2 \in$ C, $\theta \in$ [0,1]

$\because$ A set is convex if and only if its intersection with an arbitrary line $\{\hat{x} + tv | t \in R\}$ is convex.

$\therefore$ Let $H = \{x | g^T x + h = 0\}$, We have

$$(\hat{x} + tv)^T A(\hat{x} + tv) + b^T(\hat{x} + tv) + c = \alpha t^2 + \beta t + \gamma$$

where $\alpha = v^T A v, \beta = b^T v + 2\hat{x}^T A v, \gamma = c + b^T \hat{x} + \hat{x}^T A \hat{x}$

In addition, define $\delta = g^T v, \epsilon = g^T \hat{x} + h$

$\because$ Without loss of generality we can assume that $\hat{x} \in H, i.e., \epsilon = 0$. The intersection of $C \cap H$ with the line defined by $\hat{x}$ and v is

$$\{\hat{x} + tv | \alpha t^2 + \beta t + \gamma \leq 0, \delta t = 0\}.$$

$\therefore$ If $\delta = g^T v \neq 0$, the intersection is the singleton $\{\hat{x}\}$, if $\gamma \leq 0$ or it is empty. In either case it is a convex set. If $\delta = g^T v = 0$, the set reduces to

$$\{\hat{x} + tv | \alpha t^2 + \beta t + \gamma \leq 0\}$$

which is convex if $\alpha \geq 0$. Therefore $C \cap H$ is convex if

$$g^T v = 0 \Rightarrow v^T A v \geq 0.$$

$\therefore$ This is true if there exists λ such that $A + \lambda g g^T \succeq 0$; then $g^T v = 0 \Rightarrow v^T A v \geq 0$ holds. because then

$$v^T A v = v^T A v + \lambda(v^T g)(g^T v) = v^T(A + \lambda g g^T)v \geq 0$$

. for all v satisfying $g^T v = 0$

$\therefore$ The intersection of C and the hyperplane defined by $g^\top x + h = 0$ is convex if $A + \lambda g g^\top \succeq 0$ for some $\lambda \in \mathbb{R}$ is true.

b) 1) Assume $y1, y2 \in C^\circ$, we have

$$(\theta y_1 + (1-\theta)y_2)^T x = \theta y_1^T x + (1-\theta) y_2^T x \le \theta + (1-\theta) = 1$$

for all $\theta \in [0, 1]$.

$\therefore \theta y_1 + (1-\theta) y_2 \in C^\circ$

$\therefore C^\circ$ is convex.

□

Problem 4: Operations That Preserve Convexity

Suppose $\phi : \mathbb{R}^n \to \mathbb{R}^m$ and $\psi : \mathbb{R}^m \to \mathbb{R}^p$ are the linear-fractional functions

$$\phi(x) = \frac{Ax+b}{c^\top x + d}, \psi(x) = \frac{Ey+f}{g^\top y + h}, \tag{4}$$

with domains **dom** $\phi = \{x | c^\top x + d > 0\}$, **dom** $\psi = \{y | g^\top x + h > 0\}$. We associate with ϕ and ψ the matrices

$$\begin{bmatrix} A & b \\ c^\top & d \end{bmatrix}, \begin{bmatrix} A & b \\ g^\top & h \end{bmatrix}, \tag{5}$$

respectively.

Now, consider the composition Γ of ϕ and ψ, *i.e.*, $\Gamma(x) = \psi(\phi(x))$, with domain

$$\mathbf{dom}\Gamma = \{x \in \mathbf{dom}\ \phi | \phi(x) \in \mathbf{dom}\ \psi\}. \tag{6}$$

Show that Γ is linear-fractional, and that the matrix associate with it is the product

$$\begin{bmatrix} A & b \\ c^\top & d \end{bmatrix} \begin{bmatrix} A & b \\ g^\top & h \end{bmatrix}. \tag{7}$$

Solution. Write your solution here.

Problem 5: Generalized Inequalities

Let K^* be the dual cone of a convex cone K. Prove the following

1) K^* is indeed a convex cone.

2) $K_1 \subseteq K_2$ implies $K_1^* \subseteq K_2^*$.

3) K^{**} is the closure of K.

Solution. Write your solution here.